G/P_1 and G/P_2 and the Fano Plane

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The vector space of column vectors of length n

$$\mathbb{F}^n = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \;\middle|\; c_i \in \mathbb{F} \right\}$$

has basis $e_1, e_2, \ldots, e_n$ where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith.

Let

$\mathcal{L}$ be the lattice of subspaces of $\mathbb{F}^n$ partially ordered by inclusion, and.

$$Fl = \{\text{maximal chains in } \mathcal{L}\}$$

$$= \{ (0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{F}^n) \mid \dim V_i = i \}.$$

Our favorite flag is

$0 \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq \mathbb{F}^n$ ~~where $E_i = \mathrm{span}\{e_1, \ldots, e_i\}$~~

where $E_i = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\middle|\; c_j \in \mathbb{F} \right\} = \mathrm{span}\{e_1, \ldots, e_i\}$

The automorphism group of the vector space $\mathbb{F}^n$ is

$G = \mathrm{Aut}(\mathbb{F}^n) = Gl_n(\mathbb{F})$ and we write

$g = (g_{ij})$ where $ge_i = \sum_{j=1}^{n} g_{ji} e_j$ so that

$ge_i = \begin{pmatrix} | \\ g_i \\ | \end{pmatrix}$ is the ith column of g.

The stabilizer of E_i is

$$P_i = \mathrm{Stab}_G(E_i) = \left\{ \begin{pmatrix} \overbrace{\qquad * \qquad}^{i} & \overbrace{\; * \;}^{n-i} \\ 0 & * \end{pmatrix} \right\} \subseteq GL_n(\mathbb{F})$$

and the stabilizer of $0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n-1} \subseteq \mathbb{F}^n$ is

$$B = P_1 \cap P_2 \cap \cdots \cap P_{n-1} = \left\{ \begin{pmatrix} * & & & \\ & * & & * \\ & & \ddots & \\ 0 & & & * \end{pmatrix} \right\}$$

The maps

$$\begin{array}{ccc} G/P_i & \longrightarrow & \left\{ \begin{array}{l} \text{subspaces of} \\ \text{dimension } i \text{ in } \mathbb{F}^n \end{array} \right\} \\ gP_i & \longmapsto & gE_i \end{array}$$

and

$$\begin{array}{ccc} G/B & \longrightarrow & \mathcal{F}l \\ gB & \longmapsto & (0 \subseteq gE_1 \subseteq gE_2 \subseteq \cdots \subseteq gE_{n-1} \subseteq \mathbb{F}^n) \end{array}$$

are bijections and

$$gE_i = \mathrm{span}\left\{ \begin{pmatrix} | \\ g_1 \\ | \end{pmatrix}, \begin{pmatrix} | \\ g_2 \\ | \end{pmatrix}, \ldots, \begin{pmatrix} | \\ g_i \\ | \end{pmatrix} \right\}$$

is the span of the first i columns of g.

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Representatives of cosets in G/P_i

Let

$$x_i(c) = \begin{array}{cc} & \begin{array}{cccccccc} & & & i & i+1 & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array} & \left(\begin{array}{cccccccc} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 1 & c &&& \\ &&& 0 & 1 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{array}\right) \end{array} \qquad s_i = \begin{array}{cc} & \begin{array}{cccccccc} & & & i & i+1 & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array} & \left(\begin{array}{cccccccc} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 0 & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{array}\right) \end{array}$$

The Weyl group of $G = GL_n(\mathbb{C})$ is

$$W_0 = S_n = \langle s_1, \ldots, s_{n-1} \rangle = \{n \times n \text{ permutation matrices}\},$$

the subgroup of $GL_n(\mathbb{C})$ generated by $s_1, \ldots, s_{n-1}$.

If

$$W_i = W_0 \cap P_i = \langle s_1, s_2, \ldots, s_{j-1}, s_{j+1}, s_{j+2}, \ldots, s_{n-1} \rangle$$

then coset representatives of the cosets in W_0/W_i are

$$W^i = \left\{ \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{i-1} & \sigma_i & \tau_1 & \tau_2 & \cdots & \tau_{n-i} \end{pmatrix} \;\middle|\; \begin{array}{l} \sigma_1 < \sigma_2 < \cdots < \sigma_{i-1} < \sigma_i \\ \tau_1 < \tau_2 < \cdots < \tau_{n-i} \end{array} \right\}$$

and

$$G/P_i = \bigsqcup_{u \in W^i} B u P_i$$

where, if $u = s_{j_1} \cdots s_{j_\ell}$ is an minimal length expression of u as a product of the generators $s_1, \ldots, s_{n-1}$ of W_0, then

$$B u P_i = \{ x_{j_1}(c_1) s_{j_1} \cdots x_{j_\ell}(c_\ell) s_{j_\ell} P_i \mid c_1, \ldots, c_\ell \in \mathbb{F} \}.$$

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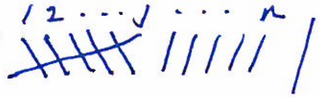
The cases G/P_1 and G/P_2

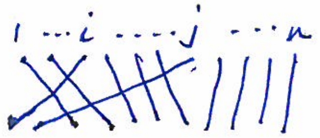
$$P_1 = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right\} \quad \text{and} \quad P_2 = \left\{ \left(\begin{array}{c|c} * \ * & \\ * \ * & * \\ \hline 0 & * \end{array} \right) \right\}$$

$W_1 = S_1 \times S_{n-1} = \langle s_2, \ldots, s_{n-1} \rangle \subseteq S_n$ and

$W_2 = S_2 \times S_{n-2} = \langle s_1, s_3, s_4, \ldots, s_{n-1} \rangle \subseteq S_n$.

$W^1 = \{$ 

$\mid j \in \{1, 2, \ldots, n\}\} = \{ s_{j-1} s_{j-2} \cdots s_2 s_1 \mid j \in \{1, 2, \ldots, n\} \}$

$W^2 = \{$ 

$\mid i, j \in \{1, 2, \ldots, n\}, \ i < j \}$

$= \{ s_{i-1} \cdots s_2 s_1 s_{j-1} s_{j-2} \cdots s_3 s_2 \mid i, j \in \{1, 2, \ldots, n\}, \ i < j \}.$

Then

$$\begin{aligned} G/P_1 &\longrightarrow \{ V_1 \subseteq \mathbb{F}^n \mid \dim V_1 = 1 \} \\ gP_1 &\longmapsto \operatorname{span}\left\{ \begin{pmatrix} | \\ g_1 \\ | \end{pmatrix} \right\} \end{aligned}$$

and

$$\begin{aligned} G/P_2 &\longrightarrow \{ V_2 \subseteq \mathbb{F}^n \mid \dim V_2 = 2 \} \\ gP_2 &\longmapsto \operatorname{span}\left\{ \begin{pmatrix} | \\ g_1 \\ | \end{pmatrix}, \begin{pmatrix} | \\ g_2 \\ | \end{pmatrix} \right\}. \end{aligned}$$

Next

$$G/P_1 = \bigsqcup_{u \in W^1} B u P_1 = \bigsqcup_{j=1}^{n} B s_{j-1} s_{j-2} \cdots s_2 s_1 P_1 \quad \text{and}$$

$$G/P_2 = \bigsqcup_{u \in W^2} B u P_2 = \bigsqcup_{\substack{i,j \in \{1,2,\ldots,n\} \\ i<j}} B s_{i-1} \cdots s_2 s_1 s_{j-1} \cdots s_3 s_2 P_2 .$$

with

$$B s_{j-1} \cdots s_2 s_1 P_1 = \{ x_{j-1}(c_{j-1}) s_{j-1} \cdots x_2(c_2) s_2 x_1(c_1) s_1 P_1 \mid c_1, \ldots, c_{j-1} \in \mathbb{F} \}$$

and

$$B s_{i-1} \cdots s_2 s_1 s_{j-1} \cdots s_3 s_2 P_2$$

$$= \left\{ \begin{array}{c} x_{i-1}(c_{i-1}) s_{i-1} \cdots x_2(c_2) s_2 x_1(c_1) s_1 x_{j-1}(d_{j-1}) s_{j-1} \cdots x_2(d_2) s_2 P_2 \\ \text{with } c_1, c_2, \ldots, c_{i-1}, d_2, d_3, \ldots, d_{j-1} \in \mathbb{F} \end{array} \right\}.$$

Note that

$$x_{j-1}(c_{j-1}) s_{j-1} \cdots x_2(c_2) s_2 x_1(c_1) s_1 = \left(\begin{array}{cccccc|c} c_1 & 1 & & & & & \\ c_2 & 0 & 1 & & & & \\ \vdots & \vdots & 0 & \ddots & & & 0 \\ \vdots & & \vdots & & \ddots & 1 & \\ c_{j-1} & 0 & 0 & \cdots & 0 & 1 & \\ 1 & 0 & 0 & \cdots & 0 & 0 & \\ \hline & & 0 & & & & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array} \right)$$

and

$$x_{i-1}(c_{i-1}) s_{i-1} \cdots x_2(c_2) s_2 x_1(c_1) s_1 x_{j-1}(d_{j-1}) s_{j-1} \cdots x_2(d_2) s_2$$

$$= \left(\begin{array}{ccccccccccc|c} c_1 & d_2 & 1 & & & & & & & & & \\ c_2 & d_3 & 0 & 1 & & & & & & & & \\ \vdots & \vdots & \vdots & 0 & \ddots & & & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 & & & & & & \\ c_{i-1} & d_i & 0 & 0 & \cdots & 0 & 1 & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & & & & 0 \\ 0 & d_{i+1} & 0 & 0 & \cdots & 0 & 0 & 1 & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & 0 & \ddots & 1 & & \\ & & & & & & & \vdots & & \ddots & & \\ 0 & d_{j-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \\ \hline & & & & & 0 & & & & & & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array} \right)$$

12.11.2012 A. Ram

The case $G = GL_3(\mathbb{F}_2)$

$$P_1 = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\} \quad \text{and} \quad P_2 = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\right\}$$

and cosets in G/P_1 have representatives

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{with } c_1, c_2 \in \mathbb{F}_2$$

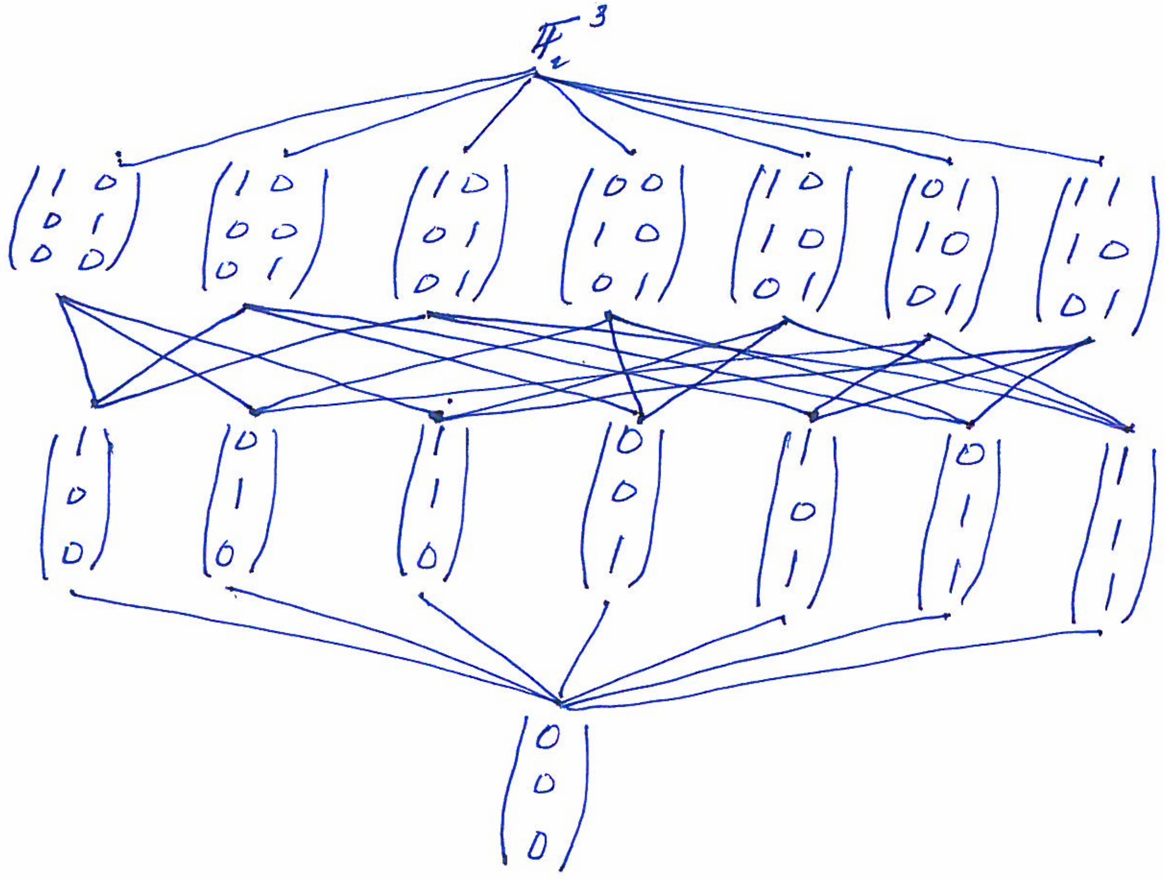
so that $\mathrm{Card}(G/P_1) = 1+2+4 = 7$.

Cosets in G/P_2 have representatives

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} c_1 & d_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{with } c_1, d_2 \in \mathbb{F}_2$$

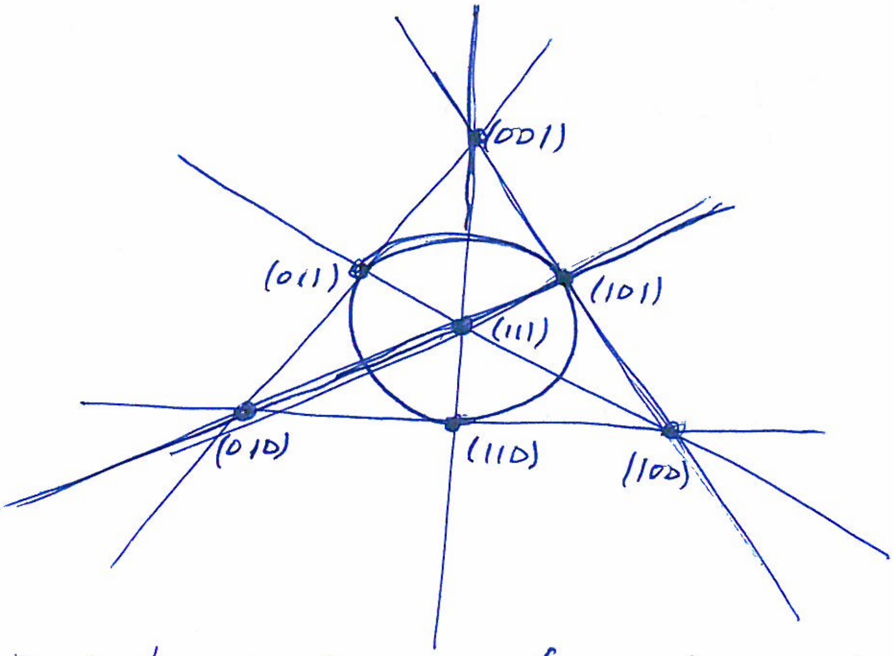
so that $\mathrm{Card}(G/P_2) = 1+2+4 = 7$.

Then the lattice $\mathcal{L}$



has representatives of G/P_1 on level 1 and representatives of G/P_2 on level 2.

Another way to encode this poset is via the following picture of the <u>Fano plane</u>



so that the inclusion of points in lines matches the poset $\mathcal{L}$.