Abelian varieties

Let $\omega_1, \dots, \omega_{2g} \in \mathbb{C}^{\ell}$ and let

$$\Omega = \mathbb{Z}\text{-span}\{\omega_1, \omega_2, \dots, \omega_{2g}\} \subseteq \mathbb{C}^{\ell}.$$

An *abelian variety* is $\mathbb{C}^g/\Omega$ which can be embedded in projective space.

Recall how G/B is embedded in projective space:

$V = H^0(G/B, \mathcal{L}_\mu)$ is a G-module

with a highest weight vector v_μ.

So B stabilizes the line $[v_\mu] = \mathbb{C} v_\mu$ in $\mathbb{P}(V)$.

Then
$$\begin{aligned} G/B &\xrightarrow{\varphi} \mathbb{P}(V) \\ gB &\longmapsto g[v_\mu] \end{aligned}$$

If μ is regular ($\langle \mu, \alpha \rangle \neq 0$ for $\alpha \in R^+$)

then $B = \mathrm{Stab}([v_\mu])$ and φ is injective.

When μ is regular, $\mathcal{L}_\mu$ is ample and so all regular functions on G/B are determined by the inclusion φ.

This could be a motivation to find:

ample line bundles $\mathcal{L}$ on $\mathbb{C}^g/\Omega$

$H^0(\mathbb{C}^g/\Omega, \mathcal{L})$ will be an irreducible rep. of the Heisenberg group and the theta function will be the "highest wt vector"!

$$\begin{tikzcd} \mathbb{C}^g \times \mathbb{C} \simeq q^*\mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \text{ "=" } \mathbb{C}^g \times_\Omega \mathbb{C} \arrow[d] \\ \mathbb{C}^g \arrow[r, "q"] & \mathbb{C}^g/\Omega \end{tikzcd}$$

The group Ω acts on $\mathbb{C}^g \times \mathbb{C}$ by

$$\gamma \cdot (z, t) = (z + \gamma,\ j(\gamma, z)\, t)$$

for some function $j(\lambda, z)$. (which must satisfy

$$j(\gamma+\mu, z) = j(\gamma, z+\mu)\, j(\mu, z) \quad \text{for this to be an } \Omega\text{-action})$$

Define a Hermitian form

$$H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}, \quad (\gamma, z) \mapsto \langle \gamma, z \rangle \quad \text{and} \quad \chi : \Omega \to U_1(\mathbb{C}) \text{ by}$$

$$j(\gamma, z) = \chi(\lambda)\, e^{\frac{\pi}{2}\langle \gamma,\gamma\rangle + \pi \langle \gamma, z\rangle}$$

Let $(*)$ be the condition

(a) $\operatorname{im}(\langle \gamma, \mu \rangle) \in \mathbb{Z}$ for $\gamma, \mu \in \Omega$

(b) $\chi(\gamma + \mu) = \chi(\gamma)\chi(\mu)\, e^{i\pi\langle \gamma, \mu\rangle}$

Then

$$\left\{ \begin{array}{l} \text{line bundles } \mathcal{L} \\ \text{on } \mathbb{C}/\Omega \end{array} \right\} \xleftrightarrow{1\text{-}1} \left\{ \begin{array}{l} \text{pairs } (H, \chi) \text{ which} \\ \text{satisfy (a) and (b)} \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} \text{ample line bundles } \mathcal{L} \\ \text{on } \mathbb{C}/\Omega \end{array} \right\} \xleftrightarrow{1\text{-}1} \left\{ \begin{array}{l} \text{pairs } (H, \chi) \text{ which} \\ \text{satisfy (a) and (b) and} \\ H \text{ is positive definite} \end{array} \right\}$$

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Theta functions

Let $\mathcal{H} = \{x+iy \mid x, y \in \mathbb{R}, y > 0\}$ be the upper half plane.

Define $\theta : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ by

$$\theta(\tau, z) = \sum_{\gamma \in \mathbb{Z}} e^{i\pi \gamma^2 \tau + 2\pi i \gamma z} \qquad \text{(from Mumford)}$$

Definition (from Kac–Peterson)

Let $\mathfrak{h}^* = \mathbb{C}^\ell$ and $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\omega_1, \ldots, \omega_\ell\}$ a $\mathbb{Z}$-lattice in $\mathfrak{h}^*$

Let

$$\mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \mathfrak{h}^* \oplus \mathbb{C}\delta = \{\tau\Lambda_0 + \mu + t\delta \mid \tau \in \mathbb{C}, \mu \in \mathfrak{h}^*, t \in \mathbb{C}\}.$$

Let $\lambda = \mu + m\Lambda_0$ with $\mu \in \mathfrak{h}^*$.

The classical theta function of degree m and characteristic μ is

$$\Theta_\lambda = \Theta_{\mu,m} : \mathcal{H} \times \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C} \quad \text{given by}$$

$$\Theta_{\mu,m}(\tau, z, t) = e^{-2\pi i m t} \sum_{\gamma \in \mathfrak{h}^*_{\mathbb{Z}} + \frac{1}{m}\mu} e^{i\pi\langle \gamma, \gamma\rangle \tau - 2\pi i \langle \gamma, z\rangle}$$

and the Riemann theta function is $\Theta_{0,1} : \mathcal{H} \times \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ given by

$$\Theta_{0,1}(\tau, z, t) = e^{-2\pi i t} \sum_{\gamma \in \mathfrak{h}^*_{\mathbb{Z}}} e^{i\pi\langle \gamma, \gamma\rangle \tau - 2\pi i \langle \gamma, z\rangle}$$

Thm (a) Θ_λ is periodic in z with periods in $\mathfrak{h}_{\mathbb{Z}} + \tau\mathfrak{h}_{\mathbb{Z}}$

(b) Θ_λ is a holomorphic function in τ.

The Heisenberg group

Mumford uses: $\mathcal{G} = U_1(\mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ with

$$(\lambda, a, b)(\lambda', a', b') = (\lambda\lambda' \exp(2\pi i b a'), a+a', b+b')$$

see p. 7. The action of $\mathcal{G}$ on $V_\infty = \{\text{entire functions } f(z)\}$ is given by $((\lambda, a, b)f)(z) = \lambda e^{(i\pi a^2\tau + 2\pi i a z)} f(z + a\tau + b)$

Mumford p 9: $\mathcal{G}_\ell = \{(\lambda, a, b) \mid \lambda \in \mu_{\ell^2},\ a, b \in \frac{1}{\ell}\mathbb{Z}\} \bmod \ell\Gamma$

$$= \mu_{\ell^2} \times \tfrac{1}{\ell}(\mathbb{Z}/\ell\mathbb{Z}) \times \tfrac{1}{\ell}(\mathbb{Z}/\ell\mathbb{Z})$$

where $\Gamma = \{(1, a, b) \in \mathcal{G} \mid a, b \in \mathbb{Z}\}$, $\ell\Gamma = \{(1, \ell a, \ell b)\} \subseteq \Gamma$

$V_\ell = \{\text{entire functions } f(z) \text{ invariant under } \ell\Gamma\}$.

$\mathcal{G}_\ell$ is a finite group acting on V_ℓ and

$$\{\Theta_{a,b} = S_b T_a \Theta = e^{2\pi i ab} T_a S_b \Theta \mid a, b \in \tfrac{1}{\ell}\mathbb{Z}\}.$$

These are the <u>Theta functions</u> with <u>characteristics</u>.