The integral form of $U_q\mathfrak{g}$

$$\mathbb{A} = \mathbb{Z}[q, q^{-1}], \quad q_i = q^{d_i} \text{ where } d_i = \frac{2}{\langle \alpha_i, \alpha_i \rangle}.$$

$U_{\mathbb{A}} = U_{\mathbb{A}}^{res}\mathfrak{g}$ is the $\mathbb{A}$-subalgebra of $U_q\mathfrak{g}$ generated by

$$(X_i^+)^{(r)}, \ (X_i^-)^{(r)} \text{ and } K_i^{\pm 1} \text{ for } 1 \le i \le n, \ r \in \mathbb{Z}_{>0}.$$

Here

$$(X_i^{\pm})^{(r)} = \frac{(X_i^{\pm})^r}{[r]!} \quad \text{where } [r] = \frac{q^r - q^{-r}}{q - q^{-1}} = q^{r-1} + q^{r-3} + \cdots + q^{-(r-3)} + q^{-(r-1)}$$

and $[r]! = [r][r-1]\cdots[3][2][1]$.

It is important to note that

$$(X_i^+)^{(r)}(X_i^-)^{(s)} = \sum_{r,s \ge t \ge 0} (X_i^-)^{(s-t)} \begin{bmatrix} K_i ; 2t-s-r \\ t \end{bmatrix}_{q_i} (X_i^+)^{(r-t)}$$

where

$$\begin{bmatrix} K_i ; c \\ r \end{bmatrix}_{q_i} = \left(\frac{K_i q_i^c - K_i^{-1} q_i^{-c}}{q_i - q_i^{-1}}\right)\left(\frac{K_i q_i^{c-1} - K_i^{-1} q_i^{-(c-1)}}{q_i^2 - q_i^{-2}}\right)\cdots\left(\frac{K_i q_i^{c-(r-1)} - K_i^{-1} q_i^{-(c-(r-1))}}{q_i^{r-1} - q_i^{-(r-1)}}\right)$$

Then $U_{\mathbb{A}}^0$ is generated by $K_i^{\pm 1}$ and $\begin{bmatrix} K_i ; 0 \\ t \end{bmatrix}$, $1 \le i \le n$, $t \in \mathbb{Z}_{\ge 0}$.

<u>For $\mathfrak{g} = sl_2$</u> $X^+, X^-, K^{\pm}$ satisfy

$$KX^+K^{-1} = q^2 X^+, \quad KX^-K^{-1} = q^{-2}X^- \quad \text{and} \quad X^+X^- - X^-X^+ = \frac{K - K^{-1}}{q - q^{-1}}.$$

Distinguished modules

The Verma module $M(\lambda)$ is $M(\lambda) = \mathcal{U}_{\mathcal{A}} v_\lambda^+$ with

$$K_i v_\lambda^+ = q_i^{\langle \lambda, \alpha_i^\vee \rangle} v_\lambda^+, \quad X_i^+ v_\lambda^+ = 0$$

$$\begin{bmatrix} K_i ; 0 \\ \ell \end{bmatrix} v_\lambda^+ = \begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle \\ \ell \end{bmatrix}_{q_i} v_\lambda^+ \quad \text{and} \quad (X_i^+)^{(\ell)} v_\lambda^+ = 0$$

Let $\lambda \in P^+$.

Let $L_q(\lambda)$ be the simple $\mathcal{U}_q\mathfrak{g}$-module of highest weight λ.

$L_{\mathcal{A}}(\lambda)$ is the $\mathcal{U}_{\mathcal{A}}$-submodule of $L_q(\lambda)$ generated by v_λ^+.

The Weyl module is

$$\Delta_\epsilon(\lambda) = L_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C} \quad \text{where} \quad q = \epsilon = e^{2\pi i/\ell},$$

the reduction ~~of $L_{\mathcal{A}}(\lambda)$~~ mod $q^\ell = 1$ of $L_{\mathcal{A}}(\lambda)$.

The simple module $L_\epsilon(\lambda)$ is the unique simple quotient of $M_\epsilon(\lambda)$ (or $\Delta_\epsilon(\lambda)$).

Lusztig conjecture

Let $\lambda \in P^+$. Then, in the Grothendieck group of finite dimensional U_ε^{res}-modules

$$V_\varepsilon^{res}(\lambda) = \sum (-1)^{\ell(w' w_\lambda)} P_{w' w_\lambda}(1)\, W_\varepsilon^{res}(\lambda_{w'})$$

where the sum is over

$$\{ w' \in \widetilde{W}_\ell \mid w' \leq w_\lambda,\ \lambda_{w'} \in P^+ \}.$$

$$A_\ell^- = \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda+\rho, \varphi^\vee \rangle \geq -\ell,\ \langle \lambda+\rho, \alpha_i^\vee \rangle \leq -1,\ 1 \leq i \leq n \}.$$

$\widetilde{W}_\ell = \langle s_0, \ldots, s_n \rangle$ where $H_{\alpha_0} = H_{\theta, -\ell}$ and $H_{\alpha_i} = H_{\alpha_i; 0}$.

If $\lambda \in \mathfrak{h}^*$ then $w_\lambda \in \widetilde{W}_\ell$ is minimal length s.t. $w_\lambda^{-1}(\lambda) \in A_\ell^-$.
If $w' \in \widetilde{W}_\ell$ then $\lambda_{w'} = w'(w_\lambda^{-1}(\lambda))$.

SL_2

Let $\lambda \in P^+$, here $P^+ = \mathbb{Z}_{\geq 0}$.
Write $\lambda = \lambda_0 + \ell \lambda_1$ where $0 \leq \lambda_0 < \ell$.
$\Delta_\varepsilon(\lambda)$ = Weyl module for $U_\varepsilon(sl_2)$ with h.w. λ.

Prop Then $\Delta_\varepsilon(\lambda) = \text{span}\{ v_0, \ldots, v_\lambda \mid v_r = f^{(r)} v_0 \}$

(a) The maximal submodule of $\Delta_\varepsilon(\lambda)$ is
$$V = \text{span}\{ v_r \in \Delta_\varepsilon(\lambda) \mid r = r_0 + \ell r_1,\ r_0 > \lambda_0 \}.$$

(b)

For sl_2

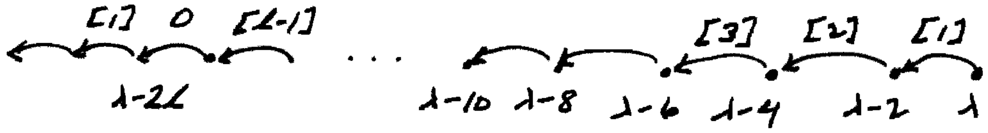
$M_{\mathcal{A}}(\lambda) = \mathcal{A}\text{-span}\{v_\lambda, v_{\lambda-2}, v_{\lambda-4}, \dots\}$ with

$$Kv_{\lambda-2r} = q^{\lambda-2r} v_{\lambda-2r} \quad \text{and} \quad X^+ v_{\lambda-2r} = [\lambda - r + 1] v_{\lambda-2r-2}$$

$$X^- v_{\lambda-2r} = [r+1] v_{\lambda-2r+2}.$$

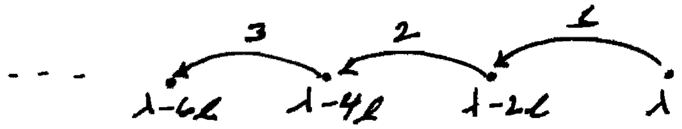
So

$$X^- v_\lambda = v_{\lambda-2}, \quad X^- v_{\lambda-2} = [2] v_{\lambda-4}, \quad X^- v_{\lambda-4} = [3] v_{\lambda-6}, \dots$$



since $[k\ell] = 0$.

also $(X^-)^{(\ell)} v_\lambda = v_{\lambda-2\ell}$, $(X^-)^{(\ell)} v_{\lambda-2\ell} = 2 v_{\lambda-4\ell}$, ...



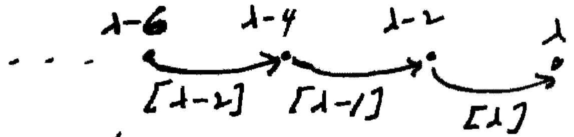
since

$$\frac{[k\ell+1][k\ell+2]\cdots[k\ell+\ell]}{[1][2]\cdots[\ell]} = \frac{[1][2]\cdots[\ell-1]}{[1][2]\cdots[\ell-1]} \frac{[(k+1)\ell]}{[\ell]}$$

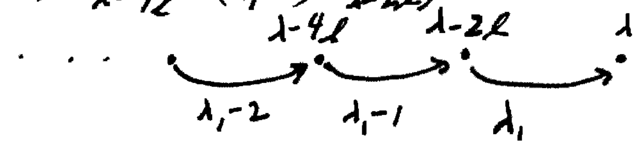
$$= \frac{[(k+1)\ell]}{[\ell]} = k+1.$$

Next

$$X^+ v_{\lambda-2} = [\lambda] v_\lambda, \quad X^+ v_{\lambda-4} = [\lambda-1] v_{\lambda-2}, \quad X^+ v_{\lambda-6} = [\lambda-2] v_{\lambda-4}$$



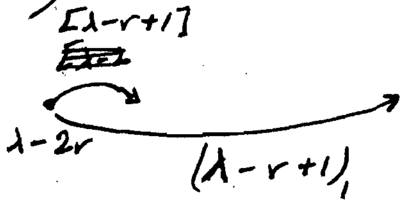
and $(X^+)^{(\ell)} v_{\lambda-2\ell} = \lambda_1 v_\lambda$, $(X^+)^{(\ell)} v_{\lambda-4\ell} = (\lambda_1 - 1) v_{\lambda-2\ell}$, ...



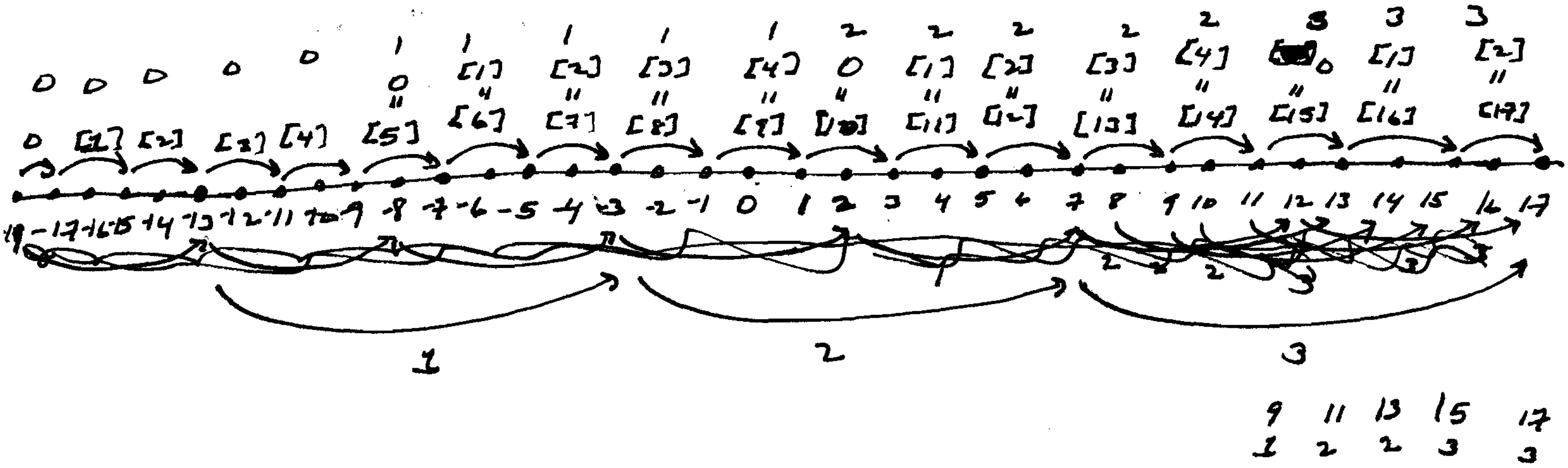
since

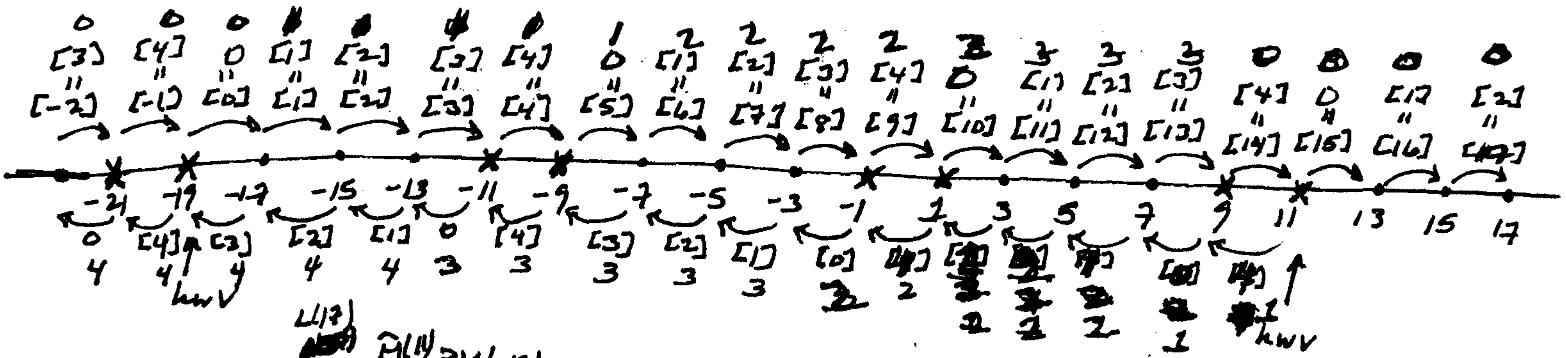
$$\frac{[\lambda_0+\ell\lambda_1][\lambda_0+\ell\lambda_1-1]\cdots[\lambda_0+\ell\lambda_1-(\ell-1)]}{[1][2]\cdots[\ell]} = \frac{[\ell\lambda_1]}{[\ell]} \frac{[1]\cdots[\ell-1]}{[1]\cdots[\ell-1]} = \lambda_1.$$

So highest weight vectors in $M(\lambda)$ are

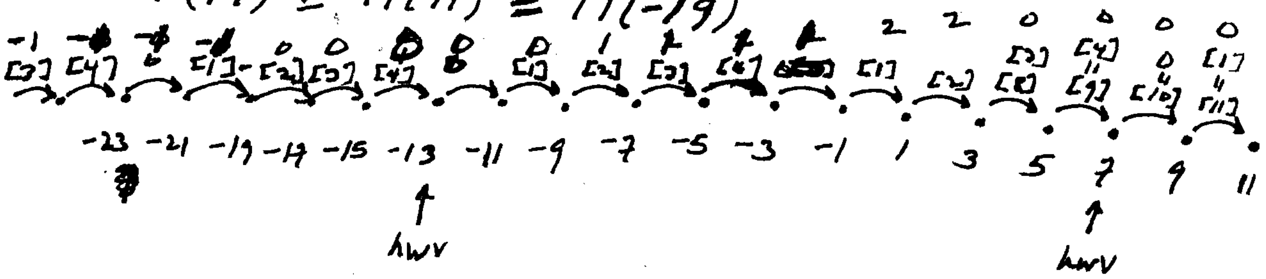


So $(\lambda-r+1)_0 = 0$ and $(\lambda-r+1)_1 = 0$.

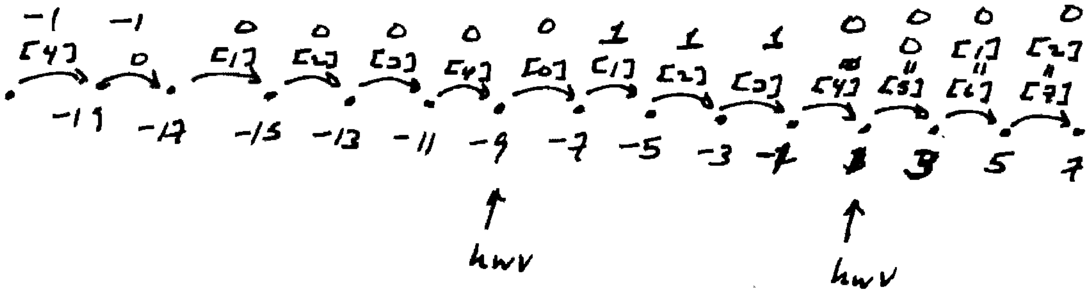




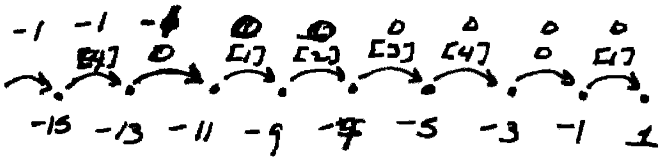
So $M(17) \supseteq M(11) \supseteq M(-19)$



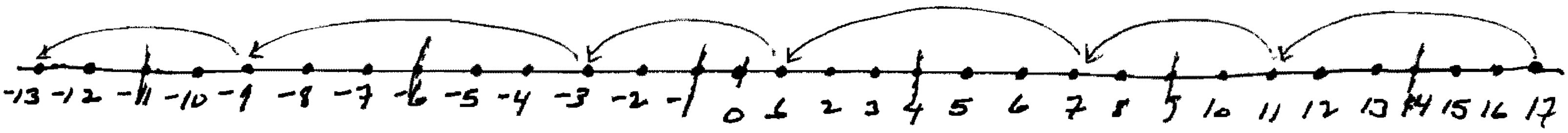
$M(11) \supseteq M(7) \supseteq M(-13)$

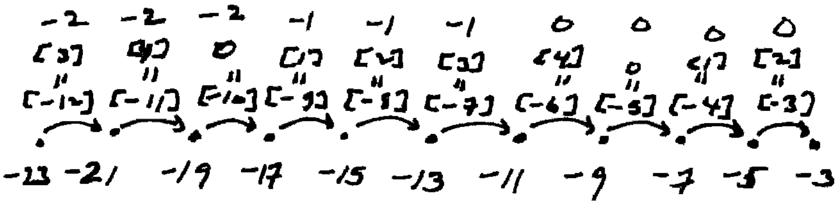


$$M(7) \geq M(1) \geq M(-9)$$

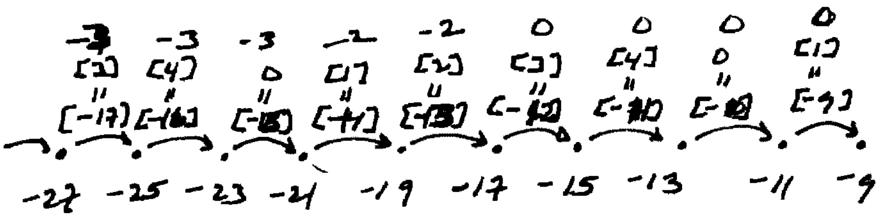


$$M(1) \geq M(-3)$$



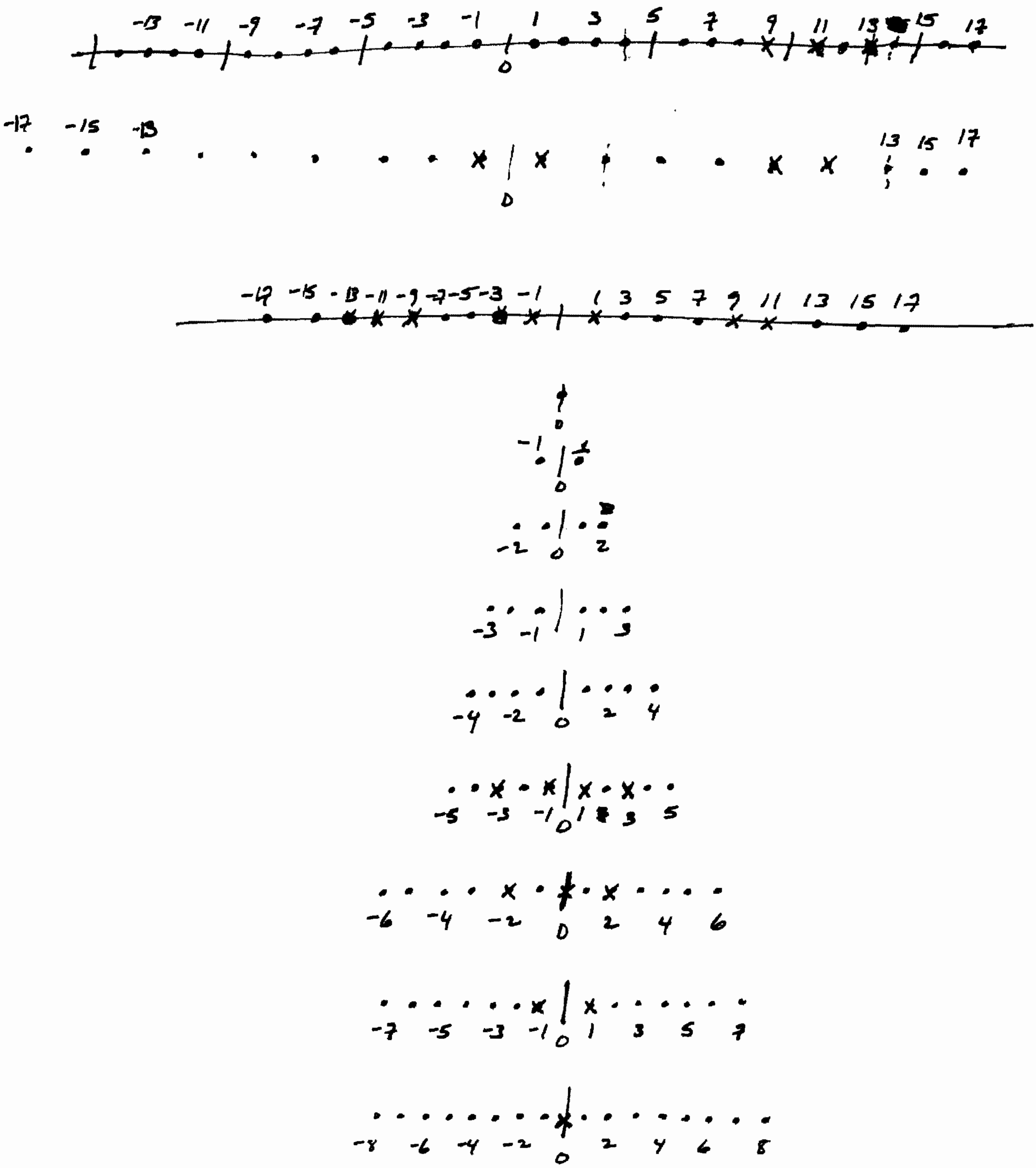


$$M(-3) \geq M(-9)$$



$$M(-9) \geq M(-13).$$

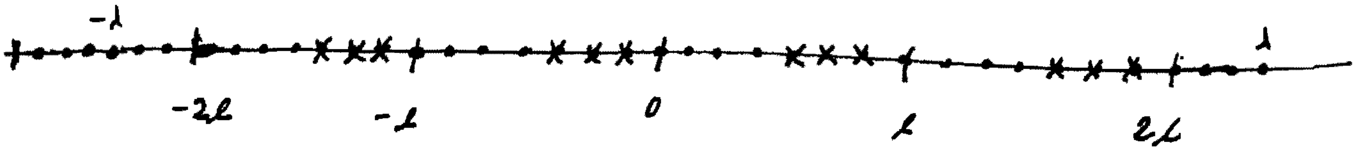
$$\frac{q^{-k\ell}-q^{k\ell}}{q^{\ell}-q^{\ell}} = -\frac{q^{k\ell}-q^{-k\ell}}{q^{\ell}-q^{-\ell}} = -k.$$



-13 -11 -9 -7 -5 -3 -1 1 3 5 7 9 11 13 15 17
0
-17 -15 -13 13 15 17
0
-17 -15 -13 -11 -9 -7 -5 -3 -1 1 3 5 7 9 11 13 15 17
0
-1 1
0
-2 0 2
-3 -1 1 3
-4 -2 0 2 4
-5 -3 -1 0 1 3 5
-6 -4 -2 0 2 4 6
-7 -5 -3 -1 0 1 3 5 7
-8 -6 -4 -2 0 2 4 6 8

$$\Delta_\varepsilon(\lambda) = \text{span}\{v_{-\lambda}, v_{-(\lambda-2)}, \ldots, v_{\lambda-2}, v_\lambda\}.$$

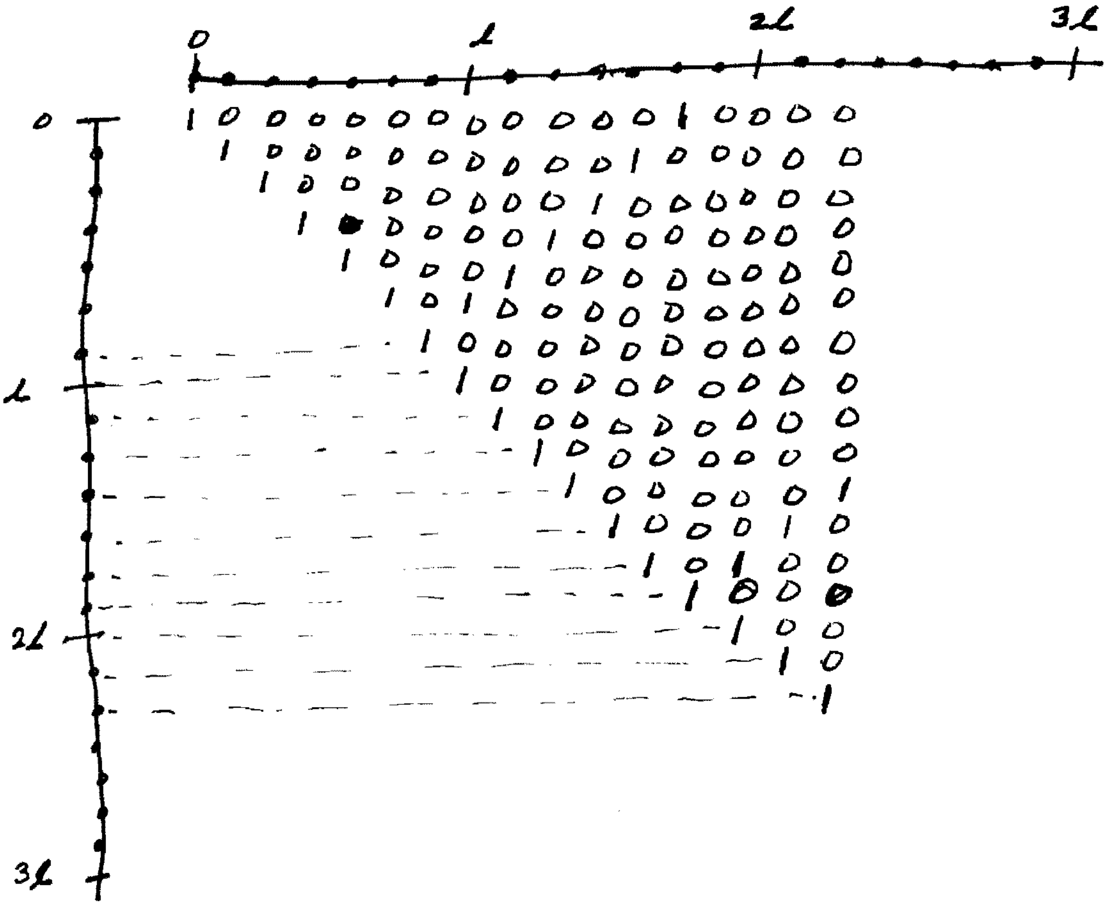
$$\lambda = \ell\lambda_1 + \lambda_0$$



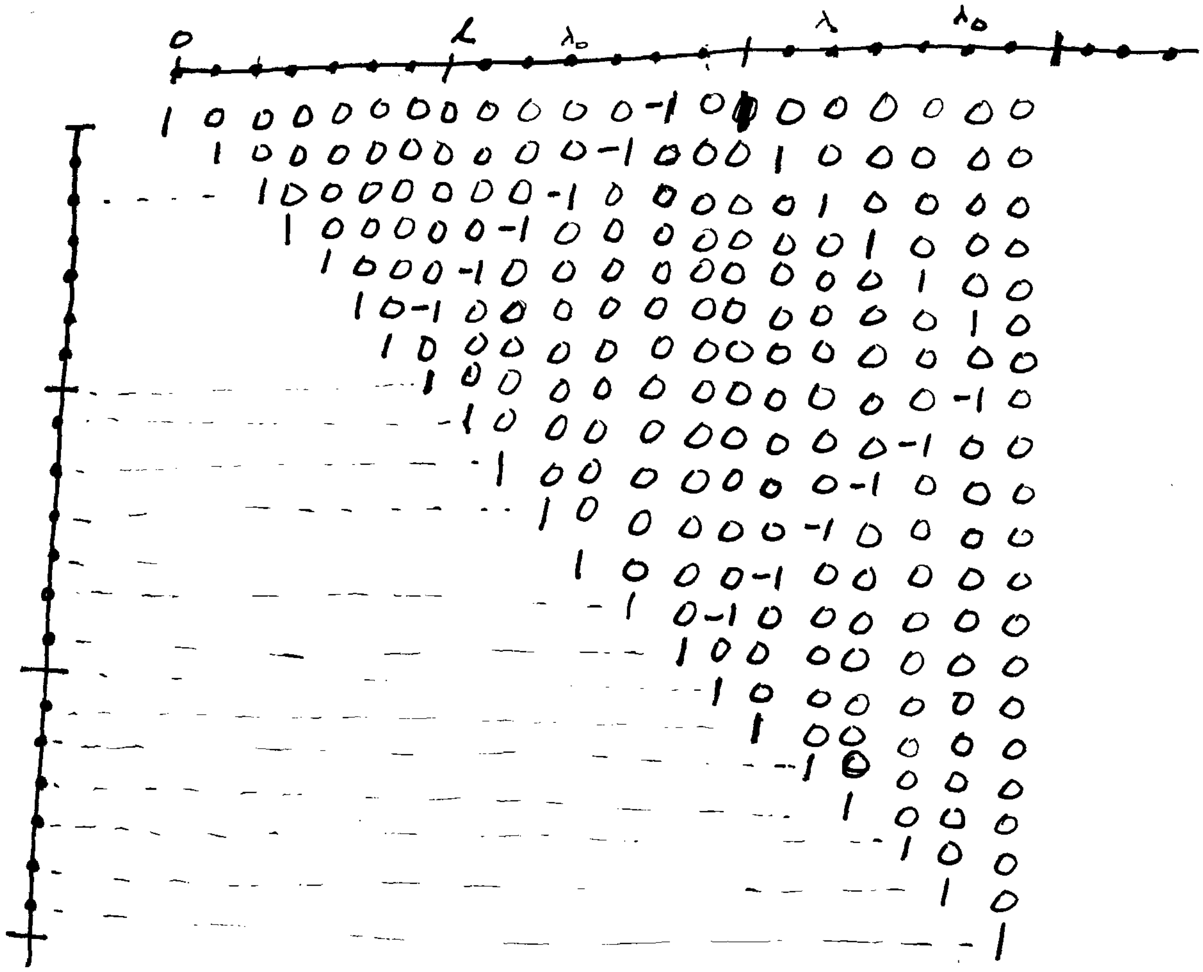
$$V = \text{span}\{ v_{r_0+\ell r_1} \mid 0 \le r_0 < \ell,\ r_1 \in \mathbb{Z},\ r_0 > \lambda_0 \}$$

Then

$$\Delta_\varepsilon(\lambda) = \begin{cases} L_\varepsilon(\lambda) + L_\varepsilon(\lambda - 2(\lambda_0+1)), & \lambda \ge \ell \text{ and } \lambda_0 < \ell - 1 \\ L_\varepsilon(\lambda), & \text{if } \lambda < \ell \text{ or } \lambda_0 = \ell - 1. \end{cases}$$



The inverse of this matrix is



i.e.

$$L_\epsilon(\lambda) \overset{?}{=} \Delta_\epsilon(\lambda) - \Delta_\epsilon(\lambda - 2(\lambda_0+1))$$

$$= L_\epsilon(\lambda) + L_\epsilon(\lambda - 2(\lambda_0+1))$$

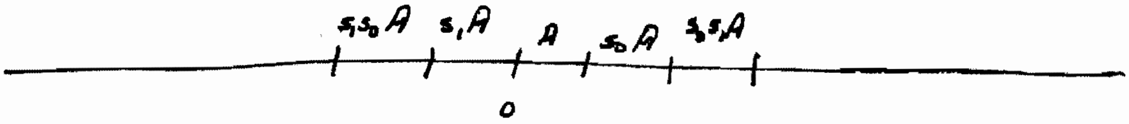
$$- \left(L_\epsilon(\lambda - 2(\lambda_0+1)) + L_\epsilon(\lambda - 4(\lambda_0+1)) + \cdots\right)$$

Label the blocks by $v \in A_\ell^-$.

$$L_\epsilon(\check{v} \circ \nu) = \sum (-1)^{\ell(wv)} P_{w,v}(1)\, \Delta_\epsilon(w \circ \nu)$$

where the sum is over $\{ w \in W_\ell \mid w \le v,\ w \circ \nu \in P^+ \}$.

KL polynomials for $\tilde{A}_1$



$$T_i^2 = 1 + (q - q^{-1}) T_i .$$

So $\overline{T_{s_i}} = T_{s_i}^{-1} = T_{s_i} - (q - q^{-1})$. Thus

$C_{s_i} = T_{s_i} + q^{-1}$ since $\overline{C_{s_i}} = T_i - (q-q^{-1}) + q = T_{s_i} + q^{-1} = C_{s_i}$.

Claim

$$C_{\underbrace{s_1 s_0 s_1 \cdots}_{r}} = T_{\underbrace{s_1 s_0 s_1 \cdots}_{r}} + q^{-1}\Big(T_{\underbrace{s_1 s_0 \cdots}_{r-1}} + T_{\underbrace{s_0 s_1 \cdots}_{r-1}}\Big) + q^{-2}\Big(T_{\underbrace{s_1 s_0 \cdots}_{r-2}} + T_{\underbrace{s_0 s_1 \cdots}_{r-2}}\Big) + \cdots + q^{-r}$$

Proof To show $C_{s_1 s_0 s_1 \cdots}$ is bar invariant.

$$C_{s_1} C_{\underbrace{s_0 s_1 \cdots}_{r-1}} = (T_{s_1} + q^{-1})\Big(T_{\underbrace{s_0 s_1 \cdots}_{r-1}} + q^{-1}\big(T_{\underbrace{s_1 s_0 \cdots}_{r-2}} + T_{\underbrace{s_0 s_1 \cdots}_{r-2}}\big) + q^{-2}\big(T_{\underbrace{s_1 s_0 \cdots}_{r-3}} + T_{\underbrace{s_0 s_1 \cdots}_{r-3}}\big) + \cdots + q^{-(r-1)}\Big)$$

$$= T_{\underbrace{s_1 s_0 s_1 \cdots}_{r}} + q^{-1} T_{\underbrace{s_1 s_0 s_1 \cdots}_{r-1}} + q^{-2} T_{\underbrace{s_1 s_0 \cdots}_{r-2}} + \cdots + q^{-(r-1)} T_{s_1}$$

$$+ \quad q^{-1} T_{\underbrace{s_0 s_1 \cdots}_{r-1}} + q^{-2} T_{\underbrace{s_0 s_1 \cdots}_{r-2}} + \cdots + q^{-(r-1)} T_{s_0} + q^{-r}$$

$$+ \, q^{-2} T_{\underbrace{s_1 s_0 \cdots}_{r-2}} + q^{-3} T_{\underbrace{s_1 s_0 \cdots}_{r-3}} + \cdots + q^{-(r-2)-1} T_{s_1}$$

$$+ (1-q^{-2}) T_{\underbrace{s_1 s_0 \cdots}_{r-2}} + (1-q^{-2}) q^{-1} T_{\underbrace{s_1 s_0 \cdots}_{r-3}} + \cdots + (1-q^{-2}) q^{-(r-2)-2} T_{s_1}$$

$$+ q^{-1} T_{\underbrace{s_0 s_1 \cdots}_{r-3}} + q^{-2} T_{\underbrace{s_0 s_1 \cdots}_{r-4}} + \cdots + q^{-(r-1)}$$

$$= C_{\underbrace{s_1 s_0 s_1 \cdots}_{r}} + C_{\underbrace{s_1 s_0 s_1 \cdots}_{r-2}}$$

So $C_{\underbrace{s_1 s_0 s_1 \cdots}_{r}}$ is bar invariant. //

In this case $\Omega = \{1, \tau\}$ where $\tau = t_{\omega_1} s_1$ with

$$\tau s_1 \tau^{-1} = s_0 \quad \text{and} \quad \tau s_0 \tau^{-1} = s_1$$

Now $T_{\tau^{-1}}^{-1} = T_\tau^{-1} = T_\tau$ and so $\overline{T_\tau} = T_\tau$ and

$$C_\tau = T_\tau .$$

Then

$$C_\tau C_{\underbrace{s_1 s_0 s_1 \cdots}_{r}} = T_{\tau \underbrace{s_1 s_0 s_1 \cdots}_{r}} + \sum_{\ell=1}^{r-1} q^{-\ell} \left(T_{\tau \underbrace{s_1 s_0 \cdots}_{r-\ell}} + T_{\tau \underbrace{s_0 s_1 \cdots}_{r-\ell}} \right) + q^{-r}$$

Thus

$$P_{wv}(q^{-1}) = q^{-(\ell(w) - \ell(v))} \quad \text{if} \quad w, v \in W_{aff}.$$

Fock space Let $\ell \in \mathbb{Z}_{>0}$

$$\mathcal{F}_\ell = \bigoplus_{v \in A_\ell^-} \varepsilon_0 \tilde{H} \mathbb{1}_v$$

$$= \varepsilon_0 \tilde{H} \mathbb{1}_0 \oplus \varepsilon_0 \tilde{H} + \cdots \oplus \varepsilon_0 \tilde{H}$$

Let $|\mu\rangle = \varepsilon_0 X^\rho T_w \mathbb{1}_v$ where $\mu = \ell\rho + wv$.

So $\mu = \ell\mu_1 + \mu_0$ if μ is regular.
$\mu = \ell\mu_1$ if μ is not regular.