On $\mathfrak{Ub}^+$ for $\mathfrak{sl}_2$

1. Two dimensional bialgebras

1.1 Let $\mathfrak{g}$ be a two dimensional vector space over a field k. Then any linear map $[,]: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ which satisfies $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$ satisfies the Jacobi identity. Similarly, any map $\delta: \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$ satisfies the co-Jacobi identity. Let x_1, x_2 be a basis of $\mathfrak{g}$ and suppose that the bracket $[,]$ and the cobracket δ are given by

$$[x_1, x_2] = \alpha_1 x_1 + \alpha_2 x_2, \quad \text{and}$$
$$\delta(x_1) = \beta_1(x_1 \wedge x_2) \quad \text{and} \quad \delta(x_2) = \beta_2(x_1 \wedge x_2).$$

If we change basis so that

$$z_1 = \beta_2 x_1 - \beta_1 x_2,$$
$$z_2 = \alpha_1 x_1 + \alpha_2 x_2,$$

then we get

$$[z_1, z_2] = (\beta_2\alpha_2 + \beta_1\alpha_1) z_2, \quad \text{and}$$
$$\delta(z_1) = 0 \quad \text{and} \quad \delta(z_2) = (\alpha_2\beta_2 + \alpha_1\beta_1) x_1 \wedge x_2 = z_1 \wedge z_2,$$

where we are assuming that $\alpha_2\beta_2 + \alpha_1\beta_1 \neq 0$. One can also perform a change of basis in the case that $\alpha_2\beta_2 + \alpha_1\beta_1 = 0$ so that we may assume that $\mathfrak{g}$ has a basis H, X and that $[,]$ and δ are given by

$$[H, X] = \alpha X, \quad \text{and}$$
$$\delta(H) = 0 \quad \text{and} \quad \delta(X) = \beta(X \wedge H).$$

1.2 Let us check the 1-cocycle condition.

$$\delta([H, X]) = \delta(\alpha X) = \alpha\beta X \wedge H.$$

On the other hand

$$\begin{aligned}(\mathrm{ad}^{\otimes 2} H)\delta(X) - (\mathrm{ad}^{\otimes 2} X)\delta(H) &= (\mathrm{ad}^{\otimes 2} H)\beta(X \otimes H - H \otimes X)\\ &= \beta([H,X]\otimes X - [H,H]\otimes X + X \otimes [H,H] - H \otimes [H,X])\\ &= \beta(\alpha X \otimes H - 0 + 0 - \alpha H \otimes X)\\ &= \alpha\beta X \wedge H.\end{aligned}$$

Thus the 1-cocycle condition is automatically satisfied. So $\mathfrak{g}$ with $[,]$ and δ is a Lie bialgebra.

1.3 The matrices of the representation ad, with respect to the basis H, X such that $[H, X] = \alpha X$ are given by

$$\mathrm{ad}\ H = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$$
$$\mathrm{ad}\ X = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}.$$

If we define an inner product on $\mathfrak{g}$ by $\langle x, y\rangle = tr((\mathrm{ad}\ x)(\mathrm{ad}\ y))$ then we have

$$\langle H, H\rangle = \alpha^2, \qquad \langle H, X\rangle = 0, \qquad \langle X, X\rangle = 0.$$

1.4 If we assume that $\langle,\rangle$ is an invariant inner product on $\mathfrak{g}$ then the equations

$$\alpha\langle X, X\rangle = \langle [H,X], X\rangle = -\langle X, [H,X]\rangle = -\alpha\langle X, X\rangle,$$
$$\alpha\langle H, X\rangle = \langle H, [H,X]\rangle = -\langle [H,H], X\rangle = 0,$$

imply that if $\alpha \neq 0$ then $\langle X, X\rangle = 0$ and $\langle H, X\rangle = 0$. Thus, if $\alpha \neq 0$ then, up to normalization, the inner product given in (1.3) is the only invariant scalar product on $\mathfrak{g}$.

1.5 Since H acts diagonally on $\mathfrak{g}$ and on $\mathfrak{g}\otimes\mathfrak{g}$ via ad and $\mathrm{ad}^{\otimes 2}$ respectively it follows that the only possible invariant in $\mathfrak{g}\otimes\mathfrak{g}$ is $H \otimes H$. Since $(\mathrm{ad}^{\otimes 2} X)(H \otimes H) = -\alpha(X \otimes H + H \otimes X)$, $H \otimes H$ is not invariant. Thus there are no invariants in $\mathfrak{g}\otimes\mathfrak{g}$. It follows that the cobracket δ is not a coboundary and that there is no "r" matrix giving a bialgebra structure on $\mathfrak{g}$.

2. Quantization of $\mathfrak{U}\mathfrak{g}$

2.1 First let us begin with the Lie algebra $\mathfrak{h} = kH$, which is one dimensional and abelian, i.e. $[H, H] = 0$. The enveloping algebra $\mathfrak{U}\mathfrak{h}$ is isomorphic to $k[H]$ as an algebra. Then let us try to create a quantization of $\mathfrak{U}\mathfrak{h}$ with the same multiplication. More precisely we want to find a coproduct Δ_h on the algebra $\mathfrak{U}\mathfrak{h}[[h]]$ where the multiplication is the $k[[h]]$ linear extension of the ordinary multiplication in $\mathfrak{U}\mathfrak{h}$ to the completion $\mathfrak{U}\mathfrak{h}[[h]]$. The condition is that Δ_h must be coassociative and an algebra homomorphism. Furthermore we must have that

$$\Delta_h(H) \bmod h = H \otimes 1 + 1 \otimes H.$$

In other words, as an algebra the quantization A is a bialgebra over $k[[h]]$, complete in the h-adic topology, and generated by a single element H.

2.2 There is a natural grading on $\mathfrak{U}\mathfrak{h}$ given by the degree as a polynomial in H over k. Similarly there is a natural grading on $\mathfrak{U}\mathfrak{h}[[h]]$ such that $\deg(H) = 1$. This induces a grading on $\mathfrak{U}\mathfrak{h}[[h]] \otimes \mathfrak{U}\mathfrak{h}[[h]]$. We would like the coproduct Δ_h on $\mathfrak{U}\mathfrak{h}[[h]]$ to preserve this grading. This assumption forces us to have

$$\Delta_h(H) = H \otimes f + g \otimes H = f(H \otimes 1) + g(1 \otimes H)$$

where $f, g \in k[[h]]$. The coassociative condition then reads

$$\begin{aligned}
(id \otimes \Delta_h)\Delta_h(H) &= f(H \otimes 1 \otimes 1) + g(1 \otimes H \otimes f) + g(1 \otimes g \otimes H) \\
&= f(H \otimes 1 \otimes 1) + fg(1 \otimes H \otimes 1) + g^2(1 \otimes 1 \otimes H) \\
= (\Delta_h \otimes id)\Delta_h(H) &= f(H \otimes f \otimes 1) + f(g \otimes H \otimes 1) + g(1 \otimes 1 \otimes H) \\
&= f^2(H \otimes 1 \otimes 1) + fg(\otimes H \otimes 1) + g(1 \otimes 1 \otimes H).
\end{aligned}$$

Thus we must have that $f^2 = f$ and $g^2 = g$. The condition that $\Delta_h(H) \bmod h = H \otimes 1 + 1 \otimes H$ forces $f \bmod h = g \bmod h = 1$. Thus f and g are invertible elements of $k[[h]]$. It follows then that $f = g = 1$. Thus the only quantization of $\mathfrak{U}\mathfrak{h}$ which preserves grading is the trivial one.

2.3 Now let us construct a quantization of the two dimensional Lie bialgebra $\mathfrak{g}$ with basis H, X such that

$$\begin{gathered}
[H, X] = \beta X, \quad \text{and} \\
\delta(H) = 0, \quad \text{and} \quad \delta(X) = \beta X \wedge H.
\end{gathered}$$

There is a natural grading on $\mathfrak{U}\mathfrak{g}$ given by putting $\deg(H) = 0$ and $\deg(X) = 1$. Let A be the associative algebra over $k[[h]]$, complete in the h-adic topology, and generated by H and X with relations

$$[H, X] = HX - XH = \alpha X.$$

We want to construct a coproduct Δ_h on A such that $\mathfrak{U}\mathfrak{h}[[h]]$ as given above is a subbialgebra of A. This condition, in combination with a condition requiring that the coproduct Δ_h preserve the grading from A to $A \otimes A$ allows us to make the assumptions that

$$\begin{gathered}
\Delta_h(H) = H \otimes 1 + 1 \otimes H, \quad \text{and} \\
\Delta_h(X) = X \otimes f + g \otimes X,
\end{gathered}$$

where

$$\begin{aligned}
f &= \sum_{i \geq 0} \frac{f_i H^i}{i!}, \quad \text{and} \\
g &= \sum_{j \geq 0} \frac{g_j H^j}{j!},
\end{aligned}$$

for some elements $f_i, g_j \in k[[h]]$.

2.4 Since $\Delta_h(H) = \Delta(H)$, by linearity, $\Delta_h(f) = \Delta(f)$ and $\Delta_h(g) = \Delta(g)$. The coassociativity condition gives that

$$\begin{aligned}(id \otimes \Delta_h)\Delta_h(X) &= X \otimes \Delta(f) + g \otimes \Delta_h(X) \\ &= X \otimes \Delta(f) + g \otimes X \otimes f + g \otimes g \otimes X \\ = (\Delta_h \otimes id)\Delta_h(X) &= \Delta_h(X) \otimes f + \Delta(g) \otimes X \\ &= X \otimes f \otimes f + g \otimes X \otimes h + \Delta(g) \otimes X.\end{aligned}$$

It follows that $\delta(f) = f \otimes f$ and $\Delta(g) = g \otimes g$. We have

$$\begin{aligned}\Delta(f) &= \sum_{k\geq 0} \frac{f_k}{k!}\Delta(H^k) \\ &= \sum_{k\geq 0} \frac{f_k}{k!}(H \otimes 1 + 1 \otimes H)^k \\ &= \sum_{k\geq 0} \frac{f_k}{k!} \sum_{i+j=k} \binom{k}{i} H^i \otimes H^j \\ &= \sum_{i,j\geq 0} \frac{f_{i+j}}{i!j!} H^i \otimes H^j \\ = f \otimes f &= \sum_{i,j\geq 0} \frac{f_i f_j}{i!j!} H^i \otimes H^j.\end{aligned}$$

It follows that $f_i f_j = f_{i+j}$ for all i, j. In particular,

$$f_n = f_{1+1+\cdots+1} = f_1^n, \quad \text{and}$$
$$f_0^2 = f_0.$$

2.5 The deformation A must satisfy $(\Delta_h(X) \bmod h) = \Delta(X)$. This forces

$$f_0 \pmod h \ = 1, \quad \text{and}$$
$$(f_1 \bmod h) = (g_1 \bmod h) = 0.$$

So f_0 is invertible in $k[[h]]$. It follows that $f_0 = 1$ and that

$$f = \sum_{k\geq 0} \frac{f_1^k H^k}{k!} = exp(f_1 H), \quad \text{and}$$
$$g = exp(g_1 H).$$

2.6 The quantization A must satisfy

$$\frac{\Delta(a) - \sigma\Delta(a)}{h} \bmod h \ = \delta(a \bmod h),$$

for all $a \in A$. This forces that

$$\begin{aligned}&h^{-1}(\Delta_h(X) - \sigma\Delta_h(X)) \pmod h \\ &\quad = h^{-1}(X \otimes exp(f_1 H) + exp(g_1 H) \otimes X - exp(f_1 H) \otimes X - X \otimes exp(g_1 H)) \pmod h \\ &\quad = h^{-1}(X \otimes (exp(f_1 H) - exp(g_1 H)) + (exp(g_1 H) - exp(f_1 H)) \otimes X) \pmod h.\end{aligned}$$

Since $(f_1 \bmod h) = (g_1 \bmod h) = 0$ this is equal to

$$\beta(X \wedge H) = \delta(X) = h^{-1}(\Delta_h(X) - \sigma\Delta_h(X)) \pmod h = X \otimes (f_1 - g_1 \bmod h)H + (g_1 - f_1 \bmod h)H \otimes X.$$

It follows that $((f_1 - g_1) \bmod h^2) = \beta h$.

2.7 Since we know that $\Delta_h(exp(f_1 H)) = \Delta_h(f) = f \otimes f = exp(f_1 H) \otimes exp(f_1 H)$ and that Δ_h is an algebra homomorphism we have that

$$\begin{aligned}
&\Delta_h(Xexp(-(f_1+g_1)H/2)) \\
&\quad = Xexp(-(f_1+g_1)H/2) \otimes exp((f_1-g_1)H/2) + exp(-(f_1-g_1)H/2) \otimes Xexp(-(f_1+g_1)H/2), \quad \text{and} \\
&\Delta_h(Xexp(-g_1 H/2)) = Xexp(-g_1 H/2) \otimes exp((f_1-g_1)H/2) + exp(0) \otimes Xexp(-g_1 H/2).
\end{aligned}$$

Therefore, by a change of variables, we may choose $f \in k[[h]]$ such that $f \bmod h^2 = \beta h$ and generators X and H of A such that Δ_h is given by

$$\begin{aligned}
\Delta_h(X) &= X \otimes exp(fH) + exp(-fH) \otimes X, \quad \text{or} \\
\Delta_h(X) &= X \otimes exp(fH) + 1 \otimes X.
\end{aligned}$$

2.8 Suppose that $f \in k[[h]]$ and that $f \bmod h = 0$ and that $f \bmod h^2 = \beta \neq 0$. Then we have that

$$f = \beta h + \beta_2 h^2 + \beta_3 h^3 + \cdots,$$

where $\beta_i \in k$. Since $\beta \neq 0$ we have

$$f = \beta h(1 + \beta^{-1} hQ),$$

where Q is some element of $k[[h]]$. Since $1 + \beta^{-1}Q$ is invertible in $k[[h]]$ it follows that we may replace f by βh after a change of variable. Thus we may assume that the quantization is of the form

$$\begin{aligned}
\Delta_h(X) &= X \otimes exp(\beta hH) + exp(-\beta hH) \otimes X, \quad \text{or} \\
\Delta_h(X) &= X \otimes exp(\beta hH) + 1 \otimes X.
\end{aligned}$$

2.9 Define an element $K \in A$ by the equation

$$K = exp(\beta hH).$$

The element K is invertible with inverse $K^{-1} = exp(-\beta hH)$. Using the relation $[H, X] = \alpha X$ we have that

$$H^k X = H^{k-1}(HX - XH + XH) = H^{k-1}(\alpha X + XH) = H^{k-1}X(\alpha + H).$$

By induction it follows that $H^k X = X(\alpha + H)^k$. Thus

$$\begin{aligned}
KX &= \sum_{k \geq 0} \frac{\beta^k h^k H^k X}{k!} \\
&= \sum_{k \geq 0} X \frac{\beta^k h^k (\alpha + H)^k}{k!} \\
&= Xexp(\beta h(\alpha + H)) \\
&= exp(\alpha\beta h)XK.
\end{aligned}$$

It follows that

$$KXK^{-1} = exp(\alpha\beta h)X.$$

2.10 We want to define a Hopf algebra structure on A. Let us assume that Δ_h is given by $\Delta_h(H) = 1 \otimes H + H \otimes 1$ and $\Delta_h(X) = X \otimes K + K^{-1} \otimes X$. Then the counit condition gives that

$$H = m(id \otimes \varepsilon)\Delta(H) = H\varepsilon(1) + 1\varepsilon(H),$$

forcing $\varepsilon(H) = 0$. It follows that $\varepsilon(K) = 1$. Using the counit condition again we have that

$$X = m(id \otimes \varepsilon)\Delta(X) = X\varepsilon(K) + K^{-1}\varepsilon(X) = X \cdot 1 + K^{-1}\varepsilon(X).$$

It follows that $\varepsilon(X) = 0$. The antipode condition gives that

$$0 = \varepsilon(H) = HS(1) + 1 \cdot S(H) = H + S(H).$$

Thus $S(H) = -H$. We get that $S(K) = K^{-1}$. Using the antipode condition again gives

$$0 = \varepsilon(X) = m(id \otimes S)\Delta(X) = XS(K) + K^{-1}S(X) = XK^{-1} + K^{-1}S(X).$$

Thus $S(X) = -KXK^{-1} = -\exp(\alpha\beta h)X$. So the Hopf structure on A is given by

$$\begin{array}{ccc} \varepsilon(H) = 0, & \varepsilon(K) = 1, & \varepsilon(X) = 0 \\ S(H) = -H, & S(K) = K^{-1}, & S(X) = -\exp(\alpha\beta h)X. \end{array}$$

If we assume that the coproduct is of the form $\Delta_h(X) = X \otimes K + 1 \otimes K$ then similar calculations will show that the counit and the antipode are given by

$$\begin{array}{ccc} \varepsilon(H) = 0, & \varepsilon(K) = 1, & \varepsilon(X) = 0 \\ S(H) = -H, & S(K) = K^{-1}, & S(X) = -XK^{-1}. \end{array}$$

3. The double $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ of $\mathfrak{g}$

3.1 Let us compute the double of $(\mathfrak{g}, \delta)$. Let $H^*, Y \in \mathfrak{g}^*$ be the dual basis to H, X. Then

$$\begin{aligned} \langle [H^*, Y], X\rangle &= \langle H^* \otimes Y, \delta(X)\rangle \\ &= \beta\langle H^* \otimes Y, X \otimes H - H \otimes X\rangle \\ &= -\beta. \end{aligned}$$

Since $\delta(H) = 0$, $\langle [H^*, Y], H\rangle = 0$. Thus we get that $[H^*, Y] = -\beta X^*$. The remainder of the brackets are computed by using the invariance of the inner product, giving

$$\begin{aligned} [H, X] &= \alpha X, \qquad & [H, Y] &= -\alpha Y, \quad [H, H*] = 0, \\ [H^*, X] &= \beta X, & [H^*, Y] &= -\beta Y, \\ [X, Y] &= \beta H + \alpha H^*, \end{aligned}$$

as the multiplication table for $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$.

3.2 Make the following change of basis,

$$X = X, \quad Y = Y, \quad K_1 = (\alpha\beta)^{-1}(\alpha H^* + \beta H), \quad K_2 = (\alpha\beta)^{-1}(\alpha H^* - \beta H).$$

Then $[\mathfrak{g} \oplus \mathfrak{g}^*, K_2] = 0$ and

$$[K_1, X] = 2X, \quad [K_1, Y] = -2Y, \quad [X, Y] = \alpha\beta K_1.$$

3.3 Since $\mathfrak{g}$ has basis $\{X, H\}$ and $\mathfrak{g}^*$ has dual basis $\{Y, H^*\}$, let

$$r = X \otimes Y + H \otimes H^*.$$

Then

$$\begin{aligned} \phi(X) &= (\mathrm{ad}^{\otimes 2} X)(r) = \beta(X \wedge H), \\ \phi(H) &= (\mathrm{ad}^{\otimes 2} H)(r) = 0, \\ \phi(H^*) &= (\mathrm{ad}^{\otimes 2} H^*)(r) = 0, \\ \phi(Y) &= (\mathrm{ad}^{\otimes 2} Y)(r) = \alpha(Y \wedge H), \end{aligned}$$

gives a Lie bialgebra structure on $D(\mathfrak{g})$.

3.4 We have that

$$H = \frac{\alpha}{2}(K_1 - K_2), \qquad \text{and} \qquad H^* = \frac{\beta}{2}(K_1 + K_2).$$

Substituting gives

$$r = X \otimes Y + \frac{\alpha\beta}{4}(K_1 - K_2) \otimes (K_1 + K_2).$$

Set $K_2 = 0$, i.e., project modulo the ideal spanned by K_2, to get

$$\bar{r} = X \otimes Y + \frac{\alpha\beta}{4}(K_1 \otimes K_1).$$

Then, one has that $\bar{r}$ determines a Lie bialgebra structure on $D(\mathfrak{g})/K_2$ which has basis $\{X, Y, K_1\}$, bracket given by

$$[K_1, X] = 2X, \quad [K_1, Y] = -2Y, \quad [X, Y] = \alpha\beta K_1,$$

and cobracket given by

$$\begin{aligned} \bar{\phi}(X) &= (\mathrm{ad}^{\otimes 2} X)(\bar{r}) = \frac{\alpha\beta}{2}(X \wedge H), \\ \bar{\phi}(H) &= (\mathrm{ad}^{\otimes 2} H)(\bar{r}) = 0, \\ \bar{\phi}(Y) &= (\mathrm{ad}^{\otimes 2} Y)(\bar{r}) = \frac{\alpha\beta}{2}(Y \wedge H). \end{aligned}$$

4. References

This example appears in Drinfel'd's paper as Examples 3.1 and 6.1.

[D] V.G. Drinfeld, *Quantum Groups*, Vol. 1 of *Proceedings of the International Congress of Mathematicians, Berkeley, California, USA, 1986.* Academic Press, 1987, pp. 798-820.

The derivation of the quantization of $\mathfrak{U}\mathfrak{b}^+$ appears in the following paper.

[CP] V. Chari and A. Pressley, *Introduction to quantum groups*, Proceedings of the Hyderabad Conference in Algebraic Groups, Madras: Manoj Prakashan (1991).