Radicals

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1. Definitions

Finiteness conditions

Let R be a ring with identity and let M be an R-module.

The module M is **noetherian** if it satisfies ACC on submodules.

The module M is **artinian** if it satisfies DCC on submodules.

The module M is **finitely generated** if there is a finite subset $\{m_1, \ldots, m_k\}$ of M such that $M = \text{span-}\{m_1, \ldots, m_k\}$, the submodule generated by $\{m_1, \ldots, m_k\}$.

A **composition series** of M is a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i+1} is simple.

Proposition 1.1. *Let N be a submodule of M.*

(a) *M is noetherian if and only if N and M/N are noetherian.*

(b) *M is artinian if and only if N and M/N are artinian.*

(c) *M has a finite composition series if and only if N and M/N have finite composition series.*

(d) *If M is finitely generated then M/N is finitely generated.*

Proof. ∎

Proposition 1.2.

(a) *M is noetherian and artinian if and only if M has a finite composition series.*

(b) *M is noetherian if and only if every submodule of M is finitely generated.*

(c) *If R is noetherian and M is finitely generated then M is noetherian.*

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Proof. (a) follows from Theorem ???, below.

(b) $\Leftarrow$: Assume that every submodule of M is finitely generated. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain. Then $\bigcup N_i$ is a finitely generated submodule of M Let $x_1, \ldots, x_k$ be generators and let $\ell_1, \ldots, \ell_k$ be such that $x_i \in N_{\ell_i}$. Then $x_1, \ldots, x_k \in N_r$ where $r = \max\{\ell_1, \ldots, \ell_k\}$. So $\bigcup N_i = N_r$ and $N_r = N_{r+1} = N_\ell$ for all $\ell > r$. So M is noetherian.

(b) $\Rightarrow$: Assume that M is noetherian and let N be a submodule of M. Then

$$\{P \subseteq N \mid P \text{ is finitely generated}\}$$

has a maximal element $P_{\max}$. If $P_{\max} \neq N$ let $x \in N \backslash P_{\max}$. Then $P \subsetneq \langle P_{\max}, x\rangle \subseteq N$ and $\langle P_{\max}, x\rangle$ is finitely generated, which is a contradiction to the maximality of $P_{\max}$. So $P_{\max} = N$. So every submodule of M is finitely generated. ▮

Theorem 1.3. *Let M be an R-module.*

(a) Any two series

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \qquad \text{and} \qquad 0 \subseteq M_1' \subseteq M_2' \subseteq \cdots \subseteq M_s' = M$$

can be refined to have the same length and the same composition factors.

(b) M has a finite composition series if and only if M any series can be refined to a composition series.

(c) If M has a finite composition series then any two composition series for M have the same length.

Proof. In the series (*) change $M_i \subseteq M_{i+1}$ to

$$M_i = (M_0' + M_i) \cap M_{i+1} \subseteq (M_1' + M_i) \cap M_{i+1} \subseteq \cdots \subseteq (M_s' + M_i) \cap M_{i+1} = M_{i+1},$$

and change $M_j' \subseteq M_{j+1}'$ to

$$M_j = (M_0 + M_j') \cap M_{j+1}' \subseteq (M_1 + M_j') \cap M_{j+1}' \subseteq \cdots \subseteq (M_r + M_j') \cap M_{j+1}' = M_{j+1}'.$$

Claim:

$$\frac{(M_j' + M_{i-1}) \cap M_i}{(M_{j-1}' + M_{i-1}) \cap M_i} \cong \frac{(M_i + M_{j-1}') \cap M_j'}{(M_{i-1} + M_{j-1}') \cap M_j'}.$$

Lemma 1.4. (Modular law) *If A, B, C are submodules of M and $B \subseteq C$ then*

$$C + (A \cap B) = (C + A) \cap B.$$

Proof. If $k + b \in C + (A \cap B)$ then $k + b \in (C + A) \cap B$. So $C + (A \cap B) \subseteq (C + A) \cap B$.

If $b = k + a \in (C + A) \cap B$ then $b = k + a = k + (b - k) \in C + (A \cap B)$ and so $(C + A) \cap B \subseteq C + (A \cap B)$. ▮

Lemma 1.5. Zassenhaus isomorphism *If* $V \subseteq U$ *and* $V' \subseteq U'$ *are submodules of* M *then*

$$\frac{(U+V')\cap U'}{(V+V')\cap U'} \cong \frac{U\cap U'}{(U\cap V')+(U'\cap V)} \cong \frac{(U'+V)\cap U}{(V'+V)\cap U}.$$

Proof. ∎

Examples.

(1) Let $\mathbb{F}$ be a field. A finite dimensional vector space is both noetherian and artinian. An infinite dimensional vector space V has $\mathrm{Rad}(V)=0$, $\mathrm{soc}(V)$ and is neither noetherian or artinian.

(2) Let $R=\mathbb{Z}$. Then every submodule of ${}_RR$ is generated by one element. The ring $\mathbb{Z}$ is noetherian but not artinian: $\mathbb{Z} \supseteq p\mathbb{Z} \supseteq p^2\mathbb{Z} \supseteq \cdots$.

$$\mathrm{Rad}(\mathbb{Z}) = \bigcap_{L_{\max}} L_{\max} = \bigcap_{p \text{ prime}} p\mathbb{Z} = 0.$$

Radicals and socles

If $m \in M$, $\mathrm{ann}(m) = \{r \in R \mid rm = 0\}$.

The **annihilator** of M is $\mathrm{ann}(M) = \{r \in R \mid rM = 0\}$.

The **radical** of M is

$$\mathrm{Rad}(M) = \bigcap_{P_{\max}} P_{\max}, \qquad \text{the intersection of the maximal proper submodules of } M.$$

The **socle** of M is

$$\mathrm{soc}(M) = \sum_{P_{\min}} P_{\min}, \qquad \text{the sum of the simple submodules of } M.$$

The **head** of M is $M/\mathrm{Rad}(M)$.

The **socle series** of M is

$$0 = \mathrm{soc}^0(M) \subseteq \mathrm{soc}^1(M) \subseteq \cdots$$

where $\mathrm{soc}^1(M) = M$ and $\mathrm{soc}^i(M)$ is determined by

$$\frac{\mathrm{soc}^i(M)}{\mathrm{soc}^{i-1}(M)} = \mathrm{soc}\left(\frac{M}{\mathrm{soc}^{i-1}(M)}\right).$$

The **radical series** of M is

$$0 = \mathrm{Rad}^0(M) \supseteq \mathrm{Rad}^1(M) \supseteq \cdots \qquad \text{where} \quad \mathrm{Rad}^i(M) = \mathrm{Rad}(\mathrm{Rad}^{i-1}(M)).$$

The **socle length** of M is the smallest positive integer n such that $\mathrm{soc}^n(M) = M$ and $\mathrm{soc}^{n-1}(M) \neq M$.

The **radical length** of M is the smallest positive integer n such that $\mathrm{Rad}^n(M) = 0$ and $\mathrm{Rad}^{n-1}(M) \neq 0$.

The **socle layers** of M are $\mathrm{soc}^k(M)/\mathrm{soc}^{k-1}(M)$.
The **radical layers** of M are $\mathrm{Rad}^{k-1}(M)/\mathrm{Rad}^k(M)$.

Proposition 1.6. *If M has socle length n them M has radical length n and*

$$\mathrm{soc}^j(M) \supseteq \mathrm{Rad}^{n-j}(M), \qquad 0 \le j \le n.$$

Proof. ∎

Proposition 1.7. *Let R be a ring.*

(a) $\mathrm{Rad}(R) = \bigcap_{L_{\max}} L_{\max}$, *the intersection of the maximal left ideals of R.*

(b) $\mathrm{Rad}(R) = \bigcap_{I_{\mathrm{prim}}} I_{\mathrm{prim}}$, *the intersection of the primitive two-sided ideals of R.*

(c) $\mathrm{Rad}(R) = \{x \in R \mid 1 - axb$ *is invertible for all* $a, b \in R\}$.

(d) $\mathrm{Rad}(R)$ *contains all nilpotent ideals.*

Proof. (a) This is a restatement of the definition of $\mathrm{Rad}(R)$, since the submodules of ${}_RR$ are the left ideals of R.

(b) If M is a simple R-module and $m \in M$ then $\mathrm{ann}(m) = \{r \in R \mid rm = 0\}$ is a maximal left ideal of R because $R/\mathrm{ann}(m) \cong M$. The primitive ideal

$$\mathrm{ann}(M) = \{r \in R \mid rM = 0\} = \bigcap_{m \in M} \mathrm{ann}(m).$$

(c) Let $s \in \mathrm{Rad}(R)$. Then $R(1-x) = R$ since $1-x$ is not in any maximal left ideal. So $t(1-x) = 1$ for some $t \in R$. So $1 - t = -tx \in \mathrm{Rad}(R)$. So $1 - (1 - t) = t$ has a left inverse, which must be $1 - x$. So $1 - x$ is invertible in R. By (b) $\mathrm{Rad}(R)$ is an indeal and so $1 - axb$ is invertible for every $a, b \in R$. So $\mathrm{Rad}(R) = \{x \in R \mid 1 - axb$ is invertible for all $a, b \in R\}$

Assume $1-axb$ is invertible for all $a, b \in R$. Let $L_{\max}$ be a maximal left ideal not containing x. Then $1 = ax+\ell$ for some $a \in R$, $p \in L_{\max}$. So $1-ax \in L_{\max}$. So $L_{\max} = R$ which is a contradiction. So x is an element of every maximal left ideal. So $\{x \in R \mid 1 - axb$ is invertible for all $a, b \in R\} \subseteq \mathrm{Rad}(R)$.

(d) Let N be a nilpotent ideal with $N^k = 0$. If $x \in N$ then $x^k \in N^k = 0$ and so $x^k = 0$. Then $(1 + x + x^2 + \cdots + x^{k-1})(1 - x) = 1$ and so $1 - x$ is invertible. Thus, since N is an ideal, $1 - axb$ is invertible for every $a, b \in R$. Thus, by (c), $N \subseteq \mathrm{Rad}(R)$. ∎

The proof of (bb) and (bc) of the following theorem uses:

Lemma 1.8. (Nakayama's lemma) *If M is a finitely generated R-module and* $\mathrm{Rad}(R)M = M$ *them $M = 0$.*

Proof. Assume $M \neq 0$. Let $m_1, \ldots, m_k$ be a minimal generating set for M. Since $\mathrm{Rad}(R)M = M$,

$$m_k = \sum_{i=1}^{k} a_i m_i, \qquad \text{with } a_i \in \mathrm{Rad}(R).$$

So $(1-a_k)m_k = \sum_{i=1}^{k-1} a_i m_i$. But $1-a_k$ has a left inverse in R. So $m_k = \sum_{i=1}^{k-1}(1-a_k)^{-1}a_i m_i$, which contradicts the minimality. So $M=0$. ∎

Theorem 1.9. *Let R be an artinian ring. Then*

(a) $\mathrm{Rad}(R)$ *is the largest nilpotent ideal of R.*

(b) If M is a finitely generated R-module then

(ba) M is noetherian and artinian,

(bb) $\mathrm{Rad}(M) = \mathrm{Rad}(R)M$,

(bc) $\mathrm{soc}(M) = \{m \in M \mid \mathrm{Rad}(R)m = 0\}$.

(c) R is noetherian.

Proof. (a) Let n be such that $\mathrm{Rad}(R)^n = \mathrm{Rad}(R)^{2n}$. If $\mathrm{Rad}(R)^n \neq 0$ then there is a minimal left ideal with $\mathrm{Rad}(R)^n I \neq 0$ (since $\mathrm{Rad}(R)^n\mathrm{Rad}(R)^n \neq 0$). Let $x \in I$, $x \neq 0$, be such that $\mathrm{Rad}(R)^n x \neq 0$. By minimality, $I = \mathrm{Rad}(R)^n x = \mathrm{Rad}(R)^n\mathrm{Rad}(R)^n x$. So $x = ax$, with $a \in \mathrm{Rad}(R)$. So $(1-a)x = 0$. Since $1-a$ is invertible in R, $x = 0$. But this is a contradiction. So $\mathrm{Rad}(R)^n = 0$. So $\mathrm{Rad}(R)$ is a nilpotent ideal.

(ba) Let $M_i = \mathrm{Rad}(R)^i M$. Then, since M is finitely generated and R is artinian, there is a surjective homomorphism

$$R \oplus \cdots \oplus R \longrightarrow M.$$

Thus M is artinian. So M_i/M_{i+1} is artinian and $\mathrm{Rad}(R)$ acts by 0. So M_i/M_{i+1} is a $R/\mathrm{Rad}(R)$-module and thus M_i/M_{i+1} is a finite direct sum of simple submodules. So, by (a), M has a composition series and is both noetherian and artinian.

(bb) By Nakayama's lemma, $\mathrm{Rad}(R)\big(M/N_{\max}\big) = 0$ for every maximal proper submodule $N_{\max} \subseteq M$. So $\mathrm{Rad}(R)M \subseteq N_{\max}$ for every $N_{\max}$. So $\mathrm{Rad}(R)M \subseteq \mathrm{Rad}(M)$.

Since M/M_1 is a finite direct sum of simple modules, $\mathrm{Rad}(M/M_1) = 0$. So $\mathrm{Rad}(M) \subseteq M_1 = \mathrm{Rad}(R)M$.

(bc) The set

$$N = \{m \in M \mid \mathrm{Rad}(R)m = 0\}$$

is a submodule of M and $\mathrm{Rad}(R)N = 0$. Since N is artinian, N is a finite direct sum of simple submodules. So $\mathrm{soc}(M) \supseteq N$.

Nakayama's lemma implies that if S is a simple module, then $\mathrm{Rad}(R)S = 0$. So $\mathrm{Rad}(R)\mathrm{soc}(M) = 0$. So $\mathrm{soc}(M) \subseteq N$.

(c) follows from (ba). ∎

Semisimplicity

Proposition 1.10. *Let M be an R-module. Then M has a simple submodule.*

Proof. ∎

Proposition 1.11. (Schur's lemma)

(a) If R^λ and R^μ are simple R-modules then

$$\mathrm{Hom}_R(R^\lambda, R^\mu) = 0, \quad \text{if } R^\lambda \not\cong R^\mu, \qquad \text{and} \qquad \mathrm{End}_R(R^\lambda) = \mathbb{D}_\lambda \quad \text{is a division ring.}$$

(b) If $M = \bigoplus_{\lambda\in\hat{A}} (R^\lambda)^{\oplus m_\lambda}$ is a finite direct sum of simple modules then

$$\mathrm{End}_R(M) = \bigoplus_{\lambda\in\hat{A}} M_{m_\lambda}(\mathbb{D}_\lambda),$$

where $\mathbb{D}_\lambda = \mathrm{End}_R(R^\lambda)$ are division rings.

Proof. (a) Let $\phi: R^\lambda \to R^\mu$ be a homomorphism. Then, since R^λ and R^μ are simple, $\ker\phi$ is either 0 or R^λ, and $\mathrm{im}\,\phi$ is either 0 or R^μ. So ϕ is either 0 or an isomorphism.

(b) If $M = \bigoplus_{\lambda\in\hat{A}} \bigoplus_{i=1}^{m_\lambda} R^{\lambda,i}$, with $R^{\lambda,i} \cong R^\lambda$, for $1 \le i \le m_\lambda$, then

$$\mathrm{End}_R(M) = \bigoplus_{\lambda\in\hat{A}} \bigoplus_{i,j=1}^{m_\lambda} \mathrm{End}_R(R^{\lambda,i}, R^{\lambda,j}) = \bigoplus_{\lambda\in\hat{A}} M_{m_\lambda}(\mathbb{D}_\lambda).$$

∎

Proposition 1.12. *Let M be an R-module.*

- *(a) $\mathrm{soc}(M) = M$ if and only if for every submodule $N \subseteq M$ there is a submodule $N' \subseteq M$ with $M = N \oplus N'$.*
- *(b) Let N be a submodule of M. If $\mathrm{soc}(M) = M$ then $\mathrm{soc}(N) = N$ and $\mathrm{soc}(M/N) = M/N$.*

Proof. (a) $\Leftarrow$: If $\mathrm{soc}(M) \neq M$ then $M = \mathrm{soc}(M) \oplus N'$. Let N be a simple submodule of N' (the existence of N is nontrivial and uses Zorn's lemma, see Theorem ???). Then $\mathrm{soc}(M) + N \neq \mathrm{soc}(M)$, but this is a contradiction to the definition of $\mathrm{soc}(M)$.

(a) $\Rightarrow$: Let N be a submodule of M and let $N' = \sum_{P\cap N=0} P$ be the sum of the simple submodules P of M such that $P \cap N = 0$. Then $N \cap N' = 0$ since, for a simple submodule P of M, $P \cap N = P$ or $P \cap N = 0$. Since $N + N' \supseteq \mathrm{soc}(M) = M$, $N + N' = M$. So $M = N \oplus N'$. ∎

Proposition 1.13. *The following are equivalent:*

- *(a) M is a finite direct sum of simple submodules.*
- *(b) M is artinian and $\mathrm{soc}(M) = M$.*
- *(c) M is noetherian and $\mathrm{soc}(M) = M$.*
- *(d) M has a finite composition series and $\mathrm{soc}(M) = M$.*
- *(e) M is finitely generated and $\mathrm{soc}(M) = M$.*
- *(f) M is artinian and $\mathrm{Rad}(M) = 0$.*

Proof. The implications (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c), (a) $\Leftrightarrow$ (d) follow directly from Proposition ??? and Proposition ???a.

(a) $\Rightarrow$ (f) follows directly from the definitions.

(f) $\Rightarrow$ (a): Let N_i be a finite (by DCC) number of maximal proper submodules such that

$$\mathrm{Rad}(M) = \bigcap N_i = 0.$$

Then

$$\begin{array}{rrcl} \phi\colon & M & \longrightarrow & M/N_1 \oplus \cdots \oplus M/N_k \\ & m & \longmapsto & (m + N_1, \ldots, m + N_k) \end{array}$$

has $\ker \phi = 0$. So $M \cong \mathrm{im}(M)$ which is a submodule of the semisimple module $M/N_1 \oplus \cdots M/N_k$. So $M \cong \mathrm{im}(M)$ is finite length and $\mathrm{soc}(M) = M$. So M is a direct sum of simple submodules.

(c) $\Rightarrow$ (e) since M is noetherian implies that M is finitely generated.

(e) $\Rightarrow$ (c): Let N be a submodule of M and let N' be a complement. Then $N \cong M/N'$ and thus, since M is finitely generated, N is finitely generated. Thus every submodule of M is finitely generated. So M is noetherian. ∎

Theorem 1.14. (Artin-Wedderburn) *The following are equivalent:*

(a) R is artinian and $\mathrm{Rad}(R) = 0$,

(b) ${}_RR$ is a finite direct sum of simple modules

(c) $R \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{D}_\lambda)$, where $\hat{A}$ is a finite index set, d_λ are positive integers, and $\mathbb{D}_\lambda$ are division rings.

Proof. (a) $\Leftrightarrow$ (b) is a consquence of Proposition ???.

(a) $\Leftarrow$ (c) is a consquence of the fact that the simple $M_{d_\lambda}(\mathbb{D}_\lambda)$ module is $\mathbb{D}_\lambda^{d_\lambda}$ the vector space of column vectors of length d_λ.

(a) $\Rightarrow$ (c): The map

$$\begin{array}{ccc} R^{\mathrm{op}} & \longrightarrow & \mathrm{End}_R({}_RR) \\ r & \longmapsto & \phi_r \end{array} \qquad \text{where} \qquad \phi_r(x) = xr, \quad \text{for } x \in R,$$

is a ring isomorphism. Thus, by Schur's lemma,

$$R^{\mathrm{op}} \cong \mathrm{End}_R({}_RR) \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{D}_\lambda), \qquad \text{and thus} \qquad R \cong \left(\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{D}_\lambda) \right)^{\mathrm{op}} \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{D}_\lambda).$$

∎

Radicals and finiteness conditions for rings

Let R be a ring.

The ring R is a **noetherian** if ${}_RR$ is noetherian.

The ring R is a **artinian** if ${}_RR$ is artinian.

A **left ideal** of R is a submodule of ${}_RR$.

An ideal I of R is **primitive** if $I = \mathrm{ann}(M)$ for a simple R-module M.

Simple and almost simple rings

The ring R is **primitive** if 0 is a primitive ideal.

The ring R is a **semiprimitive** if $\mathrm{Rad}(R) = 0$.

The ring R is **simple** if its only ideals are 0 and R.

The ring R is **prime** if A, B are ideals with $AB = 0$ then $A = 0$ or $B = 0$.

An ideal P is **prime** if R/P is a prime ring.

The ring R is **semiprime** if 0 is the only nilpotent ideal.

Proposition 1.15. *Let R be a ring and let $\mathrm{Spec}(R)$ be the set of prime ideals of R.*

(a) R is semiprime if and only if $\displaystyle\bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathfrak{p} = 0$.

(b) R is primitive if and only if R is a dense subring of $\mathrm{End}_{\mathbb{D}}(U)$ for some $\mathbb{D}$-vector space U.

(c) R is artinian and semiprime if and only if R is artinian and semiprimitive.

(d) R is artinian and primitive if and only if R is artinian and simple.

(e) R is artinian and primitive if and only if $R \cong M_n(\mathbb{D})$, for some $n \in \mathbb{Z}_{\geq 1}$, $\mathbb{D}$ a division ring.

Proof. ∎

Burnside's theorem and Jacobson density

A subring R of $\mathrm{End}_{\mathbb{D}}(U)$ is **dense** if for every $\alpha \in \mathrm{End}_{\mathbb{D}}(U)$ and every finitely generated $V \subseteq U$ there is an $r \in R$ with $\mathrm{Res}^U_V(r) = \mathrm{Res}^U_V(\alpha)$. Define a topology on $\mathrm{End}_{\mathbb{D}}(U)$ by making

$$U(\alpha, V) = \{\beta \in \mathrm{End}_{\mathbb{D}}(U) \mid \mathrm{Res}^U_V(\beta) = \mathrm{Res}^U_V(\alpha)\} \qquad \text{open}$$

for each $\alpha \in \mathrm{End}_{\mathbb{D}}(U)$ and each finitely generated $V \subseteq U$. Then R is **dense** in $\mathrm{End}_{\mathbb{D}}(U)$ if $\mathrm{End}_{\mathbb{D}}(U)$ is the closure of R, $\overline{R} = \mathrm{End}_{\mathbb{D}}(U)$.

Example. Consider an infinite dimensional vector space U with basis $u_1, u_2, \ldots$. Then

$$\begin{aligned}\mathrm{End}_{\mathbb{C}}(U) &\cong M_\infty(\mathbb{C}) \\ &= \{\text{infinite matrices with a finite number of nonzero entries in each column}\}.\end{aligned}$$

Let

$$\begin{aligned} I &= \{\text{finite rank elements of } M_\infty(\mathbb{C})\} \\ &= \{\alpha \in \mathrm{End}_{\mathbb{C}}(U) \mid \text{im } \alpha \text{ is finite dimensional}\}.\end{aligned}$$

and let

$$R = \{n \cdot 1 + \ell \mid n \in \mathbb{Z}, \ell \in I\}.$$

Then R is a dense subring of $\mathrm{End}_{\mathbb{C}}(U)$,

$$\mathbb{C} = \mathrm{End}_R(U) \quad \text{and} \quad R \neq \mathrm{End}_{\mathbb{C}}(U).$$

Theorem 1.16. *Let U be a simple R-module and let $\mathrm{Im}(R)$ be the image of R in $\mathrm{End}(U)$. Let $\mathbb{D} = \mathrm{End}_R(U)$, a division ring.*

(a) $\mathrm{Im}(R)$ *is a dense subring of* $\mathrm{End}_{\mathbb{D}}(U)$.

(b) If R *is artinian the* $\mathrm{Im}(R) = \mathrm{End}_{\mathbb{D}}(U)$.

Proof. We will show that if $x_1, \ldots, x_n \in U$ and $\alpha \in \mathrm{End}_{\mathbb{D}}(U)$ then there is an $r \in R$ with $rx_i = \alpha x_i$ for $1 \le i \le n$. The proof is by induction on n using the following lemma:

Lemma 1.17. *Let* $u \in U$. *Then*

$$\mathrm{ann}(x_1, \ldots, x_n)u = 0 \qquad \Longleftrightarrow \qquad u \in \mathbb{D}\text{-}span\{x_1, \ldots, x_n\}.$$

Assume the lemma and assume that $x_1, \ldots, x_n \in U$ and $\alpha \in \mathrm{End}_{\mathbb{D}}(U)$ are given. By the induction assumption, there is $r' \in R$ such that

$$r'x_i = \alpha x_i, \qquad \text{for } 1 \le i \le n-1.$$

If $x_n \notin \mathbb{D}\text{-span}\{x_1, \ldots, x_n\}$ then, by the lemma, $\mathrm{ann}(x_1, \ldots, x_{n-1})x_n \neq 0$. Since $\mathrm{ann}(x_1, \ldots, x_{n-1})x_n$ is a nonzero R-submodule of U and U is simple $\mathrm{ann}(x_1, \ldots, x_{n-1})x_n = U$. So

$$\ell x_n = (\alpha - r')x_n, \quad \text{for some } \ell \in \mathrm{ann}(x_1, \ldots, x_{n-1}).$$

Then

$$(r' + \ell)x_i = x_i, \quad \text{for } 1 \le i \le n-1, \qquad \text{and} \qquad (r' + \ell)x_n = x_n.$$

(b) Let R be artinian and let U be a simple module. Let I be a minimal element of

$$\{\mathrm{ann}(x_1, \ldots, x_k) \mid x_1, \ldots, x_k \in U\}$$

and let $x_1, \ldots, x_k$ be the finite subset of U such that $I = \mathrm{ann}(x_1, \ldots, x_k)$. Let $u \in I$. If $\mathrm{ann}(x_1, \ldots, x_k)u \neq 0$ then $\mathrm{ann}(x_1, \ldots, x_k, u) \subseteq I$ and $\mathrm{ann}(x_1, \ldots, x_k, u) \neq I$, a contradiction to the minimality of I. So $\mathrm{ann}(x_1, \ldots, x_n)u = 0$. So $u \in \mathbb{D}\text{-span}\{x_1, \ldots, x_n\}$. So U is finite dimensional. Now (c) follows from (b). ∎

Proof of the lemma. $\Leftarrow$: trivial.

$\Rightarrow$: Assume $\mathrm{ann}(x_1, \ldots, x_n)u = 0$. The proof is by induction on n.

Case 1. If $\mathrm{ann}(x_1, \ldots, x_{n-1})x_n = 0$ then $x_n \in \mathrm{span}\{x_1, \ldots, x_{n-1}\}$ and so $\mathrm{ann}(x_1, \ldots, x_{n-1})u = 0$. So $u \in \mathrm{span}\{x_1, \ldots, x_{n-1}\}$.

Case 2. If $\mathrm{ann}(x_1, \ldots, x_{n-1})x_n \neq 0$ then $\mathrm{ann}(x_1, \ldots, x_{n-1})x_n = U$. Define an R-module homomorphism

$$\begin{array}{rccl} \alpha\colon & U & \longrightarrow & U \\ & \ell x_n & \longmapsto & \ell u, \qquad \text{for } \ell \in \mathrm{ann}(x_1, \ldots, x_{n-1}). \end{array}$$

If $\ell x_n = \kappa x_n$ then $\ell - \kappa \in \mathrm{ann}(x_1, \ldots, x_{n-1}) \cap \mathrm{ann}(x_n) = \mathrm{ann}(x_1, \ldots, x_n)$. So $(\ell - \kappa)u = 0$ and $\ell u = \kappa u$ which shows that α is well defined. So $\alpha \in \mathbb{D} = \mathrm{End}_R(U)$.

Now $\mathrm{ann}(x_1, \ldots, x_{n-1})(u - \alpha x_n) = 0$ and so, by the induction hypothesis, $u - \alpha x_n \in \mathrm{span}\{x_1, \ldots, x_{n-1}\}$. So $u \in \mathrm{span}\{x_1, \ldots, x_n\}$. ∎

3. Radicals of algebras

Let A be an algebra over a field $\mathbb{F}$.

The radical of A is the intersection of the maximal left ideals of A,

$$\mathrm{Rad}(A) = \bigcap_{L_{\max}} L_{\max}.$$

Proposition 3.1. *Assume A satisfies the descending chain condition on left ideals. Then A is completely reducible if and only if* $\mathrm{Rad}(A) = 0$.

Proof. ∎

A *nilpotent ideal* is an ideal I such that $I^k = 0$ for some $k \in \mathbb{Z}_{>0}$. A *nilpotent element* is an element $x \in A$ such that $x^k = 0$ for some $k \in \mathbb{Z}_{>0}$.

If $\vec{t}\colon A \to \mathbb{C}$ is a trace on A then

$$\mathrm{Rad}(\vec{t}) = \{a \in A \mid \vec{t}(ab) = 0 \text{ for all } b \in A\}.$$

Proposition 3.2.

(e) $\mathrm{Rad}(A) = \mathrm{Rad}(\vec{t})$, *if $\vec{t}$ is the trace of a faithful representation of A.*

Proof. ∎

6. References

[Bou1] N. Bourbaki, *Algebra I*, Chapters 1–3, Elements of Mathematics, Springer-Verlag, Berlin, 1990.

[Bou2] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitre IV, V, VI*, Eléments de Mathématique, Hermann, Paris (1968).