The Gauss-Manin connection

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$$\nabla_{A/S} : \mathbb{H}^1_{dR}(A/S) \longrightarrow \mathbb{H}^1_{dR}(A/S) \otimes \Omega^1_S \quad \text{is} \quad \nabla_{A/S} = d_1$$

where $d_1 : E_1^{0,1} \to E_1^{1,1}$ is the differential of the spectral sequence $(E_r^{p,q}, d_r)$ of the filtration of $\pi_* \Omega^*_{A/k}$ given by

$$L^p(\pi_* \Omega^\bullet_{A/k}) = \pi_*(\pi^* \Omega^p_{S/k} \otimes_{\mathcal{O}_A} \Omega^{\bullet - p}_{A/k})$$

For generalities on spectral sequences associated to a filtration see [Bou, Alg. X §2 Ex 16], [Weibel, Theorem 5.5.1], [SU p107]. The paragraph on [SU, p. 107] is amazingly concise and comprehensive:

Associated to a filtered complex $(F^p C^\bullet)$ is a spectral sequence $(E_r^{p,q}, d_r : E_r^{p,q} \to E_r^{p+r, q-r+1})$ which converges to $H(C^\bullet)$.

This spectral sequence has

$$E_1^{p,q} = H^{p+q}\left(\frac{F^p C^\bullet}{F^{p+1} C^\bullet}\right)$$

Hence

$$E_1^{p,q} = H^{p+q}\left(Gr^p\left(\pi_* \Omega^\bullet_{A/k}\right)\right)$$

$$= H^{p+q}\left(\frac{L^p\left(\pi_*\Omega^\bullet_{A/k}\right)}{L^{p+1}\left(\pi_*\Omega^\bullet_{A/k}\right)}\right)$$

$$= H^{p+q}\left(\frac{\pi_*\left(\pi^*\Omega^p_{S/k}\otimes_{\mathcal{O}_A}\Omega^{\bullet-p}_{A/k}\right)}{\pi_*\left(\pi^*\Omega^{p+1}_{S/k}\otimes_{\mathcal{O}_A}\Omega^{\bullet-(p+1)}_{A/k}\right)}\right)$$

$$= H^{p+q}\pi_*\left(\pi^*\Omega^p_{S/k}\otimes_{\mathcal{O}_A}\Omega^{\bullet-p}_{A/k}\right)$$

$$= \Omega^p_{S/k}\otimes_{\mathcal{O}_S} H^{p+q}\pi_*\left(\Omega^{\bullet-p}_{A/S}\right)$$

$$= \Omega^p_{S/k}\otimes_{\mathcal{O}_S} H^q\pi_*\left(\Omega^\bullet_{A/S}\right)$$

$$= \Omega^p_{S/k}\otimes_{\mathcal{O}_S} \mathbb{H}^q_{dR}(A/S)$$

The deRahm complex

Fix a morphism of schemes

$\pi : A \to S$,

$\Delta : A \to A \times_S A$, and

I the ideal of $\mathcal{O}_{A\times_S A}$ defining ~~im~~ Δ in $A \times_S A$.

The *de Rahm complex* is

$$\Omega^{\bullet}_{A/S} = (\mathcal{O}_A \to \Omega^1_{A/S} \to \Omega^2_{A/S} \to \cdots)$$

given by

$$\Omega^1_{A/S} = \Delta^*(I/I^2), \qquad \Omega^p_{A/S} = \Lambda^p(\Omega^1_{A/S})$$

and the differentials

$d : \mathcal{O}_A \to \Omega^1_{A/S}$ given by $df = -1\otimes f + f\otimes 1$, and

$d : \Omega^p_{A/S} \to \Omega^{p+1}_{A/S}$ given by

$$d(f\, dg_1 \wedge \cdots \wedge dg_p) = df \wedge dg_1 \wedge \cdots \wedge dg_p$$

for $f \in \mathcal{O}_A$.

The *relative de Rahm cohomology* of A is given by

$$H^q_{dR}(A/S) = H^q \pi_*(\Omega^{\bullet}_{A/S}).$$