26.12.2010 ①

Quiver Hecke algebra representations and characters

Let $\alpha \in Q^+$ and R_α the KLR Quiver Hecke algebra.

Let $\tau : R_\alpha \to R_\alpha$ be the

antiautomorphism given by
$$\begin{aligned} \tau : R_\alpha &\longrightarrow R_\alpha \\ y_i &\longmapsto y_i \\ e_\nu &\longmapsto \cancel{e_\nu} \\ \psi_r &\longmapsto \psi_r . \end{aligned}$$

Let $Proj(R_\alpha)$ = category of projective $\mathbb{Z}$-graded R_α-modules

$Rep(R_\alpha)$ = category of finite dimensional $\mathbb{Z}$graded R_α-modules

Define
$$P = \bigoplus_{\alpha \in Q^+} [Proj(R_\alpha)] \quad \text{and} \quad R = \bigoplus_{\alpha \in Q^+} [Rep(R_\alpha)]$$

where $[Proj(R_\alpha)]$ is the Grothendieck group of R_α, and $[Rep(R_\alpha)]$ is the Grothendieck group of R_α.

Define
$$\begin{aligned} m : P \otimes P &\longrightarrow P \\ M \otimes N &\longmapsto Ind_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M \otimes N) \\ P \otimes Q &\longmapsto PQ \end{aligned} \qquad \begin{aligned} \circ : R \otimes R &\longrightarrow R \\ M \otimes N &\longmapsto Ind_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M \otimes N) \\ M \otimes N &\longmapsto M \circ N \end{aligned}$$

$$\begin{aligned} r : P\cancel{\otimes P} &\longrightarrow P \otimes P \\ M &\longmapsto Res_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M) \end{aligned} \qquad \begin{aligned} \Delta : R &\longrightarrow R \otimes R \\ M &\longmapsto Res_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M) \end{aligned}$$

should there be a sum over splittings?

Dualities

26.12.2010 (2)

Define $\# : \mathrm{Proj}(R_\alpha) \longrightarrow \mathrm{Proj}(R_\alpha)$

$$M \longmapsto M^\#$$

by

$$M^\# = \mathrm{HOM}_{R_\alpha}(M, R_\alpha) \quad \text{with} \quad (x\varphi)(m) = \varphi(m)\tau(x)$$

where

$$\mathrm{HOM}_{R_\alpha}(M,N) = \bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}_{R_\alpha}(M,N)_l$$

with $\varphi \in \mathrm{Hom}_{R_\alpha}(M,N)_l$ of degree l so that $\varphi : M_i \to N_{i+m}$.

Define $\circledast : \mathrm{Rep}(R_\alpha) \longrightarrow \mathrm{Rep}(R_\alpha)$

$$M \longmapsto M^\circledast$$

by

$$M^\circledast = \mathrm{HOM}_{\mathbb{F}}(M, \mathbb{F}) \quad \text{with} \quad (x\varphi)(m) = \varphi(\tau(x)m)$$

Define $\langle\, , \rangle : [\mathrm{Proj}(R_\alpha)] \times [\mathrm{Rep}(R_\alpha)] \longrightarrow \mathcal{A}$ by

$$\langle [P], [M] \rangle = q\dim(P^\tau \otimes_{R_\alpha} M),$$

where P^τ is the right R_α-module P with action

$$px = \tau(x)p.$$

~~Define~~ Character maps

Define

$$ch : P \longrightarrow \mathcal{U}^- \qquad \mathring{ch} : R \longrightarrow \mathring{\mathcal{U}}^-$$

$$P_i \longmapsto p_i \qquad L_i \longmapsto f_i$$

which satisfy

$$M \longmapsto \sum_{\ell \in \mathbb{Z}} \sum_{u} \dim(e_u M[\ell]) q^{\ell} \dots$$

$$ch([P][Q]) = ch([P])\, ch([Q])$$

$$ch([r(P)]) = r(ch([P]))$$

$$ch([P^\#]) = \overline{ch([P])}$$

$$\mathring{ch}([M] \circ [N]) = \mathring{ch}([M]) \circ ch([N])$$

$$\mathring{ch}(\Delta(M)) = \Delta(\mathring{ch}(M))$$

$$ch(M^{\circledast}) = \overline{ch(M)}$$

$ch : \bigoplus_\alpha Proj(R_\alpha) \to \mathcal{U}^-$ and $\mathring{ch} : \bigoplus_\alpha Rep(R_\alpha) \to \mathring{\mathcal{U}}^-$

$$ch(PQ) = ch(P)\, ch(Q) \qquad \mathring{ch}(M \circ N) = \mathring{ch}(M) \circ ch(N)$$

$$ch(r(P)) = r\, ch(P) \qquad \mathring{ch}(\Delta(M)) = \Delta(\mathring{ch}(M))$$

$$ch(P^\#) = \overline{ch(P)} \qquad \mathring{ch}(M^{\circledast}) = \overline{\mathring{ch}(M)}$$

$$ch(P_i) = p_i \qquad \mathring{ch}(F_i) = f_i.$$

and $ch : P \to \mathcal{U}^-$ and $\mathring{ch} : R \to \mathring{\mathcal{U}}^-$

(i.e. factor through to Grothendieck groups).

canonical basis
= projective indecomposables

dual canonical basis
= simples

Definitions from Kleshchev's Chapt. 5

$\tilde{e}_a M = \mathrm{soc}(e_a M)$ $\quad \tilde{f}_a M = \mathrm{hd}(\mathrm{ind}_{n,1}^{n+1}(M \boxtimes L(a)))$

$e_a = \mathrm{res}_{n-1}^{n-1,1} \circ \Delta_a : \mathcal{H}_n\text{-mod} \longrightarrow \mathcal{H}_{n-1}\text{-mod}$

$\Delta_a M =$ generalized a-eigenspace of x_n on M

$$= \bigoplus_{\underline{a} \in F^n,\, a_n = a} M_{\underline{a}}$$

Then

$$\Delta_a : \mathcal{H}_n\text{-mod} \longrightarrow \mathcal{H}_{n-1,1}\text{-mod}$$

and

$$\mathrm{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L(a^m), \Delta_{a^m} M) \simeq$$

$$\simeq \mathrm{Hom}_{\mathcal{H}_n}(\mathrm{ind}_{n-m,m}^{n} N \boxtimes L(a^m), M).$$

Let

$$\varepsilon_a(M) = \max\{m \geq 0 \mid \Delta_{a^m} M \neq 0\}.$$