<u>Theorem</u> $\mathbb{C}$ is algebraically closed

<u>Proof</u> 1 d'Alembert-Gauss [Bou, Top. Ch VIII §1, no.1, Theorem 1].

<u>To show</u>: (a) If $a \in \mathbb{R}_{\geq 0}$ then there exists $\sqrt{a} \in \mathbb{R}$.

(b) If $p(t) \in \mathbb{R}[t]$ and $\deg p$ is odd then there exists $\alpha \in \mathbb{R}$ such that $p(\alpha) = 0$.

(b) Assume $p(t) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with n odd and $a_n \neq 0$.

If $x \in \mathbb{R}$ and $x \neq 0$ then
$$p(x) = a_n x^n g(x), \text{ where } g(x) = 1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}.$$
$$\lim_{x\to\infty} g(x) = 1 \quad\text{and}\quad \lim_{x\to-\infty} g(x) = 1.$$

So there exists $a \in \mathbb{R}_{>0}$ such that
$$\operatorname{sign}(a_n) = \operatorname{sign}(f(a)) \text{ and } \operatorname{sign}(-a_n) = \operatorname{sign}(f(-a))$$
Thus, by Bolzano's theorem, [Bou, Top IV §6, no.1 Theorem 2] there exists $\alpha \in [-a, a]_{\mathbb{R}}$ such that $f(\alpha) = 0$. //.

<u>Proof 2</u> [Bou, Top. Ch VIII §2. Exercise 2]

Let $f(t) \in \mathbb{C}[t]$ such that $f(t) \neq 0$.

To show: There exists $r \in \mathbb{R}_{>0}$ such that if $z \in \mathbb{C}$ and $|z| \geq r$ then $|f(z)| > |f(0)|$

Use [Exercise 1] and Weierstrass' theorem [Bou, Top. Ch IV §6 no. 1, Theorem 1] to show $\mathbb{C}$ is algebraically closed.

[Bou Top. Ch. VIII §2, Exercise 1]

Let $a \in \mathbb{C}$, $a \neq 0$ and $n \in \mathbb{Z}_{>0}$.

~~To~~ To show (a) If $r \in \mathbb{R}_{>0}$ such that $r^n \leq |a|$ then there exists $z \in \mathbb{C}$ such that $|z| = r$ and

$$|a + z^n| = |a| - r^n.$$

(b) If $f(t) \in \mathbb{C}[t]$ and $\deg(f) > 0$ ~~then~~ and $z_0 \in \mathbb{C}$, with $f(z_0) \neq 0$ then there exists $z \in B_\varepsilon(z_0)$ such that

$$|f(z_0)| > |f(z)|.$$