Calibrated representations of affine Hecke algebras

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Abstract. This paper introduces the notion of calibrated representations for affine Hecke algebras and classifies and constructs all finite dimensional irreducible calibrated representations. The main results are that (1) irreducible calibrated representations are indexed by placed skew shapes, (2) the dimension of an irreducible calibrated representation is the number of standard Young tableaux corresponding to the placed skew shape and (3) each irreducible calibrated representation is constructed explicitly by formulas which describe the action of each generator of the affine Hecke algebra on a specific basis in the representation space. This construction is a generalization of A. Young's seminormal construction of the irreducible representations of the symmetric group. In this sense Young's construction has been generalized to arbitrary Lie type.

0. Introduction

The affine Hecke algebra was introduced by Iwahori and Matsumoto [IM] as a tool for studying the representations of a p-adic Lie group. In some sense, all irreducible principal series representations of the p-adic group can be determined by classifying the representations of the corresponding affine Hecke algebra. Unfortunately, it is not so easy to determine the irreducible representations of the affine Hecke algebra.

Kazhdan and Lusztig [KL] (see also the important work of Ginzburg [CG]) gave a geometric classification of the irreducible representations of the affine Hecke algebra. This classification is a q-analogue of Springer's construction of the irreducible representations of the Weyl group on the cohomology of unipotent varieties. In the q-case, K-theory takes the place of cohomology and the irreducible representations of the affine Hecke algebra are constructed as quotients of the K-theory of special subvarieties of the flag variety. Although the classification of Kazhdan and Lusztig is an incredible tour-de-force it is difficult to obtain combinatorial information from this geometric construction. For example, it is difficult to determine the dimensions of the irreducible modules.

In this paper I give a new construction of a large family of irreducible modules of the affine Hecke algebra. The basis vectors are labeled by generalized standard Young tableaux and the action of each generator on each basis element is given explicitly. This construction is a generalization of Young's seminormal construction of the irreducible representations of the symmetric group. In order to obtain this generalization I have had to generalize the concept of standard Young tableaux to arbitrary Lie type.

The modules which I construct I have termed "calibrated" modules. Specifically, a calibrated module is a module which has a basis of simultaneous eigenvectors for all the elements of a large

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commutative algebra inside the affine Hecke algebra. This is analogous to the situation which occurs for representations of complex semisimple Lie algebras where every finite dimensional module is a direct sum of its weight spaces. In contrast to the complex semisimple Lie algebra case, it is *never* true that all irreducible representations of the affine Hecke algebra are calibrated.

The irreducible really calibrated modules for the affine Hecke algebra are indexed by placed skew shapes, where "placed skew shape" is a generalization of the usual skew shape from combinatorial representation theory and symmetric function theory. As is to be expected, these new generalized skew shapes and standard Young tableaux reduce to the classical objects in the Type A case. This reduction is given in [Ra2].

Remarks on the results in this paper

(1) It is quite a surprise that the seminormal construction of A. Young fits so nicely into general Lie type. Up to now, the general feeling has been that Young's results are very special to the symmetric group and the Type A case. The direct generalization of Young's seminormal construction to arbitrary Lie type which is obtained in this paper shows that this is not the case at all. The results of [Ra4] indicate how Young's natural basis can also be generalized to arbitrary Lie type.

The seminormal representations of A. Young have been previously generalized to Iwahori-Hecke algebras of Type A by Hoefsmit [H] and Wenzl [Wz] independently, to Iwahori-Hecke algebras of types B and D by Hoefsmit [H] and to the cyclotomic Hecke algebras $H_{r,1,n}$ by Ariki and Koike [AK]. All of these generalizations use classical standard Young tableaux and similar formulas for the action of the generators of the Hecke algebra. Using certain surjective homomorphisms [A] from the affine Hecke algebras of type A to the algebras $H_{r,1,n}$ one can easily show that these earlier constructions are type A special cases of the general type construction given in this paper.

In a previous paper [Ra1] I gave a method for generalizing Young's theory of seminormal representations to general Lie type. I now believe that this earlier idea was not the "proper" way to proceed. The method here is much more natural and yields a cleaner and more beautiful theory. Young's classical formulas for the seminormal representations of the symmetric group S_n work in general Lie type with no change at all!! The only previously missing ingredient was a good general type definition of standard Young tableaux.

(2) In the classical theory of representations of the symmetric group S_n the "skew shape representations" are particularly well behaved S_n-modules. On the other hand there never seemed to be any a priori raison d'etre for skew shape representations via which one could generalize this concept to Weyl groups and Iwahori-Hecke algebras of other Lie types. The results in this paper show that skew shape representations do arise in a perfectly natural way. They correspond to *irreducible representations of affine Hecke algebras*. In the type A cases one recovers the classical skew shape representations by restricting to the Iwahori-Hecke algebra inside the affine Hecke algebra.

(3) The two main techniques used in this paper are generalizations of the techniques of Matsumoto [Ma] and Rodier [Ro]. In particular, the τ-operators and the calibration graphs $\Gamma(t)$ introduced in Section 2 are generalizations of the intertwining operators of Matsumoto and of the graphs used by Rodier, respectively. I have changed the role of the intertwining operators by having them be "left" operators instead of "right" operators. This means that they are no longer intertwining but there are other benefits to using these operators in this fashion.

(4) Heckman and Opdam [HO1-2] introduced a new "harmonic analysis" approach to the representations of the affine Hecke algebra. In their work they also used the sets $Z(t)$ and $P(t)$ which we use in section 3. These sets arise naturally in their work as the zeros and poles of a certain Harish-Chandra c-function. In this paper these sets describe the behaviour of the τ-

operators mentioned in remark (3). This means that there is a strong connection between the c-function and these operators. The approach of Heckman-Opdam becomes difficult when one needs to compute the residues of the c-function at certain singular points. In some cases these difficulties can be surmounted by using the methods of this paper.

(5) The Kazhdan-Lusztig construction of the irreducible representations of the affine Hecke algebras shows that the structure of these representations is intimately connected with the geometry of certain subvarieties $\mathcal{B}_{s,u}$ of the flag variety. It is my hope that the methods which I have used here for studying representations of affine Hecke algebras, which are mostly combinatorial in nature, will be useful for studying the geometry of the $\mathcal{B}_{s,u}$ varieties used in the Kazhdan-Lusztig construction. I am hoping that the standard Young tableaux introduced here can be used as index sets for the connected components of these varieties.

Acknowledgements

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1. The affine Hecke algebra

Let R be a reduced irreducible root system in $\mathbb{R}^n$, fix a set of positive roots R^+ and let $\{\alpha_1, \ldots, \alpha_n\}$ be the corresponding simple roots in R. Let W be the Weyl group corresponding to R. Let s_i denote the simple reflection in W corresponding to the simple root α_i and recall that W can be presented by generators $s_1, s_2, \ldots, s_n$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } 1 \le i \le n, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}, && \text{for } i \ne j, \end{aligned}$$

where $m_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$.

Fix $q \in \mathbb{C}^*$ such that q is not a root of unity. The *Iwahori-Hecke algebra H* is the associative algebra over $\mathbb{C}$ defined by generators $T_1, T_2, \ldots, T_n$ and relations

$$\begin{aligned} T_i^2 &= (q - q^{-1}) T_i + 1, && \text{for } 1 \le i \le n, \\ \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}, && \text{for } i \ne j, \end{aligned} \tag{1.1}$$

where m_{ij} are the same as in the presentation of W. For $w \in W$ define $T_w = T_{i_1} \cdots T_{i_p}$ where $s_{i_1} \cdots s_{i_p} = w$ is a reduced expression for w. By [Bou, Ch. IV §2 Ex. 23], the element T_w does not depend on the choice of the reduced expression. The algebra H has dimension $|W|$ and the set $\{T_w\}_{w \in W}$ is a basis of H.

The *fundamental weights* are the elements $\omega_1, \ldots, \omega_n$ of $\mathbb{R}^n$ given by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}, \qquad \text{where} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

and δ_{ij} is the Kronecker delta. The *weight lattice* is the W-invariant lattice in $\mathbb{R}^n$ given by

$$P = \sum_{i=1}^{n} \mathbb{Z}\omega_i.$$

Let X be the abelian group P except written multiplicatively. In other words,

$$X = \{X^\lambda \mid \lambda \in P\}, \quad \text{and} \quad X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda, \quad \text{for } \lambda, \mu \in P.$$

Let $\mathbb{C}[X]$ denote the group algebra of X. There is a W-action on X given by

$$wX^\lambda = X^{w\lambda} \qquad \text{for } w \in W,\ X^\lambda \in X,$$

which we extend linearly to a W-action on $\mathbb{C}[X]$.

The *affine Hecke algebra* $\tilde{H}$ associated to R and P is the algebra given by

$$\tilde{H} = \mathbb{C}\text{-span}\{T_w X^\lambda \mid w \in W, X^\lambda \in X\}$$

where the multiplication of the T_w is as in the Iwahori-Hecke algebra H, the multiplication of the X^λ is as in $\mathbb{C}[X]$ and we impose the relation

$$X^\lambda T_i = T_i X^{s_i\lambda} + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}, \qquad \text{for } 1 \le i \le n \text{ and } X^\lambda \in X. \tag{1.2}$$

This formulation of the definition of $\tilde{H}$ is due to Lusztig [Lu2] following work of Bernstein and Zelevinsky. The elements $T_w X^\lambda$, $w \in W$, $X^\lambda \in X$, form a basis of $\tilde{H}$.

Theorem 1.3. *(Bernstein, Zelevinsky, Lusztig [Lu1, 8.1]) The center of $\tilde{H}$ is $\mathbb{C}[X]^W = \{f \in \mathbb{C}[X] \mid wf = f\}$.*

2. Weight spaces and calibration graphs

Weights

Let

$$T = \{\text{group homomorphisms } t\colon X \to \mathbb{C}^*\}.$$

The *torus* T is an abelian group with a W-action given by $(wt)(X^\lambda) = t(X^{w^{-1}\lambda})$. Any element $t \in T$ is determined by the values $t(X^{\omega_1}), t(X^{\omega_2}), \ldots, t(X^{\omega_n})$. For any element $t \in T$ define the *polar decomposition*

$$t = t_r t_c, \qquad t_r, t_c \in T \text{ such that } t_r(X^\lambda) \in \mathbb{R}_{>0}, \text{ and } |t_c(X^\lambda)| = 1,$$

for all $X^\lambda \in X$. There is a unique $\mu \in \mathbb{R}^n$ and a unique $\nu \in \mathbb{R}^n/P$ such that

$$t_r(X^\lambda) = e^{\langle \mu,\lambda\rangle} \qquad \text{and} \qquad t_c(X^\lambda) = e^{2\pi i\langle \nu,\lambda\rangle}, \qquad \text{for all } \lambda \in P. \tag{2.1}$$

In this way we identify the sets $T_r = \{t \in T \mid t = t_r\}$ and $T_c = \{t \in T \mid t = t_c\}$ with $\mathbb{R}^n$ and $\mathbb{R}^n/P$, respectively.

Weight spaces

Let M be a finite dimensional $\tilde{H}$-module. For each $t \in T$ the *t-weight space of* M and the *generalized t-weight space* are the subspaces

$$M_t = \{m \in M \mid X^\lambda m = t(X^\lambda)m \text{ for all } X^\lambda \in X\} \qquad \text{and}$$

$$M_t^{\mathrm{gen}} = \{m \in M \mid \text{for each } X^\lambda \in X,\ (X^\lambda - t(X^\lambda))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\},$$

respectively. If $M_t^{\mathrm{gen}} \neq 0$ then $M_t \neq 0$. In general $M \neq \bigoplus_{t\in T} M_t$, but we do have

$$M = \bigoplus_{t\in T} M_t^{\mathrm{gen}}.$$

This is a decomposition of M into Jordan blocks for the action of $\mathbb{C}[X]$. Define the *support* of M to be

$$\mathrm{supp}(M) = \{t \in T \mid M_t^{\mathrm{gen}} \neq 0\}. \tag{2.2}$$

Principal series modules

Let $t \in T$ and let $\mathbb{C}v_t$ be the one dimensional $\mathbb{C}[X]$-module corresponding to the character $t\colon X \to \mathbb{C}^*$. Specifically, $\mathbb{C}v_t$ is the one dimensional vector space with basis $\{v_t\}$ and $\mathbb{C}[X]$-action given by

$$X^\lambda v_t = t(X^\lambda)v_t, \qquad \text{for all } X^\lambda \in X.$$

The *principal series representation* corresponding to t is

$$M(t) = \tilde{H} \otimes_{\mathbb{C}[X]} \mathbb{C}v_t. \tag{2.3}$$

The set $\{T_w \otimes v_t \mid w \in W\}$ is a basis for the $\tilde{H}$-module $M(t)$ and $\dim(M(t)) = |W|$.

If $w \in W$ and $X^\lambda \in X$ then the defining relation (1.2) for $\tilde{H}$ implies that

$$X^\lambda(T_w \otimes v_t) = t(X^{w\lambda})(T_w \otimes v_t) + \sum_{u<w} a_u(T_u \otimes v_t),$$

where the sum is over $u < w$ in the Bruhat-Chevalley order and $a_u \in \mathbb{C}$. It follows that the eigenvalues of X on $M(t)$ are of the form wt, $w \in W$, and by counting the multiplicity of each eigenvalue we have

$$M(t) = \bigoplus_{wt\in Wt} M(t)_{wt}^{\mathrm{gen}} \qquad \text{where} \quad \dim(M(t)_{wt}^{\mathrm{gen}}) = |W_t|, \quad \text{for all } w \in W. \tag{2.4}$$

Theorem 2.5. *(Kato's irreducibility criterion [Ka]) Let $t \in T$ and define $P(t) = \{\alpha > 0 \mid t(X^\alpha) = q^{\pm 2}\}$. The principal series module $M(t)$ is irreducible if and only if $P(t) = \emptyset$.*

Remark. Kato actually proves a more general result and thus needs a further condition for irreducibility. We have simplified matters by specifying the weight lattice P in our construction of the affine Hecke algebra. One can use any W-invariant lattice in $\mathbb{R}^n$ and Kato works in this more general situation. When the one uses the weight lattice P, a result of Steinberg [St, 4.2, 5.3] says that the stabilizer W_t of a point $t \in T$ under the action of W is always a reflection group. Because of this Kato's criterion takes a simpler form.

Irreducible modules

Proposition 2.6. *Let M be a finite dimensional $\tilde{H}$-module.*

(a) For some $t \in T$, M_t is nonzero.

(b) If M is irreducible and $M_t \neq 0$ then M is a quotient of $M(t)$.

(c) If M is irreducible then $\dim(M) \leq |W|$.

Proof. (a) As an $X(T)$-module M contains a simple submodule and this submodule must be one-dimensional since all irreducible representations of a commutative algebra are one-dimensional. Thus, there is a nonzero weight vector in M.

(b) Let m_t be a nonzero vector in M_t. Then there is a unique $\tilde{H}$-module homomorphism determined by

$$\begin{array}{cccc} \phi\colon & M(t) & \longrightarrow & M \\ & v_t & \longmapsto & m_t \end{array}$$

where v_t is as in the construction of $M(t)$ in (2.3) This map is surjective since M is irreducible. Thus M is a quotient of $M(t)$.

(c) follows from (b) since $\dim(M(t)) = |W|$. ∎

It follows from Proposition (2.6b) and (2.4) that the support $\mathrm{supp}(M)$ of an irreducible $\tilde{H}$-module M is contained in a single Weyl group orbit in T. Since M is irreducible and $\tilde{H}$ has countable dimension, Dixmier's version of Schur's lemma implies that $Z(\tilde{H})$ acts on M by scalars. Let $t \in T$ be such that

$$pM = t(p)M, \qquad \text{for all } p \in Z(\tilde{H}).$$

Since $Z(\tilde{H}) = \mathbb{C}[X(T)]^W$ it follows that $t(p(X)) = (wt)(p(X))$ for all $w \in W$. The W-orbit Wt of t is the *central character* of M. We shall often abuse notation and refer to any weight $s \in Wt$ as "the central character" of M.

Remarks.

(a) The algebra $\mathbb{C}[X]^W$ is the polynomial ring

$$\mathbb{C}[X]^W = \mathbb{C}[\chi^{\omega_1}, \ldots, \chi^{\omega_n}],$$

where $\omega_1, \ldots, \omega_n$ are the fundamental weights in P and $\chi^{\omega_1}, \ldots, \chi^{\omega_n}$, are the corresponding Weyl characters. Thus, in order to specify the W-orbit of a weight $t \in T$ it is sufficient to specify the n complex numbers $t(\chi^{\omega_1}), t(\chi^{\omega_2}), \ldots, t(\chi^{\omega_n})$.

(b) If $t \in T$ is real, i.e. $t = t_r$, then every element of Wt is also real. Then there is a unique dominant $\gamma \in \mathbb{R}^n$ such that the element $s \in T$ given by

$$s(X^\lambda) = e^{\langle \gamma, \lambda\rangle}, \qquad \text{for all } \lambda \in P,$$

is in Wt and this element is a canonical representative of the W-orbit Wt. In this way the dominant elements of $\mathbb{R}^n$ index the real central characters.

The τ operators

The maps $\tau_i \colon M_t^{\mathrm{gen}} \to M_{s_it}^{\mathrm{gen}}$ defined below are local operators on M in the sense that they act on each weight space M_t^{gen} of M separately. The operator τ_i is only defined on weight spaces M_t^{gen} such that $t(X^{\alpha_i}) \neq 1$.

Proposition 2.7. *Let $t \in T$ such that $t(X^{\alpha_i}) \neq 1$ and let M be a finite dimensional $\tilde{H}$-module. Define*

$$\begin{array}{rccc} \tau_i\colon & M_t^{\mathrm{gen}} & \longrightarrow & M_{s_it}^{\mathrm{gen}} \\ & m & \longmapsto & \left(T_i - \dfrac{q-q^{-1}}{1-X^{-\alpha_i}}\right) m. \end{array}$$

(a) The map $\tau_i \colon M_t^{\mathrm{gen}} \longrightarrow M_{s_it}^{\mathrm{gen}}$ is well defined.

(b) As operators on M_t^{gen}, $\quad X^\lambda \tau_i = \tau_i X^{s_i\lambda}$, for all $X^\lambda \in X$.

(c) As operators on M_t^{gen}, $\quad \tau_i\tau_i = \dfrac{(q-q^{-1}X^{\alpha_i})(q-q^{-1}X^{-\alpha_i})}{(1-X^{\alpha_i})(1-X^{-\alpha_i})}$.

(d) Both maps $\tau_i \colon M_t^{\mathrm{gen}} \to M_{s_it}^{\mathrm{gen}}$ and $\tau_i \colon M_{s_it}^{\mathrm{gen}} \to M_t^{\mathrm{gen}}$ are invertible if and only if $t(X^{\alpha_i}) \neq q^{\pm 2}$.

(e) Let $1 \le i \neq j \le n$ and let m_{ij} be as in (1.1). Then

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}\ \text{factors}} = \underbrace{\tau_j\tau_i\tau_i\cdots}_{m_{ij}\ \text{factors}},$$

whenever both sides are well defined operators on M_t^{gen}.

Proof. (a) Note that $(q-q^{-1})/(1-X^{-\alpha_i})$ is not a well defined element of $\tilde{H}$ or $\mathbb{C}[X]$ since it is not a polynomial in $X^{-\alpha_i}$. Because of this we will be careful to view $(q-q^{-1})/(1-X^{-\alpha_i})$ only as a local operator. Let us describe this operator more precisely.

The element X^{α_i} acts on M_t^{gen} by $t(X^{\alpha_i})$ times a unipotent transformation. As an operator on M_t^{gen}, $1-X^{-\alpha_i}$ is invertible since it has determinant $(1-t(X^{-\alpha_i}))^d$ where $d = \dim(M_t^{\mathrm{gen}})$. Since this determinant is nonzero $(q-q^{-1})/(1-X^{-\alpha_i}) = (q-q^{-1})(1-X^{-\alpha_i})^{-1}$ is a well defined operator on M_t^{gen}. Thus the definition of of τ_i makes sense.

The following calculation shows that τ_i maps M_t^{gen} into $M_{s_it}^{\mathrm{gen}}$. For the purposes of this calculation we are viewing all elements of $\mathbb{C}[X]$ as local operators on M_t^{gen} and we abuse notation and denote the operator $(1-X^{-\alpha_i})^{-1}$ by $1/(1-X^{-\alpha_i})$. We are able to do this without any problems because $\mathbb{C}[X]$ is commutative.

$$\begin{aligned} X^\lambda\tau_i m &= \left(T_iX^{s_i\lambda} + (q-q^{-1})\frac{X^\lambda - X^{s_i\lambda}}{1-X^{-\alpha_i}} - X^\lambda\frac{q-q^{-1}}{1-X^{-\alpha_i}}\right)m \\ &= \left(T_iX^{s_i\lambda} - (q-q^{-1})\frac{X^{s_i\lambda}}{1-X^{-\alpha_i}}\right)m \\ &= \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right)X^{s_i\lambda}m \\ &= \tau_iX^{s_i\lambda}m. \end{aligned}$$

(b) follows from the previous calculation.

(c) If $t(X^{-\alpha_i}) \neq 1$ then both $\tau_i\colon M_t^{\mathrm{gen}} \to M_{s_it}^{\mathrm{gen}}$ and $\tau_i\colon M_{s_it}^{\mathrm{gen}} \to M_t^{\mathrm{gen}}$ are well defined. If $m \in M_t^{\mathrm{gen}}$ then

$$\begin{aligned}
\tau_i\tau_i m &= \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right) m \\
&= \left(T_i^2 - \frac{q-q^{-1}}{1-X^{-\alpha_i}}T_i - T_i\frac{q-q^{-1}}{1-X^{-\alpha_i}} + \frac{(q-q^{-1})^2}{(1-X^{-\alpha_i})^2}\right) m \\
&= \Bigg((q-q^{-1})T_i + 1 - \frac{q-q^{-1}}{1-X^{-\alpha_i}}T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}T_i \\
&\qquad -(q-q^{-1})^2\frac{(1-X^{-\alpha_i})^{-1}-(1-X^{\alpha_i})^{-1}}{1-X^{-\alpha_i}} + \frac{(q-q^{-1})^2}{(1-X^{-\alpha_i})^2}\Bigg) m \\
&= \left((q-q^{-1})T_i + 1 - (q-q^{-1})T_i + (q-q^{-1})^2\frac{-1-X^{-\alpha_i}+1}{(1-X^{-\alpha_i})^2}\right) m \\
&= \left(1 + \frac{(q-q^{-1})^2}{(1-X^{-\alpha_i})(1-X^{\alpha_i})}\right) m \\
&= \frac{2-X^{\alpha_i}-X^{-\alpha_i}+(q^2-2+q^{-2})}{(1-X^{\alpha_i})(1-X^{-\alpha_i})} m \\
&= \frac{(q-q^{-1}X^{\alpha_i})(q-q^{-1}X^{-\alpha_i})}{(1-X^{\alpha_i})(1-X^{-\alpha_i})} m.
\end{aligned}$$

(d) The operator X^{α_i} acts on M_t^{gen} as $t(X^{\alpha_i})$ times a unipotent transformation. Similarly for $X^{-\alpha_i}$. Thus, as an operator on M_t^{gen} $\det((q-q^{-1}X^{\alpha_i})(q-q^{-1}X^{-\alpha_i})) = 0$ if and only if $t(X^{\alpha_i}) = q^{\pm 2}$. Thus part (c) implies that $\tau_i\tau_i$ is invertible if and only if $t(X^{\alpha_i}) \neq q^{\pm 2}$. The statement follows.

(e) Let us begin with a slight diversion which will be helpful in the proof. Let $t \in T$ be a generic element of T and let $M(t)$ be the corresponding principal series module. Since t is generic, $W_t = \{1\}$ and

$$M(t) = \bigoplus_{w\in W} M(t)_{wt}, \qquad \text{and} \qquad \dim(M(t)_{wt}) = 1,$$

for all $w \in W$. We have $M(t)_{wt}^{\mathrm{gen}} = M(t)_{wt}$ since $M(t)_{wt}$ is nonzero whenever $M(t)_{wt}^{\mathrm{gen}}$ is nonzero and we know that $\dim(M(t)_{wt}^{\mathrm{gen}}) = 1$. Let $w \in W$ such that $\ell(s_iw) = \ell(w)+1$. Since t is generic $(wt)(X^{\alpha_i}) \neq q^{\pm 2}$ for all $w \in W$. Thus by part (d), the map $\tau_i\colon M(t)_{wt}^{\mathrm{gen}} \to M(t)_{s_iwt}^{\mathrm{gen}}$ is a bijection. Using the vector $v_t \in M(t)_t$ and the maps τ_i we can construct a basis $\{v_{wt}\}_{w\in W}$ of $M(t)$ given by

$$v_{s_iwt} = \tau_i v_{wt}, \qquad \text{if } \ell(s_iw) = \ell(w)+1.$$

This basis is uniquely determined by the conditions

(2.8a) $\quad X^\lambda v_{wt} = (wt)(X^\lambda)v_{wt}, \qquad$ for all $w \in W$ and $X^\lambda \in X(T)$,

(2.8b) $\quad v_{wt} = T_w \otimes v_t + \sum_{u<w} a_{wu}(t)(T_u \otimes v_t), \qquad$ where $a_{wu}(t) \in \mathbb{C}$.

Now we proceed to the proof of the statement. We may assume that $\tilde{H}$ is the affine Hecke algebra corresponding to a rank two root system R generated by simple roots α_i and α_j. Let w_0 be the longest element of W. Every element $w \in W$, $w \neq w_0$ has unique minimal length expression as a product of generators s_i and s_j. Let T_w be the corresponding product of the T_i's and T_j's. Using the defining relation (1.2) for $\tilde{H}$ we expand to derive

$$\underbrace{\cdots \left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)\left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)}_{m_{ij}\ \text{factors}} = \underbrace{\cdots T_iT_jT_i}_{m_{ij}\ \text{factors}} + \sum_{w<w_0} T_wP_w, \tag{2.9}$$

where the sum is over $w \in W$ such that $w \neq w_0$ and P_w are rational functions of the X^α, $\alpha \in R$. Similarly,

$$\underbrace{\cdots \left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)\left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)}_{m_{ij}\ \text{factors}} = \underbrace{\cdots T_jT_iT_j}_{m_{ij}\ \text{factors}} + \sum_{w<w_0} T_wQ_w, \tag{2.10}$$

where, as before, the sum is over $w \in W$ such that $w \neq w_0$ and Q_w are rational functions of the X^α, $\alpha \in R$. We shall show that $P_w = Q_w$.

Let $t \in T$ be generic and let $M(t)$ be the corresponding principal series module for $\tilde{H}$. By the analysis in the previous paragraph we have

$$\begin{aligned} v_{w_0t} = \underbrace{\cdots \tau_i\tau_j\tau_i}_{m_{ij}\ \text{factors}} v_t &= \underbrace{\cdots T_iT_jT_i}_{m_{ij}\ \text{factors}} v_t + \sum_{w<w_0} T_wP_wv_t \\ &= T_{w_0} \otimes v_t + \sum_{w<w_0} t(P_w)T_w \otimes v_t. \end{aligned}$$

and it follows from (2.8b) that $t(P_w) = a_{w_0w}(t)$ for all $w \in W$, $w \neq w_0$. One shows similarly that $t(Q_w) = a_{w_0w}(t)$ for all $w \in W$, $w \neq w_0$.

We have shown that, for each $w \in W$, $t(P_w) = t(Q_w)$ for all generic $t \in T$. Since P_w and Q_w are rational functions which coincide on all generic points it follows that

$$P_w = Q_w \qquad \text{for all } w \in W,\ w \neq w_0. \tag{2.11}$$

Thus,

$$\begin{aligned} \underbrace{\cdots \tau_i\tau_j\tau_i}_{m_{ij}\ \text{factors}} &= \underbrace{\cdots \left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)\left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)}_{m_{ij}\ \text{factors}} \\ &= \underbrace{\cdots \left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)\left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)}_{m_{ij}\ \text{factors}} = \underbrace{\cdots \tau_j\tau_i\tau_j}_{m_{ij}\ \text{factors}}, \end{aligned}$$

whenever both sides are well defined operators on M_t^{gen}. ∎

The calibration graph

Let $t \in T$. Define a graph $\Gamma(t)$ with

$$\begin{array}{rccc} \text{Vertices:} & Wt, & & \\ \text{Edges:} & wt \longleftrightarrow s_i wt, & \text{if} & (wt)(X^{\alpha_i}) \neq q^{\pm 2}. \end{array}$$

Proposition 2.12. *If M is a finite dimensional $\tilde{H}$-module then*

$$\dim(M_t^{\mathrm{gen}}) = \dim(M_{t'}^{\mathrm{gen}})$$

if t and t' are in the same connected component of the calibration graph.

Proof. It follows from Proposition (2.7d) that if there is an edge $wt \leftrightarrow s_i wt$ in $\Gamma(t)$ then the map $\tau_i \colon M_{wt}^{\mathrm{gen}} \to M_{s_i wt}^{\mathrm{gen}}$ is a bijection. Thus, $\dim(M_{wt}^{\mathrm{gen}}) = \dim(M_{s_i wt}^{\mathrm{gen}})$ if t and $s_i t$ are connected in the calibration graph $\Gamma(t)$. ∎

Corollary 2.13. *If M is an irreducible $\tilde{H}$-module with central character t then the support* $\mathrm{supp}(M)$ *is a union of connected components of the calibration graph $\Gamma(t)$.*

The connected components of $\Gamma(t)$

Let $t \in T$ and define

$$Z(t) = \{\alpha > 0 \mid t(X^\alpha) = 1\}, \qquad \text{and} \qquad P(t) = \{\alpha > 0 \mid t(X^\alpha) = q^{\pm 2}\}.$$

If $J \subseteq P(t)$ define

$$\mathcal{F}^{(t,J)} = \{w \in W \mid R(w) \cap Z(t) = \emptyset, \quad R(w) \cap P(t) = J\},$$

where $R(w) = \{\alpha > 0 \mid w\alpha < 0\}$ is the inversion set of w. Define a placed shape to be a pair (t, J) such that $t \in T$, $J \subseteq P(t)$ and $\mathcal{F}^{(t,J)} \neq \emptyset$. The elements of the set $\mathcal{F}^{(t,J)}$ are called standard tableaux of shape (t, J).

Theorem 2.14. *The connected components of the calibration graph $\Gamma(t)$ are given by the partition of the vertices according to the sets*

$$\mathcal{F}^{(t,J)} t, \qquad \textit{such that } J \subseteq P(t) \textit{ and } \mathcal{F}^{(t,J)} \neq \emptyset.$$

Proof. Let us begin by introducing appropriate notation. The chamber

$$C = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in R^+\}$$

is a fundamental chamber for the action of W on $\mathbb{R}^n$ and the complement $\mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha)$ of the hyperplanes

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = 0\}, \qquad \alpha \in R^+,$$

in $\mathbb{R}^n$ is the disjoint union of the chambers $\{w^{-1}C \mid w \in W\}$. A chamber $w^{-1}C$ is on the *positive side* of the hyperplane H_α if $\langle x, \alpha\rangle > 0$ for all $x \in w^{-1}C$. The chambers adjacent to $w^{-1}C$ are the chambers $w^{-1}s_iC$, $1 \le i \le n$, and the common face of $w^{-1}C$ and $w^{-1}s_iC$ is contained in the hyperplane $H_{w^{-1}\alpha_i}$.

Now let t be as in the statement of the Theorem. A result of Steinberg [St, 3.15, 4.2, 5.3] says that the stabilizer of t is

$$W_t = \langle s_\alpha \mid \alpha \in Z(t)\rangle,$$

the subgroup of W generated by the reflections in the hyperplanes orthogonal to the roots in $Z(t)$. The elements of the orbit Wt can be identified with the cosets in W/W_t and these can be identified with the chambers of $\mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha)$ which are on the positive side of the hyperplanes H_α, $\alpha \in Z(t)$. Under this bijection the element $wt \in Wt$ is identified with the chamber $w^{-1}C$. The elements wt and s_iwt are not connected by an edge in $\Gamma(t)$ if and only if the hyperplane $H_{w^{-1}\alpha_i}$ containing the common face of the corresponding (adjacent) chambers $w^{-1}C$ and $w^{-1}s_iC$ is the hyperplane H_β for some root $\beta \in P(t)$. In this way we can identify the graph $\Gamma(t)$ with the graph with

Vertices: chambers of $\mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha)$ which are on the positive side of the hyperplanes H_α, $\alpha \in Z(t)$,
Edges: faces of the chambers which are not contained in the hyperplanes H_β, $\beta \in P(t)$.

The (closures of the) chambers $w^{-1}C$ which are on the positive side of the hyperplanes H_α, $\alpha \in Z(t)$, form a convex region in $\mathbb{R}^n$. This region is a disjoint union of smaller convex regions bounded by the hyperplanes H_β, $\beta \in P(t)$. Each of the smaller regions is on the positive side of some of the hyperplanes H_β, $\beta \in P(t)$, and on the negative side of others. In fact, it is determined by the set $J \subseteq P(t)$ such that it is on negative side of the hyperplanes H_β, $\beta \in J$. These smaller convex regions correspond to the connected components of $\Gamma(t)$ and thus the connected components of $\Gamma(t)$ are given by the sets $\{wt \mid w \in \mathcal{F}^{(t,J)}\}$ where $J \subseteq P(t)$ and

$$\mathcal{F}^{(t,J)} = \left\{ w \in W \;\middle|\; \begin{array}{r} w^{-1}C \text{ is on the positive side of the hyperplanes } H_\alpha,\ \alpha \in Z(t), \\ \text{on the positive side of the hyperplanes } H_\alpha,\ \alpha \in P(t)\setminus J, \\ \text{on the negative side of the hyperplanes } H_\beta,\ \beta \in J \end{array} \right\}.$$

Since the chamber $w^{-1}C$ is on the positive side of a hyperplane H_α if and only if $w\alpha > 0$ it follows that

$$\mathcal{F}^{(t,J)} = \{w \in W \mid R(w) \cap Z(t) = \emptyset, \quad R(w) \cap P(t) = J\}. \qquad \blacksquare$$

3. An $\tilde{H}$-module construction

Let α_i and α_j be simple roots in R and let R_{ij} be the rank two root subsystem of R which is generated by α_i and α_j. Let W_{ij} be the Weyl group of R_{ij}, the subgroup of W generated by the simple reflections s_i and s_j. A weight $t \in T$ is *calibratable* for R_{ij} if one of the following two conditions holds:

(a) $t(X^\alpha) \ne 1$ for all $\alpha \in R_{ij}$,

(b) R_{ij} is of type C_2 or G_2 (we may assume that α_i is the long root and α_j is the short root), $ut(X^{\alpha_i}) = q^2$ and $ut(X^{\alpha_j}) = 1$ for some $u \in W_{ij}$, and $t(X^{\alpha_i}) \ne 1$ and $t(X^{\alpha_j}) \ne 1$.

A *placed skew shape* is a placed shape (t, J) such that for all $w \in \mathcal{F}^{(t,J)}$ and all pairs α_i, α_j of simple roots in R the weight wt is calibratable for R_{ij}.

Theorem 3.1. *Let (t, J) be a placed skew shape and let $\mathcal{F}^{(t,J)}$ be the set of standard tableaux of shape (t, J). Define*

$$\tilde{H}^{(t,J)} = \mathbb{C}\text{-span}\{v_w \mid w \in \mathcal{F}^{(t,J)}\},$$

so that the symbols v_w are a labeled basis of the vector space $\tilde{H}^{(t,J)}$. Then the following formulas make $\tilde{H}^{(t,J)}$ into an irreducible $\tilde{H}$-module: For each $w \in \mathcal{F}^{(t,J)}$,

$$\begin{aligned} X^\lambda v_w &= (wt)(X^\lambda) v_w, && \text{for } X^\lambda \in X, \text{ and} \\ T_i v_w &= (T_i)_{ww} v_w + (q^{-1} + (T_i)_{ww}) v_{s_i w}, && \text{for } 1 \le i \le n, \end{aligned}$$

where $(T_i)_{ww} = \dfrac{q - q^{-1}}{1 - (wt)(X^{-\alpha_i})}$ *, and we set $v_{s_i w} = 0$ if $s_i w \notin \mathcal{F}^{(t,J)}$.*

Proof. Since (t, J) is a placed skew shape $(wt)(X^{-\alpha_i}) \neq 1$ for all $w \in \mathcal{F}^{(t,J)}$ and all simple roots α_i. This implies that the coefficient $(T_i)_{ww}$ is well defined for all i and $w \in \mathcal{F}^{(t,J)}$.

By construction, the nonzero weight spaces of $\tilde{H}^{(t,J)}$ are $(\tilde{H}^{(t,J)})^{\mathrm{gen}}_{wt} = (\tilde{H}^{(t,J)})_{wt}$ where $w \in \mathcal{F}^{(t,J)}$. These weight spaces have dimension 1 and are all in a single connected component of the calibration graph $\Gamma(t)$. If N is a proper submodule of $\tilde{H}^{(t,J)}$ then we would have $N_{wt} \neq 0$ and $N_{w't} = 0$ for some $w \neq w'$, $w, w' \in \mathcal{F}^{(t,J)}$. But since wt and $w't$ are in the same connected component of $\Gamma(t)$ this would contradict Proposition 2.12. Thus $\tilde{H}^{(t,J)}$ is irreducible if it is an $\tilde{H}$-module.

It remains to show that the defining relations for $\tilde{H}$ are satisfied.

(a) Let $w \in \mathcal{F}^{(t,J)}$. Then

$$\begin{aligned} &\left(X^{s_i\lambda} T_i + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}\right) v_w \\ &\qquad = \left((wt)(X^{s_i\lambda}) \frac{(q - q^{-1})}{1 - (wt)(X^{-\alpha_i})} + (q - q^{-1}) \frac{(wt)(X^\lambda) - (wt)(X^{s_i\lambda})}{1 - (wt)(X^{-\alpha_i})}\right) v_w \\ &\qquad\qquad + (s_i wt)(X^{s_i\lambda})(T_i)_{s_i w, w} v_{s_i w} \\ &\qquad = \frac{(q - q^{-1})}{1 - (wt)(X^{-\alpha_i})}(wt)(X^\lambda) v_w + (T_i)_{s_i w, w}(wt)(X^\lambda) v_{s_i w} \\ &\qquad = T_i X^\lambda v_w. \end{aligned}$$

(b) Let $w \in \mathcal{F}^{(t,J)}$. Using the fact that $(T_i)_{ww} + (T_i)_{s_i w, s_i w} = q - q^{-1}$ we have

$$\begin{aligned} T_i^2 v_w &= ((T_i)_{ww}^2 + (q^{-1} + (T_i)_{ww})(q^{-1} + (T_i)_{s_i w, s_i w})) v_w \\ &\qquad + (q^{-1} + (T_i)_{ww})((T_i)_{ww} + (T_i)_{s_i w, s_i w}) v_{s_i w} \\ &= (T_i)_{ww}((T_i)_{ww} + (T_i)_{s_i w, s_i w}) v_w + q^{-1}(q^{-1} + (T_i)_{ww} + (T_i)_{s_i w, s_i w})) v_w \\ &\qquad + (q^{-1} + (T_i)_{ww})(q - q^{-1}) v_{s_i w} \\ &= (T_i)_{ww}(q - q^{-1}) v_w + (q^{-1} + (T_i)_{ww})(q - q^{-1}) v_{s_i w} + q^{-1}(q^{-1} + q - q^{-1}) v_w \\ &= ((q - q^{-1}) T_i + 1) v_w. \end{aligned}$$

(c) The braid relation. Let α_i and α_j be simple roots in R and let $w \in \mathcal{F}^{(t,J)}$. Since (t,J) is a placed skew shape wt is calibratable. There are two distinct cases to consider.

Case 1: When wt is R_{ij}-regular, i.e. $(uwt)(X^{\alpha_i}) \neq 1$ and $(uwt)(X^{\alpha_j}) \neq 1$ for all $u \in W_{ij}$.

Let us extend our notation v_σ to all $\sigma \in W$ by assuming that $v_\sigma = 0$ whenever $\sigma \notin \mathcal{F}^{(t,J)}$. Then, for all $u \in W_{ij}$, the definition of the action of T_i allows us to write

$$\begin{aligned}\left(T_i - \frac{(q-q^{-1})}{1-(wt)(X^{-\alpha_i})}\right) v_{uw} &= \left(q^{-1} + \frac{(q-q^{-1})}{1-(wt)(X^{-\alpha_i})}\right) v_{s_i uw} \\ &= \left(\frac{q-q^{-1}(wt)(X^{-\alpha_i})}{1-(wt)(X^{-\alpha_i})}\right) v_{s_i uw}\end{aligned}$$

whenever $\ell(s_i u) > \ell(u)$ and $(uwt)(X^{-\alpha_i}) \neq 1$. (This is correct because if $\ell(s_i u) > \ell(u)$ and $v_{uw} = 0$ then $v_{s_i uw} = 0$.)

Since wt is R_{ij}-regular all of the factors in the following product are well defined and, by [Bou, Ch. VI §1 Cor. 2 to Prop. 17],

$$\begin{aligned}&\prod_{\alpha \in R_{ij}^+} \frac{q-q^{-1}(wt)(X^{-\alpha})}{1-(wt)(X^{-\alpha})} \\ &= \underbrace{\cdots \left(\frac{q-q^{-1}(s_j s_i wt)(X^{-\alpha_i})}{1-(s_j s_i wt)(X^{-\alpha_i})}\right)\left(\frac{q-q^{-1}(s_i wt)(X^{-\alpha_i})}{1-(s_i wt)(X^{-\alpha_i})}\right)\left(\frac{q-q^{-1}(wt)(X^{-\alpha_i})}{1-(wt)(X^{-\alpha_i})}\right)}_{m_{ij} \text{ factors}}\end{aligned}$$

where the product is over all positive roots in the root subsystem R_{ij} spanned by α_i and α_j. Thus we get that

$$\begin{aligned}&\left(\prod_{\alpha \in R_{ij}^+} \frac{q-q^{-1}(wt)(X^{-\alpha})}{1-(wt)(X^{-\alpha})}\right) v_{w_0 w} \\ &= \underbrace{\cdots \left(T_i - \frac{q-q^{-1}}{1-(s_j s_i wt)(X^{-\alpha_i})}\right)\left(T_j - \frac{q-q^{-1}}{1-(s_i wt)(X^{-\alpha_i})}\right)\left(T_i - \frac{q-q^{-1}}{1-(wt)(X^{-\alpha_i})}\right)}_{m_{ij} \text{ factors}} v_w \\ &= \underbrace{\cdots \left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)\left(T_j - \frac{q-q^{-1}}{1-X^{\alpha_j}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{\alpha_i}}\right)}_{m_{ij} \text{ factors}} v_w \\ &= \underbrace{\cdots T_i T_j T_i}_{m_{ij} \text{ factors}} v_w + \sum_{u<u_0} T_u P_u v_w,\end{aligned}$$

where the notation in the last line is exactly the same as in (2.9). By similar reasoning we obtain

$$\left(\prod_{\alpha \in R_{ij}^+} \frac{q-q^{-1}(wt)(X^{-\alpha})}{1-(wt)(X^{-\alpha})}\right) v_{w_0 w} = \underbrace{\cdots T_j T_i T_j}_{m_{ij} \text{ factors}} v_w + \sum_{u<u_0} T_u Q_u v_w,$$

where the Q_u are as in (2.10). By (2.11), $P_u = Q_u$ for all $u \neq u_0$ in W_{ij} and thus

$$\underbrace{\cdots T_i T_j T_i}_{m_{ij} \text{ factors}} v_w = \underbrace{\cdots T_j T_i T_j}_{m_{ij} \text{ factors}} v_w.$$

Case 2: Let $u \in W_{ij}$ be of minimal length such that $(uwt)(X^{\alpha_i}) = q^2$ and $(uwt)(X^{\alpha_j}) = 1$. The only possibilities are the following.

Type C_2:

(1) $u = s_i$: Then
$$X^{\alpha_i} v_w = q^{-2} v_w, \qquad X^{\alpha_j} v_w = q^2 v_w,$$
$$T_i v_w = -q^{-1} v_w, \qquad T_j v_w = q v_w.$$

(2) $u = s_i s_j$: Then
$$X^{\alpha_i} v_w = q^2 v_w, \qquad X^{\alpha_j} v_w = q^{-2} v_w,$$
$$T_i v_w = q v_w, \qquad T_j v_w = -q^{-1} v_w.$$

Type G_2:

(1) $u = s_i$: Then
$$X^{\alpha_i} v_w = q^{-2} v_w, \qquad X^{\alpha_2} v_w = q^2 v_w,$$
$$T_i v_w = -q^{-1} v_w, \qquad T_j v_w = q v_w.$$

(2) $u = s_i s_j$ or $u = s_i s_j s_i$. Then both w and $s_i w$ are in $\mathcal{F}^{(t,J)}$ and the action of X^{α_i}, X^{α_j}, T_i and T_j on $\mathbb{C}$-span$\{v_w, v_{s_i w}\}$ is given by the matrices:
$$X^{\alpha_i} = \begin{pmatrix} q^4 & 0 \\ 0 & q^{-2} \end{pmatrix}, \qquad X^{\alpha_j} = \begin{pmatrix} q^{-4} & 0 \\ 0 & q^2 \end{pmatrix},$$
$$T_i = \begin{pmatrix} \dfrac{q - q^{-1}}{1 - q^{-4}} & \dfrac{q - q^3}{1 - q^4} \\ \dfrac{q - q^{-5}}{1 - q^{-4}} & \dfrac{q - q^{-1}}{1 - q^4} \end{pmatrix}, \qquad T_j = \begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

(3) $u = s_i s_j s_i s_j$: Then
$$X^{\alpha_i} v_w = q^2 v_w, \qquad X^{\alpha_j} v_w = q^{-2} v_w,$$
$$T_i v_w = q v_w, \qquad T_j v_w = -q^{-1} v_w.$$

For each of these cases one checks the braid relations by direct computation. For type G_2 case (2) the calculations can be simplified by observing that $T_1 T_2 T_1 = T_2 T_1 T_2$ as operators on $\mathbb{C}$-span$\{v_w, v_{s_i w}\}$. From this it follows that $T_1 T_2 T_1 T_2 T_1 T_2 = T_2 T_1 T_2 T_1 T_2 T_1$ as operators on $\mathbb{C}$-span$\{v_w, v_{s_i w}\}$. ∎

4. Calibrated Representations

A finite dimensional $\tilde{H}$-module
$$M \text{ is } \textit{calibrated} \text{ if } M_t^{\mathrm{gen}} = M_t, \text{ for all } t \in T.$$

A calibrated module M is *really calibrated* if $t = t_r$ for all $t \in T$, i.e. $t_c = 1$ in the polar decomposition (2.1).

Suppose that $t \in T$ is regular, i.e. the stabilizer W_t of t under the action of W is trivial. Then $M(t) = \bigoplus_{w \in W} M_{wt}$, since each M_{wt}^{gen} is one dimensional and $M_{wt} \neq 0$ whenever $M_{wt}^{\mathrm{gen}} \neq 0$. Thus $M(t)$ is calibrated when t is regular. So any quotient of $M(t)$ is calibrated and, by Proposition 2.6b, any irreducible $\tilde{H}$-module M with regular central character is calibrated.

Classification of irreducible calibrated modules

We shall eventually show that the modules $\tilde{H}^{(t,J)}$ constructed in Theorem 3.1 are all the irreducible calibrated $\tilde{H}$-modules. The following Proposition shows that the formulas which define the $\tilde{H}$-modules in Theorem 3.1 are more or less forced.

Proposition 4.1. *Let M be a calibrated $\tilde{H}$-module and assume that for all $t \in T$ such that $M_t \neq 0$,*

$$\text{(A1)} \quad t(X_i^\alpha) \neq 1 \text{ for all } 1 \le i \le n, \qquad \text{and} \qquad \text{(A2)} \quad \dim(M_t) = 1.$$

For each $b \in T$ such that $M_b \neq 0$ let v_b be a nonzero vector in M_b. The vectors $\{v_b\}$ form a basis of M. Let $(T_i)_{cb} \in \mathbb{C}$ and $b(X^\lambda) \in \mathbb{C}$ be given by

$$T_i v_b = \sum_c (T_i)_{cb} v_c \qquad \text{and} \qquad X^\lambda v_b = b(X^\lambda) v_b.$$

Then

(a) $(T_i)_{bb} = \dfrac{q-q^{-1}}{1-b(X^{-\alpha_i})}$, *for all v_b in the basis,*

(b) If $(T_i)_{cb} \neq 0$ then $c = s_i b$,

(c) $(T_i)_{b,s_ib}(T_i)_{s_ib,b} = (q^{-1} + (T_i)_{bb})(q^{-1} + (T_i)_{s_ib,s_ib})$.

Proof. The defining equation for $\tilde{H}$,

$$X^\lambda T_i - T_i X^{s_i\lambda} = (q-q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}},$$

forces

$$\sum_c \left(c(X^\lambda)(T_i)_{cb} - (T_i)_{cb} b(X^{s_i\lambda}) \right) v_c = (q-q^{-1}) \frac{b(X^\lambda) - b(X^{s_i\lambda})}{1 - b(X^{-\alpha_i})} v_b$$

Comparing coefficients gives

$$c(X^\lambda)(T_i)_{cb} - (T_i)_{cb} b(X^{s_i\lambda}) = 0, \qquad \text{if } b \neq c, \text{ and}$$

$$b(X^\lambda)(T_i)_{bb} - (T_i)_{bb} b(X^{s_i\lambda}) = (q-q^{-1}) \frac{b(X^\lambda) - b(X^{s_i\lambda})}{1 - b(X^{-\alpha_i})}.$$

These relations give:

$$\text{If} \quad (T_i)_{cb} \neq 0 \quad \text{then} \quad b(X^{s_i\lambda}) = c(X^\lambda) \quad \text{for all } X^\lambda \in X, \text{ and}$$

$$(T_i)_{bb} = \frac{q-q^{-1}}{1-b(X^{-\alpha_i})} \quad \text{if } b(X^{-\alpha_i}) \neq 1 \text{ and } b(X^\lambda) \neq b(X^{s_i\lambda}) \text{ for some } X^\lambda \in X.$$

By assumption (A1), $b(X^{\alpha_i}) \neq 1$ for all i. For each fundamental weight ω_i, $X^{\omega_i} \in X$ and $b(X^{s_i\omega_i}) = b(X^{\omega_i - \alpha_i}) \neq b(X^{\omega_i})$ since $b(X^{\alpha_i}) \neq 1$. Thus we conclude that

$$T_i v_b = (T_i)_{bb} v_b + (T_i)_{s_ib,b} v_{s_ib}, \qquad \text{with} \qquad (T_i)_{bb} = \frac{q-q^{-1}}{1-b(X^{-\alpha_i})}.$$

This completes the proof of (a) and (b). By the definition of $\tilde{H}$ the vector

$$T_i^2 v_b = ((T_i)_{bb}^2 + (T_i)_{b,s_ib}(T_i)_{s_ib,b})v_b + ((T_i)_{bb} + (T_i)_{s_ib,s_ib})(T_i)_{s_ib,b}v_{s_ib}$$

must equal

$$((q-q^{-1})T_i + 1)v_b = ((q-q^{-1})(T_i)_{bb} + 1)v_b + (q-q^{-1})(T_i)_{s_ib,b}v_{s_ib}.$$

Using the formula for $(T_i)_{bb}$ and $(T_i)_{s_ib,s_ib}$ we find $(T_i)_{bb} + (T_i)_{s_ib,s_ib} = (q-q^{-1})$. So, by comparing coefficients of v_b, we obtain the equation

$$(T_i)_{b,s_ib}(T_i)_{s_ib,b} = (q - (T_i)_{bb})((T_i)_{bb} + q^{-1}) = (q^{-1} + (T_i)_{bb})(q^{-1} + (T_i)_{s_ib,s_ib}). \qquad \blacksquare$$

Proposition 4.2. *Let M be an irreducible calibrated module. Then, for all $t \in T$ such that $M_t \neq 0$,*
(a) $t(X^{\alpha_i}) \neq 1$ *for all* $1 \leq i \leq n$, *and*
(b) $\dim(M_t) = 1$.

Proof. (a) The proof is by contradiction. Assume that $t(X^{\alpha_i}) = 1$. Let $\tilde{H}A_1$ be the subalgebra of $\tilde{H}$ generated by T_i and X^{α_i} and view M as an $\tilde{H}A_1$-module by restriction. Let m_t be a nonzero element of M_t. There is an $\tilde{H}A_1$-module homomorphism

$$\begin{array}{rccc} \phi\colon & M(t) & \longrightarrow & M \\ & v_t & \longmapsto & m_t \end{array}$$

where $M(t)$ is the (two dimensional) principal series module for $\tilde{H}A_1$ and v_t is the generator of $M(t)$. It is easy to check that when $t(X^{\alpha_i}) = 1$ the module $M(t)$ is an irreducible $\tilde{H}A_1$-module. Thus the map ϕ is injective and we can view $M(t)$ as a submodule of M. A direct check shows that $M(t)$ is not calibrated and thus it follows that M is not calibrated. This is a contradiction to the assumption that M is calibrated. Thus $t(X^{\alpha_i}) \neq 1$.

(b) The proof is by contradiction. Assume that $t \in T$ is such that $\dim(M_t) > 1$. Let m_t be a nonzero element of M_t. Since M is calibrated, the action of any τ_i on any weight vector m is a linear combination of the action of T_i and a multiple of the identity. Thus, since M is irreducible, we must be able to generate the rest of M_t by applying τ-operators to m_t. Since $\dim(M_t) > 1$ there must be a sequence of τ-operators such that

$$n_t = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}m_t$$

is a nonzero vector in M_t which is not a multiple of m_t. Assume that the sequence $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_p}$ is chosen so that p is minimal.

Let us defer, momentarily, the proof of the following claim.

Claim: The element $s_{i_1}s_{i_2}\cdots s_{i_p} = 1$ in W.

The claim implies that there is some $1 < k \leq p$ such that $s_{i_1}s_{i_2}\cdots s_{i_k}$ is not reduced and we can use the braid relations to rewrite this word as $s_{i'_1}\cdots s_{i'_{k-2}}s_{i_k}s_{i_k}$. By Proposition 2.7e the τ_i operators also satisfy the braid relations and so

$$n_t = \tau_{i'_1}\tau_{i'_2}\cdots\tau_{i'_{k-2}}\tau_{i_k}\tau_{i_k}\cdots\tau_{i_p}m_t.$$

By Proposition 2.7b, the operator $\tau_{i_k}\tau_{i_k}$ is equal to a constant times the identity map and thus

$$n_t = c\tau_{i'_1}\tau_{i'_2}\cdots\tau_{i'_{k-2}}\tau_{i_{k+1}}\cdots\tau_{i_p}m_t,$$

where c is some constant. The constant c must be nonzero since n_t is not 0. But the expression

$$c^{-1}n_t = \tau_{i'_1}\tau_{i'_2}\cdots\tau_{i'_{k-2}}\tau_{i_{k+1}}\cdots\tau_{i_p}m_t$$

is shorter than the original expression of n_t and this contradicts the minimality of p. It follows that $\dim(M_t) \le 1$.

Proof of the claim. By [St, 3.15, 4.2, 5.3] the stabilizer W_t of t under the action of W is

$$W_t = \langle s_\alpha \mid \alpha \in Z(t)\rangle \qquad \text{where} \qquad Z(t) = \{\alpha > 0 \mid t(X^\alpha) = 1\}.$$

The elements of the orbit Wt can be identified with the cosets of W/W_t and these can be identified with the chambers of $\mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha)$ which are on the positive side of all the hyperplanes H_α for $\alpha \in Z(t)$. Specifically, the element $t \in Wt$ corresponds to the chamber C and the element wt of Wt corresponds to the chamber $w^{-1}C$.

For any $1 \le j \le p$ we have that $(s_{i_{j+1}}\cdots s_{i_p}t)(X^{\alpha_{i_j}}) \ne 1$, since $\tau_{i_j}\cdots\tau_{i_p}m_t$ is well defined. This means that $s_{i_j}(s_{i_{j+1}}\cdots s_{i_p}t)(X^{\omega_{i_j}}) = (s_{i_{j+1}}\cdots s_{i_p}t)(X^{\omega_{i_j}-\alpha_{i_j}}) \ne (s_{i_{j+1}}\cdots\ s_{i_p}t)(X^{\omega_{i_j}})$ and thus that $s_{i_j}(s_{i_{j+1}}\cdots s_{i_p}t) \ne s_{i_{j+1}}\cdots s_{i_p}t$. So $s_{i_j}\cdots s_{i_p}t$ and $s_{i_{j+1}}\cdots s_{i_p}t$ both correspond to chambers on the positive side of all the hyperplanes H_α, $\alpha \in Z(t)$. These two chambers have a common face and this face is contained in the hyperplane $H_{s_{i_p}\ldots s_{i_{j+1}}\alpha_j}$.

In this way we can identify the sequence $t, s_{i_p}t, s_{i_{p-1}}s_{i_p}t, \ldots, s_{i_1}\cdots s_{i_p}t$ with a sequence of chambers where successive chambers in the sequence are adjacent (share a face) and all the chambers in the sequence are on the positive side of all the hyperplanes H_α, $\alpha \in Z(t)$. Since $s_{i_1}\ldots s_{i_p}t = t$, the first and last chamber in this sequence are the same. It follows that $s_{i_1}\cdots s_{i_p} = 1$ in W. ∎

Proposition 4.3. *Let M be an irreducible calibrated $\tilde{H}$-module. Suppose that M_t and M_{s_it} are both nonzero. Then the map $\tau_i\colon M_t \to M_{s_it}$ is a bijection.*

Proof. By Proposition 4.2b, $\dim(M_t) = \dim(M_{s_it}) = 1$, and thus it is sufficient to show that τ_i is not the zero map. Let v_t be a nonzero vector in M_t. Since M is irreducible there must be a sequence of τ operators such that

$$v_{s_it} = \tau_{i_1}\cdots\tau_{i_p}v_t$$

is a nonzero element of M_{s_it}. Let p be minimal such that this is the case. We have $\tau_i\tau_{i_1}\cdots\tau_{i_p}v_t \in M_t$. Using the claim which was proved in the proof of Proposition 4.2 we have that $s_is_{i_1}\cdots s_{i_p} = 1$ in W. For notational convenience $i_0 = i$. Let $0 \le k < p$ be maximal such that $s_{i_k}s_{i_{k+1}}\cdots s_{i_p}$ is not reduced. If $k \ne 0$ then we can use the braid relations to get

$$v_{s_it} = \tau_{i_1}\cdots\tau_{i_k}\tau_{i_k}\tau_{i'_{k+2}}\cdots\tau_{i'_p}v_t.$$

By Proposition 2.7c $\tau_{i_k}\tau_{i_k}$ is a multiple of the identity and so

$$v_{s_it} = c\tau_{i_1}\cdots\tau_{i_{k-1}}\tau_{i'_{k+2}}\cdots\tau_{i'_p}v_t.$$

But this contradicts the minimality of p. Thus we must have $k = 0$, $p = 1$ and

$$v_{s_i t} = \tau_i v_t.$$

Thus, since $v_{s_i t} \neq 0$, $\tau_i \neq 0$. ∎

Proposition 4.4. *If M is a calibrated $\tilde{H}$-module and $t \in T$ is such that $M_t \neq 0$ then t is calibratable for all R_{ij} generated by simple roots α_i and α_j in R.*

Proof. Let $\tilde{H}_{ij}$ be the subalgebra of $\tilde{H}$ generated by T_i, T_j, X^{α_i}, and X^{α_j} and view M as an H_{ij} module by restriction. The irreducible representations of rank two affine Hecke algebras have been classified and constructed explicitly in [Ra3]. From this classification it is easy to check that the only weights t which appear in calibrated $\tilde{H}_{ij}$-modules are those that are calibratable for R_{ij}. Thus, if $M_t \neq 0$, then t must be calibratable for R_{ij}. ∎

Theorem 4.5. *Let M be an irreducible calibrated $\tilde{H}$-module. Let t be the central character of M and let $J = R(w) \cap P(t)$ for any $w \in W$ such that $M_{wt} \neq 0$. Then (t, J) is a placed skew shape and*

$$M \cong \tilde{H}^{(t,J)},$$

where $H^{(t,J)}$ is the module defined in Theorem 3.1.

Proof. Proposition 4.3 shows that if M_t and $M_{s_i t}$ are both nonzero then both $\tau_i \colon M_t \to M_{s_i t}$ and $\tau_i \colon M_{s_i t} \to M_t$ are bijections. Thus, by Proposition 2.7d $t(X^{\alpha_i}) \neq q^{\pm 2}$ and so there is an edge $t \leftrightarrow s_i t$ in the calibration graph. This shows that $\mathrm{supp}(M)$ is a single connected component of the calibration graph $\Gamma(t)$. Then (t, J) (as defined in the statement of the Theorem) is the corresponding placed shape. By Proposition 4.4 (t, J) must be a placed *skew* shape. Propositions 4.1 and 4.2 show that there is at most one calibrated $\tilde{H}$-module M such that $\mathrm{supp}(M)$ is the connected component of $\Gamma(t)$ labeled by (t, J). Thus we must have that $M \cong \tilde{H}^{(t,J)}$. ∎

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