Partition Algebras

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0. Introduction

A centerpiece of representation theory is the Schur-Weyl duality, which says that,

(a) the general linear group $GL_n(\mathbb{C})$ and the symmetric group S_k both act on tensor space

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ factors}}, \qquad \text{with} \qquad \dim(V) = n,$$

(b) these two actions commute and

(c) each action generates the full centralizer of the other, so that

(d) as a $(GL_n(\mathbb{C}), S_k)$-bimodule, the tensor space has a multiplicity free decomposition,

$$V^{\otimes k} \cong \bigoplus_{\lambda} L_{GL_n}(\lambda) \otimes S_k^{\lambda}, \tag{0.1}$$

where the $L_{GL_n}(\lambda)$ are irreducible $GL_n(\mathbb{C})$-modules and the S_k^λ are irreducible S_k-modules.

The decomposition in (0.1) essentially makes the study of the representations of $GL_n(\mathbb{C})$ and the study of representations of the symmetric group S_k two sides of the same coin.

The group $GL_n(\mathbb{C})$ has interesting subgroups,

$$\begin{array}{ccccccc} GL_n(\mathbb{C}) & \supseteq & O_n(\mathbb{C}) & \supseteq & S_n & \supseteq & S_{n-1}, \\ \mathbb{C}S_k & \subseteq & \mathbb{C}B_k(n) & \subseteq & \mathbb{C}A_k(n) & \subseteq & \mathbb{C}A_{k+\frac{1}{2}}(n), \end{array}$$

and corresponding centralizer algebras,

which are combinatorially defined in terms of the "multiplication of diagrams" (see Section 1) and which play exactly analogous "Schur-Weyl duality" roles with their corresponding subgroup of $GL_n(\mathbb{C})$. The Brauer algebras $\mathbb{C}B_k(n)$ were introduced in 1937 by R. Brauer [Bra]. The partition

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algebras $\mathbb{C}A_k(n)$ arose in the early 1990s in the work of P. Martin [Ma1-4] and later, independently, in the work of V. Jones [Jo]. Martin and Jones discovered the partiton algebra as a generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics. The partition algebras $\mathbb{C}A_{k+\frac{1}{2}}(n)$ apear in [Ma4] and [MR], and their existence and importance was pointed out to us by C. Grood [Gr]. In this paper we follow the method of [Ma4] and show that if the algebras $\mathbb{C}A_{k+\frac{1}{2}}(n)$ are given the same stature as the algebras $A_k(n)$, then well-known methods from the theory of the "basic construction" (see Section 4) allow for easy analysis of the whole tower of algebras

$$\mathbb{C}A_0(n) \subseteq \mathbb{C}A_{\frac{1}{2}}(n) \subseteq \mathbb{C}A_1(n) \subseteq \mathbb{C}A_{1\frac{1}{2}}(n) \subseteq \cdots,$$

all at once.

Let $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. In this paper we prove:

(a) A presentation by generators and relations for the algebras $\mathbb{C}A_\ell(n)$.

(b) $\mathbb{C}A_\ell(n)$ has

$$\text{an ideal} \qquad \mathbb{C}I_\ell(n), \qquad \text{with} \qquad \frac{\mathbb{C}A_\ell(n)}{\mathbb{C}I_\ell(n)} \cong \mathbb{C}S_\ell,$$

such that $\mathbb{C}I_\ell(n)$ is isomorphic to a "basic construction" (see Section 4). Thus the structure of the ideal $\mathbb{C}I_\ell(n)$ can be analyzed with the general theory of the basic construction and the structure of the quotient $\mathbb{C}A_\ell(n)/(\mathbb{C}I_\ell(n))$ follows from the general theory of the representations of the symmetric group.

(c) The algebras $\mathbb{C}A_\ell(n)$ are in "Schur-Weyl duality" with the symmetric groups S_n and S_{n-1} on $V^{\otimes k}$.

(d) The general theory of the basic construction provides a construction of "Specht modules" for the partition algebras, i.e. integral lattices in the (generically) irreducible $\mathbb{C}A_\ell(n)$-modules.

(e) Except for a few special cases, the algebras $\mathbb{C}A_\ell(n)$ are semisimple if and only if $\ell \leq (n+1)/2$.

(f) There are "Murphy elements" M_i for the partition algebras that play exactly analogous roles to the classical Murphy elements for the group algebra of the symmetric group. In particular, the M_i commute with each other in $\mathbb{C}A_\ell(n)$, and when $\mathbb{C}A_\ell(n)$ is semisimple each irreducible $\mathbb{C}A_\ell(n)$-module has a unique, up to constants, basis of simultaneous eigenvectors for the M_i.

The primary new results in this paper are (a) and (f). There has been work towards a presentation theorem for the partition monoid by Fitzgerald and Leech [FL], and it is possible that by now they have proved a similar presentation theorem. The statement in (b) has appeared implicitly and explicitly throughout the literature on the partition algebra, depending on what one considers as the definition of a "basic construction". The treatment of this connection between the partition algebras and the basic construction is explained very nicely and thoroughly in [Ma4]. We consider this connection an important part of the understanding of the structure of the partition algebras. The Schur-Weyl duality for the partition algebras $\mathbb{C}A_k(n)$ appears in [Ma1], [Ma4], and [MR] and was one of the motivations for the introduction of these algebras in [Jo]. The Schur-Weyl duality for $\mathbb{C}A_{k+\frac{1}{2}}(n)$ appears in [Ma4] and [MW]. Most of the previous literature (for example [Ma3], [MW1-2], [DW]) on the partition algebras has studied the structure of the partition algebras using the "Specht" modules of (d). Our point here is that their existence follows from the general theory of the basic construction. This is a special case of the fact that quasi-hereditary algebras are iterated sequences of basic constructions, as proved by Dlab and Ringel [DR]. The statements about the semsimplicity of $\mathbb{C}A_\ell(n)$ have mostly, if not completely, appeared in the work of Martin and Saleur [Ma3], [MS]. The Murphy elements for the partition algebras are new. Their form was conjectured by Owens [Ow], who proved that the sum of the first k of them is a central element in

$\mathbb{C}A_k(n)$. Here we prove all of Owens' theorems and conjectures (by a different technique than he was using). We have not taken the next natural step and provided formulas for the action of the generators of the partition algebra in the "seminormal" representations. We hope that someone will do this in the near future.

The "basic construction" is a fundamental tool in the study of algebras such as the partition algebra. Of course, like any fundamental construct, it appears in the literature and is rediscovered over and over in various forms. For example, one finds this construction in Bourbaki [Bou1, Ch. 2, §4.2 Remark 1], in [Bro1-2], in [GHJ, Ch. 2], and in the wonderful paper of Dlab and Ringel [DR] where it is explained that this construction is also the algebraic construct that "controls" the theory of quasi-hereditary algebras, recollement and highest weight categories [CPS] and some aspects of the theory of perverse sheaves [MiV].

Though this paper contains new results in the study of partition algebras we have made a distinct effort to present this material in a "survey" style so that it may be accessible to nonexperts and to newcomers to the field. For this reason we have included, in Sections 4 and 5, expositions, from scratch, of

(a) the theory of the basic construction (see also [GHJ, Ch. 2]), and
(b) the theory of semisimple algebras, in particular, Maschke's theorem, the Artin-Wedderburn theorem and the Tits deformation theorem (see also [CR, §3B and §68]).

Here the reader will find statements of the main theorems which are in exactly the correct form for our applications (generally difficult to find in the literature), and short slick proofs of all the results on the basic construction and on semisimple algebras that we need for the study of the partition algebras.

There are two sets of results on partition algebras that we have not had the space to treat in this paper:
(a) The "Frobenius formula," "Murnaghan-Nakayama" rule, and orthogonality rule for the irreducible characters given by Halverson [Ha] and Farina-Halverson [FH], and
(b) The cellularity of the partition algebras proved by Xi [Xi] (see also Doran and Wales [DW]).

The techniques in this paper apply, in exactly the same fashion, to the study of other diagram algebras; in particular, the planar partition algebras $\mathbb{C}P_k(n)$, the Temperley-Lieb algebras $\mathbb{C}T_k(n)$, and the Brauer algebras $\mathbb{C}B_k(n)$. It was our original intent to include in this paper results (mostly known) for these algebras analogous to those which we have proved for the algebras $\mathbb{C}A_\ell(n)$, but the restrictions of time and space have prevented this. While perusing this paper, the reader should keep in mind that the techniques we have used do apply to these other algebras.

1. The Partition Monoid

For $k \in \mathbb{Z}_{>0}$, let

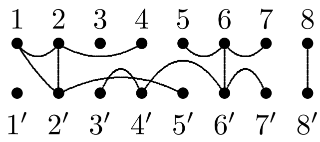
$$\begin{aligned} A_k &= \left\{\text{set partitions of } \{1,2,\ldots,k,1',2',\ldots,k'\}\right\}, \qquad \text{and} \\ A_{k+\frac12} &= \left\{d \in A_{k+1} \;\middle|\; (k+1) \text{ and } (k+1)' \text{ are in the same block}\right\}. \end{aligned} \tag{1.1}$$

The *propagating number* of $d \in A_k$ is

$$pn(d) = \begin{pmatrix} \text{the number of blocks in } d \text{ that contain both an element} \\ \text{of } \{1,2,\ldots,k\} \text{ and an element of } \{1',2',\ldots,k'\} \end{pmatrix}. \tag{1.2}$$

For convenience, represent a set partition $d \in A_k$ by a graph with k vertices in the top row, labeled $1,\ldots,k$ left to right, and k vertices in the bottom row, labeled $1',\ldots,k'$ left to right, with vertex

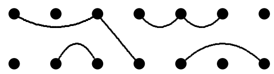
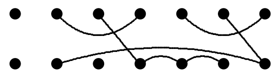
i and vertex j connected by a path if i and j are in the same block of the set partition d. For example,



represents $\{\{1,2,4,2',5'\},\{3\},\{5,6,7,3',4',6',7'\},\{8,8'\},\{1'\}\}$,

and has propagating number 3. The graph representing d is not unique.

Define the composition $d_1 \circ d_2$ of partition diagrams $d_1, d_2 \in A_k$ to be the set partition $d_1 \circ d_2 \in A_k$ obtained by placing d_1 above d_2 and identifying the bottom dots of d_1 with the top dots of d_2, removing any connected components that live entirely in the middle row. For example,

if $d_1 =$  and $d_2 =$  then

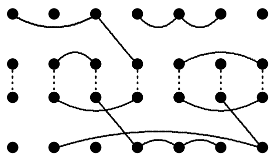
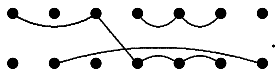
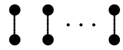
$d_1 \circ d_2 =$  $=$ .

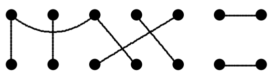
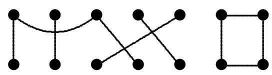
Diagram multiplication makes A_k into an associative monoid with identity, $1 =$ . The propagating number satisfies

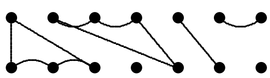
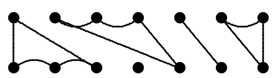
$$pn(d_1 \circ d_2) \leq \min(pn(d_1), pn(d_2)). \tag{1.3}$$

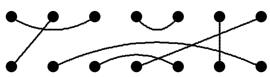
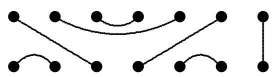
A set partition is *planar* [Jo] if it can be represented as a graph without edge crossings inside of the rectangle formed by its vertices. For each $k \in \frac{1}{2}\mathbb{Z}_{>0}$, the following are submonoids of the partition monoid A_k:

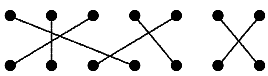
$$\begin{gathered} S_k = \{d \in A_k \mid pn(d) = k\}, \quad I_k = \{d \in A_k \mid pn(d) < k\}, \quad P_k = \{d \in A_k \mid d \text{ is planar}\}, \\ B_k = \{d \in A_k \mid \text{all blocks of } d \text{ have size } 2\}, \quad \text{and} \quad T_k = P_k \cap B_k. \end{gathered} \tag{1.4}$$

Examples are

 $\in I_7$,  $\in I_{6+\frac{1}{2}}$,

 $\in P_7$,  $\in P_{6+\frac{1}{2}}$,

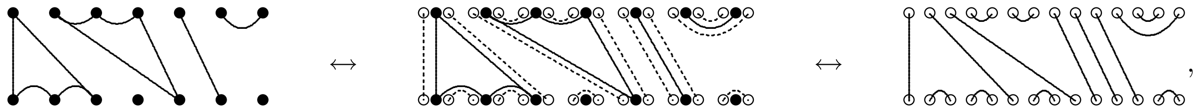
 $\in B_7$,  $\in T_7$,

 $\in S_7$.

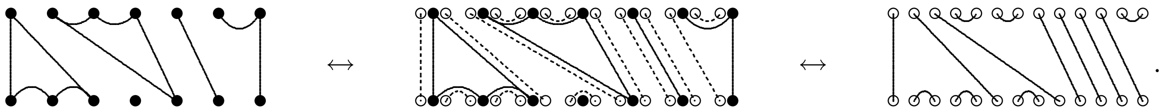
For $k \in \frac{1}{2}\mathbb{Z}_{>0}$, there is an isomorphism of monoids

$$P_k \overset{1-1}{\longleftrightarrow} T_{2k}, \tag{1.5}$$

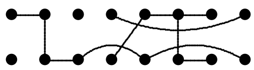
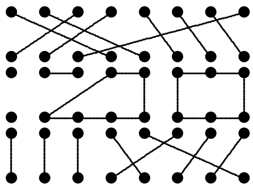
which is best illustrated by examples. For $k = 7$ we have



and for $k = 6 + \frac{1}{2}$ we have



Let $k \in \mathbb{Z}_{>0}$. By permuting the vertices in the top row and in the bottom row each $d \in A_k$ can be written as a product $d = \sigma_1 t \sigma_2$, with $\sigma_1, \sigma_2 \in S_k$ and $t \in P_k$, and so

$A_k = S_k\, P_k\, S_k.$ For example,  = . (1.6)

For $\ell \in \mathbb{Z}_{>0}$, define

$$\begin{aligned} &\text{the } \textit{Bell number,} && B(\ell) = (\text{the number of set partitions of } \{1, 2, \ldots, \ell\}), \\ &\text{the } \textit{Catalan number,} && C(\ell) = \frac{1}{\ell+1}\binom{2\ell}{\ell} = \binom{2\ell}{\ell} - \binom{2\ell}{\ell+1}, \\ &(2\ell)!! = (2\ell-1)\cdot(2\ell-3)\cdots 5\cdot 3\cdot 1, && \text{and} \quad \ell! = \ell\cdot(\ell-1)\cdots 2\cdot 1, \end{aligned} \tag{1.7}$$

with generating functions (see [Sta, 1.24f, and 6.2]),

$$\begin{aligned} &\sum_{\ell\geq 0} B(\ell)\frac{z^\ell}{\ell!} = \exp(e^z - 1), && \sum_{\ell\geq 0} C(\ell-1) z^\ell = \frac{1-\sqrt{1-4z}}{2z}, \\ &\sum_{\ell\geq 0} (2(\ell-1))!!\frac{z^\ell}{\ell!} = \frac{1-\sqrt{1-2z}}{z}, && \sum_{\ell\geq 0} \ell!\frac{z^\ell}{\ell!} = \frac{1}{1-z}. \end{aligned} \tag{1.8}$$

Then

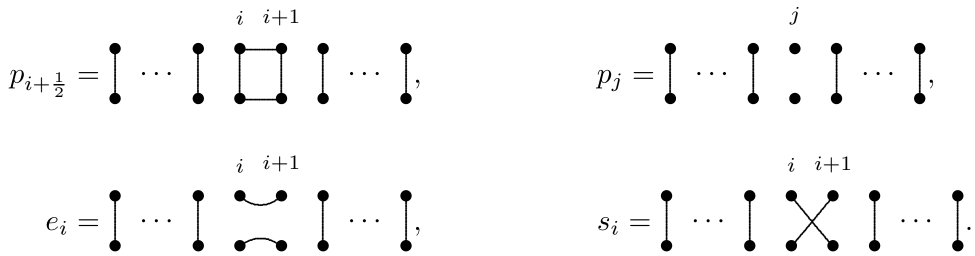
$$\begin{aligned} &\text{for } k \in \tfrac{1}{2}\mathbb{Z}_{>0}, && \mathrm{Card}(A_k) = B(2k) && \text{and} && \mathrm{Card}(P_k) = \mathrm{Card}(T_{2k}) = C(2k)\,, \\ &\text{for } k \in \mathbb{Z}_{>0}, && \mathrm{Card}(B_k) = (2k)!!, && \text{and} && \mathrm{Card}(S_k) = k!. \end{aligned} \tag{1.9}$$

Presentation of the Partition Monoid

In this section, for convenience, we will write

$$d_1 d_2 = d_1 \circ d_2, \qquad \text{for } d_1, d_2 \in A_k$$

Let $k \in \mathbb{Z}_{>0}$. For $1 \le i \le k-1$ and $1 \le j \le k$, define



(1.10)

Note that $e_i = p_{i+\frac{1}{2}} p_i p_{i+1} p_{i+\frac{1}{2}}$.

Theorem 1.11.
(a) The monoid T_k is presented by generators $e_1, \ldots, e_{k-1}$ and relations

$$e_i^2 = e_i, \qquad e_i e_{i\pm 1} e_i = e_i, \qquad \text{and} \qquad e_i e_j = e_j e_i, \quad \text{for } |i-j| > 1.$$

(b) The monoid P_k is presented by generators $p_{\frac{1}{2}}, p_1, p_{\frac{3}{2}}, \ldots, p_k$ and relations

$$p_i^2 = p_i, \qquad p_i p_{i\pm\frac{1}{2}} p_i = p_i, \qquad \text{and} \qquad p_i p_j = p_j p_i, \quad \text{for } |i-j| > 1/2.$$

(c) The group S_k is presented by generators $s_1, \ldots, s_{k-1}$ and relations

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{and} \qquad s_i s_j = s_j s_i, \quad \text{for } |i-j| > 1.$$

(d) The monoid A_k is presented by generators $s_1, \ldots, s_{k-1}$ and $p_{\frac{1}{2}}, p_1, p_{\frac{3}{2}}, \ldots, p_k$ and relations in (b) and (c) and

$$s_i p_i p_{i+1} = p_i p_{i+1} s_i = p_i p_{i+1}, \qquad s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}, \qquad s_i p_i s_i = p_{i+1},$$
$$s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_i = p_{i+\frac{3}{2}}, \qquad \text{and} \qquad s_i p_j = p_j s_i, \qquad \text{for } j \ne i - \tfrac{1}{2}, i, i+\tfrac{1}{2}, i+1, i+\tfrac{3}{2}.$$

Proof. Parts (a) and (c) are standard. See [GHJ, Prop. 2.8.1] and [Bou2, Ch. IV §1.3, Ex. 2], respectively. Part (b) is a consequence of (a) and the monoid isomorphism in (1.5).

(d) The right way to think of this is to realize that A_k is defined as a presentation by the generators $d \in A_k$ and the relations which specify the composition of diagrams. To prove the presentation in the statement of the theorem we need to establish that the generators and relations in each of these two presentations can be derived from each other. Thus it is sufficient to show that

(1) The generators in (1.10) satisfy the relations in Theorem (1.11).

(2) Every set partition $d \in A_k$ can be written as a product of the generators in (1.10).

(3) Any product $d_1 \circ d_2$ can be computed using the relations in Theorem (1.11).

(1) is established by a direct check using the definition of the multiplication of diagrams. (2) follows from (b) and (c) and the fact (1.6) that $A_k = S_k P_k S_k$. The bulk of the work is in proving (3).

Step 1. First note that the relations in (a–d) imply the following relations:

(e1) $p_{i+\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}s_is_{i-1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}s_is_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_is_is_{i-1}$
$= p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_is_{i-1} = p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}s_{i-1} = p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}$.

(e2) $p_is_ip_i = s_is_ip_is_ip_i = s_ip_{i+1}p_i = p_{i+1}p_i$.

(f1) $p_ip_{i+\frac{1}{2}}p_{i+1} = p_ip_{i+\frac{1}{2}}s_ip_{i+1} = p_ip_{i+\frac{1}{2}}p_is_i = p_is_i$.

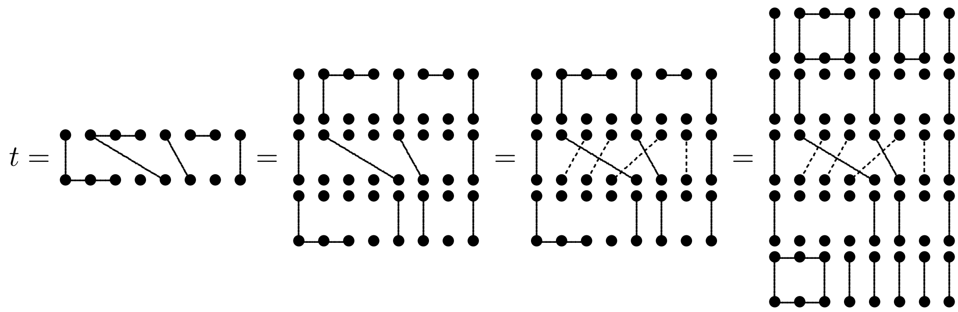
(f2) $p_{i+1}p_{i+\frac{1}{2}}p_i = p_{i+1}s_ip_{i+\frac{1}{2}}p_i = s_ip_ip_{i+\frac{1}{2}}p_i = s_ip_i$.

Step 2. Analyze how elements of P_k can be efficiently expressed in terms of the generators.

Let $t \in P_k$. The blocks of t partition $\{1, \ldots, k\}$ into *top blocks* and partition $\{1', \ldots, k'\}$ into *bottom blocks*. In t, some top blocks are connected to bottom blocks by an edge, but no top block is connected to two bottom blocks, for then by transitivity the two bottom blocks are actually a single block. Draw the diagram of t, such that if a top block connects to a bottom block, then it connects with a single edge joining the leftmost vertices in each block. The element $t \in P_k$ can be decomposed in *block form* as

$$t = (p_{i_1+\frac{1}{2}} \cdots p_{i_r+\frac{1}{2}})(p_{j_1} \cdots p_{j_s})\tau(p_{\ell_1} \cdots p_{\ell_m})(p_{r_1+\frac{1}{2}} \cdots p_{r_n+\frac{1}{2}}) \tag{1.12}$$

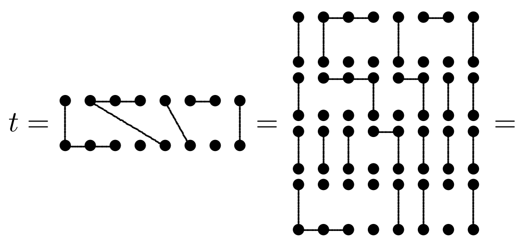
with $\tau \in S_k$, $i_1 < i_2 < \cdots < i_r$, $j_1 < j_2 < \cdots < j_s$, $\ell_1 < \ell_2 < \cdots < \ell_m$, and $r_1 < r_2 < \cdots < r_n$. The left product of p_is corresponds to the top blocks of t, the right product of p_is corresponds to the bottom blocks of t and the permutation τ corresponds to the propagation pattern of the edges connecting top blocks of t to bottom blocks of t. For example,



$$= (p_{2\frac{1}{2}}p_{3\frac{1}{2}}p_{6\frac{1}{2}})(p_3p_4p_6p_7)\tau(p_2p_3p_4p_7)(p_{1\frac{1}{2}}p_{2\frac{1}{2}}),$$
$$= (p_{2\frac{1}{2}}p_{3\frac{1}{2}}p_{6\frac{1}{2}})(p_3p_4p_6p_7)s_2s_3s_5s_4(p_2p_3p_4p_7)(p_{1\frac{1}{2}}p_{2\frac{1}{2}}),$$

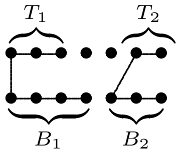
The dashed edges of τ are "non-propagating" edges, and they may be chosen so that they do not cross each other. The propagating edges of τ do not cross, since t is planar.

Using the relations *(f1)* and *(f2)*, the non-propagating edges of τ can be "removed", leaving a planar diagram which is written in terms of the generators p_i and $p_{i+\frac{1}{2}}$. In our example, this process will replace τ by $p_{2\frac{1}{2}}p_2p_{3\frac{1}{2}}p_3p_{5\frac{1}{2}}p_5p_{4\frac{1}{2}}p_4$, so that



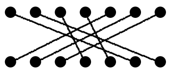
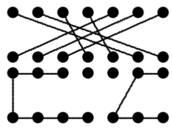
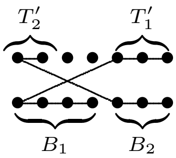
$$(p_{2\frac{1}{2}}p_{3\frac{1}{2}}p_{6\frac{1}{2}})(p_3p_4p_6p_7)$$
$$\cdot\ p_{2\frac{1}{2}}p_2p_{3\frac{1}{2}}p_3p_{5\frac{1}{2}}p_5p_{4\frac{1}{2}}p_4$$
$$\cdot\ (p_2p_3p_4p_7)(p_{1\frac{1}{2}}p_{2\frac{1}{2}}).$$

Step 3: If $t \in P_k$ and $\sigma_1 \in S_k$ which permutes the the top blocks of the planar diagram t, then there is a permutation σ_2 of the bottom blocks of t such that $\sigma_1 t \sigma_2$ is planar. Furthermore, this can be accomplished using the relations. For example, suppose

$t =$ 

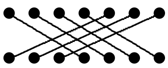
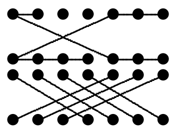
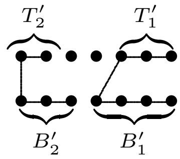
$$= \underbrace{(p_{1\frac12}p_{2\frac12})(p_2p_3)}_{T_1}\underbrace{(p_{6\frac12})(p_7)}_{T_2}\, p_4p_5s_5\, \underbrace{(p_2p_3p_4)(p_{1\frac12}p_{2\frac12}p_{3\frac12})}_{B_1}\underbrace{(p_6p_7)(p_{5\frac12}p_{6\frac12})}_{B_2}$$

is a planar diagram with top blocks T_1 and T_2 connected respectively to bottom blocks B_1 and B_2 and

$\sigma_1 =$  so that $\sigma_1 t =$  $=$ 

$= t'$,

then transposition of B_1 and B_2 can be accomplished with the permutation

$\sigma_2 =$  so that $\sigma_1 t \sigma_2 = t'\sigma_2 =$  $=$ 

is planar. It is possible to accomplish these products using the relations from the statement of the theorem. In our example, with $\sigma_1 = s_2s_1s_3s_2s_4s_3s_5s_2s_4s_6s_1s_3s_5s_2s_4s_3$ and with $\sigma_2 = s_4s_5s_6s_3s_4s_5s_2s_3s_4s_1s_2s_3$,

$$\sigma_1 T_1T_2\, p_4p_5\, s_5\, B_1B_2\sigma_2 = (\sigma_1 T_1 T_2\, p_4p_5\, \sigma_1^{-1})(\sigma_1 s_5 \sigma_2)(\sigma_2^{-1}B_1B_2\sigma_2) = T_2'T_1'\, p_3p_4\, s_4\, B_2'B_1',$$

where $T_2'T_1' = (p_{1\frac12}p_2)(p_{5\frac12}p_{6\frac12}p_6p_7)$ and $B_2'B_1' = (p_2p_3p_{1\frac12}p_{2\frac12})(p_5p_6p_7p_{4\frac12}p_{5\frac12}p_{6\frac12})$.

Step 4: Let $t, b \in P_k$ and let $\pi \in S_k$. Then $t\pi b = tx\sigma$ where $x \in P_k$ and $\sigma \in S_k$, and this transformation can be acccomplished using the relations in (b), (c) and (d).

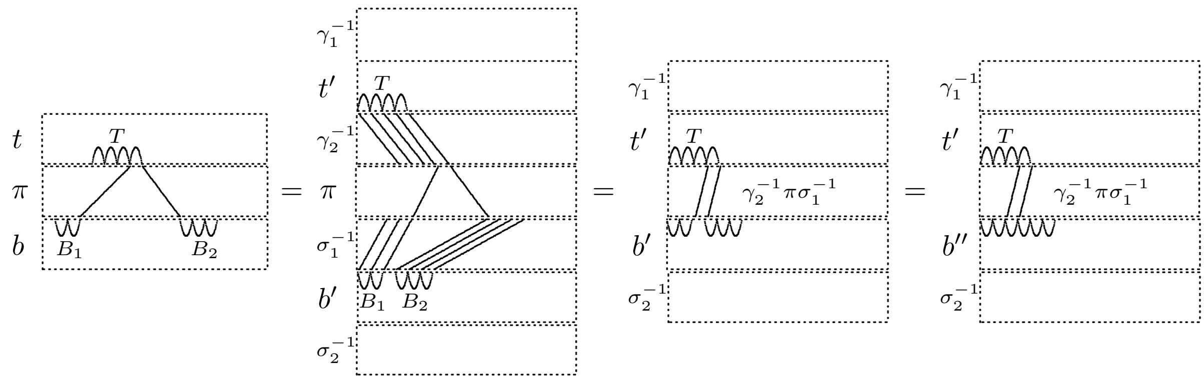
Suppose T is a block of bottom dots of t containing more than one dot and which is connected, by edges of π, to two top blocks B_1 and B_2 of b. Using Step 3 find permutations $\gamma_1, \gamma_2 \in S_k$ and $\sigma_1, \sigma_2 \in S_k$ such that

$$t' = \gamma_1 t \gamma_2 \qquad \text{and} \qquad b' = \sigma_1 b \sigma_2$$

are planar diagrams with T as the leftmost bottom block of t' and B_1 and B_2 as the two leftmost top blocks of b'. Then

$$t\pi b = \gamma_1^{-1}t'\gamma_2^{-1}\pi\sigma_1^{-1}b'\sigma_2^{-1} = \gamma_1^{-1}t'(\gamma_2^{-1}\pi\sigma_1^{-1})b'\sigma_2^{-1} = \gamma_1^{-1}t'(\gamma_2^{-1}\pi\sigma_1^{-1})b''\sigma_2^{-1} = t\pi\sigma_1^{-1}b''\sigma_2^{-1},$$

where b'' is a planar diagram with fewer top blocks than b has. This is best seen from the following picture, where $t\pi b$ equals



and the last equality uses the relations $p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}^2$ and fourth relation in (d) (multiple times). Then $t\pi b = \gamma_1^{-1}t'\gamma_2^{-1}\pi\sigma_1^{-1}b''\sigma_2^{-1} = t\pi' b''\sigma_2^{-1}$, where $\pi' = \pi\sigma_1^{-1}$.

By iteration of this process it is sufficient to assume that in proving Step 4 we are analyzing $t\pi b$ where each bottom block of t connects to a single top block of b. Then, since π is a permutation, the bottom blocks of t must have the same sizes as the top blocks of b and π is the permutation that permutes the bottom blocks of t to the top blocks of b. Thus, by Step 1, there is $\sigma \in S_k$ such that $x = \pi b\sigma^{-1}$ is planar and

$$t\pi b = t(\pi b\sigma^{-1})\sigma = tx\sigma.$$

Completion of the proof: If $d_1, d_2 \in A_k$ then use the decomposition $A_k = S_k P_k S_k$ (from (1.6)) to write d_1 and d_2 in the form

$$d_1 = \pi_1 t\pi_2 \qquad \text{and} \qquad d_2 = \sigma_1 b\sigma_2, \qquad \text{with } t, b \in P_k, \pi_1, \pi_2, \sigma_1, \sigma_2 \in S_k,$$

and use (b) and (c) to write these products in terms of the generators. Let $\pi = \pi_2\sigma_1$. Then Step 4 tell us that the relations give $\sigma \in S_k$ and $x \in P_k$ such that

$$d_1d_2 = \pi_1 t\pi_2\sigma_1 b\sigma_2 = \pi_1 t\pi b\sigma_2 = \pi_1 tx\sigma\sigma_2,$$

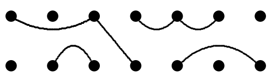
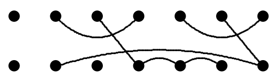
Using Step 2 and that $A_k = S_kP_kS_k$, this product can be identified with the product diagram d_1d_2. Thus, the relations are sufficient to compose any two elements of A_k. ∎

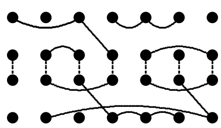
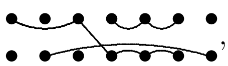
2. Partition Algebras

For $k \in \frac{1}{2}\mathbb{Z}_{>0}$ and $n \in \mathbb{C}$, the *partition algebra* $\mathbb{C}A_k(n)$ is the associative algebra over $\mathbb{C}$ with basis A_k,

$$\mathbb{C}A_k(n) = \mathbb{C}\text{span-}\{d \in A_k\}, \qquad \text{and multiplication defined by} \qquad d_1d_2 = n^{\ell}(d_1 \circ d_2),$$

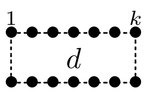
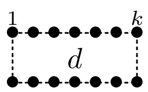
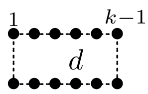
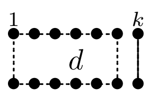
where, for $d_1, d_2 \in A_k$, $d_1 \circ d_2$ is the product in the monoid A_k and ℓ is the number of blocks removed from the the middle row when constructing the composition $d_1 \circ d_2$. For example,

if $d_1 =$  and $d_2 =$  then

$d_1d_2 =$  $= n^2$ , (2.1)

since two blocks are removed from the middle row. There are inclusions of algebras given by

$$\mathbb{C}A_{k-\frac{1}{2}} \hookrightarrow \mathbb{C}A_k \qquad\qquad \mathbb{C}A_{k-1} \hookrightarrow \mathbb{C}A_{k-\frac{1}{2}}$$

 $\mapsto$  and  $\mapsto$ . (2.2)

For $d_1, d_2 \in A_k$, define

$$d_1 \leq d_2, \qquad \text{if the set partition } d_2 \text{ is coarser than the set partition } d_1,$$

i.e., i and j in the same block of d_1 implies that i and j are in the same block of d_2. Let $\{x_d \in \mathbb{C}A_k | d \in A_k\}$ be the basis of $\mathbb{C}A_k$ uniquely defined by the relation

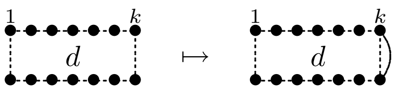
$$d = \sum_{d' \leq d} x_{d'}, \qquad \text{for all } d \in A_k. \tag{2.3}$$

Under any linear extension of the partial order $\leq$ the transition matrix between the basis $\{d \mid d \in A_k\}$ of $\mathbb{C}A_k(n)$ and the basis $\{x_d \mid d \in A_k\}$ of $\mathbb{C}A_k(n)$ is upper triangular with 1s on the diagonal and so the x_d are well defined.

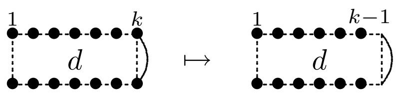
The maps $\varepsilon_{\frac{1}{2}}: \mathbb{C}A_k \to \mathbb{C}A_{k-\frac{1}{2}}$, $\varepsilon^{\frac{1}{2}}: \mathbb{C}A_{k-\frac{1}{2}} \to \mathbb{C}A_{k-1}$ *and* $tr_k: \mathbb{C}A_k \to \mathbb{C}$

Let $k \in \mathbb{Z}_{>0}$. Define linear maps

$\varepsilon_{\frac{1}{2}}: \quad \mathbb{C}A_k \longrightarrow \mathbb{C}A_{k-\frac{1}{2}}$ and $\varepsilon^{\frac{1}{2}}: \quad \mathbb{C}A_{k-\frac{1}{2}} \longrightarrow \mathbb{C}A_{k-1}$



and

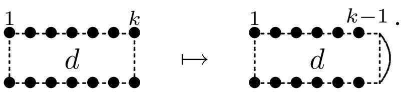


so that $\varepsilon_{\frac{1}{2}}(d)$ is the same as d except that the block containing k and the block containing k' are combined, and $\varepsilon^{\frac{1}{2}}(d)$ has the same blocks as d except with k and k' removed. There is a factor of n in $\varepsilon^{\frac{1}{2}}(d)$ if the removal of k and k' reduces the number of blocks by 1. For example,

and

The map $\varepsilon^{\frac{1}{2}}$ is the composition $\mathbb{C}A_{k-\frac{1}{2}} \hookrightarrow \mathbb{C}A_k \xrightarrow{\varepsilon_1} \mathbb{C}A_{k-1}$. The composition of $\varepsilon_{\frac{1}{2}}$ and $\varepsilon^{\frac{1}{2}}$ is the map

$\varepsilon_1: \quad \mathbb{C}A_k \longrightarrow \mathbb{C}A_{k-1}$ (2.4)



By drawing diagrams it is straightforward to check that, for $k \in \mathbb{Z}_{>0}$,

$$\begin{aligned} \varepsilon_{\frac{1}{2}}(a_1 b a_2) &= a_1 \varepsilon_{\frac{1}{2}}(b) a_2, \quad \text{for } a_1, a_2 \in A_{k-\frac{1}{2}}, b \in A_k \\ \varepsilon^{\frac{1}{2}}(a_1 b a_2) &= a_1 \varepsilon^{\frac{1}{2}}(b) a_2, \quad \text{for } a_1, a_2 \in A_{k-1}, b \in A_{k-\frac{1}{2}} \\ \varepsilon_1(a_1 b a_2) &= a_1 \varepsilon_1(b) a_2, \quad \text{for } a_1, a_2 \in A_{k-1}, b \in A_k. \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} p_{k+\frac{1}{2}} b p_{k+\frac{1}{2}} &= \varepsilon_{\frac{1}{2}}(b) p_{k+\frac{1}{2}} = p_{k+\frac{1}{2}} \varepsilon_{\frac{1}{2}}(b), \quad \text{for } b \in A_k \\ p_k b p_k &= \varepsilon^{\frac{1}{2}}(b) p_k = p_k \varepsilon^{\frac{1}{2}}(b), \quad \text{for } b \in A_{k-\frac{1}{2}} \\ e_k b e_k &= \varepsilon_1(b) e_k = e_k \varepsilon_1(b), \quad \text{for } b \in A_k. \end{aligned} \tag{2.6}$$

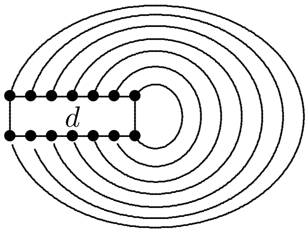
Define $tr_k: \mathbb{C}A_k \to \mathbb{C}$ and $tr_{k-\frac{1}{2}}: \mathbb{C}A_{k-\frac{1}{2}} \to \mathbb{C}$ by the equations

$$\mathrm{tr}_k(b) = \mathrm{tr}_{k-\frac{1}{2}}(\varepsilon_{\frac{1}{2}}(b)), \quad \text{for } b \in A_k, \quad \text{and} \quad tr_{k-\frac{1}{2}}(b) = \mathrm{tr}_{k-1}(\varepsilon^{\frac{1}{2}}(b)), \quad \text{for } b \in A_{k-\frac{1}{2}}, \tag{2.7}$$

so that

$$tr_k(b) = \varepsilon_1^k(b), \quad \text{for } b \in A_k, \qquad \text{and} \qquad tr_{k-\frac{1}{2}}(b) = \varepsilon_1^{k-1}\varepsilon^{\frac{1}{2}}(b), \quad \text{for } b \in A_{k-\frac{1}{2}}. \tag{2.8}$$

Pictorially $tr_k(d) = n^c$ where c is the number of connected components in the closure of the diagram d,

$tr_k(d) =$ , for $d \in A_k$. (2.9)

The ideal $\mathbb{C}I_k(n)$

For $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ define

$$\mathbb{C}I_k(n) = \mathbb{C}\text{-span}\{d \in I_k\}. \tag{2.10}$$

By (1.3),

$$\mathbb{C}I_k(n) \text{ is an ideal of } \mathbb{C}A_k(n) \qquad \text{and} \qquad \mathbb{C}A_k(n)/\mathbb{C}I_k(n) \cong \mathbb{C}S_k, \tag{2.11}$$

since the set partitions with propagating number k are exactly the permutations in the symmetric group S_k (by convention $S_{\ell+\frac{1}{2}} = S_\ell$ for $\ell \in \mathbb{Z}_{>0}$, see (2.2)).

View $\mathbb{C}I_k(n)$ as an algebra (without identity). Since $\mathbb{C}A_k(n)/\mathbb{C}I_k \cong \mathbb{C}S_k$ and $\mathbb{C}S_k$ is semisimple, $\text{Rad}(\mathbb{C}A_k(n)) \subseteq \mathbb{C}I_k(n)$. Since $\mathbb{C}I_k(n)/\text{Rad}(\mathbb{C}A_k(n))$ is an ideal in $\mathbb{C}A_k(n)/\text{Rad}(\mathbb{C}A_k(n))$ the quotient $\mathbb{C}I_k(n)/\text{Rad}(\mathbb{C}A_k(n))$ is semisimple. Therefore $\text{Rad}(\mathbb{C}I_k(n)) \subseteq \text{Rad}(\mathbb{C}A_k(n))$. On the other hand, since $\text{Rad}(\mathbb{C}A_k(n))$ is an ideal of nilpotent elements in $\mathbb{C}A_k(n)$, it is an ideal of nilpotent elements in $\mathbb{C}I_k(n)$ and so $\text{Rad}(\mathbb{C}I_k(n)) \supseteq \text{Rad}(\mathbb{C}A_k(n))$. Thus

$$\text{Rad}(\mathbb{C}A_k(n)) = \text{Rad}(\mathbb{C}I_k(n)). \tag{2.12}$$

Let $k \in \mathbb{Z}_{\geq 0}$. By (2.5) the maps

$$\varepsilon_{\frac{1}{2}} : \mathbb{C}A_k \longrightarrow \mathbb{C}A_{k-\frac{1}{2}} \qquad \text{and} \qquad \varepsilon^{\frac{1}{2}} : \mathbb{C}A_{k-\frac{1}{2}} \longrightarrow \mathbb{C}A_{k-1}$$

are $(\mathbb{C}A_{k-\frac{1}{2}}, \mathbb{C}A_{k-\frac{1}{2}})$-bimodule and $(\mathbb{C}A_{k-1}, \mathbb{C}A_{k-1})$-bimodule homomorphisms, respectively. The corresponding *basic constructions* (see Section 4) are the algebras

$$\mathbb{C}A_k(n) \otimes_{\mathbb{C}A_{k-\frac{1}{2}}(n)} \mathbb{C}A_k(n) \qquad \text{and} \qquad \mathbb{C}A_{k-\frac{1}{2}}(n) \otimes_{\mathbb{C}A_{k-1}(n)} \mathbb{C}A_{k-\frac{1}{2}}(n) \tag{2.13}$$

with products given by

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon_{\frac{1}{2}}(b_2 b_3) b_4, \qquad \text{and} \qquad (c_1 \otimes c_2)(c_3 \otimes c_4) = c_1 \otimes \varepsilon^{\frac{1}{2}}(c_2 c_3) c_4, \tag{2.14}$$

for $b_1, b_2, b_3, b_4 \in \mathbb{C}A_k(n)$, and for $c_1, c_2, c_3, c_4 \in \mathbb{C}A_{k-\frac{1}{2}}(n)$.

Let $k \in \frac{1}{2}\mathbb{Z}_{>0}$. Then, by the relations in (2.6) and the fact that

$$\text{every } d \in I_k \qquad \text{can be written as} \qquad d = d_1 p_k d_2, \qquad \text{with } d_1, d_2 \in A_{k-\frac{1}{2}}, \tag{2.15}$$

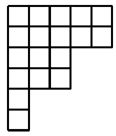
the maps

$$\begin{array}{ccc} \mathbb{C}A_{k-\frac{1}{2}}(n) \otimes_{\mathbb{C}A_{k-1}(n)} \mathbb{C}A_{k-\frac{1}{2}}(n) & \longrightarrow & \mathbb{C}I_k(n) \\ b_1 \otimes b_2 & \longmapsto & b_1 p_k b_2 \end{array} \tag{2.16}$$

are algebra isomorphisms. Thus the ideal $\mathbb{C}I_k(n)$ is always isomorphic to a basic construction (in the sense of Section 4).

Representations of the symmetric group

A partition λ is a collection of boxes in a corner. We shall conform to the conventions in [Mac] and assume that gravity goes up and to the left, i.e.,



Numbering the rows and columns in the same way as for matrices, let

$$\begin{aligned} \lambda_i &= \text{the number of boxes in row } i \text{ of } \lambda, \\ \lambda'_j &= \text{the number of boxes in column } j \text{ of } \lambda, \quad \text{and} \\ |\lambda| &= \text{the total number of boxes in } \lambda. \end{aligned} \tag{2.17}$$

Any partition λ can be identified with the sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ and the *conjugate partition* to λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$. The *hook length* of the box b of λ is

$$h(b) = (\lambda_i - i) + (\lambda'_j - j) + 1, \quad \text{if } b \text{ is in position } (i,j) \text{ of } \lambda. \tag{2.18}$$

Write $\lambda \vdash n$ if λ is a partition with n boxes. In the example above $\lambda = (553311)$ and $\lambda \vdash 18$.

See [Mac, §I.7] for details on the representation theory of the symmetric group. The irreducible $\mathbb{C}S_k$-modules S_k^λ are indexed by the elements of

$$\hat{S}_k = \{\lambda \vdash n\} \qquad \text{and} \qquad \dim(S_k^\lambda) = \frac{k!}{\prod_{b\in\lambda} h(b)}. \tag{2.19}$$

For $\lambda \in \hat{S}_k$, and $\mu \in \hat{S}_{k-1}$,

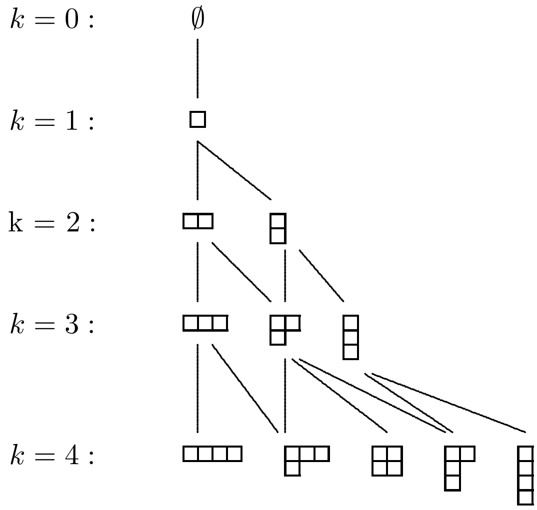
$$\mathrm{Res}^{S_k}_{S_{k-1}}(S_k^\lambda) \cong \bigoplus_{\lambda/\nu=\square} S^\nu_{k-1} \qquad \text{and} \qquad \mathrm{Ind}^{S_k}_{S_{k-1}}(S^\mu_{k-1}) \cong \bigoplus_{\nu/\mu=\square} S^\nu_k. \tag{2.20}$$

where the first sum is over all partitions ν that are obtained from λ by removing a box, and the second sum is over all partitions ν which are obtained from μ by adding a box (this result follows, for example, from [Mac, §I.7 Ex. 22(d)]).

The *Young lattice* is the graph $\hat{S}$ given by setting

$$\begin{array}{c} \text{vertices on level } k\text{:} \qquad \hat{S}_k = \{\text{partitions } \lambda \text{ with } k \text{ boxes}\}, \quad \text{and} \\ \text{an edge } \lambda \to \mu,\ \lambda \in \hat{S}_k,\ \mu \in \hat{S}_{k+1} \text{ if } \mu \text{ is obtained from } \lambda \text{ by adding a box.} \end{array} \tag{2.21}$$

It encodes the decompositions in (2.20). The first few levels of $\hat{S}$ are given by



For $\mu \in \hat{S}_k$ define

$$\hat{S}_k^\mu = \left\{ T = (T^{(0)}, T^{(1)}, \ldots, T^{(k)}) \;\middle|\; \begin{array}{l} T^{(0)} = \emptyset\,,\; T^{(k)} = \mu\,, \quad \text{and,} \quad \text{for each } \ell, \\ T^{(\ell)} \in \hat{S}_\ell \text{ and } T^{(\ell)} \to T^{(\ell+1)} \text{ is an edge in } \hat{S} \end{array} \right\}$$

so that $\hat{S}_k^\mu$ is the set of paths from $\emptyset \in \hat{S}_0$ to $\mu \in \hat{S}_k$ in the graph $\hat{S}$. In terms of the Young lattice

$$\dim(S_k^\mu) = \mathrm{Card}(\hat{S}_k^\mu). \tag{2.22}$$

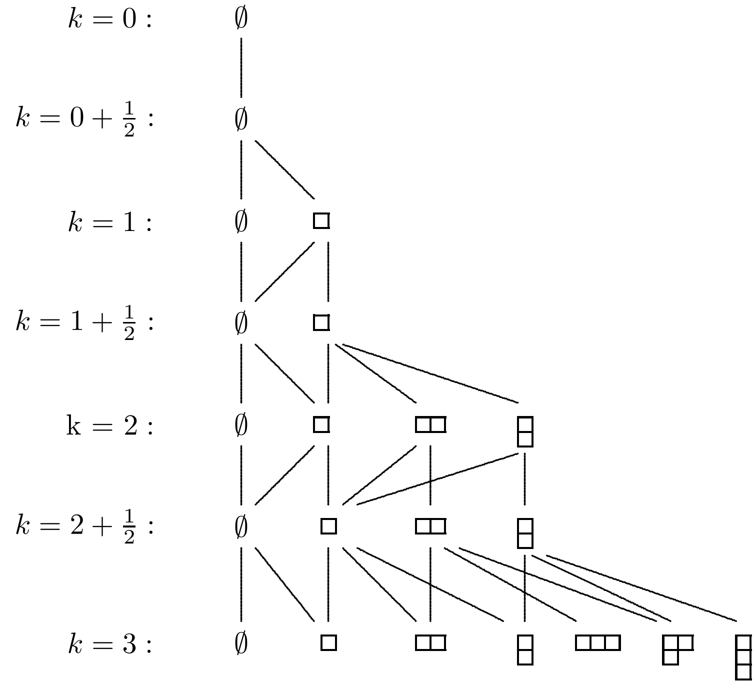
This is a translation of the classical statement (see [Mac, §I.7.6(ii)]) that $\dim(S_k^\mu)$ is the number of standard Young tableaux of shape λ (the correspondence is obtained by putting the entry ℓ in the box of λ which is added at the ℓth step $T^{(\ell-1)} \to T^{(\ell)}$ of the path).

Structure of the algebra $\mathbb{C}A_k(n)$

Build a graph $\hat{A}$ by setting

vertices on level k: $\hat{A}_k = \{\text{partitions } \mu \mid k - |\mu| \in \mathbb{Z}_{\geq 0}\}$,
vertices on level $k + \frac{1}{2}$: $\hat{A}_{k+\frac{1}{2}} = \hat{A}_k = \{\text{partitions } \mu \mid k - |\mu| \in \mathbb{Z}_{\geq 0}\}$,
an edge $\lambda \to \mu$, $\lambda \in \hat{A}_k$, $\mu \in \hat{A}_{k+\frac{1}{2}}$ if $\lambda = \mu$ or if μ is obtained from λ by removing a box,
an edge $\mu \to \lambda$, $\mu \in \hat{A}_{k+\frac{1}{2}}$, $\lambda \in \hat{A}_{k+1}$, if $\lambda = \mu$ or if λ is obtained from μ by adding a box. (2.23)

The first few levels of $\hat{A}$ are given by



The following result is an immediate consequence of the Tits deformation theorem, Theorems 5.12 and 5.13 in this paper (see also [CR (68.17)]).

Theorem 2.24.
(a) For all but a finite number of $n \in \mathbb{C}$ the algebra $\mathbb{C}A_k(n)$ is semisimple.
(b) If $\mathbb{C}A_k(n)$ is semisimple then the irreducible $\mathbb{C}A_k(n)$-modules,

$$A_k^\mu \quad \text{are indexed by elements of the set} \quad \hat{A}_k = \{ \text{ partitions } \mu \mid k - |\mu| \in \mathbb{Z}_{\geq 0}\}, \qquad \text{and}$$

$$\dim(A_k^\mu) = (\text{number of paths from } \emptyset \in \hat{A}_0 \text{ to } \mu \in \hat{A}_k \text{ in the graph } \hat{A}).$$

Let

$$\hat{A}_k^\mu = \left\{ T = (T^{(0)}, T^{(\frac{1}{2})}, \ldots, T^{(k-\frac{1}{2})}, T^{(k)}) \;\middle|\; \begin{array}{l} T^{(0)} = \emptyset\,,\ T^{(k)} = \mu\,, \quad \text{and,} \quad \text{for each } \ell, \\ T^{(\ell)} \in \hat{A}_\ell \text{ and } T^{(\ell)} \to T^{(\ell+\frac{1}{2})} \text{ is an edge in } \hat{A} \end{array} \right\}$$

so that $\hat{A}_k^\mu$ is the set of paths from $\emptyset \in \hat{A}_0$ to $\mu \in \hat{A}_k$ in the graph $\hat{A}$. If $\mu \in \hat{S}_k$ then $\mu \in \hat{A}_k$ and $\mu \in \hat{A}_{k+\frac{1}{2}}$ and, for notational convenience in the following theorem,

$$\begin{array}{lll} \text{identify} & P = (P^{(0)}, P^{(1)}, \ldots, P^{(k)}) \in \hat{S}_k^\mu & \text{with the corresponding} \\ & P = (P^{(0)}, P^{(0)}, P^{(1)}, P^{(1)}, \ldots, P^{(k-1)}, P^{(k-1)}, P^{(k)}) & \in \hat{A}_k^\mu, \qquad \text{and} \\ & P = (P^{(0)}, P^{(0)}, P^{(1)}, P^{(1)}, \ldots, P^{(k-1)}, P^{(k-1)}, P^{(k)}, P^{(k)}) & \in \hat{A}_{k+\frac{1}{2}}^\mu. \end{array}$$

For $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $n \in \mathbb{C}$ such that $\mathbb{C}A_\ell(n)$ is semisimple let $\chi^\mu_{A_\ell(n)}$, $\mu \in \hat{A}_\ell$, be the irreducible characters of $\mathbb{C}A_\ell(n)$. Let $\mathrm{tr}_\ell : \mathbb{C}A_\ell(n) \to \mathbb{C}$ be the traces on $\mathbb{C}A_\ell(n)$ defined in (2.8) and define constants $\mathrm{tr}_\ell^\mu(n)$, $\mu \in \hat{A}_\ell$, by

$$\mathrm{tr}_\ell = \sum_{\mu \in \hat{A}_\ell} \mathrm{tr}_\ell^\mu(n) \chi^\mu_{A_\ell(n)}. \tag{2.25}$$

Theorem 2.26.

(a) Let $n \in \mathbb{C}$ and let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Assume that

$$\mathrm{tr}^{\lambda}_{\ell}(n) \neq 0, \qquad \text{for all } \lambda \in \hat{A}_\ell,\ \ell \in \tfrac{1}{2}\mathbb{Z}_{\geq 0},\ \ell < k.$$

Then the partition algebras

$$\mathbb{C}A_\ell(n) \text{ are semisimple for all } \ell \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}, \ell \leq k. \tag{2.27}$$

For each $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $\ell \leq k - \frac{1}{2}$, define

$$\varepsilon^{\lambda}_{\mu} = \frac{\mathrm{tr}^{\lambda}_{\ell-\frac{1}{2}}(n)}{\mathrm{tr}^{\mu}_{\ell-1}(n)} \qquad \text{for each edge } \mu \to \lambda,\ \mu \in \hat{A}_{\ell-1},\ \lambda \in \hat{A}_{\ell-\frac{1}{2}}, \text{ in the graph } \hat{A}.$$

Inductively define elements in $\mathbb{C}A_\ell(n)$ by

$$e^{\mu}_{PQ} = \frac{1}{\sqrt{\varepsilon^{\tau}_{\mu}\varepsilon^{\gamma}_{\mu}}}\, e^{\tau}_{P^- T} p_\ell e^{\gamma}_{TQ^-}, \qquad \text{for } \mu \in \hat{A}_\ell,\ |\mu| \leq \ell - 1,\ P, Q \in \hat{A}^{\mu}_{\ell}, \tag{2.28}$$

where $\tau = P^{(\ell-\frac{1}{2})}$, $\gamma = Q^{(\ell-\frac{1}{2})}$, $R^- = (R^{(0)}, \ldots, R^{(\ell-\frac{1}{2})})$ for $R = (R^{(0)}, \ldots, R^{(\ell-\frac{1}{2})}, R^{(\ell)}) \in \hat{A}^{\mu}_{\ell}$ and T is an element of $\hat{A}^{\mu}_{\ell-1}$ (the element e^{λ}_{PQ} does not depend on the choice of T). Then define

$$e^{\lambda}_{PQ} = (1 - z)s^{\lambda}_{PQ}, \quad \text{for} \quad \lambda \in \hat{S}_\ell, P, Q \in \hat{S}^{\lambda}_{\ell}, \qquad \text{where} \quad z = \sum_{\substack{\mu \in \hat{A}_\ell \\ |\mu| \leq \ell - 1}} \sum_{P \in \hat{A}^{\mu}_{\ell}} e^{\mu}_{PP} \tag{2.29}$$

and $\{s^{\lambda}_{PQ} \mid \lambda \in \hat{S}_\ell, P, Q \in \hat{S}^{\lambda}_{\ell}\}$ is any set of matrix units for the the group algebra of the symmetric group $\mathbb{C}S_\ell$. Together, the elements in (2.28) and (2.29) form a set of matrix units in $\mathbb{C}A_\ell(n)$.

(b) Let $n \in \mathbb{Z}_{\geq 0}$ and let $k \in \frac{1}{2}\mathbb{Z}_{>0}$ be minimal such that $\mathrm{tr}^{\lambda}_{k}(n) = 0$ for some $\lambda \in \hat{A}_k$. Then $\mathbb{C}A_{k+\frac{1}{2}}(n)$ is not semisimple.

(c) Let $n \in \mathbb{Z}_{\geq 0}$ and $k \in \frac{1}{2}\mathbb{Z}_{>0}$. If $\mathbb{C}A_k(n)$ is not semisimple then $\mathbb{C}A_{k+j}(n)$ is not semisimple for $j \in \mathbb{Z}_{>0}$.

Proof. (a) Assume that $\mathbb{C}A_{\ell-1}(n)$ and $\mathbb{C}A_{\ell-\frac{1}{2}}(n)$ are both semisimple and that $\mathrm{tr}^{\mu}_{\ell-1}(n) \neq 0$ for all $\mu \in \hat{A}_{\ell-1}$. If $\lambda \in \hat{A}_{\ell-\frac{1}{2}}$ then $\varepsilon^{\lambda}_{\mu} \neq 0$ if and only if $\mathrm{tr}^{\lambda}_{\ell-\frac{1}{2}}(n) \neq 0$, and, since the ideal $\mathbb{C}I_\ell(n)$ is isomorphic to the basic construction $\mathbb{C}A_{\ell-\frac{1}{2}}(n) \otimes_{\mathbb{C}A_{\ell-1}(n)} \mathbb{C}A_{\ell-\frac{1}{2}}(n)$ (see (2.13)) it then follows from Theorem 4.28 that $\mathbb{C}I_\ell(n)$ is semisimple if and only if $\mathrm{tr}^{\lambda}_{\ell-\frac{1}{2}}(n) \neq 0$ for all $\lambda \in \hat{A}_{\ell-\frac{1}{2}}$. Thus, by (2.12), *if $\mathbb{C}A_{\ell-1}(n)$ and $\mathbb{C}A_{\ell-\frac{1}{2}}(n)$ are both semisimple and $\mathrm{tr}^{\mu}_{\ell-1}(n) \neq 0$ for all $\mu \in \hat{A}_{\ell-1}$* then

$$\mathbb{C}A_\ell(n) \qquad \text{is semisimple if and only if} \qquad \mathrm{tr}^{\lambda}_{\ell-\frac{1}{2}}(n) \neq 0 \text{ for all } \lambda \in \hat{A}_{\ell-\frac{1}{2}}. \tag{2.30}$$

By Theorem 4.28, when $\mathrm{tr}^{\lambda}_{\ell-\frac{1}{2}}(n) \neq 0$ for all $\lambda \in \hat{A}_{\ell-\frac{1}{2}}$, the algebra $\mathbb{C}I_\ell(n)$ has matrix units given by the formulas in (2.28). The element z in (2.29) is the central idempotent in $\mathbb{C}A_\ell(n)$ such that $\mathbb{C}I_\ell(n) = z\mathbb{C}A_\ell(n)$. Hence the complete set of elements in (2.28) and (2.29) form a set of matrix units for $\mathbb{C}A_\ell(n)$. This completes the proof of (a) and (b) follows from Theorem 4.28(b).

(c) Part (g) of Theorem 4.28 shows that if $\mathbb{C}A_{\ell-1}(n)$ is not semisimple then $\mathbb{C}A_\ell(n)$ is not semisimple. ∎

Specht modules

Let A be an algebra. An idempotent is a nonzero element $p \in A$ such that $p^2 = p$. A *minimal idempotent* is an idempotent p which cannot be written as a sum $p = p_1 + p_2$ with $p_1p_2 = p_2p_1 = 0$. If p is an idempotent in A and $pAp = \mathbb{C}p$ then p is a minimal idempotent of A since, if $p = p_1 + p_2$ with $p_1^2 = p_1$, $p_2^2 = p_2$ and $p_1p_2 = p_2p_1 = 0$ then $pp_1p = kp$ for some constant p and so $kp_1 = kpp_1 = pp_1pp_1 = p_1$ giving that either $p_1 = 0$ or $k = 1$, in which case $p_1 = pp_1p = p$.

Let p be an idempotent in A. Then the map

$$\begin{matrix} (pAp)^{\mathrm{op}} & \xrightarrow{\sim} & \mathrm{End}_A(Ap) \\ pbp & \mapsto & \phi_{pbp} \end{matrix}, \qquad \text{where} \qquad \phi_{pbp}(ap) = (ap)(pbp) = apbp, \quad \text{for } ap \in Ap, \tag{2.31}$$

is a ring isomorphism.

If p is a minimal idempotent of A and Ap is a semisimple A-module then Ap must be a simple A-module. To see this suppose that Ap is not simple so that there are A-submodules V_1 and V_2 of Ap such that $Ap = V_1 \oplus V_2$. Let $\phi_1, \phi_2 \in \mathrm{End}_A(Ap)$ be the A-invariant projections on V_1 and V_2. By (2.31) ϕ_1 and ϕ_2 are given by right multiplication by $p_1 = p\tilde{p}_1p$ and $p_2 = p\tilde{p}_2p$, respectively, and it follows that $p = p_1 + p_2$, $V_1 = Ap_1$, $V_2 = Ap_2$, and $Ap = Ap_1 \oplus Ap_2$. Then $p_1^2 = \phi_1(p_1) = \phi_1^2(p) = p_1$ and $p_1p_2 = \phi_2(p_1) = \phi_2(\phi_1(p)) = 0$. Similarly $p_2^2 = p_2$ and $p_2p_1 = 0$. Thus p is not a minimal idempotent.

If p is an idempotent in A and Ap is a simple A-module then

$$pAp = \mathrm{End}_A(Ap)^{\mathrm{op}} = \mathbb{C}(p \cdot 1 \cdot p) = \mathbb{C}p,$$

by (2.31) and Schur's lemma (Theorem 5.3).

The group algebra of the symmetric group S_k over the ring $\mathbb{Z}$ is

$$S_{k,\mathbb{Z}} = \mathbb{Z}S_k \qquad \text{and} \qquad \mathbb{C}S_k = \mathbb{C} \otimes_{\mathbb{Z}} S_{k,\mathbb{Z}}\,, \tag{2.32}$$

where the tensor product is defined via the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of k. Define subgroups of S_k by

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell} \qquad \text{and} \qquad S_{\lambda'} = S_{\lambda'_1} \times \cdots \times S_{\lambda'_r}, \tag{2.33}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_r)$ is the conjugate partition to λ, and let

$$\mathbf{1}_\lambda = \sum_{w \in S_\lambda} w \qquad \text{and} \qquad \varepsilon_{\lambda'} = \sum_{w \in S_{\lambda'}} (-1)^{\ell(w)} w. \tag{2.34}$$

Let τ be the permutation in S_k that takes the row reading tableau of shape λ to the column reading tableau of shape λ. For example for $\lambda = (553311)$,

$$\tau = (2,7,8,12,9,16,14,4,15,10,18,6)(3,11)(5,17), \quad \text{since} \quad \tau \cdot \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 \\ \cline{1-3} 14 & 15 & 16 \\ \cline{1-3} 17 \\ \cline{1-1} 18 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|c|} \hline 1 & 7 & 11 & 15 & 17 \\ \hline 2 & 8 & 12 & 16 & 18 \\ \hline 3 & 9 & 13 \\ \cline{1-3} 4 & 10 & 14 \\ \cline{1-3} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}.$$

The *Specht module* for S_k is the $\mathbb{Z}S_k$-module

$$S^\lambda_{k,\mathbb{Z}} = \text{im}\ \Psi_{S_k} = (\mathbb{Z}S_k)\, p_\lambda, \qquad \text{where } p_\lambda = \mathbf{1}_\lambda \tau \varepsilon_{\lambda'} \tau^{-1}, \text{ and} \tag{2.35}$$

where Ψ_{S_k} is the $\mathbb{Z}S_k$-module homomorphism given by

$$\begin{array}{rcccccc} \Psi_{S_k}\colon & (\mathbb{Z}S_k)\mathbf{1}_\lambda & \xrightarrow{\iota} & \mathbb{Z}S_k & \xrightarrow{\pi} & (\mathbb{Z}S_k)\tau\varepsilon_{\lambda'}\tau^{-1} \\ & b\,\mathbf{1}_\lambda & \longmapsto & b\,\mathbf{1}_\lambda & \longmapsto & b\,\mathbf{1}_\lambda\tau\varepsilon_{\lambda'}\tau^{-1} \end{array} \tag{2.36}$$

By induction and restriction rules for the representations of the symmetric groups, the $\mathbb{C}S_k$-modules $(\mathbb{C}S_k)\mathbf{1}_\lambda$ and $(\mathbb{C}S_k)\tau\varepsilon_{\lambda'}\tau^{-1}$ have only one irreducible component in common and it follows (see [Mac,§I.7, Ex. 15]) that

$$S^\lambda_k = \mathbb{C} \otimes_{\mathbb{Z}} S^\lambda_{k,\mathbb{Z}} \qquad \text{is the irreducible } \mathbb{C}S_k\text{-module indexed by } \lambda, \tag{2.37}$$

once one shows that Ψ_{S_k} is not the zero map.

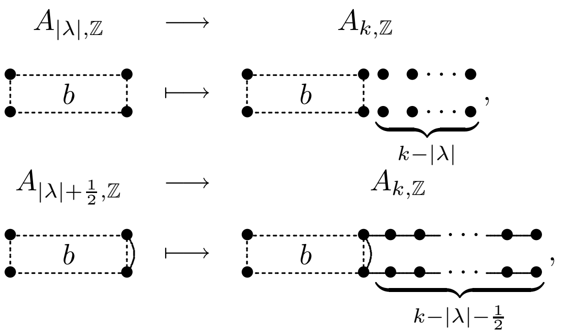
Let $k \in \frac{1}{2}\mathbb{Z}_{>0}$. For an indeterminate x, define the $\mathbb{Z}[x]$-algebra by

$$A_{k,\mathbb{Z}} = \mathbb{Z}[x]\text{-span}\{d \in A_k\} \tag{2.38}$$

with multiplication given by replacing n with x in (2.1). For each $n \in \mathbb{C}$,

$$\mathbb{C}A_k(n) = \mathbb{C} \otimes_{\mathbb{Z}[x]} A_{k,\mathbb{Z}}, \quad \text{where the } \mathbb{Z}\text{-module homomorphism} \quad \begin{array}{rccc} \text{ev}_n\colon & \mathbb{Z}[x] & \to & \mathbb{C} \\ & x & \mapsto & n \end{array} \tag{2.39}$$

is used to define the tensor product. Let λ be a partition with $\leq k$ boxes. Let $b \otimes p_k^{\otimes(k-|\lambda|)}$ denote the image of $b \in A_{|\lambda|,\mathbb{Z}}$ under the map given by



if k is an integer, and

if $k - \frac{1}{2}$ is an integer.

For $k \in \frac{1}{2}\mathbb{Z}_{>0}$, define an $A_{k,\mathbb{Z}}$-module homomorphism

$$\begin{array}{rcccccc} \Psi_{A_k}\colon & A_{k,\mathbb{Z}}t_\lambda & \xrightarrow{\psi_1} & A_{k,\mathbb{Z}}s_{\lambda'} & \xrightarrow{\psi_2} & A_{k,\mathbb{Z}}/I_{|\lambda|,\mathbb{Z}} \\ & bt_\lambda & \longmapsto & bt_\lambda s_{\lambda'} & \longmapsto & \overline{bt_\lambda s_{\lambda'}}\,, \end{array} \tag{2.40}$$

where $I_{|\lambda|,\mathbb{Z}}$ is the ideal

$$I_{|\lambda|,\mathbb{Z}} = \mathbb{Z}[x]\text{-span}\left\{\ d \in A_k \mid d \text{ has propagating number} < |\lambda|\ \right\}$$

and $t_\lambda, s_{\lambda'} \in A_{k,\mathbb{Z}}$ are defined by

$$t_\lambda = \mathbf{1}_\lambda \otimes p_k^{\otimes(k-|\lambda|)} \quad \text{and} \quad s_{\lambda'} = \tau\varepsilon_{\lambda'}\tau^{-1} \otimes p_k^{\otimes(k-|\lambda|)}. \tag{2.41}$$

The *Specht module* for $\mathbb{C}A_k(n)$ is the $A_{k,\mathbb{Z}}$-module

$$A^\lambda_{k,\mathbb{Z}} = \mathrm{im}\ \Psi_{A_k} = \left(\text{image of } A_{k,\mathbb{Z}}e_\lambda \text{ in } A_{k,\mathbb{Z}}/I_{|\lambda|,\mathbb{Z}}\right), \qquad \text{where} \qquad e_\lambda = p_\lambda \otimes p_k^{\otimes(k-|\lambda|)}. \tag{2.42}$$

Proposition 2.43. *Let $k \in \frac{1}{2}\mathbb{Z}_{>0}$, and let λ be a partition with $\le k$ boxes. If $n \in \mathbb{C}$ such that $\mathbb{C}A_k(n)$ is semisimple, then*

$$A^\lambda_k(n) = \mathbb{C} \otimes_{\mathbb{Z}[x]} A^\lambda_{k,\mathbb{Z}} \qquad \textit{is the irreducible } \mathbb{C}A_k(n)\textit{-module indexed by } \lambda,$$

where the tensor product is defined via the $\mathbb{Z}$-module homomorphism in (2.39).

Proof. Let $r = |\lambda|$. Since

$$\mathbb{C}A_r(n)/\mathbb{C}I_r(n) \cong \mathbb{C}S_r$$

and p_λ is a minimal idempotent of $\mathbb{C}S_r$, it follows from (4.20) that $\overline{e_\lambda}$, the image of e_λ in $(\mathbb{C}A_k(n))/(\mathbb{C}I_r(n))$, is a minimal idempotent in $(\mathbb{C}A_k(n))/(\mathbb{C}I_r(n))$. Thus

$$\left(\frac{\mathbb{C}A_k(n)}{\mathbb{C}I_r(n)}\right)\overline{e}_\lambda \qquad \text{is a simple } (\mathbb{C}A_k(n))/(\mathbb{C}I_r(n))\text{-module.}$$

Since the projection $\mathbb{C}A_k(n) \to (\mathbb{C}A_k(n))/(\mathbb{C}I_r(n))$ is surjective, any simple $(\mathbb{C}A_k(n))/(\mathbb{C}I_r(n))$-module is a simple $\mathbb{C}A_k(n)$-module. ∎

3. Schur-Weyl Duality for Partition Algebras

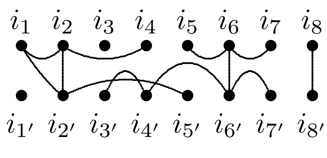
Let $n \in \mathbb{Z}_{>0}$ and let V be a vector space with basis $v_1, \ldots, v_n$. Then the tensor product

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}} \qquad \text{has basis} \qquad \{\ v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\ \}.$$

For $d \in A_k$ and values $i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'} \in \{1, \ldots, n\}$ define

$$(d)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = \begin{cases} 1, & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are in the same block of } d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

For example, viewing $(d)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}}$ as the diagram d with vertices labeled by the values $i_1, \ldots, i_k$ and $i_{1'}, \ldots, i_{k'}$, we have



$$= \delta_{i_1i_2}\delta_{i_1i_4}\delta_{i_1i_{2'}}\delta_{i_1i_{5'}}\delta_{i_5i_6}\delta_{i_5i_7}\delta_{i_5i_{3'}}\delta_{i_5i_{4'}}\delta_{i_5i_{6'}}\delta_{i_5i_{7'}}\delta_{i_8i_{8'}}.$$

With this notation, the formula

$$d(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{1 \le i_{1'},\ldots i_{k'} \le n} (d)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}} \tag{3.2}$$

defines actions

$$\Phi_k : \mathbb{C}A_k \longrightarrow \mathrm{End}(V^{\otimes k}) \qquad \text{and} \qquad \Phi_{k+\frac{1}{2}} : \mathbb{C}A_{k+\frac{1}{2}} \longrightarrow \mathrm{End}(V^{\otimes k}) \tag{3.3}$$

of $\mathbb{C}A_k$ and $\mathbb{C}A_{k+\frac{1}{2}}$ on $V^{\otimes k}$, where the second map $\Phi_{k+\frac{1}{2}}$ comes from the fact that if $d \in A_{k+\frac{1}{2}}$, then d acts on the subspace

$$V^{\otimes k} \cong V^{\otimes k} \otimes v_n = \mathbb{C}\text{-span}\,\{v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n \mid 1 \le i_1, \ldots, i_k \le n\} \quad \subseteq V^{\otimes (k+1)}. \tag{3.4}$$

In other words, the map $\Phi_{k+\frac{1}{2}}$ is obtained from Φ_{k+1} by restricting to the subspace $V^{\otimes k} \otimes v_n$ and identifying $V^{\otimes k}$ with $V^{\otimes k} \otimes v_n$.

The group $GL_n(\mathbb{C})$ acts on the vector spaces V and $V^{\otimes k}$ by

$$g v_i = \sum_{j=1}^{n} g_{ji} v_j, \qquad \text{and} \qquad g(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = g v_{i_1} \otimes g v_{i_2} \otimes \cdots \otimes g v_{i_k}, \tag{3.5}$$

for $g = (g_{ij}) \in GL_n(\mathbb{C})$. View $S_n \subseteq GL_n(\mathbb{C})$ as the subgoup of permutation matrices and let

$$\mathrm{End}_{S_n}(V^{\otimes k}) = \left\{ b \in \mathrm{End}(V^{\otimes k}) \mid b\sigma v = \sigma b v \text{ for all } \sigma \in S_n \text{ and } v \in V^{\otimes k} \right\}.$$

Theorem 3.6. *Let $n \in \mathbb{Z}_{>0}$ and let $\{x_d \mid d \in A_k\}$ be the basis of $\mathbb{C}A_k(n)$ defined in (2.3). Then*
(a) $\Phi_k : \mathbb{C}A_k(n) \to End(V^{\otimes k})$ has

$$\text{im}\ \Phi_k = End_{S_n}(V^{\otimes k}) \qquad \textit{and} \qquad \ker \Phi_k = \mathbb{C}\textit{-span}\{x_d \mid d \textit{ has more than } n \textit{ blocks}\}, \qquad \textit{and}$$

(b) $\Phi_{k+\frac{1}{2}} : \mathbb{C}A_{k+\frac{1}{2}}(n) \to End(V^{\otimes k})$ has

$$\text{im}\ \Phi_{k+\frac{1}{2}} = End_{S_{n-1}}(V^{\otimes k}) \qquad \textit{and} \qquad \ker \Phi_{k+\frac{1}{2}} = \mathbb{C}\textit{-span}\{x_d \mid d \textit{ has more than } n \textit{ blocks}\}.$$

Proof. (a) As a subgroup of $GL_n(\mathbb{C})$, S_n acts on V via its permutation representation and S_n acts on $V^{\otimes k}$ by

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_k)}. \tag{3.7}$$

Then $b \in \mathrm{End}_{S_n}(V^{\otimes k})$ if and only if $\sigma^{-1} b \sigma = b$ (as endomorphisms on $V^{\otimes k}$) for all $\sigma \in S_n$. Thus, using the notation of (3.1), $b \in \mathrm{End}_{S_n}(V^{\otimes k})$ if and only if

$$b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = (\sigma^{-1} b \sigma)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = b^{\sigma(i_1),\ldots,\sigma(i_k)}_{\sigma(i_{1'}),\ldots,\sigma(i_{k'})}, \qquad \text{for all } \sigma \in S_n.$$

It follows that the matrix entries of b are constant on the S_n-orbits of its matrix coordinates. These orbits decompose $\{1, \ldots, k, 1', \ldots, k'\}$ into subsets and thus correspond to set partitions $d \in A_k$. It follows from (2.3) and (3.1) that for all $d \in A_k$,

$$(\Phi_k(x_d))^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = \begin{cases} 1, & \text{if } i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Thus $\Phi_k(x_d)$ has 1s in the matrix positions corresponding to d and 0s elsewhere, and so b is a linear combination of $\Phi_k(x_d), d \in A_k$. Since $x_d, d \in A_k$, form a basis of $\mathbb{C}A_k$, im $\Phi_k = \mathrm{End}_{S_n}(V^{\otimes k})$.

If d has more than n blocks, then by (3.8) the matrix entry $(\Phi_k(x_d))^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = 0$ for all indices $i_1,\ldots,i_k,i_{1'},\ldots,i_{k'}$, since we need a distinct $i_j \in \{1,\ldots,n\}$ for each block of d. Thus, $x_d \in \ker\Phi_k$. If d has $\le n$ blocks, then we can find an index set $i_1,\ldots,i_k,i_{1'},\ldots,i_{k'}$ with $(\Phi_k(x_d))^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = 1$ simply by choosing a distinct index from $\{1,\ldots,n\}$ for each block of d. Thus, if d has $\le n$ blocks then $x_d \notin \ker\Phi_k$, and so $\ker\Phi_k = \mathbb{C}\text{-span}\{x_d | d \text{ has more than } n \text{ blocks}\}$.

(b) The vector space $V^{\otimes k}\otimes v_n \subseteq V^{\otimes(k+1)}$ is a submodule both for $\mathbb{C}A_{k+\frac12} \subseteq \mathbb{C}A_{k+1}$ and $\mathbb{C}S_{n-1} \subseteq \mathbb{C}S_n$. If $\sigma \in S_{n-1}$, then $\sigma(v_{i_1}\otimes\cdots\otimes v_{i_k}\otimes v_n) = v_{\sigma(i_1)}\otimes\cdots\otimes v_{\sigma(i_k)}\otimes v_n$. Then as above $b \in \mathrm{End}_{S_{n-1}}(V^{\otimes k})$ if and only if

$$b^{i_1,\ldots,i_k,n}_{i_{1'},\ldots,i_{k'},n} = b^{\sigma(i_1),\ldots,\sigma(i_k),n}_{\sigma(i_{1'}),\ldots,\sigma(i_{k'}),n}, \qquad \text{for all } \sigma \in S_{n-1}.$$

The S_{n-1} orbits of the matrix coordinates of b correspond to set partitions $d \in A_{k+\frac12}$; that is vertices i_{k+1} and $i_{(k+1)'}$ must be in the same block of d. The same argument as part (a) can be used to show that $\ker\Phi_{k+\frac12}$ is the span of x_d with $d \in A_{k+\frac12}$ having more than n blocks. We always choose the index n for the block containing $k+1$ and $(k+1)'$. ∎

The maps $\varepsilon_{\frac12} : End(V^{\otimes k}) \to End(V^{\otimes k})$ *and* $\varepsilon^{\frac12} : End(V^{\otimes k}) \to End(V^{\otimes(k-1)})$

If $b \in \mathrm{End}(V^{\otimes k})$ let $b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} \in \mathbb{C}$ be the coefficients in the expansion

$$b(v_{i_1}\otimes\cdots\otimes v_{i_k}) = \sum_{1\le i_{1'},\ldots i_{k'}\le n} b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} v_{i_{1'}}\otimes\cdots\otimes v_{i_{k'}}. \tag{3.9}$$

Define linear maps

$$\varepsilon_{\frac12} : \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes k}) \qquad \text{and} \qquad \varepsilon^{\frac12} : \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes(k-1)}) \qquad \text{by}$$

$$\varepsilon_{\frac12}(b)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}}\delta_{i_k i_{k'}} \qquad \text{and} \qquad \varepsilon^{\frac12}(b)^{i_1,\ldots,i_{k-1}}_{i_{1'},\ldots,i_{(k-1)'}} = \sum_{j,\ell=1}^{n} b^{i_1,\ldots,i_{k-1},j}_{i_{1'},\ldots,i_{(k-1)'},\ell}. \tag{3.10}$$

The composition of $\varepsilon_{\frac12}$ and $\varepsilon^{\frac12}$ is the map

$$\varepsilon_1 : \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes(k-1)}) \qquad \text{given by} \qquad \varepsilon_1(b)^{i_1,\ldots,i_{k-1}}_{i_{1'},\ldots,i_{(k-1)'}} = \sum_{j=1}^{n} b^{i_1,\ldots,i_{k-1},j}_{i_{1'},\ldots,i_{(k-1)'},j}, \tag{3.11}$$

and

$$\mathrm{Tr}(b) = \varepsilon_1^k(b), \qquad \text{for } b \in \mathrm{End}(V^{\otimes k}). \tag{3.12}$$

The relation between the maps $\varepsilon^{\frac12}$, $\varepsilon_{\frac12}$ in (3.10) and the maps $\varepsilon^{\frac12}$, $\varepsilon_{\frac12}$ in Section 2 is given by

$$\begin{aligned} &\Phi_{k-\frac12}(\varepsilon_{\frac12}(b)) = \varepsilon_{\frac12}(\Phi_k(b))\big|_{V^{\otimes(k-1)}\otimes v_n}, && \text{for } b \in \mathbb{C}A_k(n), \\ &\Phi_{k-1}(\varepsilon^{\frac12}(b)) = \tfrac{1}{n}\varepsilon^{\frac12}(\Phi_k(b)), && \text{for } b \in \mathbb{C}A_{k-\frac12}(n), \quad \text{and} \\ &\Phi_{k-1}(\varepsilon_1(b)) = \varepsilon_1(\Phi_k(b)), && \text{for } b \in \mathbb{C}A_k(n), \end{aligned} \tag{3.13}$$

where, on the right hand side of the middle equality b is viewed as an element of $\mathbb{C}A_k$ via the natural inclusion $\mathbb{C}A_{k-\frac12}(n) \subseteq \mathbb{C}A_k(n)$. Then

$$\mathrm{Tr}(\Phi_k(b)) = \varepsilon_1^k(\Phi_k(b)) = \Phi_0(\varepsilon_1^k(b)) = \varepsilon_1^k(b) = \mathrm{tr}_k(b), \tag{3.14}$$

and, by (3.4), if $b \in \mathbb{C}A_{k-\frac{1}{2}}(n)$ then

$$\mathrm{Tr}(\Phi_{k-\frac{1}{2}}(b)) = \mathrm{Tr}(\Phi_k(b)\big|_{V^{\otimes(k-1)}\otimes v_n}) = \tfrac{1}{n}\,\mathrm{Tr}(\Phi_k(b)) = \tfrac{1}{n}\,\mathrm{tr}_k(b) = \tfrac{1}{n}\,\mathrm{tr}_{k-\frac{1}{2}}(b). \tag{3.15}$$

The representations $(\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n})^k(\mathbf{1}_n)$ *and* $\mathrm{Res}_{S_{n-1}}^{S_n}(\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n})^k(\mathbf{1}_n)$

Let $\mathbf{1}_n = S_n^{(n)}$ be the trivial representation of S_n and let $V = \mathbb{C}\text{-span}\{v_1, \ldots, v_n\}$ be the permutation representation of S_n given in (3.5). Then

$$V \cong \mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n}(\mathbf{1}_n). \tag{3.16}$$

More generally, for any S_n-module M,

$$\begin{aligned}\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n}(M) &\cong \mathrm{Ind}_{S_{n-1}}^{S_n}(\mathrm{Res}_{S_{n-1}}^{S_n}(M)\otimes \mathbf{1}_{n-1})\\ &\cong \mathrm{Ind}_{S_{n-1}}^{S_n}(\mathrm{Res}_{S_{n-1}}^{S_n}(M)\otimes \mathrm{Res}_{S_{n-1}}^{S_n}(\mathbf{1}_n))\\ &\cong M\otimes \mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n}(\mathbf{1}_n) \cong M\otimes V,\end{aligned} \tag{3.17}$$

where the third isomorphism comes from the "tensor identity,"

$$\begin{array}{ccc}\mathrm{Ind}_{S_{n-1}}^{S_n}(\mathrm{Res}_{S_{n-1}}^{S_n}(M)\otimes N) & \xrightarrow{\sim} & M\otimes \mathrm{Ind}_{S_{n-1}}^{S_n}N\\ g\otimes(m\otimes n) & \mapsto & gm\otimes(g\otimes n)\end{array}, \tag{3.18}$$

for $g \in S_n$, $m \in M$, $n \in N$, and the fact that $\mathrm{Ind}_{S_{n-1}}^{S_n}(W) = \mathbb{C}S_n \otimes_{S_{n-1}} W$. By iterating (3.17) it follows that

$$(\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n})^k(\mathbf{1}) \cong V^{\otimes k} \qquad \text{and} \qquad \mathrm{Res}_{S_{n-1}}^{S_n}(\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n})^k(\mathbf{1}) \cong V^{\otimes k} \tag{3.19}$$

as S_n-modules and S_{n-1}-modules, respectively.

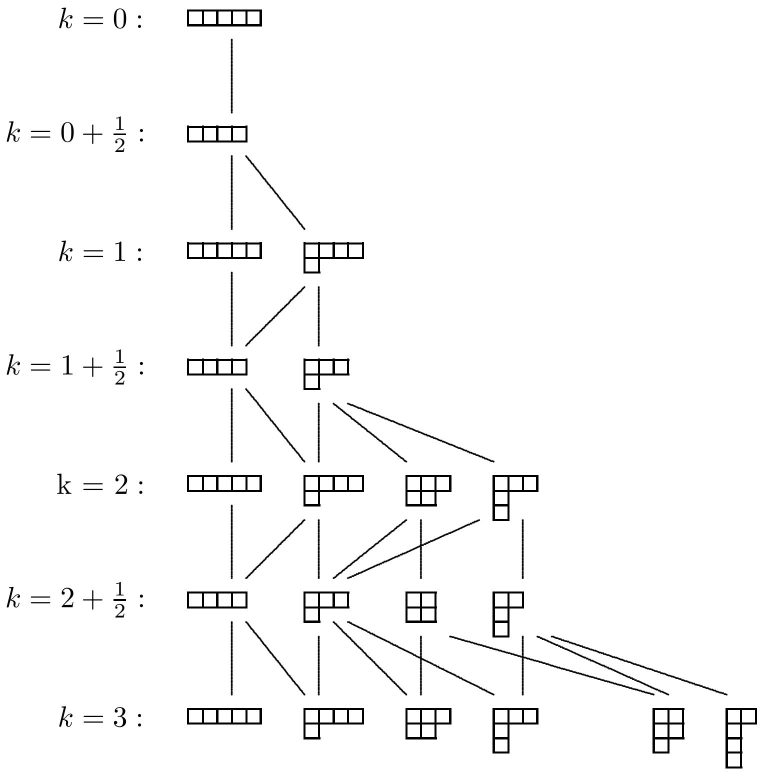
If

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \qquad \text{define} \qquad \lambda_{>1} = (\lambda_2, \ldots, \lambda_\ell) \tag{3.20}$$

to be the same partition as λ except with the first row removed. Build a graph $\hat{A}(n)$ which encodes the decomposition of $V^{\otimes k}$, $k \in \mathbb{Z}_{\geq 0}$, by letting

$$\begin{array}{ll}\text{vertices on level } k\text{:} & \hat{A}_k(n) = \{\lambda \vdash n \mid k - |\lambda_{>1}| \in \mathbb{Z}_{\geq 0}\},\\ \text{vertices on level } k+\tfrac{1}{2}\text{:} & \hat{A}_{k+\frac{1}{2}}(n) = \{\lambda \vdash n-1 \mid k - |\lambda_{>1}| \in \mathbb{Z}_{\geq 0}\}, \quad \text{and}\\ \multicolumn{2}{l}{\text{an edge } \lambda \to \mu, \text{ if } \mu \in \hat{A}_{k+\frac{1}{2}}(n) \text{ is obtained from } \lambda \in \hat{A}_k(n) \text{ by removing a box,}}\\ \multicolumn{2}{l}{\text{an edge } \mu \to \lambda, \text{ if } \lambda \in \hat{A}_{k+1}(n) \text{ is obtained from } \mu \in \hat{A}_{k+\frac{1}{2}}(n) \text{ by adding a box.}}\end{array} \tag{3.21}$$

For example, if $n=5$ then the first few levels of $\hat{A}(n)$ are



The following theorem is a consequence of Theorem 3.5 and the Centralizer Theorem, Theorem 5.4, (see also [GW, Theorem 3.3.7]).

Theorem 3.22. *Let $n,k\in\mathbb{Z}_{\geq 0}$. Let S_n^λ denote the irreducible S_n-module indexed by λ.*

(a) As $(\mathbb{C}S_n, \mathbb{C}A_k(n))$-bimodules,

$$V^{\otimes k}\cong\bigoplus_{\lambda\in\hat{A}_k(n)} S_n^\lambda\otimes A_k^\lambda(n),$$

where the vector spaces $A_k^\lambda(n)$ are irreducible $\mathbb{C}A_k(n)$-modules and

$$\dim(A_k^\lambda(n)) = (\textit{number of paths from } (n)\in\hat{A}_0(n) \textit{ to } \lambda\in\hat{A}_k(n) \textit{ in the graph } \hat{A}(n)).$$

(b) As $(\mathbb{C}S_{n-1}, \mathbb{C}A_{k+\frac{1}{2}}(n))$-bimodules,

$$V^{\otimes k}\cong\bigoplus_{\mu\in\hat{A}_{k+\frac{1}{2}}(n)} S_{n-1}^\mu\otimes A_{k+\frac{1}{2}}^\mu(n),$$

where the vector spaces $A_{k+\frac{1}{2}}^\mu(n)$ are irreducible $\mathbb{C}A_{k+\frac{1}{2}}(n)$-modules and

$$\dim(A_{k+\frac{1}{2}}^\mu(n)) = (\textit{number of paths from } (n)\in\hat{A}_0(n) \textit{ to } \mu\in\hat{A}_{k+\frac{1}{2}}(n) \textit{ in the graph } \hat{A}(n)).$$

Determination of the polynomials $\mathrm{tr}^\mu(n)$

Let $n \in \mathbb{Z}_{>0}$. For a partition λ, let

$$\lambda_{>1} = (\lambda_2, \ldots, \lambda_\ell), \qquad \text{if} \qquad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell),$$

i.e., remove the first row of λ to get $\lambda_{>1}$. Then, for $n \geq 2k$, the maps

$$\begin{array}{ccc} \hat{A}_k(n) & \longleftrightarrow & \hat{A}_k \\ \lambda & \longmapsto & \lambda_{>1} \end{array} \quad \text{are bijections} \tag{3.23}$$

which provide an isomorphism between levels 0 to n of the graphs $\hat{A}(n)$ and $\hat{A}$.

Proposition 3.24. *For $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $n \in \mathbb{C}$ such that $\mathbb{C}A_k(n)$ is semisimple, let $\chi^\mu_{A_k(n)}$, $\mu \in \hat{A}_k$, be the irreducible characters of $\mathbb{C}A_\ell(n)$ and let $\mathrm{tr}_k\colon \mathbb{C}A_k(n) \to \mathbb{C}$ be the trace on $\mathbb{C}A_k(n)$ defined in (2.25). Use the notations for partitions in (2.17). For $k > 0$ the coefficients in the expansion*

$$\mathrm{tr}_k = \sum_{\mu \in \hat{A}_k} \mathrm{tr}^\mu(n) \chi^\mu_{A_k(n)}, \qquad \textit{are} \qquad \mathrm{tr}^\mu(n) = \Big(\prod_{b \in \mu} \frac{1}{h(b)}\Big) \prod_{j=1}^{|\mu|} (n - |\mu| - (\mu_j - j)).$$

If $n \in \mathbb{C}$ is such that $\mathbb{C}A_{k+\frac{1}{2}}(n)$ is semisimple then for $k > 0$ the coefficients in the expansion

$$\mathrm{tr}_{k+\frac{1}{2}} = \sum_{\mu \in \hat{A}_{k+\frac{1}{2}}} \mathrm{tr}^\mu_{\frac{1}{2}}(n) \chi^\mu_{A_{k+\frac{1}{2}}(n)}, \qquad \textit{are} \qquad \mathrm{tr}^\mu_{\frac{1}{2}}(n) = \Big(\prod_{b \in \mu} \frac{1}{h(b)}\Big) \cdot n \cdot \prod_{j=1}^{|\mu|} (n - 1 - |\mu| - (\mu_j - j)).$$

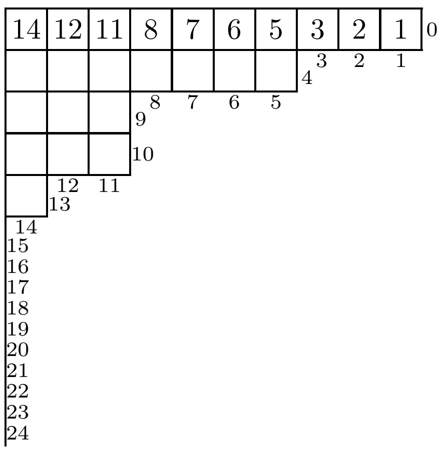
Proof. Let λ be a partition with n boxes. Beginning with the vertical edge at the end of the first row, label the boundary edges of λ sequentially with $0, 1, 2 \ldots, n$. Then the

$$\begin{aligned} \text{vertical edge label for row } i &= (\text{number of horizontal steps}) + (\text{number of vertical steps}) \\ &= (\lambda_1 - \lambda_i) + (i - 1) = (\lambda_1 - 1) - (\lambda_i - i), \qquad \text{and the} \\ \text{horizontal edge label for column } j &= (\text{number of horizontal steps}) + (\text{number of vertical steps}) \\ &= (\lambda_1 - j + 1) + (\lambda'_j - 1) = (\lambda_1 - 1) + (\lambda'_j - j) + 1. \end{aligned}$$

Hence

$$\begin{aligned} \{1, 2, \ldots, n\} &= \{(\lambda_1 - 1) - (\lambda'_j - j) + 1 \mid 1 \leq j \leq \lambda_1\} \sqcup \{(\lambda_1 - 1) - (\lambda_i - i) \mid 2 \leq i \leq n - \lambda_1 + 1\} \\ &= \{h(b) \mid b \text{ is in row 1 of } \lambda\} \sqcup \{(\lambda_1 - 1) - (\lambda_i - i) \mid 2 \leq i \leq n - \lambda_1 + 1\}. \end{aligned}$$

For example, if $\lambda = (10, 7, 3, 3, 1) \vdash 24$, then the boundary labels of λ and the hook numbers in the first row of λ are



.

Thus, since $\lambda_1 = n - |\lambda_{>1}|$,

$$\dim(S_n^\lambda) = \frac{n!}{\prod_{b\in\lambda} h(b)} = \Big(\prod_{b\in\lambda_{>1}} \frac{1}{h(b)}\Big) \prod_{i=2}^{|\lambda_{>1}|+1} (n - |\lambda_{>1}| - (\lambda_i - (i-1))). \tag{3.25}$$

Let $n \in \mathbb{Z}_{>0}$ and let $\chi^\lambda_{S_n}$ denote the irreducible characters of the symmetric group S_n. By taking the trace on both sides of the equality in Theorem 3.22,

$$\mathrm{Tr}(b, V^{\otimes k}) = \sum_{\lambda\in\hat{A}_k(n)} \chi^\lambda_{S_n}(1)\chi_{A_k(n)}(b) = \sum_{\lambda\in\hat{A}_k(n)} \dim(S_n^\lambda)\chi_{A_k(n)}(b), \qquad \text{for } b \in \mathbb{C}A_k(n).$$

Thus the equality in (3.25) and the bijection in (3.23) provide the expansion of tr_k for all $n \in \mathbb{Z}_{\geq 0}$ such that $n \geq 2k$. The statement for all $n \in \mathbb{C}$ such that $\mathbb{C}A_k(n)$ is semisimple is then a consequence of the fact that any polynomial is determined by its evaluations at an infinite number of values of the parameter. The proof of the expansion of $\mathrm{tr}_{k+\frac12}$ is exactly analogous. ∎

Note that the polynomials $\mathrm{tr}^\mu(n)$ and $\mathrm{tr}^\mu_{\frac12}(n)$ (of degrees $|\mu|$ and $|\mu|+1$, respectively) do not depend on k. By Proposition 3.24,

$$\begin{aligned}
&\{\text{roots of } \mathrm{tr}^\mu_{\frac12}(n) \mid \mu \in \hat{A}_{\frac12}\} = \{0\},\\
&\{\text{roots of } \mathrm{tr}^\mu_{1}(n) \mid \mu \in \hat{A}_{1}\} = \{1\},\\
&\{\text{roots of } \mathrm{tr}^\mu_{1\frac12}(n) \mid \mu \in \hat{A}_{1\frac12}\} = \{0,2\}, \qquad \text{and}\\
&\{\text{roots of } \mathrm{tr}^\mu_{k}(n) \mid \mu \in \hat{A}_{k}\} = \{0,1,\dots,2k-1\}, \qquad \text{for } k \in \tfrac12\mathbb{Z}_{\geq 0},\ k \geq 2.
\end{aligned} \tag{3.26}$$

For example, the first few values of tr^μ and $\mathrm{tr}^\mu_{\frac12}$ are

$$\begin{aligned}
&\mathrm{tr}^{\emptyset}(n) = 1, && \mathrm{tr}^{\emptyset}_{\frac12}(n) = n\\
&\mathrm{tr}^{\square}(n) = n-1, && \mathrm{tr}^{\square}_{\frac12}(n) = n(n-2),\\
&\mathrm{tr}^{\square\square}(n) = \tfrac12 n(n-3), && \mathrm{tr}^{\square\square}_{\frac12}(n) = \tfrac12 n(n-1)(n-4),\\
&\mathrm{tr}^{\substack{\square\\\square}}(n) = \tfrac12 (n-1)(n-2), && \mathrm{tr}^{\substack{\square\\\square}}_{\frac12}(n) = \tfrac12 n(n-2)(n-3),\\
&\mathrm{tr}^{\square\square\square}(n) = \tfrac16 n(n-1)(n-5), && \mathrm{tr}^{\square\square\square}_{\frac12}(n) = \tfrac16 n(n-1)(n-2)(n-6),\\
&\mathrm{tr}^{\substack{\square\square\\\square}}(n) = \tfrac16 n(n-2)(n-4), && \mathrm{tr}^{\substack{\square\square\\\square}}_{\frac12}(n) = \tfrac16 n(n-1)(n-3)(n-5),\\
&\mathrm{tr}^{\substack{\square\\\square\\\square}}(n) = \tfrac16 (n-1)(n-2)(n-3), && \mathrm{tr}^{\substack{\square\\\square\\\square}}_{\frac12}(n) = \tfrac16 n(n-2)(n-3)(n-4),
\end{aligned}$$

Theorem 3.27. *Let $n \in \mathbb{Z}_{\geq 2}$ and $k \in \frac12\mathbb{Z}_{\geq 0}$. Then*

$$\mathbb{C}A_k(n) \textit{ is semisimple} \quad \textit{if and only if} \quad k \leq \tfrac{n+1}{2}.$$

Proof. By Theorem 2.26(a) and the observation (3.26) it follows that $\mathbb{C}A_k(n)$ is semisimple if $n \geq 2k-1$.

Suppose n is even. Then Theorems 2.26(a) and 2.26(b) imply that

$$\mathbb{C}A_{\frac{n}{2}+\frac{1}{2}}(n) \text{ is semisimple} \quad \text{and} \quad \mathbb{C}A_{\frac{n}{2}+1}(n) \text{ is not semisimple,}$$

since $(n/2) \in \hat{A}_{\frac{n}{2}+\frac{1}{2}}$ and $\mathrm{tr}_{\frac{1}{2}}^{(n/2)}(n) = 0$. Since $(n/2) \in \hat{A}_{\frac{n}{2}+1}(n)$, the $A_{\frac{n}{2}+1}(n)$-module $A_{\frac{n}{2}+1}^{(n/2)}(n) \neq 0$. Since the path $(\emptyset, \ldots, (n/2), (n/2), (n/2)) \in \hat{A}_{\frac{n}{2}+1}^{(n/2)}$ does not correspond to an element of $\hat{A}_{\frac{n}{2}+1}^{(n/2)}(n)$,

$$\mathrm{Card}(\hat{A}_{\frac{n}{2}+1}^{(n/2)}) \neq \mathrm{Card}(\hat{A}_{\frac{n}{2}+1}^{(n/2)}(n)).$$

Thus, Tits deformation theorem (Theorem 5.13) implies that $\mathbb{C}A_{\frac{n}{2}+1}(n)$ is cannot be semisimple. Now it follows from Theorem Theorem 2.26(c) that $\mathbb{C}A_k(n)$ is not semisimple for $k \geq \frac{n}{2} + \frac{1}{2}$.

If n is odd then Theorems 2.26(a) and 2.26(b) imply that

$$\mathbb{C}A_{\frac{n}{2}+\frac{1}{2}}(n) \text{ is semisimple} \quad \text{and} \quad \mathbb{C}A_{\frac{n}{2}+1}(n) \text{ is not semisimple,}$$

since $(n/2) \in \hat{A}_{\frac{n}{2}+\frac{1}{2}}$ and $\mathrm{tr}^{(n/2)}(n) = 0$. Since $(\frac{n}{2} - \frac{1}{2}) \in \hat{A}_{\frac{n}{2}+1}(n)$, the $A_{\frac{n}{2}+1}(n)$-module $A_{\frac{n}{2}+1}^{(\frac{n}{2}-\frac{1}{2})}(n) \neq 0$. Since the path $(\emptyset, \ldots, (\frac{n}{2} - \frac{1}{2}), (\frac{n}{2} + \frac{1}{2}), (\frac{n}{2} - \frac{1}{2})) \in \hat{A}_{\frac{n}{2}+1}^{(\frac{n}{2}-\frac{1}{2})}$ does not correspond to an element of $\hat{A}_{\frac{n}{2}+1}^{(\frac{n}{2}-\frac{1}{2})}(n)$, and since

$$\mathrm{Card}(\hat{A}_{\frac{n}{2}+1}^{(\frac{n}{2}-\frac{1}{2})}) \neq \mathrm{Card}(\hat{A}_{\frac{n}{2}+1}^{(\frac{n}{2}-\frac{1}{2})}(n))$$

the Tits deformation theorem implies that $\mathbb{C}A_{\frac{n}{2}+1}(n)$ is not semisimple. Now it follows from Theorem 2.26(c) that $\mathbb{C}A_k(n)$ is not semisimple for $k \geq \frac{n}{2} + \frac{1}{2}$. ∎

Murphy elements for $\mathbb{C}A_k(n)$

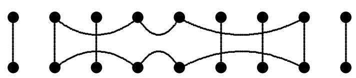
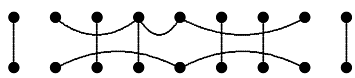
Let κ_n be the element of $\mathbb{C}S_n$ given by

$$\kappa_n = \sum_{1 \leq \ell < m \leq n} s_{\ell m}\,, \tag{3.28}$$

where $s_{\ell m}$ is the transposition in S_n which switches ℓ and m. Let $S \subseteq \{1, 2, \ldots, k\}$ and let $I \subseteq S \cup S'$. Define $b_S, d_I \in A_k$ by

$$b_S = \{S \cup S', \{\ell, \ell'\}_{\ell \notin S}\} \qquad \text{and} \qquad d_{I \subseteq S} = \{I, I^c, \{\ell, \ell'\}_{\ell \notin S}\}. \tag{3.29}$$

For example, in A_9, if $S = \{2, 4, 5, 8\}$ and $I = \{2, 4, 4', 5, 8\}$ then

$b_S =$  and $d_{I \subseteq S} =$ .

For $S \subseteq \{1, 2, \ldots, k\}$ define

$$p_S = \sum_I \frac{1}{2}(-1)^{\#(\{\ell,\ell'\} \subseteq I) + \#(\{\ell,\ell'\} \subseteq I^c)} d_I, \tag{3.30}$$

where the sum is over $I \subseteq S \cup S'$ such that $I \neq \emptyset$, $I \neq S \cup S'$, $I \neq \{\ell, \ell'\}$ and $I \neq \{\ell, \ell'\}^c$. For $S \subseteq \{1, \ldots, k+1\}$ such that $k+1 \in S$ define

$$\tilde{p}_S = \sum_I \frac{1}{2}(-1)^{\#(\{\ell,\ell'\}\subseteq I)+\#(\{\ell,\ell'\}\subseteq I^c)} d_I, \tag{3.31}$$

where the sum is over all $I \subseteq S \cup S'$ such that $\{k+1, (k+1)'\} \subseteq I$ or $\{k+1, (k+1)'\} \subseteq I^c$, $I \neq S \cup S'$, $I \neq \{k+1, (k+1)'\}$ and $I \neq \{k+1, (k+1)'\}^c$.

Let $Z_1 = 1$ and, for $k \in \mathbb{Z}_{>1}$, let

$$Z_k = \binom{k}{2} + \sum_{\substack{S\subseteq\{1,\ldots,k\}\\ |S|\geq 1}} p_S + \sum_{\substack{S\subseteq\{1,\ldots,k\}\\ |S|\geq 2}} (n-k+|S|))(-1)^{|S|} b_S. \tag{3.32}$$

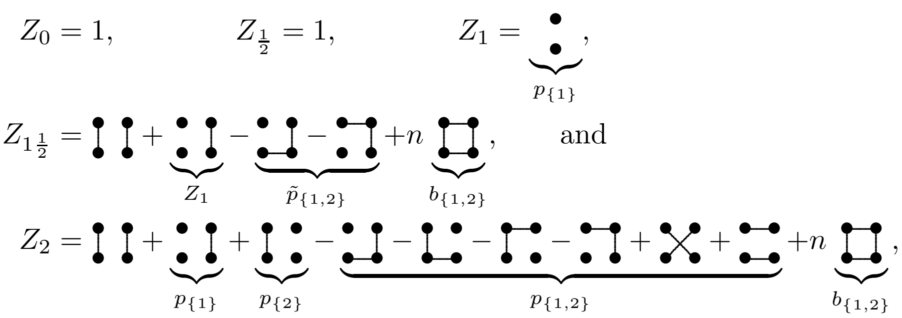
View $Z_k \in \mathbb{C}A_k \subseteq \mathbb{C}A_{k+\frac{1}{2}}$ using the embedding in (2.2), and define $Z_{\frac{1}{2}} = 1$ and

$$Z_{k+\frac{1}{2}} = k + Z_k + \sum_{\substack{|S|\geq 2\\ k+1\in S}} \tilde{p}_S + (n-(k+1)+|S|)(-1)^{|S|} b_S, \tag{3.33}$$

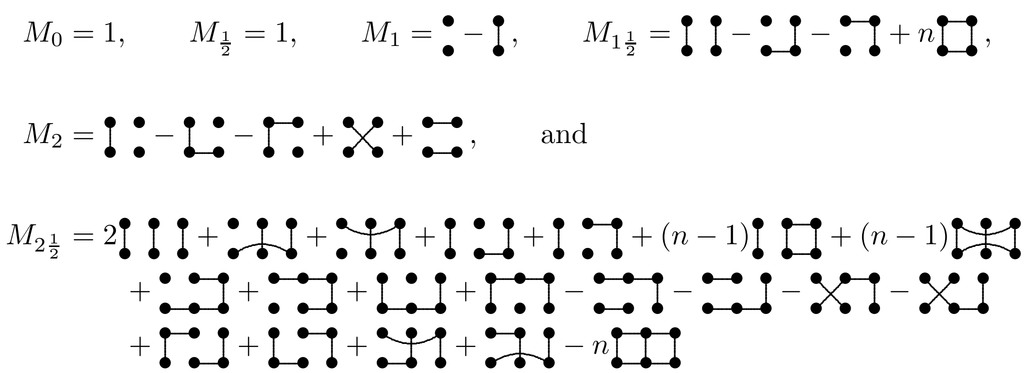
where the sum is over $S \subseteq \{1, \ldots, k+1\}$ such that $k+1 \in S$ and $|S| \geq 2$. Define

$$M_{\frac{1}{2}} = 1, \qquad \text{and} \qquad M_k = Z_k - Z_{k-\frac{1}{2}}, \quad \text{for } k \in \tfrac{1}{2}\mathbb{Z}_{>0}. \tag{3.34}$$

For example, the first few Z_k are



and the first few M_k are



Part (a) of the following theorem is well known.

Theorem 3.35.
(a) For $n \in \mathbb{Z}_{\geq 0}$, κ_n is a central element of $\mathbb{C}S_n$. If λ is a partition with n boxes and S_n^λ is the irreducible S_n-module indexed by the partition λ,

$$\kappa_n = \sum_{b \in \lambda} c(b), \qquad \text{as operators on } S_n^\lambda.$$

(b) Let $n, k \in \mathbb{Z}_{\geq 0}$. Then, as operators on $V^{\otimes k}$, where $\dim(V) = n$,

$$Z_k = \kappa_n - \binom{n}{2} + kn, \qquad \text{and} \qquad Z_{k+\frac{1}{2}} = \kappa_{n-1} - \binom{n}{2} + (k+1)n - 1.$$

(c) Let $n \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$. Then Z_k is a central element of $\mathbb{C}A_k(n)$, and, if $n \in \mathbb{C}$ is such that $\mathbb{C}A_k(n)$ is semisimple and $\lambda \vdash n$ with $|\lambda_{>1}| \leq k$ boxes then

$$Z_k = kn - \binom{n}{2} + \sum_{b \in \lambda} c(b), \qquad \text{as operators on } A_k^\lambda,$$

where A_k^λ is the irreducible $\mathbb{C}A_k(n)$-module indexed by the partition λ. Furthermore, $Z_{k+\frac{1}{2}}$ is a central element of $\mathbb{C}A_{k+\frac{1}{2}}(n)$, and, if n is such that $\mathbb{C}A_{k+\frac{1}{2}}(n)$ is semisimple and $\lambda \vdash n$ is a partition with $|\lambda_{>1}| \leq k$ boxes then

$$Z_{k+\frac{1}{2}} = kn + n - 1 - \binom{n}{2} + \sum_{b \in \lambda} c(b), \qquad \text{as operators on } A_{k+\frac{1}{2}}^\lambda,$$

where $A_{k+\frac{1}{2}}^\lambda$ is the irreducible $\mathbb{C}A_{k+\frac{1}{2}}(n)$-module indexed by the partition λ.

Proof. (a) The element κ_n is the class sum corresponding to the conjugacy class of transpositions and thus κ_n is a central element of $\mathbb{C}S_n$. The constant by which κ_n acts on S_n^λ is computed in [Mac, Ch. 1 §7 Ex. 7].

(c) The first statement follows from parts (a) and (b) and Theorems 3.6 and 3.22 as follows. By Theorem 3.6, $\mathbb{C}A_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$ if $n \geq 2k$. Thus, by Theorem 3.22, if $n \geq 2k$ then Z_k acts on the irreducible $\mathbb{C}A_k(n)$-module $A_k^\lambda(n)$ by the constant given in the statement. This means that Z_k is a central element of $\mathbb{C}A_k(n)$ for all $n \geq k$. Thus, for $n \geq 2k$, $dZ_k = Z_k d$ for all diagrams $d \in A_k$. Since the coefficients in dZ_k (in terms of the basis of diagrams) are polynomials in n, it follows that $dZ_k = Z_k d$ for all $n \in \mathbb{C}$.

If $n \in \mathbb{C}$ is such that $\mathbb{C}A_k(n)$ is semisimple let $\chi^\lambda_{\mathbb{C}A_k(n)}$ be the irreducible characters. Then Z_k acts on $A_k^\lambda(n)$ by the constant $\chi^\lambda_{\mathbb{C}A_k(n)}(Z_k)/\dim(A_k^\lambda(n))$. If $n \geq k$ this is the constant in the statement, and therefore it is a polynomial in n, determined by its values for $n \geq 2k$.

The proof of the second statement is completely analogous using $\mathbb{C}A_{k+\frac{1}{2}}$, S_{n-1}, and the second statement in part (b).

(b) Let $s_{ii} = 1$ so that

$$2\kappa_n + n = n + 2\sum_{1 \leq i < j \leq n} s_{ij} = \sum_{i=1}^n s_{ii} + \sum_{1 \leq i < j \leq n} (s_{ij} + s_{ji}) = \sum_{i=j} s_{ij} + \sum_{i \neq j} s_{ij} = \sum_{i,j=1}^n s_{ij}.$$

Then

$$(2\kappa_n + n)(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \left(\sum_{i,j=1}^{n} s_{ij}\right)(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{i,j=1}^{n} s_{ij} v_{i_1} \otimes \cdots \otimes s_{ij} v_{i_k}$$

$$= \sum_{i,j=1}^{n} (1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}) v_{i_1} \otimes \cdots \otimes (1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}) v_{i_k}$$

and expanding this sum gives that $(2\kappa_n + n)(v_{i_1} \otimes \cdots \otimes v_{i_k})$ is equal to

$$\sum_{S \subseteq \{1,\ldots,k\}} \sum_{i_{1'},\ldots,i_{k'}} \sum_{i,j=1}^{n} \left(\prod_{\ell \in S^c} \delta_{i_\ell i_{\ell'}}\right) \cdot \\ \cdot \sum_{I \subseteq S \cup S'} (-1)^{\#(\{\ell,\ell'\} \subseteq I) + \#(\{\ell,\ell'\} \subseteq I^c)} \left(\prod_{\ell \in I} \delta_{i_\ell i}\right) \left(\prod_{\ell \in I^c} \delta_{i_\ell j}\right) (v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}) \tag{3.36}$$

where $S^c \subseteq \{1, \ldots, k\}$ corresponds to the tensor positions where 1 is acting, and where $I \subseteq S \cup S'$ corresponds to the tensor positions that must equal i and I^c corresponds to the tensor positions that must equal j.

When $|S| = 0$ the set I is empty and the term corresponding to S in (3.36) is

$$\sum_{i,j=1}^{n} \sum_{i_{1'},\ldots,i_{k'}} \Big(\prod_{\ell \in \{1,\ldots,k\}} \delta_{i_\ell i_{\ell'}} \Big)(v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}) = n^2 (v_{i_1} \otimes \cdots \otimes v_{i_k}).$$

Assume $|S| \geq 1$ and separate the sum according to the cardinality of I. Note that the sum for I is equal to the sum for I^c, since the whole sum is symmetric in i and j. The sum of the terms in (3.36) which come from $I = S \cup S'$ is equal to

$$\sum_{i_{1'},\ldots,i_{k'}} n \sum_{i=1}^{n} \left(\prod_{\ell \in S^c} \delta_{i_\ell i_{\ell'}}\right) (-1)^{|S|} \left(\prod_{\ell \in S \cup S'} \delta_{i_\ell i}\right) (v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}) = n(-1)^{|S|} b_S (v_{i_1} \otimes \cdots \otimes v_{i_k}).$$

We get a similar contribution from the sum of the terms with $I = \emptyset$.

If $|S| > 1$ then the sum of the terms in (3.36) which come from $I = \{\ell, \ell'\}$ is equal to

$$\sum_{i_{1'},\ldots,i_{k'}} \sum_{i,j=1}^{n} \left(\prod_{r \in S^c} \delta_{i_r i_{r'}}\right) (-1)^{|S|} \delta_{i_\ell i} \delta_{i_{\ell'} i} \left(\prod_{r \neq \ell} \delta_{i_r j} \delta_{i_{r'} j}\right) (v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}) \\ = (-1)^{|S|} b_{S - \{\ell\}} (v_{i_1} \otimes \cdots \otimes v_{i_k}).$$

and there is a corresponding contribution from $I = \{\ell, \ell'\}^c$. The remaining terms can be written as

$$\sum_{i_{1'},\ldots,i_{k'}} \sum_{i,j=1}^{n} \Big(\prod_{\ell \in S^c} \delta_{i_\ell i_{\ell'}}\Big) \sum_{I \subseteq S \cup S'} (-1)^{\#(\{\ell,\ell'\} \subseteq I) + \#(\{\ell,\ell'\} \subseteq I^c)} \Big(\prod_{\ell \in I} \delta_{i_\ell i}\Big) \Big(\prod_{\ell \in I^c} \delta_{i_\ell j}\Big)(v_{i_{1'}} \otimes v_{i_{k'}}) \\ = 2 p_S (v_{i_1} \otimes \cdots \otimes v_{i_k}).$$

Putting these cases together gives that $2\kappa_n + n$ acts on $v_{i_1}\otimes\cdots\otimes v_{i_k}$ the same way that

$$\sum_{|S|=0} n^2 + \sum_{|S|=1}\big(2n(-1)^1 b_S + 2p_S\big) + \sum_{|S|=2}\Big(2n(-1)^2 b_S + 2p_S + \sum_{\ell\in S}(-1)^2 2b_{S-\{\ell\}}\Big)$$
$$+ \sum_{|S|>2}\Big(2n(-1)^{|S|} b_S + 2p_S + \sum_{\ell\in S}(-1)^{|S|} 2b_{S-\{\ell\}}\Big)$$

acts on $v_{i_1}\otimes\cdots\otimes v_{i_k}$. Note that $b_S = 1$ if $|S| = 1$. Hence $2\kappa_n + n$ acts on $v_{i_1}\otimes\cdots\otimes v_{i_k}$ the same way that

$$n^2 + \sum_{|S|=1}(-2n + 2p_S) + \sum_{|S|=2}(2nb_S + 2 + 2p_S) + \sum_{|S|>2}\Big((-1)^{|S|}2nb_S + 2p_S + \sum_{\ell\in S}(-1)^{|S|}2b_{S-\{\ell\}}\Big)$$
$$= n^2 - 2nk + 2\binom{k}{2} + \sum_{|S|\ge 1} 2p_S + \sum_{|S|\ge 2} 2(n-k+|S|)(-1)^{|S|} b_S$$

acts on $v_{i_1}\otimes\cdots\otimes v_{i_k}$, and so $Z_k = \kappa_n + (n - n^2 + 2nk)/2$ as operators on $V^{\otimes k}$. This proves the first statement.

For the second statement, since $(1-\delta_{in})(1-\delta_{jn}) = \begin{cases} 0, & \text{if } i = n \text{ or } j = n,\\ 1, & \text{otherwise,}\end{cases}$

$$(2\kappa_{n-1} + (n-1))(v_{i_1}\otimes\cdots\otimes v_{i_k}\otimes v_n) = \left(\sum_{i,j=1}^{n-1} s_{ij}\right)(v_{i_1}\otimes\cdots\otimes v_{i_k}\otimes v_n)$$
$$= \left(\sum_{i,j=1}^{n} s_{ij}(1-\delta_{in})(1-\delta_{jn})\right)(v_{i_1}\otimes\cdots\otimes v_{i_k}\otimes v_n)$$
$$= \sum_{i,j=1}^{n} s_{ij}v_{i_1}\otimes\cdots\otimes s_{ij}v_{i_k}\otimes(1-\delta_{in})(1-\delta_{jn})v_n,$$
$$= \sum_{i,j=1}^{n}(1 - E_{ii} - E_{jj} + E_{ij} + E_{ji})v_{i_1}\otimes\cdots\otimes(1 - E_{ii} - E_{jj} + E_{ij} + E_{ji})v_{i_k}$$
$$\otimes(1 - E_{ii} - E_{jj} + E_{ii}E_{jj})v_n$$
$$= \big(\sum_{i,j} s_{ij}\big)(v_{i_1}\otimes\cdots\otimes v_{i_k})\otimes v_n$$
$$+ \sum_{i,j=1}^{n}(1 - E_{ii} - E_{jj} + E_{ij} + E_{ji})v_{i_1}\otimes\cdots\otimes(1 - E_{ii} - E_{jj} + E_{ij} + E_{ji})v_{i_k}\otimes(-E_{ii} - E_{jj})v_n$$
$$+ \sum_{i,j=1}^{n}(1 - E_{ii} - E_{jj} + E_{ij} + E_{ji})v_{i_1}\otimes\cdots\otimes(1 - E_{ii} - E_{jj} + E_{ij} + E_{ji})v_{i_k}\otimes E_{ii}E_{jj}v_n$$

The first sum is known to equal $(2\kappa_n + n)(v_{i_1}\otimes\cdots\otimes v_{i_k})$ by the computation proving the first statement, and the last sum has only one nonzero term, the term corresponding to $i = j = n$. Expanding the middle sum gives

$$\sum_{\substack{S\subseteq\{1,\ldots,k+1\}\\ k+1\in S}}\ \sum_{i_{1'},\ldots,i_{k'}}\ \sum_{i,j=1}^{n}\Big(\prod_{\ell\in S}\delta_{i_\ell,i_{\ell'}}\Big)\cdot$$
$$\cdot\sum_{I}(-1)^{\#(\{\ell,\ell'\}\subseteq I)+\#(\{\ell,\ell'\}\subseteq I^c)}\left(\prod_{\ell\in I}\delta_{i_\ell i}\right)\left(\prod_{\ell\in I^c}\delta_{i_\ell j}\right)(v_{i_{1'}}\otimes\cdots\otimes v_{i_{k'}})$$

where the inner sum is over all $I \subseteq \{1, \ldots, k+1\}$ such that $\{k+1, (k+1)'\} \subseteq I$ or $\{k+1, (k+1)'\} \subseteq I^c$. As in part (a) this sum is treated in four cases: (1) when $|S| = 0$, (2) when $I = S \cup S'$ or $I = \emptyset$, (3) when $I = \{\ell, \ell'\}$ or $I = \{\ell, \ell'\}^c$, and (4) the remaining cases. Since $k+1 \in S$, the first case does not occur, and cases (2), (3) and (4) are as in part (a) giving

$$\sum_{\substack{|S|=1 \\ k+1 \in S}} -2n + \sum_{\substack{|S|=2 \\ k+1 \in S}} (2nb_S + 2p_S + 2) + \sum_{\substack{|S|>2 \\ k+1 \in S}} \Big(2n(-1)^{|S|} b_S + 2p_S + 2\sum_{\ell \in S} (-1)^{|S|} b_{S-\{\ell\}}\Big).$$

Combining this with the terms $(2\kappa_n + n)(v_{i_1} \otimes \cdots \otimes v_{i_k}) \otimes v_n$ and $1 \otimes (v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n)$ gives that $2\kappa_{n-1} + (n-1)$ acts on $v_{i_1} \otimes \cdots \otimes v_{i_k}$ as

$$(2\kappa_n + n) + 1 - 2n + 2k + \sum_{\substack{|S| \geq 2 \\ k+1 \in S}} 2\tilde{p}_S + 2(n - (k+1) + |S|)(-1)^{|S|} b_S.$$

Thus $\kappa_{n-1} - \kappa_n$ acts on $v_{i_1} \otimes \cdots \otimes v_{i_k}$ as

$$\frac{1}{2}\left(n - (n-1) + 1 - 2n + 2k + \sum_{\substack{|S| \geq 2 \\ k+1 \in S}} 2\tilde{p}_S + 2(n - (k+1) + |S|)(-1)^{|S|} b_S \right),$$

so, as operators on $V^{\otimes k}$, we have $Z_{k+\frac{1}{2}} = k + Z_k + (\kappa_{n-1} - \kappa_n) - 1 + n - k = Z_k + (\kappa_{n-1} - \kappa_n) + n - 1$. By the first statement in part (c) of this theorem we get $Z_{k+\frac{1}{2}} = (\kappa_n - \binom{n}{2} + kn) + (\kappa_{n-1} - \kappa_n) + n - 1 = \kappa_{n-1} - \binom{n}{2} + kn + n - 1$. ∎

Theorem 3.37. *Let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and let $n \in \mathbb{C}$.*
(a) The elements $M_{\frac{1}{2}}, M_1, \ldots, M_{k-\frac{1}{2}}, M_k$, all commute with each other in $\mathbb{C}A_k(n)$.

(b) Assume that $\mathbb{C}A_k(n)$ is semisimple. Let $\mu \in \hat{A}_k$ so that μ is a partition with $\leq k$ boxes, and let $A_k^\mu(n)$ be the irreducible $\mathbb{C}A_k(n)$-module indexed by μ. Then there is a unique, up to multiplication by constants, basis $\{v_T \mid T \in \hat{A}_k^\mu\}$ of $A_k^\mu(n)$ such that, for all $T = (T^{(0)}, T^{(\frac{1}{2})}, \ldots, T^{(k)}) \in \hat{A}_k^\mu$, and $\ell \in \mathbb{Z}_{\geq 0}$ such that $\ell \leq k$,

$$M_\ell v_T = \begin{cases} c(T^{(\ell)}/T^{(\ell-\frac{1}{2})}) v_T, & \text{if } T^{(\ell)}/T^{(\ell-\frac{1}{2})} = \square, \\ (n - |T^{(\ell)}|) v_T, & \text{if } T^{(\ell)} = T^{(\ell-\frac{1}{2})}, \end{cases}$$

and

$$M_{\ell+\frac{1}{2}} v_T = \begin{cases} \big(n - c(T^{(\ell)}/T^{(\ell+\frac{1}{2})})\big) v_T, & \text{if } T^{(\ell)}/T^{(\ell+\frac{1}{2})} = \square, \\ |T^{(\ell)}| v_T, & \text{if } T^{(\ell)} = T^{(\ell+\frac{1}{2})}, \end{cases}$$

where λ/μ denotes the box where λ and μ differ.

Proof. (a) View $Z_0, Z_{\frac{1}{2}}, \ldots, Z_k \in \mathbb{C}A_k$. Then $Z_k \in Z(\mathbb{C}A_k)$, so $Z_k Z_\ell = Z_\ell Z_k$ for all $0 \leq \ell \leq k$. Since $M_\ell = Z_\ell - Z_{\ell-\frac{1}{2}}$, we see that the M_ℓ commute with each other in $\mathbb{C}A_k$.

(b) The basis is defined inductively. If $k = 0, \frac{1}{2}$ or 1, then $\dim(A_k^\lambda(n)) = 1$, so up to a constant there is a unique choice for the basis. For $k > 1$, we consider the restriction $\mathrm{Res}^{\mathbb{C}A_k(n)}_{\mathbb{C}A_{k-\frac{1}{2}}(n)}(A_k^\lambda(n))$. The branching rules for this restriction are multiplicity free, meaning that each $\mathbb{C}A_{k-\frac{1}{2}}(n)$-irreducible that shows up in $A_k^\lambda(n)$ does so exactly once. By induction, we can choose a basis for each $\mathbb{C}A_{k-\frac12}(n)$-irreducible, and the union of these bases forms a basis for $A_k^\lambda(n)$. For $\ell < k$, $M_\ell \in \mathbb{C}A_{k-\frac12}(n)$, so M_ℓ acts on this basis as in the statment of the theorem. It remains only to check the statement for M_k. Let k be an integer, and let $\lambda \vdash n$ and $\gamma \vdash (n-1)$ such that $\lambda_{>1} = T^{(k)}$ and $\gamma_{>1} = T^{(k-\frac12)}$. Then by Theorem 3.35(c), $M_k = Z_k - Z_{k-\frac12}$ acts on v_T by the constant,

$$\left(\sum_{b\in\lambda} c(b) - \binom{n}{2} + kn\right) - \left(\sum_{b\in\gamma} c(b) - \binom{n}{2} + kn - 1\right) = c(\lambda/\gamma) + 1,$$

and $M_{k+\frac12} = Z_{k+\frac12} - Z_k$ acts on $v_T \in A^\lambda_{k+\frac12}(n)$ by the constant

$$\left(\sum_{b\in\gamma} c(b) - \binom{n}{2} + kn + n - 1\right) - \left(\sum_{b\in\lambda} c(b) - \binom{n}{2} + kn\right) = -c(\lambda/\gamma) + n - 1.$$

The result now follows from (3.23) and the observation that

$$c(\lambda/\gamma) = \begin{cases} c(T^{(k)}/T^{(k-\frac12)}) - 1, & \text{if } T^{(k)} = T^{(k+\frac12)} + \square, \\ n - |T^{(k)}| - 1, & \text{if } T^{(k)} = T^{(k+\frac12)}. \end{cases}$$

∎

4. The Basic Construction

In this section we shall assume that all algebras are finite dimensional algebras over an algebraically closed field $\mathbb{F}$. The fact that $\mathbb{F}$ is algebraically closed is only for convenience, to avoid the division rings that could arise in the decomposition of $\bar{A}$ just before (4.8) below.

Let $A \subseteq B$ be an inclusion of algebras. Then $B \otimes_{\mathbb{F}} B$ is an (A,A)-bimodule where A acts on the left by left multiplication and on the right by right multiplication. Fix an (A,A)-bimodule homomorphism

$$\varepsilon : B \otimes_{\mathbb{F}} B \longrightarrow A. \tag{4.1}$$

The *basic construction* is the algebra $B \otimes_A B$ with product given by

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon(b_2 \otimes b_3) b_4, \qquad \text{for } b_1, b_2, b_3, b_4 \in B. \tag{4.2}$$

More generally, let A be an algebra and let L be a left A-module and R a right A-module. Let

$$\varepsilon : L \otimes_{\mathbb{F}} R \longrightarrow A, \tag{4.3}$$

be an (A,A)-bimodule homomorphism. The *basic construction* is the algebra $R \otimes_A L$ with product given by

$$(r_1 \otimes \ell_1)(r_2 \otimes \ell_2) = r_1 \otimes \varepsilon(\ell_1 \otimes r_2)\ell_2, \qquad \text{for } r_1, r_2 \in R \text{ and } \ell_1, \ell_2 \in L. \tag{4.4}$$

Theorem 4.18 below determines, explicitly, the structure of the algebra $R \otimes_A L$.

Let $N = \mathrm{Rad}(A)$ and let

$$\bar{A} = A/N, \qquad \bar{L} = L/NL, \qquad \text{and} \qquad \bar{R} = R/RN \tag{4.5}$$

Define an $(\bar{A},\bar{A})$-bimodule homomorphism

$$\begin{array}{rccc} \bar\varepsilon: & \bar{L}\otimes_{\mathbb{F}}\bar{R} & \longrightarrow & \bar{A} \\ & \bar\ell\otimes\bar r & \mapsto & \overline{\varepsilon(\ell\otimes r)} \end{array} \tag{4.6}$$

where $\bar\ell=\ell+NL$, $\bar r=r+RN$, and $\bar a=a+N$, for $\ell\in L, r\in R$ and $a\in A$. Then by basic tensor product relations [Bou1, Ch. II §3.3 Cor. to Prop. 2 and §3.6 Cor. to Prop. 6], the surjective algebra homomorphism

$$\begin{array}{rccc} \pi: & R\otimes_A L & \longrightarrow & \bar{R}\otimes_{\bar A}\bar{L} \\ & r\otimes\ell & \mapsto & \bar r\otimes\bar\ell \end{array} \qquad \text{has} \qquad \ker(\pi)=R\otimes_A NL. \tag{4.7}$$

The algebra $\bar{A}$ is a split semisimple algebra (an algebra isomorphic to a direct sum of matrix algebras). Fix an algebra isomorphism

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\sim} & \bigoplus_{\mu\in\hat{A}} M_{d_\mu}(\mathbb{F}) \\ a^\mu_{PQ} & \leftarrow & E^\mu_{PQ} \end{array}$$

where $\hat{A}$ is an index set for the components and E^μ_{PQ} is the matrix with 1 in the (P,Q) entry of the μth block and 0 in all other entries. Also, fix isomorphisms

$$\bar{L}\cong\bigoplus_{\mu\in\hat{A}}\overrightarrow{A}^\mu\otimes L^\mu \qquad \text{and} \qquad \bar{R}\cong\bigoplus_{\mu\in\hat{A}}R^\mu\otimes\overleftarrow{A}^\mu \tag{4.8}$$

where $\overrightarrow{A}^\mu$, $\mu\in\hat{A}$, are the simple left $\bar{A}$-modules, $\overleftarrow{A}^\mu$, $\mu\in\hat{A}$, are the simple right $\bar{A}$-modules, and L^μ, R^μ, $\mu\in\hat{A}$ are vector spaces. The practical effect of this setup is that if $\hat{R}^\mu$ is an index set for a basis $\{r^\mu_Y|Y\in\hat{R}^\mu\}$ of R^μ, $\hat{L}^\mu$ is an index set for a basis $\{\ell^\mu_X|X\in\hat{L}^\mu\}$ of L^μ, and $\hat{A}^\mu$ is an index set for bases

$$\{\overrightarrow{a}^\mu_Q \mid Q\in\hat{A}^\mu\} \text{ of } \overrightarrow{A}^\mu \qquad \text{and} \qquad \{\overleftarrow{a}^\mu_P \mid P\in\hat{A}^\mu\} \text{ of } \overleftarrow{A}^\mu \tag{4.9}$$

such that

$$a^\lambda_{ST}\overrightarrow{a}^\mu_Q=\delta_{\lambda\mu}\delta_{TQ}\overrightarrow{a}^\mu_S \qquad \text{and} \qquad \overleftarrow{a}^\mu_P a^\lambda_{ST}=\delta_{\lambda\mu}\delta_{PS}\overleftarrow{a}^\mu_T \tag{4.10}$$

then

$$\begin{array}{l} \bar{L} \text{ has basis } \{\overrightarrow{a}^\mu_P\otimes\ell^\mu_X \mid \mu\in\hat{A}, P\in\hat{A}^\mu, X\in\hat{L}^\mu\} \qquad \text{and} \\ \bar{R} \text{ has basis } \{r^\mu_Y\otimes\overleftarrow{a}^\mu_Q \mid \mu\in\hat{A}, Q\in\hat{A}^\mu, Y\in\hat{R}^\mu\}. \end{array} \tag{4.11}$$

With notations as in (4.9) and (4.11) the map $\bar\varepsilon:\bar{L}\otimes_{\mathbb{F}}\bar{R}\to\bar{A}$ is determined by the constants $\varepsilon^\mu_{XY}\in\mathbb{F}$ given by

$$\varepsilon(\overrightarrow{a}^\mu_Q\otimes\ell^\mu_X\otimes r^\mu_Y\otimes\overleftarrow{a}^\mu_P)=\varepsilon^\mu_{XY}a^\mu_{QP} \tag{4.12}$$

and ε^μ_{XY} does not depend on Q and P since

$$\begin{aligned} \varepsilon(\overrightarrow{a}^\lambda_S\otimes\ell^\lambda_X\otimes r^\mu_Y\otimes\overleftarrow{a}^\mu_T) &= \varepsilon(a^\lambda_{SQ}\overrightarrow{a}^\lambda_Q\otimes\ell^\lambda_X\otimes r^\mu_Y\otimes\overleftarrow{a}^\mu_P a^\mu_{PT}) \\ &= a^\lambda_{SQ}\varepsilon(\overrightarrow{a}^\lambda_Q\otimes\ell^\lambda_X\otimes r^\mu_Y\otimes\overleftarrow{a}^\mu_P)a^\mu_{PT} \\ &= \delta_{\lambda\mu}a^\mu_{SQ}\varepsilon^\mu_{XY}a^\mu_{QP}a^\mu_{PT}=\varepsilon^\mu_{XY}a^\mu_{ST}. \end{aligned} \tag{4.13}$$

For each $\mu \in \hat{A}$ construct a matrix

$$\mathcal{E}^\mu = (\varepsilon^\mu_{XY}) \tag{4.14}$$

and let $D^\mu = (D^\mu_{ST})$ and $C^\mu = (C^\mu_{ZW})$ be invertible matrices such that $D^\mu \mathcal{E}^\mu C^\mu$ is a diagonal matrix with diagonal entries denoted ε^μ_X,

$$D^\mu \mathcal{E}^\mu C^\mu = \mathrm{diag}(\varepsilon^\mu_X). \tag{4.15}$$

In practice D^μ and C^μ are found by row reducing $\mathcal{E}^\mu$ to its Smith normal form. The ε^μ_P are the *invariant factors* of $\mathcal{E}^\mu$.

For $\mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu$, define the following elements of $\bar{R} \otimes_{\bar{A}} \bar{L}$,

$$\bar{m}^\mu_{XY} = r^\mu_X \otimes \overrightarrow{a}^\mu_P \otimes \overleftarrow{a}^\mu_P \otimes \ell^\mu_Y, \qquad \text{and} \qquad \bar{n}^\mu_{XY} = \sum_{Q_1,Q_2} C^\mu_{Q_1X} D^\mu_{YQ_2} \bar{m}^\mu_{Q_1Q_2}. \tag{4.16}$$

Since

$$\begin{aligned}(r^\lambda_S \otimes \overrightarrow{a}^\lambda_W \otimes \overleftarrow{a}^\mu_Z \otimes \ell^\mu_T) &= (r^\lambda_S \otimes \overrightarrow{a}^\lambda_P a^\lambda_{PW} \otimes \overleftarrow{a}^\mu_Z \otimes \ell^\mu_T) \\ &= (r^\lambda_S \otimes \overrightarrow{a}^\lambda_P \otimes a^\lambda_{PW} \overleftarrow{a}^\mu_Z \otimes \ell^\mu_T) \\ &= \delta_{\lambda\mu}\delta_{WZ}(r^\lambda_S \otimes \overrightarrow{a}^\lambda_P \otimes \overleftarrow{a}^\lambda_P \otimes \ell^\lambda_T)\end{aligned} \tag{4.17}$$

the element $\bar{m}^\mu_{XY}$ does not depend on P and $\{\bar{m}^\mu_{XY} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$ is a basis of $\bar{R} \otimes_{\bar{A}} \bar{L}$.

The following theorem determines the structure of the algebras $R \otimes_A L$ and $\bar{R} \otimes_{\bar{A}} \bar{L}$. This theorem is used by W.P. Brown in the study of the Brauer algebra. Part (a) is implicit in [Bro1,§2.2] and part (b) is proved in [Bro2].

Theorem 4.18. *Let $\pi: R \otimes_A L \to \bar{R} \otimes_{\bar{A}} \bar{L}$ be as in (4.7) and let $\{k_i\}$ be a basis of $\ker(\pi) = R \otimes_A NL$. Let*

$$n^\mu_{YT} \in R \otimes_A L \quad \text{be such that} \quad \pi(n^\mu_{YT}) = \bar{n}^\mu_{YT},$$

where the elements $\bar{n}^\mu_{YT} \in \bar{R} \otimes_{\bar{A}} \bar{L}$ are as defined in (4.16).

(a) The sets $\{\bar{m}^\mu_{XY} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$ and $\{\bar{n}^\mu_{XY} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$ (see (4.16)) are bases of $\bar{R} \otimes_{\bar{A}} \bar{L}$, which satisfy

$$\bar{m}^\lambda_{ST}\bar{m}^\mu_{QP} = \delta_{\lambda\mu}\varepsilon^\mu_{TQ}\bar{m}^\mu_{SP} \qquad \text{and} \qquad \bar{n}^\lambda_{ST}\bar{n}^\mu_{QP} = \delta_{\lambda\mu}\delta_{TQ}\varepsilon^\mu_T\bar{n}^\mu_{SP},$$

where ε^μ_{TQ} and ε^μ_T are as defined in (4.12) and (4.15).

(b) The radical of the algebra $R \otimes_A L$ is

$$Rad(R \otimes_A L) = \mathbb{F}\text{-span}\{k_i, n^\mu_{YT} \mid \varepsilon^\mu_Y = 0 \text{ or } \varepsilon^\mu_T = 0\}$$

and the images of the elements

$$e^\mu_{YT} = \frac{1}{\varepsilon^\mu_T} n^\mu_{YT}, \qquad \text{for } \varepsilon^\mu_Y \neq 0 \text{ and } \varepsilon^\mu_T \neq 0,$$

are a set of matrix units in $(R \otimes_A L)/Rad(R \otimes_A L)$.

Proof. The first statement in (a) follows from the equations in (4.17). If $(C^{-1})^\mu$ and $(D^{-1})^\mu$ are the inverses of the matrices C^μ and D^μ then

$$\sum_{X,Y}(C^{-1})^\mu_{XS}(D^{-1})^\mu_{TY}\bar n_{XY} = \sum_{X,Y,Q_1,Q_2}(C^{-1})^\mu_{XS}C^\mu_{Q_1X}\bar m_{Q_1Q_2}D^\mu_{YQ_2}(D^{-1})^\mu_{TY}$$
$$= \sum_{Q_1,Q_2}\delta_{SQ_1}\delta_{Q_2T}\bar m^\mu_{Q_1Q_2} = \bar m^\mu_{ST},$$

and so the elements $\bar m^\mu_{ST}$ can be written as linear combinations of the $\bar n^\mu_{XY}$. This establishes the second statement in (a). By direct computation, using (4.10) and (4.12),

$$\begin{aligned}\bar m^\lambda_{ST}\bar m^\mu_{QP} &= (r^\lambda_S\otimes\overrightarrow{a}^\lambda_W\otimes\overleftarrow{a}^\lambda_W\otimes\ell^\lambda_T)(r^\mu_Q\otimes\overrightarrow{a}^\mu_Z\otimes\overleftarrow{a}^\mu_Z\otimes\ell^\mu_P)\\ &= r^\lambda_S\otimes\overrightarrow{a}^\lambda_W\otimes\varepsilon(\overleftarrow{a}^\lambda_W\otimes\ell^\lambda_T\otimes r^\mu_Q\otimes\overrightarrow{a}^\mu_Z)\overleftarrow{a}^\mu_Z\otimes\ell^\mu_P\\ &= \delta_{\lambda\mu}(r^\lambda_S\otimes\overrightarrow{a}^\lambda_W\otimes\varepsilon^\lambda_{TQ}\bar a^\lambda_{WZ}\overleftarrow{a}^\lambda_Z\otimes\ell^\lambda_P)\\ &= \delta_{\lambda\mu}\varepsilon^\lambda_{TQ}(r^\lambda_S\otimes\overrightarrow{a}^\lambda_W\otimes\overleftarrow{a}^\lambda_W\otimes\ell^\lambda_P) = \delta_{\lambda\mu}\varepsilon^\lambda_{TQ}\bar m^\lambda_{SP},\end{aligned}$$

and

$$\begin{aligned}\bar n^\lambda_{ST}\bar n^\mu_{UV} &= \sum_{Q_1,Q_2,Q_3,Q_4} C^\lambda_{Q_1S}D^\lambda_{TQ_2}\bar m^\lambda_{Q_1Q_2}C^\mu_{Q_3U}D^\mu_{VQ_4}\bar m^\mu_{Q_3Q_4}\\ &= \sum_{Q_1,Q_2,Q_3,Q_4}\delta_{\lambda\mu}C^\lambda_{Q_1S}D^\lambda_{TQ_2}\varepsilon^\mu_{Q_2Q_3}C^\mu_{Q_3U}D^\mu_{VQ_4}\bar m^\mu_{Q_1Q_4}\\ &= \delta_{\lambda\mu}\sum_{Q_1,Q_4}\delta_{TU}\varepsilon^\mu_T C^\mu_{Q_1S}D^\mu_{VQ_4}\bar m^\mu_{Q_1Q_4} = \delta_{\lambda\mu}\delta_{TU}\varepsilon^\mu_T\bar n^\mu_{SV}.\end{aligned}$$

(b) Let $N=\mathrm{Rad}(A)$ as in (4.5). If $r_1\otimes n_1\ell_1, r_2\otimes n_2\ell_2\in R\otimes_A NL$ with $n_1\in N^i$ for some $i\in\mathbb{Z}_{>0}$ then

$$(r_1\otimes n_1\ell_1)(r_2\otimes n_2\ell_2) = r_1\otimes\varepsilon(n_1\ell_1\otimes r_2)n_2\ell_2 = r_1\otimes n_1\varepsilon(\ell_1\otimes r_2)n_2\ell_2\in R\otimes_A N^{i+1}L.$$

Since N is a nilpotent ideal of A it follows that $\ker(\pi)=R\otimes_A NL$ is a nilpotent ideal of $R\otimes_A L$. So $\ker(\pi)\subseteq\mathrm{Rad}(R\otimes_A L)$.

Let

$$I = \mathbb{F}\text{-span}\{k_i, n^\mu_{YT}\mid \varepsilon^\mu_Y=0 \text{ or } \varepsilon^\mu_T=0\}.$$

The multiplication rule for the $\bar n_{YT}$ implies that $\pi(I)$ is an ideal of $\bar R\otimes_{\bar A}\bar L$ and thus, by the correspondence between ideals of $\bar R\otimes_{\bar A}\bar L$ and ideals of $R\otimes_A L$ which contain $\ker(\pi)$, I is an ideal of $R\otimes_A L$.

If $\bar n^\mu_{Y_1T_1},\bar n^\mu_{Y_2T_2},\bar n^\mu_{Y_3T_3}\in\{\bar n^\mu_{YT}\mid\varepsilon^\mu_Y=0\text{ or }\varepsilon^\mu_T=0\}$ then

$$\bar n^\mu_{Y_1T_1}\bar n^\mu_{Y_2T_2}\bar n^\mu_{Y_3T_3} = \delta_{T_1Y_2}\varepsilon^\mu_{Y_2}\bar n^\mu_{Y_1T_2}\bar n^\mu_{Y_3T_3} = \delta_{T_1Y_2}\delta_{T_2Y_3}\varepsilon^\mu_{Y_2}\varepsilon^\mu_{T_2}\bar n^\mu_{Y_1T_3}=0,$$

since $\varepsilon^\mu_{Y_2}=0$ or $\varepsilon^\mu_{T_2}=0$. Thus any product $n^\mu_{Y_1T_1}n^\mu_{Y_2T_2}n^\mu_{Y_3T_3}$ of three basis elements of I is in $\ker(\pi)$. Since $\ker(\pi)$ is a nilpotent ideal of $R\otimes_A L$ it follows that I is an ideal of $R\otimes_A L$ consisting of nilpotent elements. So $I\subseteq\mathrm{Rad}(R\otimes_A L)$.

Since

$$e^\lambda_{YT}e^\mu_{UV} = \frac{1}{\varepsilon^\lambda_T}\frac{1}{\varepsilon^\mu_V}n^\lambda_{YT}n^\mu_{UV} = \delta_{\lambda\mu}\delta_{TU}\frac{1}{\varepsilon^\lambda_T\varepsilon^\lambda_V}\varepsilon^\lambda_T n^\lambda_{YV} = \delta_{\lambda\mu}\delta_{TU}e^\lambda_{YV}\qquad\mathrm{mod}\ I,$$

the images of the elements e^λ_{YT} in (4.7) form a set of matrix units in the algebra $(R\otimes_A L)/I$. Thus $(R\otimes_A L)/I$ is a split semisimple algebra and so $I\supseteq\mathrm{Rad}(R\otimes_A L)$. ∎

Basic constructions for $A \subseteq B$

Let $A \subseteq B$ be an inclusion of algebras. Let $\varepsilon_1 : B \to A$ be an (A, A) bimodule homomorphism and use the (A, A)-bimodule homomorphism

$$\begin{array}{rccc} \varepsilon : & B \otimes_{\mathbb{F}} B & \longrightarrow & A \\ & b_1 \otimes b_2 & \longmapsto & \varepsilon_1(b_1 b_2) \end{array} \tag{4.19}$$

and (4.2) to define the basic construction $B \otimes_A B$. Theorem 4.28 below provides the structure of $B \otimes_A B$ in the case that both A and B are split semisimple.

Let us record the following facts,

(4.20a) If $p \in A$ and $pAp = \mathbb{F}p$ then $(p \otimes 1)(B \otimes_A B)(p \otimes 1) = \mathbb{F} \cdot (p \otimes 1)$,

(4.20b) If p is an idempotent of A and $pAp = \mathbb{F}p$ then $\varepsilon_1(1) \in \mathbb{F}$,

(4.20c) If $p \in A$, $pAp = \mathbb{F}p$ and if $\varepsilon_1(1) \neq 0$, then $\frac{1}{\varepsilon(1)}(p \otimes 1)$ is a minimal idempotent in $B \otimes_A B$,

which are justified as follows. If $p \in A$ and $pAp = \mathbb{F}p$ and $b_1, b_2 \in B$ then $(p \otimes 1)(b_1 \otimes b_2)(p \otimes 1) = (p \otimes \varepsilon_1(b_1)b_2)(p \otimes 1) = p \otimes \varepsilon_1(b_1)\varepsilon_1(b_2 p) = p\varepsilon_1(b_1)\varepsilon_1(b_2)p \otimes 1 = \xi p \otimes 1$, for some constant $\xi \in \mathbb{F}$. This establishes (a). If p is an idempotent of A and $pAp = \mathbb{F}p$ then $p\varepsilon_1(1)p = \varepsilon_1(p^2) = \varepsilon_1(1 \cdot p) = \varepsilon_1(1)p$ and so (b) holds. If $p \in A$ and $pAp = \mathbb{F}p$ then $(p \otimes 1)^2 = \varepsilon_1(1)(p \otimes 1)$ and so, if $\varepsilon_1(1) \neq 0$, then $\frac{1}{\varepsilon(1)}(p \otimes 1)$ is a minimal idempotent in $B \otimes_A B$.

Assume A and B are split semisimple. Let

$\hat{A}$ be an index set for the irreducible A-modules A^μ,

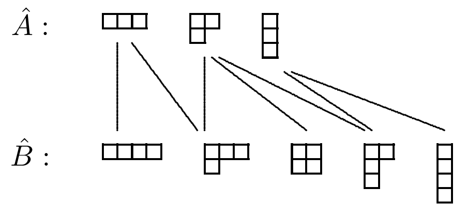
$\hat{B}$ be an index set for the irreducible B-modules B^λ, and let

$\hat{A}^\mu = \{ {\scriptstyle P \to \mu} \}$ be an index set for a basis of the simple A-module A^μ,

for each $\mu \in \hat{A}$ (the composite ${\scriptstyle P \to \mu}$ is viewed as a single symbol). We think of $\hat{A}^\mu$ as the set of "paths to μ" in the two level graph

$$\Gamma \quad \text{with} \quad \begin{array}{l} \text{vertices on level A: } \hat{A}, \quad \text{vertices on level B: } \hat{B}, \quad \text{and} \\ m_\mu^\lambda \text{ edges } \mu \to \lambda \text{ if } A^\mu \text{ appears with multiplicity } m_\mu^\lambda \text{ in } \mathrm{Res}_A^B(B^\lambda). \end{array} \tag{4.21}$$

For example, the graph Γ for the symmetric group algebras $A = \mathbb{C}S_3$ and $B = \mathbb{C}S_4$ is



If $\lambda \in \hat{B}$ then

$$\hat{B}^\lambda = \{ {\scriptstyle P \to \mu} \to \lambda \mid \mu \in \hat{A},\ {\scriptstyle P \to \mu} \in \hat{A}^\mu \text{ and } \mu \to \lambda \text{ is an edge in } \Gamma \} \tag{4.22}$$

is an index set for a basis of the irreducible B-module B^λ. We think of $\hat{B}^\lambda$ as the set of paths to λ in the graph Γ. Let

$$\{ a_{\substack{PQ \\ \mu}} \mid \mu \in \hat{A}, {\scriptstyle P \to \mu}, {\scriptstyle Q \to \mu} \in \hat{A}^\mu \} \qquad \text{and} \qquad \{ b_{\substack{PQ \\ \mu\,\nu \\ \lambda}} \mid \lambda \in \hat{B}, {\scriptstyle P \to \mu} \to \lambda, {\scriptstyle Q \to \nu} \to \lambda \in \hat{B}^\lambda \}, \tag{4.23}$$

be sets of matrix units in the algebras A and B, respectively, so that

$$a_{\substack{PQ\\ \mu}} a_{\substack{ST\\ \nu}} = \delta_{\mu\nu}\delta_{QS} a_{\substack{PT\\ \mu}} \qquad \text{and} \qquad b_{\substack{PQ\\ \mu\gamma\\ \lambda}} b_{\substack{ST\\ \tau\nu\\ \sigma}} = \delta_{\lambda\sigma}\delta_{QS}\delta_{\gamma\tau} b_{\substack{PT\\ \mu\nu\\ \lambda}}, \tag{4.24}$$

and such that, for all $\mu \in \hat{A}$, $P, Q \in \hat{A}^\mu$,

$$a^\mu_{\substack{PQ\\ \mu}} = \sum_{\mu\to\lambda} b^\lambda_{\substack{PQ\\ \mu\mu\\ \lambda}} \tag{4.25}$$

where the sum is over all edges $\mu \to \lambda$ in the graph Γ.

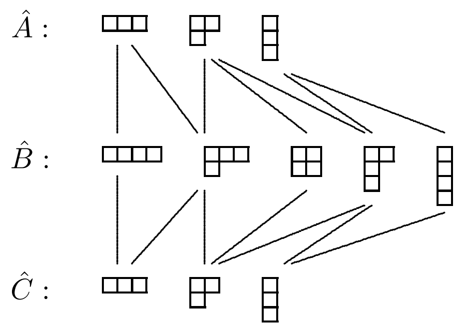
Though is not necessary for the following it is conceptually helpful to let $C = B \otimes_A B$, let $\hat{C} = \hat{A}$ and extend the graph Γ to a graph $\hat{\Gamma}$ with three levels, so that the edges between level B and level C are the reflections of the edges between level A and level B. In other words,

$$\hat{\Gamma} \quad \text{has} \quad \begin{array}{l} \text{vertices on level } C\text{:} \quad \hat{C}, \quad \text{and} \\ \text{an edge } \lambda \to \mu,\ \lambda \in \hat{B},\ \mu \in \hat{C}, \text{ for each edge } \mu \to \lambda,\ \mu \in \hat{A},\ \lambda \in \hat{B}. \end{array} \tag{4.26}$$

For each $\nu \in \hat{C}$ define

$$\hat{C}^\nu = \left\{ P{\to}\mu \to \lambda \to \nu \;\middle|\; \begin{array}{c} \mu \in \hat{A},\ \lambda \in \hat{B},\ \nu \in \hat{C},\ P{\to}\mu \in \hat{A}^\mu \text{ and} \\ \mu \to \lambda \text{ and } \lambda \to \nu \text{ are edges in } \hat{\Gamma} \end{array} \right\}, \tag{4.27}$$

so that $\hat{C}^\nu$ is the set of "paths to ν" in the graph $\hat{\Gamma}$. Continuing with our previous example, $\hat{\Gamma}$ is



Theorem 4.28. *Assume A and B are split semisimple, and let the notations and assumption be as in (4.21-4.25).*

(a) *The elements of $B \otimes_A B$ given by*

$$b_{\substack{PT\\ \mu\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\nu\\ \sigma}}$$

do not depend on the choice of $T{\to}\gamma \in \hat{A}^\gamma$ and form a basis of $B \otimes_A B$.

(b) *For each edge $\mu \to \lambda$ in Γ define a constant $\varepsilon^\lambda_\mu \in \mathbb{F}$ by*

$$\varepsilon_1\left(b_{\substack{PP\\ \mu\mu\\ \lambda}}\right) = \varepsilon^\lambda_\mu\, a_{\substack{PP\\ \mu}} \tag{4.29}$$

Then ε_μ^λ *is independent of the choice of* $P\to\mu\in\hat{A}^\mu$ *and*

$$\Big(b_{\substack{P\,T\\ \mu\,\gamma\\ \lambda}}\otimes b_{\substack{T\,Q\\ \gamma\,\nu\\ \sigma}}\Big)\Big(b_{\substack{R\,X\\ \tau\,\pi\\ \rho}}\otimes b_{\substack{X\,S\\ \pi\,\xi\\ \eta}}\Big)=\delta_{\gamma\pi}\delta_{QR}\delta_{\nu\tau}\delta_{\sigma\rho}\varepsilon_\gamma^\sigma\Big(b_{\substack{P\,T\\ \pi\,\mu\\ \gamma}}\otimes b_{\substack{T\,S\\ \gamma\,\xi\\ \eta}}\Big).$$

$$\text{Rad}(B\otimes_A B)\qquad\textit{has basis}\qquad\Big\{b_{\substack{P\,T\\ \mu\,\gamma\\ \lambda}}\otimes b_{\substack{T\,Q\\ \gamma\,\nu\\ \sigma}}\mid\varepsilon_\mu^\lambda=0\ \textit{or}\ \varepsilon_\nu^\sigma=0\Big\},$$

and the images of the elements

$$e_{\substack{P\,Q\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}}=\left(\frac{1}{\varepsilon_\gamma^\sigma}\right)\Big(b_{\substack{P\,T\\ \mu\,\gamma\\ \lambda}}\otimes b_{\substack{T\,Q\\ \gamma\,\nu\\ \sigma}}\Big),\qquad\textit{such that}\quad\varepsilon_\mu^\lambda\neq0\ \textit{and}\ \varepsilon_\nu^\sigma\neq0,$$

form a set of matrix units in $(B\otimes_A B)/\text{Rad}(B\otimes_A B)$.

(c) *Let* $tr_B: B\to\mathbb{F}$ *and* $tr_A: A\to\mathbb{F}$ *be traces on* B *and* A*, respectively, such that*

$$tr_A(\varepsilon_1(b))=tr_B(b),\qquad\textit{for all } b\in B.\tag{4.30}$$

Let χ_A^μ, $\mu\in\hat{A}$, *and* χ_B^λ, $\lambda\in\hat{B}$, *be the irreducible characters of the algebras* A *and* B*, respectively. Define constants* tr_A^μ, $\mu\in\hat{A}$, *and* tr_B^λ, $\lambda\in\hat{B}$, *by the equations*

$$\text{tr}_A=\sum_{\mu\in\hat{A}}\text{tr}_A^\mu\chi_A^\mu\qquad\textit{and}\qquad\text{tr}_B=\sum_{\lambda\in\hat{B}}\text{tr}_B^\lambda\chi_B^\lambda,\tag{4.31}$$

respectively. Then the constants ε_μ^λ *defined in (4.29) satisfy*

$$\text{tr}_B^\lambda=\varepsilon_\mu^\lambda\ \text{tr}_A^\mu.$$

(d) *In the algebra* $B\otimes_A B$,

$$1\otimes1=\sum_{\substack{P\\ \downarrow\\ \mu\\ \lambda\swarrow\ \searrow\gamma}}b_{\substack{P\,P\\ \mu\,\mu\\ \lambda}}\otimes b_{\substack{P\,P\\ \mu\,\mu\\ \gamma}}$$

(g) *By left multiplication, the algebra* $B\otimes_A B$ *is a left* B*-module. If* $\text{Rad}(B\otimes_A B)$ *is a* B*-submodule of* $B\otimes_A B$ *and* $\iota: B\to(B\otimes_A B)/\text{Rad}(B\otimes_A B)$ *is a left* B*-module homomorphism then*

$$\iota\Big(b_{\substack{R\,S\\ \tau\,\beta\\ \pi}}\Big)=\sum_{\pi\to\gamma}e_{\substack{R\,S\\ \tau\,\beta\\ \pi\,\pi\\ \gamma}}.$$

Proof. By (4.11) and (4.25),

$$\begin{array}{ccc}B&\xrightarrow{\sim}&\displaystyle\bigoplus_{\mu\in\hat{A}}\overrightarrow{A}^\mu\otimes L^\mu\\ b_{\substack{P\,Q\\ \mu\,\nu\\ \lambda}}&\longmapsto&\overrightarrow{a}_{\substack{P\\ \mu}}\otimes\ell^\mu_{\substack{Q\\ \mu\,\nu\\ \lambda}}\end{array}\qquad\text{and}\qquad\begin{array}{ccc}B&\xrightarrow{\sim}&\displaystyle\bigoplus_{\nu\in\hat{A}}R^\nu\otimes\overleftarrow{A}^\nu\\ b_{\substack{P\,Q\\ \mu\,\nu\\ \lambda}}&\longmapsto&r^\nu_{\substack{P\\ \mu\,\nu\\ \lambda}}\otimes\overleftarrow{a}_{\substack{Q\\ \nu}}\end{array}\tag{4.32}$$

as left A-modules and as right A-modules, respectively. Identify the left and right hand sides of these isomorphisms. Then, by (4.17), the elements of $C = B \otimes_A B$ given by

$$\bar m_{\substack{P Q\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = r^{\gamma}_{\substack{P\\ \mu\,\gamma\\ \lambda}} \otimes \overleftarrow{a}_{\substack{T\\ \gamma}} \otimes \overrightarrow{a}_{\substack{T\\ \gamma}} \otimes \ell^{\gamma}_{\substack{Q\\ \gamma\,\nu\\ \sigma}} = b_{\substack{P T\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{T Q\\ \gamma\,\nu\\ \sigma}} \tag{4.33}$$

do not depend on $T\to\gamma \in \hat A^{\gamma}$ and form a basis of $B \otimes_A B$.

(b) By (4.12), the map $\varepsilon\colon B \otimes_{\mathbb F} B \to A$ is determined by the values

$$\varepsilon^{\mu}_{\substack{T Q\\ \gamma\,\tau\\ \lambda\,\sigma\\ \mu}} \in \mathbb F \qquad \text{given by} \qquad \varepsilon^{\mu}_{\substack{T Q\\ \gamma\,\tau\\ \lambda\,\sigma\\ \mu}} a_{\substack{P P\\ \mu}} = \varepsilon\big(\overrightarrow{a}_{\substack{P\\ \mu}} \otimes \ell^{\mu}_{\substack{T\\ \mu\,\gamma\\ \lambda}} \otimes r^{\mu}_{\substack{Q\\ \tau\,\mu\\ \sigma}} \otimes \overleftarrow{a}_{\substack{P\\ \mu}}\big). \tag{4.34}$$

Since

$$\begin{aligned} \varepsilon^{\mu}_{\substack{T Q\\ \gamma\,\tau\\ \lambda\,\sigma\\ \mu}} a_{\substack{P P\\ \mu}} &= \varepsilon\big(b_{\substack{P T\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{Q P\\ \tau\,\mu\\ \sigma}}\big) = \varepsilon_1\big(b_{\substack{P T\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{Q P\\ \tau\,\mu\\ \sigma}}\big) \\ &= \delta_{\substack{T Q\\ \gamma\,\tau\\ \lambda\,\sigma}} \varepsilon_1\big(b_{\substack{P P\\ \mu\,\mu\\ \lambda}}\big) = \delta_{\substack{T Q\\ \gamma\,\tau\\ \lambda\,\sigma}} \varepsilon_1\big(b_{\substack{P P\\ \mu\,\mu\\ \lambda}} b_{\substack{P P\\ \mu\,\mu\\ \lambda}}\big) = \delta_{\substack{T Q\\ \gamma\,\tau\\ \lambda\,\sigma}} \varepsilon^{\mu}_{\substack{P P\\ \mu\mu\mu\\ \lambda\lambda}} a_{\substack{P P\\ \mu}}. \end{aligned}$$

the matrix $\mathcal E^{\mu}$ given by (4.14) is diagonal with entries $\varepsilon^{\lambda}_{\mu}$ given by (4.15) and, by (4.17), $\varepsilon^{\lambda}_{\mu}$ is independent of $P\to\mu \in \hat A^{\mu}$. By Theorem 4.18(a),

$$\bar m_{\substack{P Q\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} \bar m_{\substack{R S\\ \tau\,\xi\\ \rho\,\eta\\ \pi}} = \delta_{\gamma\pi} \varepsilon_{\substack{Q R\\ \nu\,\tau\\ \sigma\,\rho\\ \gamma}} \bar m_{\substack{P S\\ \mu\,\xi\\ \lambda\,\eta\\ \gamma}} = \delta_{\gamma\pi} \delta_{\substack{Q R\\ \nu\,\tau\\ \sigma\,\rho}} \varepsilon^{\sigma}_{\gamma} \bar m_{\substack{P S\\ \mu\,\xi\\ \lambda\,\eta\\ \gamma}}$$

in the algebra C. The rese of the statements in part (b) follow from Theorem 4.18(b).

(c) Evaluating the equations in (4.31) and using (4.29) gives

$$\mathrm{tr}^{\lambda}_{B} = \mathrm{tr}_B(b_{\substack{P P\\ \mu\,\mu\\ \lambda}}) = \mathrm{tr}_A(\varepsilon_1(b_{\substack{P P\\ \mu\,\mu\\ \lambda}})) = \varepsilon^{\lambda}_{\mu} \mathrm{tr}_A(a_{\substack{P P\\ \mu}}) = \varepsilon^{\lambda}_{\mu} \mathrm{tr}^{\mu}_{A}, \tag{4.35}$$

(d) Since

$$1 = \sum_{P\to\mu\to\lambda} b_{\substack{P P\\ \mu\,\mu\\ \lambda}} \qquad \text{in the algebra } B,$$

it follows from part (b) and (4.16) that

$$1 \otimes 1 = \Big(\sum_{P\to\mu\to\lambda} b_{\substack{P P\\ \mu\,\mu\\ \lambda}}\Big) \otimes \Big(\sum_{Q\to\nu\to\gamma} b_{\substack{Q Q\\ \nu\,\nu\\ \gamma}}\Big) = \sum_{\substack{P\to\mu\to\lambda\\ Q\to\nu\to\gamma}} \delta_{PQ}\delta_{\mu\nu} \Big(b_{\substack{P P\\ \mu\,\mu\\ \lambda}} \otimes b_{\substack{Q Q\\ \nu\,\nu\\ \gamma}}\Big) = \sum_{\substack{P\\ \downarrow\\ \mu\\ \lambda\swarrow\ \searrow\gamma}} \bar m_{\substack{P P\\ \mu\,\mu\\ \lambda\,\gamma\\ \mu}}$$

giving part (d).

(e) By left multiplication, the algebra $B \otimes_A B$ is a left B-module. If $\varepsilon^{\lambda}_{\gamma} \neq 0$ and $\varepsilon^{\sigma}_{\gamma} \neq 0$ then

$$b_{\substack{R S\\ \tau\,\beta\\ \pi}} e_{\substack{P Q\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = \left(\frac{1}{\varepsilon^{\sigma}_{\gamma}}\right) b_{\substack{R S\\ \tau\,\beta\\ \pi}} \Big(b_{\substack{P T\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{T Q\\ \gamma\,\nu\\ \sigma}}\Big) = \left(\frac{1}{\varepsilon^{\sigma}_{\gamma}}\right) \delta_{\substack{S P\\ \beta\,\mu\\ \pi\,\lambda}} \Big(b_{\substack{R T\\ \tau\,\gamma\\ \lambda}} \otimes b_{\substack{T Q\\ \gamma\,\nu\\ \sigma}}\Big) = \delta_{\substack{S P\\ \beta\,\mu\\ \pi\,\lambda}} e_{\substack{R Q\\ \tau\,\nu\\ \pi\,\sigma\\ \gamma}}.$$

Thus, if $\iota\colon B \to (B \otimes_A B)/\mathrm{Rad}(B \otimes_A B)$ is a left B-module homomorphism then

$$\iota\Big(b_{\substack{R\,S\\ \tau\,\beta\\ \pi}}\Big) = \iota\Big(b_{\substack{R\,S\\ \tau\,\beta\\ \pi}}\Big) \cdot 1 = b_{\substack{R\,S\\ \tau\,\beta\\ \pi}} \sum_{P\to\mu\to\lambda\to\gamma} e_{\substack{P\,P\\ \mu\,\mu\\ \lambda\,\lambda\\ \gamma}} = \sum_{P\to\mu\to\lambda\to\gamma} \delta_{\substack{S\,P\\ \beta\,\mu\\ \pi\,\lambda}} e_{\substack{R\,P\\ \tau\,\mu\\ \pi\,\lambda\\ \gamma}} = \sum_{\pi\to\gamma} e_{\substack{R\,S\\ \tau\,\beta\\ \pi\,\pi\\ \gamma}}. \qquad \blacksquare$$

5. Semisimple Algebras

Let R be a integral domain and let A_R be an algebra over R, so that A_R has an R-basis $\{b_1, \ldots, b_d\}$,

$$A_R = R\text{-span}\{b_1, \ldots, b_d\} \qquad \text{and} \quad b_i b_j = \sum_{k=1}^{d} r_{ij}^k b_k, \qquad \text{with } r_{ij}^k \in R,$$

making A_R a ring with identity. Let $\mathbb{F}$ be the field of fractions of R, let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and set

$$A = \bar{\mathbb{F}} \otimes_R A_R = \bar{\mathbb{F}}\text{-span}\{b_1, \ldots, b_d\},$$

with multiplication determined by the multiplication in A_R. Then A is an algebra over $\bar{\mathbb{F}}$.

A *trace* on A is a linear map $\vec{t}\colon A \to \bar{\mathbb{F}}$ such that

$$\vec{t}(a_1 a_2) = \vec{t}(a_2 a_1), \qquad \text{for all } a_1, a_2 \in A.$$

A trace $\vec{t}$ on A is *nondegenerate* if for each $b \in A$ there is an $a \in A$ such that $\vec{t}(ba) \neq 0$.

Lemma 5.1. *Let A be a finite dimensional algebra over a field $\mathbb{F}$, let $\vec{t}$ be a trace on A. Define a symmetric bilinear form $\langle , \rangle\colon A \times A \to \mathbb{F}$ on A by $\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2)$, for all $a_1, a_2 \in A$. Let B be a basis of A. Let $G = \big(\langle b, b' \rangle\big)_{b,b' \in B}$ be the matrix of the form $\langle , \rangle$ with respect to B. The following are equivalent:*

(1) The trace $\vec{t}$ is nondegenerate.

(2) $\det G \neq 0$.

(3) The dual basis B^* to the basis B with respect to the form $\langle , \rangle$ exists.

Proof. (2) $\Leftrightarrow$ (1): The trace $\vec{t}$ is degenerate if there is an element $a \in A$, $a \neq 0$, such that $\vec{t}(ac) = 0$ for all $c \in B$. If $a_b \in \bar{\mathbb{F}}$ are such that

$$a = \sum_{b \in B} a_b b, \qquad \text{then} \qquad 0 = \langle a, c \rangle = \sum_{b \in B} a_b \langle b, c \rangle$$

for all $c \in B$. So a exists if and only if the columns of G are linearly dependent, i.e. if and only if G is not invertible.

(3) $\Leftrightarrow$ (2): Let $B^* = \{b^*\}$ be the dual basis to $\{b\}$ with respect to $\langle , \rangle$ and let P be the change of basis matrix from B to B^*. Then

$$d^* = \sum_{b \in B} P_{db} b, \qquad \text{and} \qquad \delta_{bc} = \langle b, d^* \rangle = \sum_{d \in B} P_{dc} \langle b, c \rangle = (GP^t)_{b,c}.$$

So P^t, the transpose of P, is the inverse of the matrix G. So the dual basis to B exists if and only if G is invertible, i.e. if and only if $\det G \neq 0$. $\blacksquare$

Proposition 5.2. *Let A be an algebra and let $\vec{t}$ be a nondegenerate trace on A. Define a symmetric bilinear form $\langle , \rangle : A \times A \to \bar{\mathbb{F}}$ on A by $\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2)$, for all $a_1, a_2 \in A$. Let B be a basis of A and let B^* be the dual basis to B with respect to $\langle\ ,\ \rangle$.*
(a) Let $a \in A$. Then

$$[a] = \sum_{b \in B} bab^* \qquad \text{is an element of the center } Z(A) \text{ of } A$$

and $[a]$ does not depend on the choice of the basis B.
(b) Let M and N be A-modules and let $\phi \in Hom_{\bar{\mathbb{F}}}(M,N)$ and define

$$[\phi] = \sum_{b \in B} b\phi b^*.$$

Then $[\phi] \in Hom_A(M,N)$ and $[\phi]$ does not depend on the choice of the basis B.

Proof. (a) Let $c \in A$. Then

$$c[a] = \sum_{b \in B} cbab^* = \sum_{b \in B}\sum_{d \in B} \langle cb, d^* \rangle dab^* = \sum_{d \in B} da \sum_{b \in B} \langle d^* c, b \rangle b^* = \sum_{d \in B} dad^* c = [a]c,$$

since $\langle cb, d^* \rangle = \vec{t}(cbd^*) = \vec{t}(d^* cb) = \langle d^* c, b \rangle$. So $[a] \in Z(A)$.

Let D be another basis of A and let D^* be the dual basis to D with respect to $\langle , \rangle$. Let $P = \big(P_{db}\big)$ be the transition matrix from D to B and let P^{-1} be the inverse of P. Then

$$d = \sum_{b \in B} P_{db} b \qquad \text{and} \qquad d^* = \sum_{\tilde{b} \in B} (P^{-1})_{\tilde{b}d} \tilde{b}^*,$$

since

$$\langle d, \tilde{d}^* \rangle = \left\langle \sum_{b \in B} P_{db} b, \sum_{\tilde{b} \in B} (P^{-1})_{\tilde{b}\tilde{d}} \tilde{b}^* \right\rangle = \sum_{b, \tilde{b} \in B} P_{db} (P^{-1})_{\tilde{b}\tilde{d}} \delta_{b\tilde{b}} = \delta_{d\tilde{d}}.$$

So

$$\sum_{d \in D} dad^* = \sum_{d \in D}\sum_{b \in B} P_{db} ba \sum_{\tilde{b} \in B} (P^{-1})_{\tilde{b}d} \tilde{b}^* = \sum_{b, \tilde{b} \in B} ba\tilde{b}^* \delta_{b\tilde{b}} = \sum_{b \in B} bab^*.$$

So $[a]$ does not depend on the choice of the basis B.
The proof of part (b) is the same as the proof of part (a) except with a replaced by ϕ. ∎

Let A be an algebra and let M be an A-module. Define

$$\mathrm{End}_A(M) = \{ T \in \mathrm{End}(M) \mid Ta = aT \text{ for all } a \in A \}.$$

Theorem 5.3. *(Schur's Lemma) Let A be a finite dimensional algebra over an algebraically closed field $\bar{\mathbb{F}}$.*

(1) Let A^λ be a simple A-module. Then $End_A(A^\lambda) = \bar{\mathbb{F}} \cdot \mathrm{Id}_{A^\lambda}$.

(2) If A^λ and A^μ are nonisomorphic simple A-modules then $Hom_A(A^\lambda, A^\mu) = 0$.

Proof. Let $T\colon A^\lambda \to A^\mu$ be a nonzero A-module homomorphism. Since A^λ is simple, $\ker T = 0$ and so T is injective. Since A^μ is simple, $\mathrm{im}T = A^\mu$ and so T is surjective. So T is an isomorphism. Thus we may assume that $T\colon A^\lambda \to A^\lambda$.

Since $\bar{\mathbb{F}}$ is algebraically closed T has an eigenvector and a corresponding eigenvalue $\alpha \in \bar{\mathbb{F}}$. Then $T - \alpha \cdot Id \in \mathrm{Hom}_A(A^\lambda, A^\lambda)$ and so $T - \alpha \cdot Id$ is either 0 or an isomorphism. However, since $\det(T - \alpha \cdot Id) = 0$, $T - \alpha \cdot Id$ is not invertible. So $T - \alpha \cdot Id = 0$. So $T = \alpha \cdot Id$. So $\mathrm{End}_A(A^\lambda) = \bar{\mathbb{F}} \cdot Id$. ∎

Theorem 5.4. *(The Centralizer Theorem) Let A be a finite dimensional algebra over an algebraically closed field $\bar{\mathbb{F}}$. Let M be a semisimple A-module and set $Z = End_A(M)$. Suppose that*

$$M \cong \bigoplus_{\lambda \in \hat{M}} (A^\lambda)^{\oplus m_\lambda},$$

where $\hat{M}$ is an index set for the irreducible A-modules A^λ which appear in M and the m_λ are positive integers.

(a) $Z \cong \bigoplus_{\lambda \in \hat{M}} M_{m_\lambda}(\bar{\mathbb{F}})$.

(b) As an (A, Z)-bimodule

$$M \cong \bigoplus_{\lambda \in \hat{M}} A^\lambda \otimes Z^\lambda,$$

where the Z^λ, $\lambda \in \hat{M}$, are the simple Z-modules.

Proof. Index the components in the decomposition of M by dummy variables ϵ_i^λ so that we may write

$$M \cong \bigoplus_{\lambda \in \hat{M}} \bigoplus_{i=1}^{m_\lambda} A^\lambda \otimes \epsilon_i^\lambda.$$

For each $\lambda \in \hat{M}$, $1 \le i, j \le m_\lambda$ let $\phi_{ij}^\lambda\colon A^\lambda \otimes \epsilon_j \to A^\lambda \otimes \epsilon_i$ be the A-module isomorphism given by

$$\phi_{ij}^\lambda(m \otimes \epsilon_j^\lambda) = m \otimes \epsilon_i^\lambda, \qquad \text{for } m \in A^\lambda.$$

By Schur's Lemma,

$$\begin{aligned}\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M) &\cong \mathrm{Hom}_A\left(\bigoplus_\lambda \bigoplus_j A^\lambda \otimes \epsilon_j^\lambda, \bigoplus_\mu \bigoplus_i A^\mu \otimes \epsilon_i^\mu\right) \\ &\cong \bigoplus_{\lambda,\mu} \bigoplus_{i,j} \delta_{\lambda\mu} \mathrm{Hom}_A(A^\lambda \otimes \epsilon_j^\lambda, A^\mu \otimes \epsilon_i^\mu) \cong \bigoplus_\lambda \bigoplus_{i,j=1}^{m_\lambda} \bar{\mathbb{F}} \phi_{ij}^\lambda.\end{aligned}$$

Thus each element $z \in \mathrm{End}_A(M)$ can be written as

$$z = \sum_{\lambda \in \hat{M}} \sum_{i,j=1}^{m_\lambda} z_{ij}^\lambda \phi_{ij}^\lambda, \qquad \text{for some } z_{ij}^\lambda \in \bar{\mathbb{F}},$$

and identified with an element of $\bigoplus_\lambda M_{m_\lambda}(\bar{\mathbb{F}})$. Since $\phi^\lambda_{ij}\phi^\mu_{kl} = \delta_{\lambda\mu}\delta_{jk}\phi^\lambda_{il}$ it follows that

$$\mathrm{End}_A(M) \cong \bigoplus_{\lambda\in\hat{M}} M_{m_\lambda}(\bar{\mathbb{F}}).$$

(b) As a vector space $Z^\mu = \mathrm{span}\{\epsilon^\mu_i \mid 1 \le i \le m_\mu\}$ is isomorphic to the simple $\bigoplus_\lambda M_{m_\lambda}(\bar{\mathbb{F}})$ module of column vectors of length m_μ. The decomposition of M as $A \otimes Z$ modules follows since

$$(a \otimes \phi^\lambda_{ij})(m \otimes \epsilon^\mu_k) = \delta_{\lambda\mu}\delta_{jk}(a \otimes \epsilon^\mu_i), \qquad \text{for all } m \in A^\mu,\ a \in A, \qquad \blacksquare$$

If A is an algebra then A^{op} is the algebra A except with the opposite multiplication, i.e.

$$A^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in A\} \qquad \text{with} \qquad a_1^{\mathrm{op}}a_2^{\mathrm{op}} = (a_2a_1)^{\mathrm{op}}, \quad \text{for all } a_1, a_2 \in A.$$

The left *regular representation* of A is the vector space A with A action given by left multiplication. Here A is serving both as an algebra and as an A-module. It is often useful to distinguish the two roles of A and use the notation $\vec{A}$ for the A-module, i.e. $\vec{A}$ is the vector space

$$\vec{A} = \{\vec{b} \mid b \in A\} \qquad \text{with } A\text{-action} \qquad a\vec{b} = \overrightarrow{ab}, \qquad \text{for all } a \in A,\ \vec{b} \in \vec{A}.$$

Proposition 5.5. *Let A be an algebra and let $\vec{A}$ be the regular representation of A. Then $\mathrm{End}_A(\vec{A}) \cong A^{\mathrm{op}}$. More precisely,*

$$\mathrm{End}_A(\vec{A}) = \{\phi_b \mid b \in A\}, \qquad \text{where } \phi_b \text{ is given by} \qquad \phi_b(\vec{a}) = \overrightarrow{ab}, \quad \text{for all } \vec{a} \in \vec{A}.$$

Proof. Let $\phi \in \mathrm{End}_A(\vec{A})$ and let $b \in A$ be such that $\phi(\vec{1}) = \vec{b}$. For all $\vec{a} \in \vec{A}$,

$$\phi(\vec{a}) = \phi(a \cdot \vec{1}) = a\phi(\vec{1}) = a\vec{b} = \overrightarrow{ab},$$

and so $\phi = \phi_b$. Then $\mathrm{End}_A(\vec{A}) \cong A^{\mathrm{op}}$ since

$$(\phi_{b_1} \circ \phi_{b_2})(\vec{a}) = \overrightarrow{ab_2b_1} = \phi_{b_2b_1}(\vec{a}),$$

for all $b_1, b_2 \in A$ and $\vec{a} \in \vec{A}$. $\blacksquare$

Theorem 5.6. *Suppose that A is a finite dimensional algebra over an algebraically closed field $\mathbb{F}$ such that the regular representation $\vec{A}$ of A is completely decomposable. Then A is isomorphic to a direct sum of matrix algebras, i.e.*

$$A \cong \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\bar{\mathbb{F}}),$$

for some set $\hat{A}$ and some positive integers d_λ, indexed by the elements of $\hat{A}$.

Proof. If $\vec{A}$ is completely decomposable then, by Theorem 5.4, $\mathrm{End}_A(\vec{A})$ is isomorphic to a direct sum of matrix algebras. By Proposition 5.5,

$$A^{\mathrm{op}} \cong \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\bar{\mathbb{F}}),$$

for some set $\hat{A}$ and some positive integers d_λ, indexed by the elements of $\hat{A}$. The map

$$\begin{array}{ccc} \left(\bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\bar{\mathbb{F}})\right)^{\rm op} & \longrightarrow & \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\bar{\mathbb{F}}) \\ a & \longmapsto & a^t, \end{array}$$

where a^t is the transpose of the matrix a, is an algebra isomorphism. So A is isomorphic to a direct sum of matrix algebras. ∎

If A is an algebra then the trace tr of the regular representation is the trace on A given by

$$\mathrm{tr}(a) = \mathrm{Tr}(\vec{A}(a)), \qquad \text{for } a \in A,$$

where $\vec{A}(a)$ is the linear transformation of A induced by the action of a on A by left multiplication.

Proposition 5.7. *Let* $A = \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\bar{\mathbb{F}})$. *Then the trace* tr *of the regular representation of* A *is nondegenerate.*

Proof. As A-modules, the regular representation

$$\vec{A} \cong \bigoplus_{\lambda\in\hat{A}} (A^\lambda)^{\oplus d_\lambda},$$

where A^λ is the irreducible A-module consisting of column vectors of length d_λ. For $a \in A$ let $A^\lambda(a)$ be the linear transformation of A^λ induced by the action of a. Then the trace tr of the regular representation is given by

$$\mathrm{tr} = \sum_{\lambda\in\hat{A}} d_\lambda \chi^\lambda, \qquad \text{where} \qquad \begin{array}{cccc} \chi_A^\lambda\colon & A & \to & \bar{\mathbb{F}} \\ & a & \longmapsto & \mathrm{Tr}(A^\lambda(a)), \end{array}$$

where χ_A^λ are the irreducible characters of A. Since the d_λ are all nonzero the trace tr is nondegenerate. ∎

Theorem 5.8. *(Maschke's theorem) Let* A *be a finite dimensional algebra over a field* $\mathbb{F}$ *such that the trace* tr *of the regular representation of* A *is nondegenerate. Then every representation of* A *is completely decomposable.*

Proof. Let B be a basis of A and let B^* be the dual basis of A with respect to the form $\langle , \rangle\colon A\times A \to \bar{\mathbb{F}}$ defined by

$$\langle a_1, a_2\rangle = tr(a_1a_2), \qquad \text{for all } a_1, a_2 \in A.$$

The dual basis B^* exists because the trace tr is nondegenerate.

Let M be an A-module. If M is irreducible then the result is vacuously true, so we may assume that M has a proper submodule N. Let $p \in \mathrm{End}(M)$ be a projection onto N, i.e. $pM = N$ and $p^2 = p$. Let

$$[p] = \sum_{b\in B} bpb^*, \qquad \text{and} \qquad e = \sum_{b\in B} bb^*.$$

For all $a \in A$,

$$\mathrm{tr}(ea) = \sum_{b\in B} \mathrm{tr}(bb^*a) = \sum_{b\in B} \langle ab, b^*\rangle = \sum_{b\in B} ab\big|_b = \mathrm{tr}(a),$$

So $\mathrm{tr}((e-1)a) = 0$, for all $a \in A$. Thus, since tr is nondegenerate, $e = 1$.

Let $m \in M$. Then $pb^*m \in N$ for all $b \in B$, and so $[p]m \in N$. So $[p]M \subseteq N$. Let $n \in N$. Then $pb^*n = b^*n$ for all $b \in B$, and so $[p]n = en = 1 \cdot n = n$. So $[p]M = N$ and $[p]^2 = [p]$, as elements of $\mathrm{End}(M)$.

Note that $[1-p] = [1] - [p] = e - [p] = 1 - [p]$. So

$$M = [p]M \oplus (1 - [p])M = N \oplus [1-p]M,$$

and, by Proposition 5.2b, $[1-p]M$ is an A-module. So $[1-p]M$ is an A-submodule of M which is complementary to M. By induction on the dimension of M, N and $[1-p]M$ are completely decomposable, and therefore M is completely decomposable. ∎

Together, Theorems 5.6, 5.8 and Proposition 5.7 yield the following theorem.

Theorem 5.9. *(Artin-Wedderburn theorem) Let A be a finite dimensional algebra over an algebraically closed field $\bar{\mathbb{F}}$. Let $\{b_1, \ldots, b_d\}$ be a basis of A and let* tr *be the trace of the regular representation of A. The following are equivalent:*

(1) Every representation of A is completely decomposable.

(2) The regular representation of A is completely decomposable.

(3) $A \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\bar{\mathbb{F}})$ for some finite index set $\hat{A}$, and some $d_\lambda \in \mathbb{Z}_{>0}$.

(4) The trace of the regular representation of A is nondegenerate.

(5) $\det(\mathrm{tr}(b_i b_j)) \neq 0$.

Remark. Let R be an integral domain, and let A_R be an algebra over R with basis $\{b_1, \ldots, b_d\}$. Then $\det(\mathrm{tr}(b_i b_j))$ is an element of R and $\det(\mathrm{tr}(b_i b_j)) \neq 0$ in $\bar{\mathbb{F}}$ if and only if $\det(\mathrm{tr}(b_i b_j)) \neq 0$ in R. In particular, if $R = \mathbb{C}[x]$, then $\det(\mathrm{tr}(b_i b_j))$ is a polynomial. Since a polynomial has only a finite number of roots, $\det(\mathrm{tr}(b_i b_j))(n) = 0$ for only a finite number of values $n \in \mathbb{C}$.

Theorem 5.10. *(Tits deformation theorem) Let R be an integral domain, $\mathbb{F}$, the field of fractions of R, $\bar{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$, and $\bar{R}$, the integral closure of R in $\bar{\mathbb{F}}$. Let A_R be an R-algebra and let $\{b_1, \ldots, b_d\}$ be a basis of A_R. For $a \in A_R$ let $\vec{A}(a)$ denote the linear transformation of A_R induced by left multiplication by a. Let $t_1, \ldots, t_d$ be indeterminates and let*

$$\vec{p}(t_1, \ldots, t_d; x) = \det(x \cdot \mathrm{Id} - (t_1 \vec{A}(b_1) + \cdots t_d \vec{A}(b_d))) \in R[t_1, \ldots, t_d][x],$$

so that $\vec{p}$ is the characteristic polynomial of a "generic" element of A_R.

(a) Let $A_{\bar{\mathbb{F}}} = \bar{\mathbb{F}} \otimes_R A_R$. If

$$A_{\bar{\mathbb{F}}} \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\bar{\mathbb{F}}),$$

then the factorization of $\vec{p}(t_1, \ldots, t_d, x)$ into irreducibles in $\bar{\mathbb{F}}[t_1, \ldots, t_d, x]$ has the form

$$\vec{p} = \prod_{\lambda \in \hat{A}} (\vec{p}^\lambda)^{d_\lambda}, \qquad \textit{with} \qquad \vec{p}^\lambda \in \bar{R}[t_1, \ldots, t_d, x] \qquad \textit{and} \qquad d_\lambda = \deg(\vec{p}^\lambda).$$

If $\chi^\lambda(t_1, \ldots, t_d) \in \bar{R}[t_1, \ldots, t_d]$ is given by

$$\vec{p}^\lambda(t_1, \ldots, t_d, x) = x^{d_\lambda} - \chi^\lambda(t_1, \ldots, t_d) x^{d_\lambda - 1} + \cdots,$$

then
$$\chi^\lambda_{A_{\bar{\mathbb{F}}}}: \begin{array}{ccc} A_{\bar{\mathbb{F}}} & \longmapsto & \bar{\mathbb{F}} \\ \alpha_1 b_1 + \cdots + \alpha_d b_d & \longmapsto & \chi^\lambda(\alpha_1, \ldots, \alpha_d)\,, \end{array} \qquad \lambda \in \hat{A},$$
are the irreducible characters of $A_{\bar{\mathbb{F}}}$.

(b) Let $\mathbb{K}$ *be a field and let* $\bar{\mathbb{K}}$ *be the algebraic closure of* $\mathbb{K}$. *Let* $\gamma: R \to \mathbb{K}$ *be a ring homomorphism and let* $\bar{\gamma}: \bar{R} \to \bar{\mathbb{K}}$ *be the extension of* γ. *Let* $\chi^\lambda(t_1, \ldots, t_d) \in \bar{R}[t_1, \ldots, t_d]$ *be as in (a). If* $A_{\bar{\mathbb{K}}} = \bar{\mathbb{K}} \otimes_R A_R$ *is semisimple then*

$$A_{\bar{\mathbb{K}}} \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\bar{\mathbb{K}}), \qquad \text{and} \qquad \chi^\lambda_{A_{\bar{\mathbb{K}}}}: \begin{array}{ccc} A_{\bar{\mathbb{K}}} & \longmapsto & \bar{\mathbb{K}} \\ \alpha_1 b_1 + \cdots + \alpha_d b_d & \longmapsto & (\bar{\gamma}\chi^\lambda)(\alpha_1, \ldots, \alpha_d)\,, \end{array}$$

for $\lambda \in \hat{A}$, *are the irreducible characters of* $A_{\bar{\mathbb{K}}}$.

Proof. First note that if $\{b_1', \ldots, b_d'\}$ is another basis of A_R and the change of basis matrix $P = (P_{ij})$ is given by
$$b_i' = \sum_j P_{ij} b_j \qquad \text{then the transformation} \qquad t_i' = \sum_j P_{ij} t_j,$$
defines an isomorphism of polynomial rings $R[t_1, \ldots, t_d] \cong R[t_1', \ldots, t_d']$. Thus it follows that if the statements are true for one basis of A_R (or $A_{\bar{\mathbb{F}}}$) then they are true for every basis of A_R (resp. $A_{\bar{\mathbb{F}}}$).

(a) Using the decomposition of $A_{\bar{\mathbb{F}}}$ let $\{e^\mu_{ij}, \mu \in \hat{A}, 1 \le i, j \le d_\lambda\}$ be a basis of matrix units in $A_{\bar{\mathbb{F}}}$ and let t^μ_{ij} be corresponding variables. Then the decomposition of $A_{\bar{\mathbb{F}}}$ induces a factorization

$$\vec{p}(t^\mu_{ij}, x) = \prod_{\lambda \in \hat{A}} (\vec{p}^\lambda)^{d_\lambda}, \qquad \text{where} \qquad \vec{p}^\lambda(t^\mu_{ij}; x) = \det(x - \sum_{\mu,i,j} t^\mu_{ij} A^\lambda(e_{ij})). \tag{5.11}$$

The polynomial $\vec{p}^\lambda(t^\mu_{ij}; x)$ is irreducible since specializing the variables gives

$$\vec{p}^\lambda(t^\lambda_{j+1,j} = 1, t^\lambda_{1,n} = t, t^\mu_{i,j} = 0 \text{ otherwise}; x) = x^{d_\lambda} - t, \tag{5.12}$$

which is irreducible in $\bar{R}[t; x]$. This provides the factorization of $\vec{p}$ and establishes that $\deg(\vec{p}^\lambda) = d_\lambda$. By (5.11)
$$\vec{p}^\lambda(t^\mu_{ij}; x) = x^{d_\lambda} - \mathrm{Tr}(A^\lambda(\sum_{\mu,i,j} t^\mu_{ij} e^\mu_{ij}))x^{d_\lambda - 1} + \cdots.$$
which establishes the last statement.

Any root of $\vec{p}(t_1, \ldots, t_d, x)$ is an element of $\overline{R[t_1, \ldots, t_d]} = \bar{R}[t_1, \ldots, t_d]$. So any root of $\vec{p}^\lambda(t_1, \ldots, t_d, x)$ is an element of $\bar{R}[t_1, \ldots, t_d]$ and therefore the coefficients of $\vec{p}^\lambda(t_1, \ldots, t_d, x)$ (symmetric functions in the roots of $\vec{p}^\lambda$) are elements of $\bar{R}[t_1, \ldots, t_d]$.

(b) Taking the image of the equation (5.11) give a factorization of $\gamma(\vec{p})$,

$$\gamma(\vec{p}) = \prod_{\lambda \in \hat{A}} \gamma(\vec{p}^\lambda)^{d_\lambda}, \qquad \text{in } \bar{\mathbb{K}}[t_1, \ldots, t_d, x].$$

For the same reason as in (5.12) the factors $\gamma(\vec{p}^\lambda)$ are irreducible polynomials in $\bar{\mathbb{K}}[t_1, \ldots, t_d, x]$.

On the other hand, as in the proof of (a), the decomposition of $A_{\bar{\mathbb{K}}}$ induces a factorization of $\gamma(\vec{p})$ into irreducibles in $\bar{\mathbb{K}}[t_1, \ldots, t_d, x]$. These two factorizations must coincide, whence the result. ∎

Applying the Tits deformation theorem to the case where $R = \mathbb{C}[x]$ (so that $\mathbb{F} = \mathbb{C}(x)$) gives the following theorem. The statement in (a) is a consequence of Theorem 5.6 and the remark which follows Theorem 5.9.

Theorem 5.13. *Let $\mathbb{C}A(n)$ be a family of algebras defined by generators and relations such that the coefficients of the relations are polynomials in n. Assume that there is an $\alpha \in \mathbb{C}$ such that $\mathbb{C}A(\alpha)$ is semisimple. Let $\hat{A}$ be an index set for the irreducible $\mathbb{C}A(\alpha)$-modules $A^\lambda(\alpha)$. Then*

(a) $\mathbb{C}A(n)$ is semisimple for all but a finite number of $n \in \mathbb{C}$.

(b) If $n \in \mathbb{C}$ is such that $\mathbb{C}A(n)$ is semisimple then $\hat{A}$ is an index set for the simple $\mathbb{C}A(n)$-modules $A^\lambda(n)$ and $\dim(A^\lambda(n)) = \dim(A^\lambda(\alpha))$ for each $\lambda \in \hat{A}$.

(c) Let x be an indeterminate and let $\{b_1, \ldots, b_d\}$ be a basis of $\mathbb{C}[x]A(x)$. Then there are polynomials $\chi^\lambda(t_1, \ldots, t_d) \in \mathbb{C}[t_1, \ldots, t_d, x]$, $\lambda \in \hat{A}$, such that for every $n \in \mathbb{C}$ such that $\mathbb{C}A(n)$ is semisimple,

$$\begin{array}{rccc} \chi^\lambda_{A(n)}\colon & \mathbb{C}A(n) & \longrightarrow & \mathbb{C} \\ & \alpha_1 b_1 + \cdots + \alpha_d b_d & \longmapsto & \chi^\lambda(\alpha_1, \ldots, \alpha_d, n)\,, \end{array} \qquad \lambda \in \hat{A},$$

are the irreducible characters of $\mathbb{C}A(n)$.

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