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Spaces: What is a space?

Question? Which is more fundamental for us, the space X. or the ring of functions $\mathcal{O}_X$ on X?

$$\mathcal{O}_X = \{\text{favorite functions } f: X \to \mathbb{F}\}.$$

Then $\mathcal{O}_X$ is an $\mathbb{F}$-algebra with

$$(f+g)(x) = f(x)+g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$

for $c \in \mathbb{F}$, $f,g \in \mathcal{O}_X$ and $x \in X$.

Hilbert's Nullstellensatz There is an equivalence of categories

$$\left\{\begin{matrix}\text{affine} \\ \text{varieties}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{commutative } \mathbb{F}\text{-algebras with} \\ \text{no nilpotent elements}\end{matrix}\right\}$$

Grothendieck's version: There is an equivalence of categories:

$$\left\{\begin{matrix}\text{affine} \\ \text{schemes}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{commutative} \\ \text{rings}\end{matrix}\right\}$$

A scheme is a ringed space that is locally isomorphic to an affine scheme.

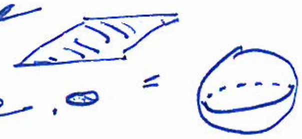
A variety is a ringed space that is locally isomorphic to an affine variety

A smooth manifold is a ringed space which is locally isomorphic to $(\mathbb{R}^n, C^\infty)$

A complex manifold is a ringed space which is locally isomorphic to $(\mathbb{C}^n, C^{an})$

A <u>topological manifold</u> is a ringed space which is locally isomorphic to $(\mathbb{R}^n, C)$

The words "locally isomorphic" mean that the gluing conditions for a sheaf are in place.



Before we think about how to glue things together it is helpful to understand something about what the pieces look like.

Hence we begin with

affine schemes, affine varieties, $(\mathbb{R}^n, C)$, $(\mathbb{R}^n, C^\infty)$
and $(\mathbb{C}^n, C^{an})$

<u>Affine schemes</u>

Our goal is to define <u>Spec</u>: A functor

$$\mathrm{Spec}: \left\{\begin{matrix}\text{commutative}\\ \text{rings}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{ringed}\\ \text{spaces}\end{matrix}\right\}.$$

Begin with

$$\mathrm{Spec}: \left\{\begin{matrix}\text{commutative}\\ \text{rings}\end{matrix}\right\} \longrightarrow \{\text{Sets}\}$$

given by

$$\mathrm{Spec}(A) = \{\text{prime ideals } x \text{ of } A\}$$

Recall: Quotient rings $A/I \longleftrightarrow$ ideals I of A.

A/x is an integral domain $\longleftrightarrow$ prime ideals x

A/x is a field $\longleftrightarrow$ maximal ideals x

The notation

$$\mathrm{Spec}(\mathcal{O}_X) = X \quad \text{where } X = \{\text{prime ideals } x \text{ of } \mathcal{O}_X\}.$$

is preferred. Given

$$f : \mathcal{O}_Y \to \mathcal{O}_X \quad \text{we get} \quad \begin{aligned} \mathrm{Spec}(f) : X &\to Y \\ x &\mapsto f^{-1}(x) \end{aligned}$$

or, given

$$f^{\#} : \mathcal{O}_Y \to \mathcal{O}_X \quad \text{we get} \quad f : X \to Y.$$

In our heads, Spec is the inverse functor to

$$\begin{aligned} \mathcal{O} : \{\text{spaces}\} &\to \left\{\begin{matrix}\text{commutative}\\ \text{rings}\end{matrix}\right\} \\ X &\longmapsto \mathcal{O}_X \end{aligned} \qquad \text{where } \mathcal{O}_X = \left\{\begin{matrix}\text{favorite functions}\\ f : X \to \mathbb{F}\end{matrix}\right\}.$$

<u>Next step:</u>

$$\begin{aligned} \mathrm{Spec} : \left\{\begin{matrix}\text{commutative}\\ \text{rings}\end{matrix}\right\} &\to \left\{\begin{matrix}\text{topological}\\ \text{spaces}\end{matrix}\right\} \\ \mathcal{O}_X &\longmapsto X \end{aligned}$$

with

$$X = \{\text{prime ideals } x \text{ of } \mathcal{O}_X\}$$

and X has closed sets

$$V(E) = \{ y \in \mathrm{Spec}(A) \mid y \supseteq E \} \quad \text{for} \quad E \subseteq \mathcal{O}_X.$$

In particular, if $f \in \mathcal{O}_X$ then

$$V(\{f\}) = \{ y \in X \mid f \in y \} \quad \text{are closed sets and}$$

$$X_f = V(\{f\}) \quad \text{are open sets.}$$

The sets X_f, $f \in \mathcal{O}_X$, form a basis for the topology on X.