

## The basic trig identities

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Define

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,\end{aligned}$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

**Example:** Explain why  $e^{ix} = \cos x + i \sin x$ , if  $i = \sqrt{-1}$ .

$$\begin{aligned}e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots, \\ &= 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \frac{i^6 x^6}{6!} + \frac{i^7 x^7}{7!} + \cdots, \\ &= 1 + ix + \frac{i^2 x^2}{2!} + \frac{i \cdot i^2 x^3}{3!} + \frac{(i^2)^2 x^4}{4!} + \frac{i \cdot (i^2)^2 x^5}{5!} + \frac{(i^2)^3 x^6}{6!} + \frac{i \cdot (i^2)^3 x^7}{7!} + \cdots, \\ &= 1 + ix + \frac{(-1)x^2}{2!} + \frac{i \cdot (-1)x^3}{3!} + \frac{(-1)^2 x^4}{4!} + \frac{i \cdot (-1)^2 x^5}{5!} + \frac{(-1)^3 x^6}{6!} + \frac{i \cdot (-1)^3 x^7}{7!} + \cdots, \\ &= 1 + ix - \frac{x^2}{2!} - i \frac{x^3}{3!} + \frac{x^4}{4!} + i \frac{x^5}{5!} - \frac{x^6}{6!} - i \frac{x^7}{7!} + \cdots, \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x.\end{aligned}$$

**Example:** Explain why  $\cos(-x) = \cos x$  and  $\sin(-x) = -\sin x$ .

$$\begin{aligned}
\cos(-x) &= 1 - \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} - \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} - \dots, \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \dots, \\
&= \cos x,
\end{aligned}$$

and

$$\begin{aligned}
\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} - \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} - \dots, \\
&= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} - \frac{x^{13}}{13!} + \dots, \\
&= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \dots\right), \\
&= -\sin x.
\end{aligned}$$

**Example:** Explain why  $\cos^2 x + \sin^2 x = 1$ .

$$\begin{aligned}
1 &= e^0 = e^{ix+(-ix)} \\
&= e^{ix}e^{-ix} = e^{ix}e^{i(-x)} \\
&= (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\
&= (\cos x + i \sin x)(\cos x + i(-\sin x)) \\
&= \cos^2 x - i \sin x \cos x + i \sin x \cos x - i^2 \sin^2 x \\
&= \cos^2 x - (-1) \sin^2 x \\
&= \cos^2 x + \sin^2 x.
\end{aligned}$$

**Example:** Explain why  $\cos(x+y) = \cos x \cos y - \sin x \sin y$ , and  $\sin(x+y) = \sin x \cos y + \cos x \sin y$ .

$$\begin{aligned}
\cos(x+y) + i \sin(x+y) &= e^{i(x+y)} \\
&= e^{ix+iy} = e^{ix}e^{iy} \\
&= (\cos x + i \sin x)(\cos y + i \sin y) \\
&= \cos x \cos y + i \cos x \sin y + i \sin x \cos y + i^2 \sin x \sin y \\
&= (\cos x \cos y + (-1) \sin x \sin y) + i(\cos x \sin y + \sin x \cos y).
\end{aligned}$$

Comparing terms on each side gives

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y, \quad \text{and} \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y.
\end{aligned}$$

Define

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots, \\ \sinh x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \dots, \\ \cosh x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \dots, \end{aligned}$$

and

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

**Example:** Explain why  $e^x = \cosh x + \sinh x$ .

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots \\ &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\right) \\ &= \cosh x + \sinh x. \end{aligned}$$

**Example:** Explain why  $\cosh(-x) = \cosh x$  and  $\sinh(-x) = -\sinh x$ .

$$\begin{aligned} \cosh(-x) &= 1 + \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} + \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} + \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} + \dots \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \dots \\ &= \cosh x, \end{aligned}$$

and

$$\begin{aligned} \sinh(-x) &= (-x) + \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} + \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} + \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} + \dots \\ &= -x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} - \frac{x^{11}}{11!} - \frac{x^{13}}{13!} - \dots \\ &= -\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \dots\right) \\ &= -\sinh x. \end{aligned}$$

**Example:** Explain why  $\cosh x = \frac{e^x + e^{-x}}{2}$  and  $\sinh x = \frac{e^x - e^{-x}}{2}$ .

$$\begin{aligned}
\frac{1}{2}(e^x + e^{-x}) &= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right. \\
&\quad \left.+ 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots\right) \\
&= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right. \\
&\quad \left.+ 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \\
&= \cosh x.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}(e^x - e^{-x}) &= \frac{1}{2}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right)\right. \\
&\quad \left.- \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots\right)\right) \\
&= \frac{1}{2}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right)\right. \\
&\quad \left.- \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right)\right) \\
&= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right. \\
&\quad \left.- 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \cdots\right) \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \\
&= \sinh x.
\end{aligned}$$

**Example:** Explain why  $\cosh^2 x - \sinh^2 x = 1$ .

$$\begin{aligned}
1 &= e^0 = e^{x+(-x)} \\
&= e^x e^{-x} \\
&= (\cosh x + \sinh x)(\cosh(-x) + \sinh(-x)) \\
&= (\cosh x + \sinh x)(\cosh x - \sinh x) \\
&= \cosh^2 x - \sinh x \cosh x + \sinh x \cosh x - \sinh^2 x \\
&= \cosh^2 x - \sinh^2 x.
\end{aligned}$$

**Example:** Explain why  $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ , and  $\sinh(x + y) = 2 \sinh x \cosh y$ .

$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right) + \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right) \\
&= \frac{e^x e^y + e^{-x} e^y + e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&\quad + \frac{e^x e^y - e^{-x} e^y - e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&= \frac{2e^x e^y + 2e^{-x} e^{-y}}{4} \\
&= \frac{e^{(x+y)} + e^{-(x+y)}}{2} \\
&= \cosh(x + y).
\end{aligned}$$

and

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right) + \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right) \\
&= \frac{e^x e^y - e^{-x} e^y + e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&\quad + \frac{e^x e^y + e^{-x} e^y - e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} \\
&= \sinh(x + y).
\end{aligned}$$