Numbers

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PICTURE

π is the distance half way around a circle of radius 1.

Measure angles according to the distance traveled on a circle of radius 1.

PICTURE

The angle θ is measured by traveling a distance θ on a circle of radius 1.

Stretch both x and y to get a circle of radius r.

PICTURE

The distance θ stretches to $r\theta$. Hence, the

$$(\text{arc length along an angle } \theta \text{ on a circle of radius } r) = r\theta.$$

The distance 2π around a circle of radius 1 stretches to $2\pi r$ around a circle of radius r. So the circumference of a circle is $2\pi r$ is the circle is radius r.

To find the area of a circle first approximate with a polygon inscribed in the circle.

PICTURE

the eight triangles form an octagon P_8 in the circle. The area of the octagon is almost the same as the area of the circle.

Unwrap the octagon.

PICTURE

The area of the octagon is the area of the 8 triangles. The area of each triangle is $\frac{1}{2}bh$. So the area of the octagon is $\frac{1}{2}Bh$.

Take the limit as the number of triangles in the interior polygon gets larger and larger (the polygon gets closer and closer to being the circle). Then

$$\begin{aligned}
\text{Area of the circle} &= \lim_{n\to\infty} \Big(\text{area of an } n\text{-sided polygon } P_n\Big) \\
&= \lim_{n\to\infty} \Big(\frac{1}{2}Bh\Big) \\
& PICTURE \text{total base} \quad \text{height of triangle} \\
&= \frac{1}{2}(2\pi r)(r) \\
& PICTURE \text{length of an unwrapped circle} \quad \text{radius of the circle} \\
&= \pi r^2.
\end{aligned}$$

So the area of a circle is πr^2 if the circle is radius r, and the

$$\begin{aligned}
(\text{area of an arc of angle } \theta \text{ for a circle of radius } r) &= \frac{\theta}{2\pi} \cdot (\text{area of the whole circle}) \\
&= \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{\theta r^2}{2}.
\end{aligned}$$

Trigonometric functions

$$\begin{aligned}
&\sin\theta \text{ is the } y\text{-coordinate of a point at distance } \theta \text{ on a circle of radius 1,} \\
&\cos\theta \text{ is the } x\text{-coordinate of a point at distance } \theta \text{ on a circle of radius 1,} \\
&\tan\theta = \frac{\sin\theta}{\cos\theta}, \\
&\cot\theta = \frac{\cos\theta}{\sin\theta}, \\
&\sec\theta = \frac{1}{\cos\theta}, \\
&\csc\theta = \frac{1}{\sin\theta},
\end{aligned}$$

Since the equation of a circle of radius 1 is $x^2 + y^2 = 1$ this forces

$$\sin^2\theta + \cos^2\theta = 1.$$

The pictures

$$PICTURE \qquad \text{and} \qquad PICTURE$$

show that

$$\sin(-\theta) = -\sin\theta \qquad \text{and} \qquad \cos(-\theta) = \cos\theta.$$

Also

$$PICTURE \qquad \text{and} \qquad PICTURE$$

show that

$$\begin{matrix} \sin 0 = 0 \\ \cos 0 = 1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} \sin\frac{\pi}{2} = 1, \\ \cos\frac{\pi}{2} = 0. \end{matrix}$$

Draw the graphs

$$\textit{PICTURE} \qquad \text{and} \qquad \textit{PICTURE},$$

by seeing how the x and y coordinates change as you walk around the circle.

There are five trig identities to remember:

$$\begin{gathered}\sin(x+y) = \sin x \cos y + \cos x \sin y, \\ \cos(x+y) = \cos x \cos y - \sin x \sin y, \\ \sin^2 x + \cos^2 x = 1, \\ \sin(-x) = -\sin x \quad \text{and} \quad \cos(-x) = \cos x,\end{gathered}$$

As well as the two triangles

$$\textit{PICTURE} \qquad \text{and} \qquad \textit{PICTURE}.$$

From these triangles,

$$\begin{array}{ll} \sin\frac{\pi}{6} = \frac{1}{2} & \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} & \cos\frac{\pi}{3} = \frac{1}{2} \\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} & \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{array}$$

Since the only trig identities I remember are identities for sines and consines I usually verify trig identities by first writing them completely in terms of sines and cosines.

Example. Verify $\dfrac{\sec B}{\cos B} - \dfrac{\tan B}{\cot B} = 1$.

$$\begin{aligned}\frac{\sec B}{\cos B} - \frac{\tan B}{\cot B} &= \frac{\left(\dfrac{1}{\cos B}\right)}{\cos B} - \frac{\left(\dfrac{\sin B}{\cos B}\right)}{\left(\dfrac{\cos B}{\sin B}\right)} \\ &= \frac{1}{\cos^2 B} - \frac{\sin^2 B}{\cos^2 B} = \frac{1-\sin^2 B}{\cos^2 B} = \frac{\cos^2 B}{\cos^2 B} = 1.\end{aligned}$$

Example. Verify $\cot\alpha - \cot\beta = \dfrac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$.

$$\begin{aligned}\text{Left Hand Side} = \cot\alpha - \cot\beta &= \frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta} \\ &= \frac{\cos\alpha\sin\beta - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}\end{aligned}$$

$$\begin{aligned}\text{Right Hand Side} = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta} &= \frac{\sin\beta\cos(-\alpha) + \cos\beta\sin(-\alpha)}{\sin\alpha\sin\beta} \\ &= \frac{\sin\beta\cos\alpha + \cos\beta(-\sin\alpha)}{\sin\alpha\sin\beta} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}.\end{aligned}$$

So

$$\text{Left Hand Side} = \text{Right Hand Side}.$$

Example. Verify $\dfrac{\tan A - \sin A}{\sec A} = \dfrac{\sin^3 A}{1+\cos A}$.

$$\frac{\tan A - \sin A}{\sec A} =^? \frac{\sin^3 A}{1+\cos A}$$

So $\quad (1+\cos A)(\tan A - \sin A) =^? \sin^3 A \sec A.$

So $\quad \tan A + \cos A \tan A - \sin A - \sin A \cos A =^? \sin^3 A \sec A.$

So $\quad \dfrac{\sin A}{\cos A} + \cos A\left(\dfrac{\sin A}{\cos A}\right) - \sin A - \sin A \cos A =^? \sin^3 A\left(\dfrac{1}{\cos A}\right).$

So $\quad \dfrac{\sin A}{\cos A} + \sin A - \sin A - \sin A \cos A =^? \sin^3 A\left(\dfrac{1}{\cos A}\right).$

So $\quad \dfrac{\sin A - \sin A \cos^2 A}{\cos A} =^? \dfrac{\sin^3 A}{\cos A}$

So $\quad \sin A - \sin A \cos^2 A =^? \dfrac{\sin^3 A}{\cos A}.$

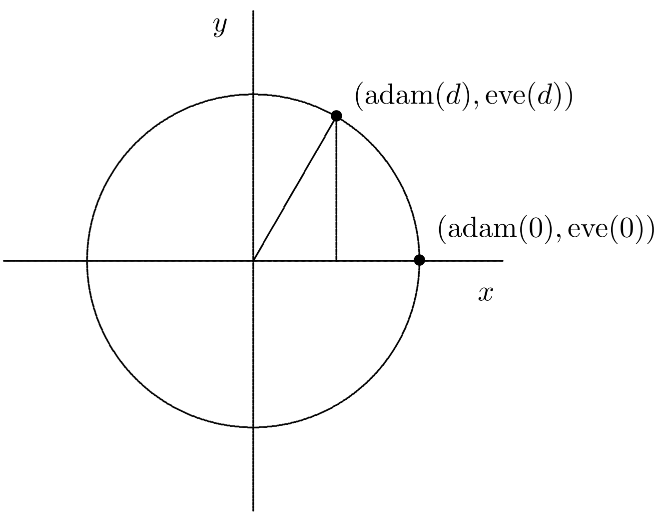
So $\quad 1 - \cos^2 A =^? \sin^2 A.$

YES, because $\sin^2 A + \cos^2 A = 1$.

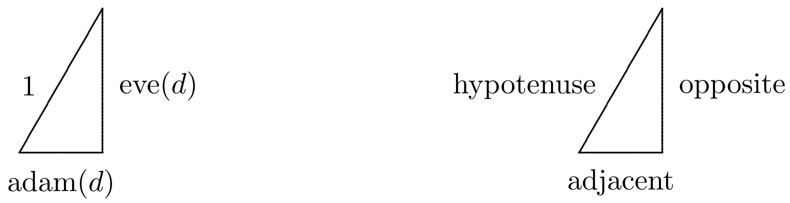
and so

adam(t) = x-coordinate of the point on a circle of radius 1 which is distance d from the point (1,0), and

eve(t) = y-coordinate of the point on a circle of radius 1 which is distance d from the point (1,0).



The triangle in this picture is



and so

$$\text{adam}(d) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{and} \qquad \text{eve}(d) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

for a right triangle with angle d.