Notes from discussions with J. Bamberg and M. Giudici 25 and 24 Oct. 2012 A. Ram ①

Incidence

An *incidence geometry* is a triple (P, L, I) where P and L are sets and $I \subseteq P \times L$.

$$\begin{tikzcd} I \subseteq P \times L \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & P \\ L \end{tikzcd}$$

A point $p \in P$ is *contained in a line* $\ell \in L$ if $(p, \ell) \in I$.

A set of points $S \subseteq P$ is *collinear* if there exists $\ell \in L$ such that if $p \in S$ then $(p, \ell) \in I$.

Often it is convenient to

identify $\ell \in L$ with the set of points $pr_1(pr_2^{-1}(\ell))$.

Subspaces

Assume that (P, L, I) is an incidence geometry such that

if $p_1, p_2 \in P$ and $p_1 \neq p_2$ then

there exists a unique $\ell \in L$ with $(p_1, \ell) \in I$ and $(p_2, \ell) \in I$.

The line $\ell = \ell(p_1, p_2)$ containing p_1 and p_2 is the *line connecting* p_1 and p_2.

A *subspace* is a subset $S \subseteq P$ such that

if $p_1, p_2 \in S$ then $pr_1(pr_2^{-1}(\ell(p_1, p_2))) \subseteq S$.

A subspace is $S \subseteq P$ which contains any line connecting two of its points.

Lattices

Let $\mathcal{L}$ be a partially ordered set and let $x, y \in \mathcal{L}$.

The join, or supremum, or least upper bound of x and y is

$$x \vee y = \sup\{x, y\} \text{ in } \mathcal{L} \text{ such that}$$

(a) $\sup\{x,y\} \geq x$ and $\sup\{x,y\} \geq y$, and

(b) If $z \in \mathcal{L}$ and $z \geq x$ and $z \geq y$ then $z \geq \sup\{x,y\}$.

The meet, or infimum, or greatest lower bound, of x and y is

$$x \wedge y = \inf\{x, y\} \text{ in } \mathcal{L} \text{ such that}$$

(a) $\inf\{x,y\} \leq x$ and $\inf\{x,y\} \leq y$, and

(b) If $z \in \mathcal{L}$ and $z \leq x$ and $z \leq y$ then $z \leq \inf\{x,y\}$.

A lattice is a partially ordered set $\mathcal{L}$ such that

if $x, y \in \mathcal{L}$ then $x \vee y$ and $x \wedge y$ exist in $\mathcal{L}$.

A modular lattice is a lattice $\mathcal{L}$ such that

if $x, z \in \mathcal{L}$ and $x \leq z$

then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

24-25.10.2012 A. Ram

Projective lattices

Let $\mathcal{L}$ be a finite lattice with a unique minimal element 0 and a unique maximal element 1.

An atom is $a \in \mathcal{L}$ such that there does not exist $a' \in \mathcal{L}$ with $0 < a' < a$.

An atomic lattice is a lattice $\mathcal{L}$ such that every element is a join of atoms.

A maximal chain is a maximal length sequence $0 < a_1 < a_2 < \cdots < a_\ell < 1$ in $\mathcal{L}$.

A lattice $\mathcal{L}$ is ranked if all maximal chains in $\mathcal{L}$ have the same length.

Let $\mathcal{L}$ be a ranked lattice and let $a \in \mathcal{L}$.
The rank of a is i if there exists a maximal chain

$$0 < a_1 < a_2 < \cdots < a_\ell < 1 \quad \text{with } a_i = a. \quad \text{Write } \mathrm{rank}(a) = i$$

A projective lattice is an atomic ranked modular lattice such that

$$\text{if } x, y \in \mathcal{L} \text{ then } \mathrm{rank}(x \vee y) + \mathrm{rank}(x \wedge y) = \mathrm{rank}(x) + \mathrm{rank}(y).$$

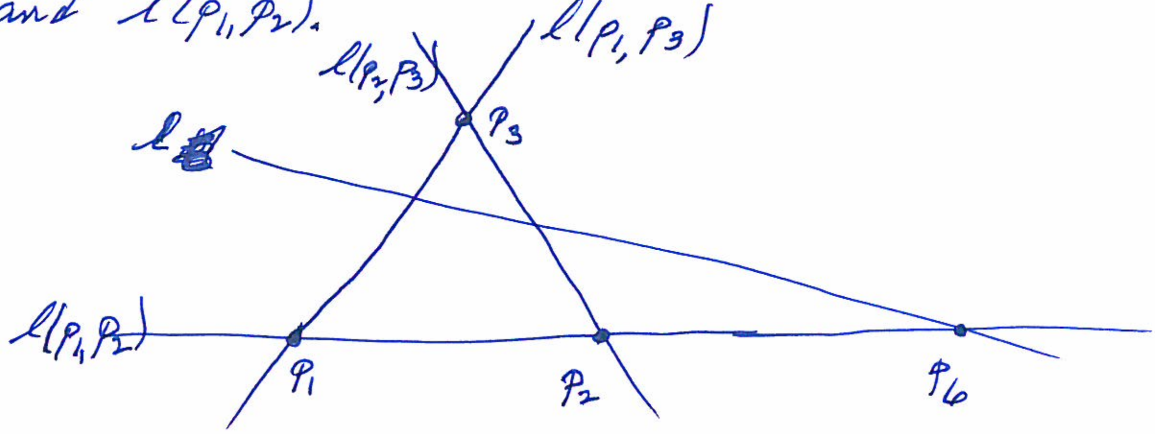
A *projective geometry* is an incidence (P, L, I)

$$\begin{array}{ccc} I \subseteq P \times L & \xrightarrow{pr_1} & P \\ \Big\downarrow{\scriptstyle pr_2} & & \\ L & & \end{array}$$

such that

(a) If $p_1, p_2 \in P$ and $p_1 \neq p_2$ then there exists a unique line $l(p_1, p_2) \in L$ containing p_1 and p_2,

(b) If $p_1, p_2, p_3 \in P$ are noncollinear and l is a line intersecting $l(p_1, p_3)$ and $l(p_2, p_3)$ then there exists $p_6 \in P$ contained in l and $l(p_1, p_2)$.



(c) any line contains at least 3 points

(d) There exist 3 noncollinear points in P

(e) any increasing sequence of subspaces has finite length.

Theorem Let $\mathcal{L}$ be the subspace lattice of (P, L, I)

Then
$$\left\{\begin{array}{c}\text{projective} \\ \text{geometries}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{projective} \\ \text{lattices}\end{array}\right\}$$
$$(P, L, I) \longmapsto \mathcal{L}$$

is a bijection.

Automorphisms

An **automorphism** of (P, L, I) is

$g \in Sym(P) \times Sym(L)$ such that $gI = I$.

Hence, An automorphism of (P, L, I) is ~~$g \in Sym(I)$ such that~~

$g \in (Sym(P) \times Sym(L)) \cap Sym(I)$

If G is the automorphism group of (P, L, I) then

$$\begin{tikzcd} I \subseteq P \times L \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & L \\ P & \end{tikzcd}$$

is G-equivariant.

A **homology** is a matrix g conjugate to

$$\begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

i.e. g is semisimple and fixes a hyperplane.

An **elation** is a matrix g conjugate to

$$\begin{pmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

i.e. g is unipotent and fixes a hyperplane.

Theorem If $\mathcal{L}$ is a projective lattice of rank $r \geq 3$ then $Aut(\mathcal{L}) = P\Gamma L_r(\mathbb{D}) = PGL_r(\mathbb{D}) \rtimes Gal(\mathbb{D}/?)$ where $\mathbb{D}$ is a division ring and

$\mathcal{L}$ is the lattice of subspaces of $\mathbb{D}^r$.