Symmetric functions
Lecture Notes

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Version: January 29, 2004

## 1. Symmetric functions

*Tableaux*

Let $\lambda$ be a partition and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ be a sequence of nonnegative integers. A *column strict tableau of shape* $\lambda$ *and weight* $\mu$ is a filling of the boxes of $\lambda$ with $\mu_1$ 1s, $\mu_2$ 2s, …, $\mu_n$ $n$s, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If $p$ is a column strict tableau write $\mathrm{shp}(p)$ and $\mathrm{wt}(p)$ for the shape and the weight of $p$ so that

$$\begin{aligned} \mathrm{shp}(p) &= (\lambda_1, \ldots, \lambda_n), \qquad \text{where} \quad \lambda_i = \text{number of boxes in row } i \text{ of } p, \quad \text{and} \\ \mathrm{wt}(p) &= (\mu_1, \ldots, \mu_n), \qquad \text{where} \quad \mu_i = \text{number of } i\,\text{s in } p. \end{aligned}$$

For example,

$$p = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & 3 & 3 & 4 \\ \cline{1-7} 3 & 3 & 3 & 4 & 4 & 4 & 5 \\ \cline{1-7} 4 & 5 & 5 & 6 \\ \cline{1-4} 6 & 7 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \text{has} \quad \begin{aligned} \mathrm{shp}(p) &= (9,7,7,4,2,1,0) \quad \text{and} \\ \mathrm{wt}(p) &= (7,6,5,5,3,2,2). \end{aligned}$$

For a partition $\lambda$ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } p \mid \mathrm{shp}(p) = \lambda \text{ and } \mathrm{wt}(p) = \mu\}, \end{aligned} \tag{1.1}$$

Define $B_n = \{b_1, \ldots, b_n\}$ and let

$$B_n^{\otimes k} = \{b_{i_1} \cdots b_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$$

* Research supported in part by National Science Foundation grant DMS-??????.

be the set of words of length $k$ in the alphabet $B$. For each $1 \le i \le n$ define *root operators*

$$\tilde{e}_i\colon B_n^{\otimes k} \longrightarrow B_n^{\otimes k} = \{0\} \qquad \text{and} \qquad \tilde{f}_i\colon B_n^{\otimes k} \longrightarrow B_n^{\otimes k} = \{0\}$$

by the following process. If $b = b_{i_1} \cdots b_{i_k}$ in $B_n^{\otimes k}$ place the value

$$\begin{aligned} +1 &\ \text{ over each } b_i, \\ -1 &\ \text{ over each } b_{i+1}, \\ 0 &\ \text{ over each } b_j,\ j \ne i, i+1. \end{aligned}$$

Ignoring 0s read this sequence of $\pm 1$ from left to right and successively remove adjacent $(+1, -1)$ pairs until the sequence is of the form

$$\underbrace{+1\ +1\ \ldots \overset{\substack{\text{cogood}\\ \downarrow}}{+1}}_{\text{conormal nodes}}\quad \underbrace{\overset{\substack{\text{good}\\ \downarrow}}{-1}\ -1\ \ldots -1}_{\text{normal nodes}}$$

The $-1$s in this sequence are the *normal nodes* and the $+1$s are the *conormal nodes*. The *good node* is the leftmost normal node and the *cogood node* is the right most conormal node. Then

$$\begin{aligned} \tilde{e}_i(b) &= \text{same as } b \text{ except with the cogood node path step changed to } b_i, \\ \tilde{f}_i(b) &= \text{same as } b \text{ except with the good node path step changed to } b_i, \end{aligned}$$

For example, if $n = 5$, $k = 30$,

$$b = b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3b_2b_1b_1b_2b_3b_3b_2b_1b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4,$$

and $i = 1$, then the parentheses in the table

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | ( | ) | $-1$ | | | ( | ) | | | $-1$ | ( | ( | ) | | | ) | $+1$ |
| 0 | 0 | 0 | $+1$ | $-1$ | $-1$ | 0 | 0 | $+1$ | $-1$ | 0 | 0 | $-1$ | $+1$ | $+1$ | $-1$ | 0 | 0 | $-1$ | $+1$ |
| $b_4$ | $b_3$ | $b_3$ | $b_1$ | $b_2$ | $b_2$ | $b_4$ | $b_4$ | $b_1$ | $b_2$ | $b_3$ | $b_3$ | $b_2$ | $b_1$ | $b_1$ | $b_2$ | $b_3$ | $b_3$ | $b_2$ | $b_1$ |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | $+1$ | $+1$ | ( | ( | ) | ) | 0 |
| 0 | 0 | 0 | $+1$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ | 0 |
| $b_4$ | $b_5$ | $b_5$ | $b_1$ | $b_1$ | $b_1$ | $b_1$ | $b_2$ | $b_2$ | $b_4$ |

indicate the $(+1, -1)$ pairings and the numbers in the top row indicate the resulting sequence of $-1$s and $+1$s. Then

$$\begin{aligned} \tilde{e}_1(b) &= b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3b_2b_1b_1b_2b_3b_3b_2\underline{b_2}b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4, \qquad \text{and} \\ \tilde{f}_1(b) &= b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3\underline{b_1}b_1b_1b_2b_3b_3b_2b_1b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4. \end{aligned}$$

If $\lambda$ is a partition of $k$ define an imbedding

$$\begin{array}{ccc} B(\lambda) & \longrightarrow & B_n^{\otimes k} \\ p & \longmapsto & b_{i_1}b_{i_2}\cdots b_{i_k} \end{array}$$

where the entries $i_1 i_2 \cdots i_k$ are the entries of $p$ read in Arabic reading order. The action of $\tilde{e}_i$ and $\tilde{f}_i$ preserves the image of $B(\lambda)$ in $B_n^{\otimes k}$ and so the set $B(\lambda)$ can be viewed as a subcrystal.

If $B$ is a normal crystal and $b \in B$ the *i-string* of $b$ is the set

$$\tilde{f}_i^{\varphi_i(b)} b \overset{i}{\longleftrightarrow} \cdots \overset{i}{\longleftrightarrow} \tilde{f}_i^2 b \overset{i}{\longleftrightarrow} \tilde{f}_i b \overset{i}{\longleftrightarrow} b \overset{i}{\longleftrightarrow} \tilde{e}_i b \overset{i}{\longleftrightarrow} \tilde{e}_i^2 b \overset{i}{\longleftrightarrow} \cdots \overset{i}{\longleftrightarrow} \tilde{e}_i^{\varepsilon_i(b)} b,$$

and (3) is equivalent to $\langle \mathrm{wt}(\tilde{e}_i^{\varepsilon_i(b)} b), \alpha_i^\vee \rangle = -\langle \mathrm{wt}(\tilde{f}_i^{\varphi(b)} b), \alpha_i^\vee \rangle$ so that *every $i$ string in a normal crystal $B$ is a model for a finite dimensional $\mathfrak{sl}_2$-module.*

If $B$ is a normal crystal define a bijection $s_i \colon B \to B$ by

$$s_i b = \begin{cases} \tilde{f}_i^{\langle \mathrm{wt}(b), \alpha_i^\vee \rangle} b, & \text{if } \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \geq 0, \\ \tilde{e}_i^{-\langle \mathrm{wt}(b), \alpha_i^\vee \rangle} b, & \text{if } \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \leq 0, \end{cases} \qquad \text{so that} \qquad \mathrm{wt}(s_i b) = s_i \mathrm{wt}(b), \qquad \text{for all } b \in B.$$

The map $s_i$ flips each $i$-string in $B$. The equality $\mathrm{wt}(s_i b) = s_i \mathrm{wt}(b)$ implies

$$\chi^B \in \mathbb{Z}[P]^W, \qquad \text{for any normal crystal } B.$$

**Proposition 1.2.** *[Kashiwara, Duke* **73** *(1994), 383-413] Let $B$ be a normal crystal. The maps $s_i \colon B \to B$ $i \in I$, define an action of $W$ on $B$.*

*Proof.* ∎

**Theorem 1.3.** *Let $B$ be a subcrystal of $\vec{B}$ such that $B_\mu$ is finite for all $\mu \in P$. Then*

$$\chi^B = \sum_{\substack{b \in B \\ b \subseteq C - \rho}} s_{\mathrm{wt}(b)},$$

*where $s_\lambda$ denotes the Weyl character corresponding to $\lambda \in P^+$.*

*Proof.* Let $\mu \in P^+$. Then

$$\chi^B a_\rho \big|_{a_{\mu+\rho}} = \left( \sum_{w \in W} (-1)^{\ell(w)} e^{w\rho} \right) \left( \sum_{p \in B} e^{\mathrm{wt}(p)} \right) \Big|_{a_{\mu+\rho}} = \sum_{\substack{w \in W \\ p \in B}} (-1)^{\ell(w)} e^{\mathrm{wt}(p) + w\rho} \Big|_{e^{\mu+\rho}}. \tag{1.4}$$

Let $p \in B$ and $w \in W$ be such that $\mathrm{wt}(p) + w\rho = \mu + \rho$. Let $t_0$ be maximal such that there is an $i \in I$ with $w\rho + p(t_0) \in H_{\alpha_i}$. If $t_0$ does not exist then $p \in C - \rho$ and $w = 1$. If $t_0$ does exist set

$$\Phi(p) = \begin{cases} \tilde{f}_i^{-\langle w\rho, \alpha_i^\vee \rangle} p, & \text{if } \langle w\rho, \alpha_i^\vee \rangle < 0, \\ \tilde{e}_i^{\langle w\rho, \alpha_i^\vee \rangle} p, & \text{if } \langle w\rho, \alpha_i^\vee \rangle > 0. \end{cases}$$

Then $\mathrm{wt}(p) + w\rho = \mathrm{wt}(\Phi(p)) + s_i w\rho$ and and the pairs $(p, w)$ and $(\Phi(p), s_i w)$ cancel in the sum (???). ∎

*Symmetric functions*

The symmetric group $S_n$ acts on the vector space

$$\mathbb{Z}^n = \mathbb{Z}\text{-span}\{x_1, \ldots, x_n\} \qquad \text{by} \qquad wx_i = x_{w(i)},$$

for $w \in S_n$, $1 \le i \le n$. This action induces an action of $S_n$ on the polynomial ring $\mathbb{Z}[X_n] = \mathbb{Z}[x_1, \ldots, x_n]$ by ring automorphisms. For a sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ of nonnegative integers let

$$x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}, \qquad \text{so that} \quad \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}\text{-span}\{x^\gamma \mid \gamma \in \mathbb{Z}^n_{\ge 0}\}.$$

The ring of *symmetric functions* is

$$\mathbb{Z}[X_n]^{S_n} = \{f \in \mathbb{Z}[X_n] \mid wf = f \text{ for all } w \in S_n\}, \tag{1.5}$$

Define the *orbit sums*, or *monomial symmetric functions*, by

$$m_\lambda = \sum_{\gamma \in S_n\lambda} x^\gamma, \qquad \text{for } \lambda \in \mathbb{Z}^n_{\ge 0},$$

where $S_n\lambda$ is the orbit of $\lambda$ under the action of $S_n$. Let

$$P^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\ge 0} \mid \lambda_1 \ge \cdots \ge \lambda_n\} \tag{1.6}$$

so that

$$\{m_\lambda \mid \lambda \in P^+\} \quad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}[X_n]^{S_n}. \tag{1.7}$$

## 2. Skew polynomials

The polynomial ring

$$\mathbb{Z}[X_n] = \mathbb{Z}[x_1, \ldots, x_n] \qquad \text{has basis} \qquad \{x^\mu \mid \mu \in \mathbb{Z}^n_{\ge 0}\}, \quad \text{where} \quad x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n},$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\ge 0}$. The vector space of *skew polynomials* is

$$A_n = \{g \in \mathbb{Z}[x_1, \ldots, x_n] \mid wg = \det(w)g \text{ for all } w \in S_n\}.$$

If $f \in \mathbb{Z}[X_n]^S_n$ and $g \in A_n$ then $fg \in A_n$ and so $A_n$ is a $\mathbb{Z}[X_n]^{S_n}$-module.

The symmetric group $S_n$ acts on $\mathbb{Z}^n_{\ge 0}$ by permuting the coordinates. If

$$\begin{aligned} \mathbb{Z}^n &= \{(\gamma_1, \ldots, \gamma) \mid \gamma_i \in \mathbb{Z}\}, \\ P^+ &= \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_n\}, \qquad \text{and} \\ P^{++} &= \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 > \gamma_2 > \cdots > \gamma_n\}, \end{aligned}$$

then $P^+$ is a set of representatives of the orbits of the $S_n$ action on $\mathbb{Z}^n$ and the map defined by

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array} \qquad \text{where} \qquad \rho = (n-1, n-2, \cdots, 2, 1, 0),$$

is a bijection.

Let

$$P^+_n = \{(\gamma_1, \ldots, \gamma) \in \mathbb{Z}^n \mid \gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_n \ge 0\}.$$

For each $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ such that $\mu_n \geq 0$,

$$a_\mu = \sum_{w \in S_n} \det(w) w x^\mu \tag{2.1}$$

is a skew polynomial. Since $a_\mu = \det(w) a_{w\mu}$ and $a_\mu = 0$ unless $\mu_1 > \mu_2 > \cdots > \mu_n$,

$$\{a_{\lambda+\rho} \mid \lambda \in P_n^+\} \qquad \text{is a basis of } A_n,$$

and thus

$$A_n = \varepsilon \cdot \mathbb{Z}[X_n], \quad \text{where} \quad \varepsilon = \sum_{w \in S_n} \det(w) w.$$

The skew element

$$a_{\lambda+\rho} = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ & \vdots & \cdots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n} \end{pmatrix} \quad \text{is divisible by} \quad \prod_{n \geq j > i \geq 1} (x_j - x_i), \tag{2.2}$$

since the factors $(x_j - x_i)$ in the product on the right hand side are coprime in $\mathbb{Z}[x_1, \ldots, x_n]$ and setting $x_i = x_j$ makes the determinant vanish so that $a_{\lambda+\rho}$ must be divisible by $x_j - x_i$. When $\lambda = 0$, comparing coefficients of the maximal terms on each side shows that the *Vandermonde determinant*

$$a_\rho = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^0 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^0 \\ \vdots & & \cdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^0 \end{pmatrix} = \prod_{n \geq j > i \geq 1} (x_j - x_i). \tag{2.3}$$

Since $\{a_{\lambda+\rho} \mid \lambda \in P_n^+\}$ is a basis of $A_n$, (???) shows that the inverse of the map

$$\begin{array}{ccc} \mathbb{Z}[x_1, \ldots, x_n]^{S_n} & \longrightarrow & A_n \\ f & \longmapsto & a_\rho f \end{array} \tag{2.4}$$

is well defined, and thus the map in (???) is an isomorphism of $\mathbb{Z}[X_n]^{S_n}$-modules.

The *Schur polynomials* are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P_n^+,$$

and since $\{a_{\lambda+\rho} \mid \lambda \in P_n^+\}$ is a basis of $A_n$ and the map in (???) is an isomorphism,

$$\{s_\lambda \mid \lambda \in P_n^+\} \qquad \text{is a basis of } \mathbb{Z}[x_1, \ldots, x_n]^{S_n}.$$

**Theorem 2.5.** *Let $\lambda \in P_n^+$. Then*

$$s_\lambda = \sum_{p \in B(\lambda)} x^{\mathrm{wt}(p)}.$$

*Proof.* Since the action of $S_n$ on $B(\lambda)$ defined in (???) satisfies $\mathrm{wt}(wp) = w\mathrm{wt}(p)$ for $w \in S_n$, $p \in B(\lambda)$, the sum on the right hand side is an element of $\mathbb{Z}[X_n]^{S_n}$.
∎

## Notes and References

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford University Press, 1995.