Combinatorics in affine flag varieties

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Dedicated to Gus Lehrer on the occasion of his 60^{th} *birthday*

Abstract

The Littelmann path model gives a realisation of the crystals of integrable representations of symmetrizable Kac-Moody Lie algebras. Recent work of Gaussent-Littelmann [GL] and others [BG] [GR] has demonstrated a connection between this model and the geometry of the loop Grassmanian. The alcove walk model is a version of the path model which is intimately connected to the combinatorics of the affine Hecke algebra. In this paper we define a refined alcove walk model which encodes the points of the affine flag variety. We show that this combinatorial indexing naturally indexes the cells in generalized Mirkovic-Vilonen intersections.

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1 Introduction

A *Chevalley group* is a group in which row reduction works. This means that it is a group with a special set of generators (the "elementary matrices") and relations which are generalisations of the usual row reduction operations. One way to efficiently encode these generators and relations is with a Kac-Moody Lie algebra $\mathfrak{g}$. From the data of the Kac-Moody Lie algebra and a choice of a commutative ring or field $\mathbb{F}$ the group $G(\mathbb{F})$ is built by generators and relations following Chevalley-Steinberg-Tits.

Of particular interest is the case where $\mathbb{F}$ is the field of fractions of $\mathfrak{o}$, the discrete valuation ring $\mathfrak{o}$ is the ring of integers in $\mathbb{F}$, $\mathfrak{p}$ is the unique maximal ideal in $\mathfrak{o}$ and $k = \mathfrak{o}/\mathfrak{p}$ is the residue field. The favourite examples are

$$\begin{array}{lll} \mathbb{F} = \mathbb{C}((t)) & \mathfrak{o} = \mathbb{C}[[t]] & k = \mathbb{C}, \\ \mathbb{F} = \mathbb{Q}_p & \mathfrak{o} = \mathbb{Z}_p & k = \mathbb{F}_p, \\ \mathbb{F} = \mathbb{F}_q((t)) & \mathfrak{o} = \mathbb{F}_q[[t]] & k = \mathbb{F}_q, \end{array}$$

where $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Z}_p$ is the ring of p-adic integers, and $\mathbb{F}_q$ is the finite field with q elements. For clarity of presentation we shall work in the first case where $\mathbb{F} = \mathbb{C}((t))$. The diagram

$$\begin{array}{ccc} \mathbb{F} & & \\ \cup\!\mid & & \\ \mathfrak{o} & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & k = \mathfrak{o}/\mathfrak{p} \end{array} \quad \text{gives} \quad \begin{array}{ccccc} G & = & G(\mathbb{C}((t))) & & \\ \cup\!\mid & & \cup\!\mid & & \\ K & = & G(\mathbb{C}[[t]]) & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & G(\mathbb{C}) \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ I & = & \mathrm{ev}_{t=0}^{-1}(B(\mathbb{C})) & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & B(\mathbb{C}) \end{array} \tag{1.1}$$

where $B(\mathbb{C})$ is the "Borel subgroup" of "upper triangular matrices" in $G(\mathbb{C})$. The *loop group* is $G = G(\mathbb{C}((t)))$, I is the standard *Iwahori subgroup* of G,

$$G(\mathbb{C})/B(\mathbb{C}) \quad \text{is the } \textit{flag variety},$$
$$G/I \quad \text{is the } \textit{affine flag variety}, \qquad \text{and} \qquad G/K \quad \text{is the } \textit{loop Grassmanian}. \tag{1.2}$$

The primary tool for the study of these varieties (ind-schemes) are the following "classical" double coset decompositions, see [St, Ch. 8] and [Mac1, §(2.6)]

Theorem 1.1. *Let W be the Weyl group of $G(\mathbb{C})$, $\widetilde{W} = W \ltimes \mathfrak{h}_{\mathbb{Z}}$ the affine Weyl group, and U^- the subgroup of "unipotent lower triangular" matrices in $G(\mathbb{F})$ and $\mathfrak{h}_{\mathbb{Z}}^+$ the set of dominant elements of $\mathfrak{h}_{\mathbb{Z}}$. Then*

Bruhat decomposition
$$G = \bigsqcup_{w \in W} BwB \qquad K = \bigsqcup_{w \in W} IwI$$

Iwahori decomposition
$$G = \bigsqcup_{w \in \widetilde{W}} IwI \qquad G = \bigsqcup_{v \in \widetilde{W}} U^- vI$$

Cartan decomposition / *Iwasawa decomposition*
$$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+} Kt_{\lambda^\vee}K \qquad G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^- t_{\mu^\vee} K$$

In this paper we shall refine the Littelmann path model (in its alcove walk form, see [Ra]) by putting labels on the paths to provide a combinatorial indexing of the points in the affine flag variety. This combinatorial method of expressing the points of G/I gives detailed information about the structure of the intersections

$$U^-vI \cap IwI \qquad \text{with} \qquad v, w \in \widetilde{W}. \tag{1.3}$$

The corresponding intersections in G/K have arisen in many contexts. Most notably, the set of *Mirković-Vilonen cycles of shape* $\lambda^\vee$ *and weight* $\mu^\vee$ is the set of irreducible components of the closure of $U^-t_{\mu^\vee}K \cap Kt_{\lambda^\vee}K$ in G/K,

$$MV(\lambda^\vee)_{\mu^\vee} = \mathrm{Irr}(\overline{U^-t_{\mu^\vee}K \cap Kt_{\lambda^\vee}K}),$$

and

$$\text{when } k = \mathbb{F}_q, \qquad \mathrm{Card}_{G/K}(U^-t_{\mu^\vee}K \cap Kt_{\lambda^\vee}K) \;\text{ is}$$

(up to some easily understood factors) the coefficient of the monomial symmetric function $m_{\mu^\vee}$ in the expansion of the Macdonald spherical function $P_{\lambda^\vee}$.

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2 Borcherds-Kac-Moody Lie algebras

This section reviews definitions and sets notations for Borcherds-Kac-Moody Lie algebras. Standard references are the book of Kac [Kac], the books of Wakimoto [Wak1][Wak2], the survey article of Macdonald [Mac3] and the handwritten notes of Macdonald [Mac2]. Specifically, [Kac, Ch. 1] is a reference for §2.1, [Kac, Ch. 3 and 5] for §2.2, and [Kac, Ch. 2] for §2.3.

2.1 Constructing a Lie algebra from a matrix

Let $A = (a_{ij})$ be an $n \times n$ matrix. Let

$$r = \mathrm{rank}(A), \qquad \ell = \mathrm{corank}(A), \qquad \text{so that} \quad r + \ell = n. \tag{2.1}$$

By rearranging rows and columns we may assume that $(a_{ij})_{1\le i,j\le r}$ is nonsingular. Define a $\mathbb{C}$-vector space

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}, \qquad \text{where} \qquad \begin{array}{l} \mathfrak{h}' \text{ has basis } h_1, \ldots, h_n, \quad \text{and} \\ \mathfrak{d} \text{ has basis } d_1, \ldots, d_\ell. \end{array} \tag{2.2}$$

Define $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ by

$$\alpha_i(h_j) = a_{ij} \qquad \text{and} \qquad \alpha_i(d_j) = \delta_{i,r+j}, \tag{2.3}$$

and let

$$\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{c}, \qquad \text{where} \qquad \mathfrak{c} = \{h \in \mathfrak{h}' \mid \alpha_i(h) = 0 \text{ for all } 1 \le i \le n\}. \tag{2.4}$$

Let $c_1 \dots, c_\ell \in \mathfrak{h}'$ be a basis of $\mathfrak{c}$ so that $h_1, \dots, h_r, c_1, \dots, c_\ell, d_1, \dots, d_\ell$ is another basis of $\mathfrak{h}$ and define $\kappa_1, \dots, \kappa_\ell \in \mathfrak{h}^*$ by

$$\kappa_i(h_j) = 0, \qquad \kappa_i(c_j) = \delta_{ij}, \qquad \text{and} \qquad \kappa_i(d_j) = 0. \tag{2.5}$$

Then $\alpha_1, \dots, \alpha_n, \kappa_1, \dots, \kappa_\ell$ form a basis of $\mathfrak{h}^*$.

Let $\mathfrak{a}$ be the Lie algebra given by generators $\mathfrak{h}, e_1, \dots, e_n, f_1, \dots, f_n$ and relations

$$[h, h'] = 0, \qquad [e_i, f_j] = \delta_{ij} h_i, \qquad [h, e_i] = \alpha_i(h) e_i, \qquad [h, f_i] = -\alpha_i(h) f_i, \tag{2.6}$$

for $h, h' \in \mathfrak{h}$ and $1 \le i, j \le n$. The *Borcherds-Kac-Moody Lie algebra* of A is

$$\mathfrak{g} = \frac{\mathfrak{a}}{\mathfrak{r}}, \qquad \text{where} \quad \mathfrak{r} \text{ is a the largest ideal of } \mathfrak{a} \text{ such that } \mathfrak{r} \cap \mathfrak{h} = 0. \tag{2.7}$$

The Lie algebra $\mathfrak{a}$ is graded by

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \qquad \text{by setting} \quad \deg(e_i) = \alpha_i, \;\; \deg(f_i) = -\alpha_i, \;\; \deg(h) = 0, \tag{2.8}$$

for $h \in \mathfrak{h}$. Any ideal of $\mathfrak{a}$ is Q-graded and so $\mathfrak{g}$ is Q-graded (see [Mac2, (1.6)] or [Mac3, p. 81]),

$$\begin{gathered} \mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right), \qquad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h) x\}, \quad \text{and} \\ R = \{\alpha \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\} \qquad \text{is the set of } \textit{roots} \text{ of } \mathfrak{g}. \end{gathered} \tag{2.9}$$

The *multiplicity* of a root $\alpha \in R$ is $\dim(\mathfrak{g}_\alpha)$ and the decomposition of $\mathfrak{g}$ in (2.9) is the decomposition of $\mathfrak{g}$ as an $\mathfrak{h}$-module (under the adjoint action). If

$\mathfrak{n}^+$ is the subalgebra generated by $e_1, \dots, e_n$, and
$\mathfrak{n}^-$ is the subalgebra generated by $f_1, \dots, f_n$,

then (see [Mac3, p. 83] or [Kac, §1.3])

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \qquad \text{and} \qquad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}, \tag{2.10}$$

where

$$R^+ = Q^+ \cap R \qquad \text{with} \qquad Q^+ = \sum_{i=1}^{n} \mathbb{Z}_{\ge 0} \alpha_i. \tag{2.11}$$

Let $\mathfrak{c}$ and $\mathfrak{d}$ be as in (2.2) and (2.4). Then

$$\begin{gathered} \mathfrak{d} \text{ acts on } \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \text{ by derivations}, \qquad \mathfrak{c} = Z(\mathfrak{g}) = Z(\mathfrak{g}'), \\ \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{a}/\mathfrak{r} = \mathfrak{g}' \rtimes \mathfrak{d}, \\ \mathfrak{g}' = \mathfrak{n}^- \oplus \mathfrak{h}' \oplus \mathfrak{n}^+ = [\mathfrak{g}, \mathfrak{g}], \\ \bar{\mathfrak{g}}' = \mathfrak{n}^- \oplus \bar{\mathfrak{h}}' \oplus \mathfrak{n}^+ = \mathfrak{g}'/\mathfrak{c}, \end{gathered} \tag{2.12}$$

and $\mathfrak{g}'$ is the universal central extension of $\mathfrak{g}'$ (see [Kac, Ex. 3.14]).

2.2 Cartan matrices, $\mathfrak{sl}_2$ subalgebras and the Weyl group

A *Cartan matrix* is an $n \times n$ matrix $A = (a_{ij})$ such that

$$a_{ij} \in \mathbb{Z}, \qquad a_{ii} = 2, \qquad a_{ij} \le 0 \text{ if } i \neq j, \qquad a_{ij} \neq 0 \text{ if and only if } a_{ji} \neq 0. \tag{2.13}$$

When A is a Cartan matrix the Lie algebra $\mathfrak{g}$ contains many subalgebras isomorphic to $\mathfrak{sl}_2$. For $1 \le i \le n$, the elements e_i and f_i act locally nilpotently on $\mathfrak{g}$ (see [Mac3, p. 85] or [Mac2, (1.19)] or [Kac, Lemma 3.5]),

$$\text{span}\{e_i, f_i, h_i\} \cong \mathfrak{sl}_2, \qquad \text{and} \qquad \tilde{s}_i = \exp(\text{ad}e_i)\exp(-\text{ad}f_i)\exp(\text{ad}e_i) \tag{2.14}$$

is an automorphism of $\mathfrak{g}$ (see [Kac, Lemma 3.8]). Thus $\mathfrak{g}$ has lots of symmetry.

The *simple reflections* $s_i \colon \mathfrak{h}^* \to \mathfrak{h}^*$ and $s_i \colon \mathfrak{h} \to \mathfrak{h}$ are given by

$$s_i\lambda = \lambda - \lambda(h_i)\alpha_i \qquad \text{and} \qquad s_ih = h - \alpha_i(h)h_i, \qquad \text{for } 1 \le i \le n, \tag{2.15}$$

$\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and

$$\tilde{s}_i\mathfrak{g}_\alpha = \mathfrak{g}_{s_i\alpha} \qquad \text{and} \qquad \tilde{s}_ih = s_ih, \qquad \text{for } \alpha \in R, \quad h \in \mathfrak{h}.$$

The *Weyl group* W is the subgroup of $GL(\mathfrak{h}^*)$ (or $GL(\mathfrak{h})$) generated by the simple reflections. The simple reflections on $\mathfrak{h}$ are reflections in the hyperplanes

$$\mathfrak{h}^{\alpha_i} = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0\}, \qquad \text{and} \qquad \mathfrak{c} = \mathfrak{h}^W = \bigcap_{i=1}^n \mathfrak{h}^{\alpha_i}.$$

The representation of W on $\mathfrak{h}$ and $\mathfrak{h}^*$ are dual so that

$$\lambda(wh) = (w^{-1}\lambda)(h), \qquad \text{for } w \in W,\ \lambda \in \mathfrak{h}^*,\ h \in \mathfrak{h}.$$

The group W is presented by generators $s_1, \ldots, s_n$ and relations

$$s_i^2 = 1 \qquad \text{and} \qquad \underbrace{s_is_js_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{s_js_is_j\cdots}_{m_{ij}\text{ factors}} \tag{2.16}$$

for pairs $i \neq j$ such that $a_{ij}a_{ji} < 4$, where $m_{ij} = 2, 3, 4, 6$ if $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively (see [Mac2, (2.12)] or [Kac, Prop. 3.13]).

The *real roots* of $\mathfrak{g}$ are the elements of the set

$$R_{\text{re}} = \bigcup_{i=1}^n W\alpha_i, \qquad \text{and} \qquad R_{\text{im}} = R \backslash R_{\text{re}} \tag{2.17}$$

is the set of *imaginary roots* of $\mathfrak{g}$. If $\alpha = w\alpha_i$ is a real root then there is a subalgebra isomorphic to $\mathfrak{sl}_2$ spanned by

$$e_\alpha = \tilde{w}e_i, \quad f_\alpha = \tilde{w}f_i, \quad \text{and} \quad h_\alpha = \tilde{w}h_i, \tag{2.18}$$

and $s_\alpha = ws_iw^{-1}$ is a reflection in W acting on $\mathfrak{h}$ and $\mathfrak{h}^*$ by

$$s_\alpha\lambda = \lambda - \lambda(h_\alpha)\alpha \qquad \text{and} \qquad s_\alpha h = h - \alpha(h)h_\alpha, \qquad \text{respectively.} \tag{2.19}$$

Let $\mathfrak{h}_\mathbb{R} = \mathbb{R}\text{-span}\{h_1, \ldots, h_n, d_1, \ldots, d_\ell\}$. The group W acts on $\mathfrak{h}_\mathbb{R}$ and the *dominant chamber*

$$C = \{\lambda^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha_i, \lambda^\vee \rangle \ge 0 \text{ for all } 1 \le i \le n\} \tag{2.20}$$

is a fundamental domain for the action of W on the *Tits cone*

$$X = \bigcup_{w \in W} wC = \{h \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha, h \rangle < 0 \text{ for a finite number of } \alpha \in R^+\}. \tag{2.21}$$

$X = \mathfrak{h}_\mathbb{R}$ if and only if W is finite (see [Kac, Prop. 3.12] and [Mac2, (2.14)]).

2.3 Symmetrizable matrices and invariant forms

A *symmetrizable matrix* is a matrix $A = (a_{ij})$ such that there exists a diagonal matrix

$$\mathcal{E} = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_i \in \mathbb{R}_{>0}, \qquad \text{such that} \qquad A\mathcal{E} \text{ is symmetric.} \tag{2.22}$$

If $\langle , \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a $\mathfrak{g}$-invariant symmetric bilinear form then

$$\langle h_i, h \rangle = \langle [e_i, f_i], h \rangle = -\langle f_i, [e_i, h] \rangle = \langle f_i, \alpha_i(h) e_i \rangle = \alpha_i(h) \langle e_i, f_i \rangle,$$

so that

$$\langle h_i, h \rangle = \alpha_i(h)\epsilon_i, \qquad \text{where} \qquad \epsilon_i = \langle e_i, f_i \rangle. \tag{2.23}$$

Conversely, if A is a symmetrizable matrix then there is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ determined by the formulas in (2.23) (see [Mac2, (3.12)] or [Kac, Theorem 2.2]).

If A is a Cartan matrix and $\langle , \rangle \colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ is a W-invariant symmetric bilinear form then

$$\langle h_i, h \rangle = -\langle s_i h_i, h \rangle = -\langle h_i, s_i h \rangle = -\langle h_i, h - \alpha_i(h) h_i \rangle = -\langle h_i, h \rangle + \alpha_i(h) \langle h_i, h_i \rangle,$$

so that

$$\langle h_i, h \rangle = \alpha_i(h)\epsilon_i, \qquad \text{where} \qquad \epsilon_i = \tfrac{1}{2}\langle h_i, h_i \rangle. \tag{2.24}$$

In particular, $\alpha_i(h_j)\epsilon_i = \langle h_i, h_j \rangle = \langle h_j, h_i \rangle = \alpha_j(h_i)\epsilon_j$ so that A is symmetrizable. Conversely, if A is a symmetrizable Cartan matrix then there is a nondegenerate W-invariant symmetric bilinear form on $\mathfrak{h}$ determined by the formulas in (2.24) (see [Mac2, (2.26)]).

If $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$ and $\langle h, [x_\alpha, y_\alpha] \rangle = -\langle [x_\alpha, h], y_\alpha \rangle = \alpha(h) \langle x_\alpha, y_\alpha \rangle$, so that

$$[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha^\vee, \qquad \text{where } \langle h, h_\alpha^\vee \rangle = \alpha(h) \text{ for all } h \in \mathfrak{h} \tag{2.25}$$

determines $h_\alpha^\vee \in \mathfrak{h}$. If $\alpha \in R_{\mathrm{re}}$ and $e_\alpha, f_\alpha, h_\alpha$ are as in (2.18) then

$$h_\alpha = [e_\alpha, f_\alpha] = \langle e_\alpha, f_\alpha \rangle h_\alpha^\vee \qquad \text{and} \qquad \langle e_\alpha, f_\alpha \rangle = \tfrac{1}{2}\langle h_\alpha, h_\alpha \rangle. \tag{2.26}$$

Let

$$\alpha^\vee = \langle e_\alpha, f_\alpha \rangle \alpha = \tfrac{1}{2}\langle h_\alpha, h_\alpha \rangle \alpha \qquad \text{so that} \qquad \alpha^\vee(h) = \langle h, h_\alpha \rangle. \tag{2.27}$$

Use the vector space isomorphism

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\sim} & \mathfrak{h}^* \\ h & \longmapsto & \langle h, \cdot \rangle \\ h_\alpha & \longmapsto & \alpha^\vee \\ h_\alpha^\vee & \longmapsto & \alpha \end{array} \qquad \text{to identify} \quad Q^\vee = \sum_{i=1}^n \mathbb{Z} h_i \quad \text{and} \quad Q^* = \sum_{i=1}^n \mathbb{Z} \alpha_i^\vee \tag{2.28}$$

and write

$$\langle \lambda^\vee, \mu \rangle = \mu(h_\lambda) \qquad \text{if} \quad \lambda^\vee = \lambda_1 \alpha_1^\vee + \cdots + \lambda_n \alpha_n^\vee \quad \text{and} \quad h_\lambda = \lambda_1 h_1 + \cdots + \lambda_n h_n. \tag{2.29}$$

3 Steinberg-Chevalley groups

This section gives a brief treatment of the theory of Chevalley groups. The primary reference is [St] and the extensions to the Kac-Moody case are found in [Ti].

Let A be a Cartan matrix and let $R_{\rm re}$ be the real roots of the corresponding Borcherds-Kac-Moody Lie algebra $\mathfrak{g}$. Let U be the enveloping algebra of $\mathfrak{g}$. For each $\alpha \in R_{\rm re}$ fix a choice of e_α in (2.18) (a choice of $\tilde{w}$). Use the notation

$$x_\alpha(t) = \exp(te_\alpha) = 1 + e_\alpha + \frac{1}{2!}t^2 e_\alpha^2 + \frac{1}{3!}t^3 e_\alpha^3 + \cdots, \qquad \text{in } U[[t]].$$

Then

$$x_\alpha(t)x_\alpha(u) = x_\alpha(t+u) \qquad \text{in } U[[t,u]].$$

Following [Ti, 3.2], a *prenilpotent pair* is a pair of roots $\alpha, \beta \in R_{\rm re}$ such that there exists $w, w' \in W$ with

$$w\alpha, w\beta \in R_{\rm re}^+ \qquad \text{and} \qquad w'\alpha, w'\beta \in -R_{\rm re}^+.$$

This condition guarantees that the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ is nilpotent. Let α, β be a prenilpotent pair and let $e_\alpha \in \mathfrak{g}_\alpha$ and $e_\beta \in \mathfrak{g}_\beta$ be as in (2.18). By [St, Lemma 15] there are unique integers $C_{\alpha\beta}^{i,j}$ such that

$$x_\alpha(t)x_\beta(u) = x_\beta(u)x_\alpha(t)x_{\alpha+\beta}(C_{\alpha,\beta}^{1,1}tu)x_{2\alpha+\beta}(C_{\alpha,\beta}^{2,1}t^2u)x_{\alpha+2\beta}(C_{\alpha,\beta}^{1,2}ut^2)\cdots.$$

Let $\mathbb{F}$ be a commutative ring. The *Steinberg group*

$$\text{St is given by generators } x_\alpha(f) \text{ for } \alpha \in R_{\rm re},\ f \in \mathbb{F},$$

and relations

$$x_\alpha(f_1)x_\alpha(f_2) = x_\alpha(f_1+f_2), \qquad \text{for } \alpha \in R_{\rm re}, \quad \text{and} \tag{3.1}$$

$$x_\alpha(f_1)x_\beta(f_2) = x_\beta(f_2)x_\alpha(f_1)x_{\alpha+\beta}(C_{\alpha,\beta}^{1,1}f_1f_2)x_{2\alpha+\beta}(C_{\alpha,\beta}^{2,1}f_1^2f_2)x_{\alpha+2\beta}(C_{\alpha,\beta}^{1,2}f_1f_2^2)\cdots \tag{3.2}$$

for prenilpotent pairs α, β. In St define

$$n_\alpha(g) = x_\alpha(g)x_{-\alpha}(-g^{-1})x_\alpha(g), \quad n_\alpha = n_\alpha(1), \quad \text{and} \quad h_{\alpha^\vee}(g) = n_\alpha(g)n_\alpha^{-1}, \tag{3.3}$$

for $\alpha \in R_{\rm re}$ and $g \in \mathbb{F}^\times$.

Let $\mathfrak{h}_\mathbb{Z}$ be a $\mathbb{Z}$-lattice in $\mathfrak{h}$ which is stable under the W-action and such that

$$\mathfrak{h}_\mathbb{Z} \supseteq Q^\vee, \qquad \text{where} \quad Q^\vee = \mathbb{Z}\text{-span}\{h_1, \ldots, h_n\}$$

with $h_1, \ldots, h_n$ as in (2.2). With

$$T \text{ given by generators } h_{\lambda^\vee}(g) \text{ for } \lambda^\vee \in \mathfrak{h}_\mathbb{Z},\ g \in \mathbb{F}^\times, \quad \text{and relations}$$

$$h_{\lambda^\vee}(g_1)h_{\lambda^\vee}(g_2) = h_{\lambda^\vee}(g_1g_2) \qquad \text{and} \qquad h_{\lambda^\vee}(g)h_{\mu^\vee}(g) = h_{\lambda^\vee+\mu^\vee}(g), \tag{3.4}$$

the *Tits group*

$$G \text{ is the group generated by } St \text{ and } T$$

with the relations coming from the third equation in (3.3) and the additional relations

$$h_{\lambda^\vee}(g)x_\alpha(f)h_{\lambda^\vee}(g)^{-1} = x_\alpha(g^{\langle\lambda^\vee,\alpha\rangle}f) \qquad \text{and} \qquad n_ih_{\lambda^\vee}(g)n_i^{-1} = h_{s_i\lambda^\vee}(g). \tag{3.5}$$

For $\alpha, \beta \in R_{\rm re}$ let $\epsilon_{\alpha\beta} = \pm 1$ be given by

$$\tilde{s}_\alpha(e_\beta) = \epsilon_{\alpha\beta} e_{s_\alpha\beta}, \qquad \text{where} \qquad \tilde{s}_\alpha = \exp(\mathrm{ad} e_\alpha)\exp(-\mathrm{ad} f_\alpha)\exp(\mathrm{ad} e_\alpha)$$

(see [CC, p.48] and [Ti, (3.3)]). By [St, Lemma 37] (see also [Ti, §3.7(a)])

$$n_\alpha(g) x_\beta(f) n_\alpha(g)^{-1} = x_{s_\alpha\beta}(\epsilon_{\alpha\beta} g^{-\langle\beta,\alpha^\vee\rangle} f), \qquad h_{\lambda^\vee}(g) x_\beta(f) h_{\lambda^\vee}(g)^{-1} = x_\beta(g^{\langle\beta,\lambda^\vee\rangle} f), \tag{3.6}$$

$$\text{and} \quad n_\alpha(g) h_{\lambda^\vee}(g') n_\alpha(g)^{-1} = h_{s_\alpha\lambda^\vee}(g'). \tag{3.7}$$

Thus G has a symmetry under the subgroup

$$N \text{ generated by } T \text{ and the } n_\alpha(g) \text{ for } \alpha \in R_{\rm re}, g \in \mathbb{F}^\times. \tag{3.8}$$

If $\mathbb{F}$ is big enough then N is the normalizer of T in G [St, Ex. (b) p. 36] and, by [St, Lemma 27], the homomorphism

$$\begin{matrix} N & \longrightarrow & W \\ n_\alpha(g) & \longmapsto & s_\alpha \end{matrix} \qquad \text{is surjective with kernel } T. \tag{3.9}$$

Remark 3.1. [Ti, §3.7(b)] If $\mathfrak{h}_\mathbb{Z} = Q^\vee$ and the first relation of (3.5) holds in St then there is a surjective homomorphism $\psi\colon \mathrm{St} \twoheadrightarrow G$. By [St, Lemma 22], the elements

$$n_\alpha h_{\lambda^\vee}(g) n_\alpha^{-1} h_{s_\alpha\lambda^\vee}(g)^{-1} \qquad \text{and} \qquad n_\alpha(g) n_\alpha^{-1} h_{\alpha^\vee}(g)^{-1}$$

automatically commute with each $x_\beta(f)$ so that $\ker(\psi) \subseteq Z(\mathrm{St})$. In many cases St is the universal central extension of G (see [Ti, 3.7(c)] and [St, Theorems 10,11,12]).

Remark 3.2. The algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ in (2.12) is generated by e_α, $\alpha \in R_{\rm re}$. A $\mathfrak{g}'$-module V is *integrable* if e_α, $\alpha \in R_{\rm re}$, act locally nilpotently so that

$$x_\alpha(c) = \exp(c e_\alpha), \qquad \text{for } \alpha \in R_{\rm re},\ c \in \mathbb{C}, \tag{3.10}$$

are well defined operators on V. The *Chevalley group* G_V is the subgroup of $GL(V)$ generated by the operators in (3.10). To do this integrally use a Kostant $\mathbb{Z}$-form and choose a lattice in the module V (see [Ti, §4.3-4] and [St, Ch. 1]). The *Kac-Moody group* is the group G_{KM} generated by symbols

$$x_\alpha(c), \quad \alpha \in R_{\rm re}, c \in \mathbb{C}, \qquad \text{with relations} \qquad x_\alpha(c_1) x_\alpha(c_2) = x_\alpha(c_1 + c_2)$$

and the additional relations coming from forcing an element to be 1 if it acts by 1 on *every* integrable $\mathfrak{g}'$ module. This is essentially the Chevalley group G_V for the case when V is the adjoint representation and so $G_{KM} \subseteq \mathrm{Aut}(\mathfrak{g}')$. There are surjective homomorphisms

$$\mathrm{St}(\mathbb{C}) \twoheadrightarrow G_{KM} \twoheadrightarrow G_V.$$

See [Kac, Exercises 3.16-19] and [Ti, Proposition 1].

Remark 3.3. [St, Lemma 28] In the setting of Remark 3.2 let T_V be the subgroup of G_V generated by $h_{\alpha^\vee}(g)$ for $\alpha \in R_{\rm re}, g \in \mathbb{F}^\times$. Then

$$h_{\alpha_1^\vee}(g_1) \cdots h_{\alpha_n^\vee}(g_n) = 1 \quad \text{if and only if} \quad g_1^{\langle\mu,\alpha_1^\vee\rangle} \cdots g_n^{\langle\mu,\alpha_n^\vee\rangle} = 1 \quad \text{for all weights } \mu \text{ of } V,$$

$$Z(G_V) = \{h_{\alpha_1^\vee}(g_1) \cdots h_{\alpha_n^\vee}(g_n) \mid g_1^{\langle\beta,\alpha_1^\vee\rangle} \cdots g_n^{\langle\beta,\alpha_n^\vee\rangle} = 1 \quad \text{for all } \beta \in R\},$$

and if $\mathbb{F}$ is big enough

$$T_V = \{h_{\omega_1^\vee}(g_1) \cdots h_{\omega_n^\vee}(g_n) \mid g_1, \ldots, g_n \in \mathbb{F}^\times\},$$

where $\omega_1^\vee, \ldots, \omega_n^\vee$ is a $\mathbb{Z}$-basis of the $\mathbb{Z}$-span of the weights of V [St, Lemma 35].

4 Labeling points of the flag variety G/B

In this section we follow [St, Ch. 8] to show that the points of the flag variety are naturally indexed by labeled walks. This is the first step in making a precise connection between the points in the flag variety and the alcove walk theory in [Ra].

Let G be a Tits group as in (3.5) over the field $\mathbb{F} = \mathbb{C}$. The *root subgroups*

$$\mathcal{X}_\alpha = \{x_\alpha(c) \mid c \in \mathbb{C}\}, \quad \text{for } \alpha \in R_{\mathrm{re}}, \qquad \text{satisfy} \qquad w\mathcal{X}_\beta w^{-1} = \mathcal{X}_{w\beta}, \tag{4.1}$$

for $w \in W$ and $\beta \in R_{\mathrm{re}}$, since $h_{\alpha^\vee}(c)\mathcal{X}_\beta h_{\alpha^\vee}(c)^{-1} = \mathcal{X}_\beta$ and $n_\alpha \mathcal{X}_\beta n_\alpha^{-1} = \mathcal{X}_{s_\alpha\beta}$. As a group $\mathcal{X}_\alpha$ is isomorphic to $\mathbb{C}$ (under addition).

The *flag variety* is G/B, where the subgroup

$$B \text{ is generated by } T \text{ and } x_\alpha(f) \text{ for } \alpha \in R^+_{\mathrm{re}},\ f \in \mathbb{C}. \tag{4.2}$$

Let $w \in W$. The *inversion set of* w is

$$R(w) = \{\alpha \in R^+_{\mathrm{re}} \mid w^{-1}\alpha \notin R^+_{\mathrm{re}}\} \qquad \text{and} \qquad \ell(w) = \mathrm{Card}(R(w)) \tag{4.3}$$

is the *length of* w. View a reduced expression $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ in the generators in (2.16) as a *walk* in W starting at 1 and ending at w,

$$1 \longrightarrow s_{i_1} \longrightarrow s_{i_1}s_{i_2} \longrightarrow \cdots \longrightarrow s_{i_1}\cdots s_{i_\ell} = w. \tag{4.4}$$

Letting $x_i(c) = x_{\alpha_i}(c)$ and $n_i = n_{\alpha_i}(1)$, the following theorem shows that

$$BwB = \{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B \mid c_1,\ldots,c_\ell \in \mathbb{C}\} \tag{4.5}$$

so that the G/B-points of BwB are in bijection with labelings of the edges of the walk by complex numbers $c_1,\ldots,c_\ell$. The elements of $R(w)$ are

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}\alpha_{i_2}, \quad \ldots, \quad \beta_\ell = s_{i_1}\cdots s_{i_{\ell-1}}\alpha_{i_\ell}, \tag{4.6}$$

and the first relation in (3.6) gives

$$x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} = x_{\beta_1}(\pm c_1)\cdots x_{\beta_\ell}(\pm c_\ell)n_w, \tag{4.7}$$

where $n_w = n_{i_1}^{-1}\cdots n_{i_\ell}^{-1}$.

Theorem 4.1. [St, Thm. 15 and Lemma 43] *Let $w \in W$ and let n_w be a representative of w in N. If*

$$R(w) = \{\beta_1,\ldots,\beta_\ell\} \qquad \textit{then} \qquad \{x_{\beta_1}(c_1)\cdots x_{\beta_\ell}(c_\ell)n_w \mid c_1,\ldots,c_\ell \in \mathbb{C}\}$$

is a set of representatives of the B-cosets in BwB.

Proof. The conceptual reason for this is that

$$\begin{aligned} BwB &= \left(\prod_{\alpha\in R^+_{\mathrm{re}}} \mathcal{X}_\alpha\right) n_wB = n_w\left(\prod_{w^{-1}\alpha\notin R^+_{\mathrm{re}}} \mathcal{X}_{w^{-1}\alpha}\right)\left(\prod_{w^{-1}\alpha\in R^+_{\mathrm{re}}} \mathcal{X}_{w^{-1}\alpha}\right)B \\ &= n_w\left(\prod_{w^{-1}\alpha\notin R^+_{\mathrm{re}}} \mathcal{X}_{w^{-1}\alpha}\right)B = \left(\prod_{\alpha\in R(w)} \mathcal{X}_\alpha\right)n_wB \\ &= \{x_{\beta_1}(c_1)\cdots x_{\beta_\ell}(c_\ell)n_wB \mid c_1,\ldots,c_\ell \in \mathbb{F}\}. \end{aligned}$$

Since R^+_{re} may be infinite there is a subtlety in the decomposition and ordering of the product of $\mathcal{X}_\alpha$ in the second "equality" and it is necessary to proceed more carefully. Choose a reduced decomposition $w = s_{i_1}\cdots s_{i_\ell}$ and let $\beta_1,\ldots,\beta_\ell$ be the ordering of $R(w)$ from (4.6).

Step 1: Since $R(w)\subseteq R^+_{\mathrm{re}}$ there is an inclusion

$$\{x_{\beta_1}(c_1)\cdots x_{\beta_\ell}(c_\ell)n_wB \mid c_1,\ldots,c_\ell\in\mathbb{C}\}\subseteq BwB.$$

To prove equality proceed by induction on ℓ.

Base case: Suppose that $w=s_j$. Let $\alpha\in R^+_{\mathrm{re}}$ and $c,d\in\mathbb{C}$. If $c=0$ or α,α_j is a prenilpotent pair then, by relation (3.2),

$$x_\alpha(d)x_{\alpha_j}(c)n_j^{-1}B = x_{\alpha_j}(c')n_j^{-1}B,\qquad \text{for some } c'\in\mathbb{C}. \tag{4.8}$$

If α,α_j is not a prenilpotent pair and $c\neq 0$ then $\alpha,-\alpha_j$ is a prenilpotent pair and, by (3.2),

$$x_\alpha(d)x_{\alpha_j}(c)n_j^{-1}B = x_\alpha(d)x_{-\alpha_j}(c^{-1})B = x_{-\alpha_j}(c^{-1})B = x_{\alpha_j}(c)n_j^{-1}B.$$

Thus $\{x_{\alpha_j}(c)n_j^{-1}B\mid c\in\mathbb{C}\}$ is B-invariant and so $Bs_jB=\{x_{\alpha_j}(c)n_j^{-1}B\mid c\in\mathbb{C}\}$.

Induction step: If $w=s_{i_1}\cdots s_{i_\ell}$ is reduced and if $\ell(ws_j)>\ell(w)$ then, by induction,

$$Bws_jB\subseteq BwB\cdot Bs_jB = \{x_{\beta_1}(c_1)\cdots x_{\beta_\ell}(c_\ell)x_{w\alpha_j}(c)n_wn_j^{-1}B \mid c_1,\ldots,c_\ell,c\in\mathbb{F}\},$$

so that $Bws_jB=\{x_{\beta_1}(c_1)\cdots x_{\beta_{\ell+1}}(c_{\ell+1})n_{ws_j}B\mid c_1,\ldots,c_{\ell+1}\in\mathbb{C}\}$ with $\beta_{\ell+1}=w\alpha_j$.

Step 2: Prove that $BwB=BvB$ if and only if $w=v$ by induction on $\ell(w)$.

Base case: Suppose that $\ell(w)=0$. Then $BwB=BvB$ implies that $v\in B$ so that there is a representative n_v of v such that $n_v\in B\cap N$. Then $vR^+_{\mathrm{re}}\subseteq R^+_{\mathrm{re}}$ since $n_v\mathcal{X}_\alpha n_v^{-1}=\mathcal{X}_{v\alpha}\in B$ for $\alpha\in R^+_{\mathrm{re}}$. So $\ell(v)=0$. Thus, by (2.16), $v=1$.

Induction step: Assume $BwB=BvB$ and s_j is such that $\ell(ws_j)<\ell(w)$. Since $BvB{\cdot}Bs_jB\subseteq BvB\cup Bvs_jB$ (see [St, Lemma 25],

$$Bws_jB\subseteq BwB\cdot Bs_jB = BvB\cdot Bs_jB\subseteq BvB\cup Bvs_jB = BwB\cup Bvs_jB.$$

Thus, by induction, $ws_j=w$ or $ws_j=vs_j$. Since $ws_j\neq w$, it follows that $w=v$.

Step 3: Let us show that if $x_{\alpha_{i_1}}(c_1)n_{i_1}^{-1}\cdots x_{\alpha_{i_\ell}}(c_\ell)n_{i_\ell}^{-1}B = x_{\alpha_{i_1}}(c_1')n_{i_1}^{-1}\cdots x_{\alpha_{i_\ell}}(c_\ell')n_{i_\ell}^{-1}B$, then $c_i=c_i'$ for $i=1,2,\ldots,\ell$. The left hand side of

$$x_{\alpha_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B = n_{i_1}x_{i_1}(c_1'-c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell')n_{i_\ell}^{-1}B$$

is in $Bs_{i_2}\cdots s_{i_\ell}B$. If $c_1'\neq c_1$ then $n_{i_1}^{-1}x_{i_1}(c_1'-c_1)n_{i_1}\in Bs_{i_1}B$ and the right hand side is contained in

$$n_{i_1}^{-1}x_{i_1}(c_1'-c_1)n_{i_1}Bs_{i_2}\cdots s_{i_\ell}B\subseteq Bs_{i_1}B\cdot Bs_{i_2}\cdots s_{i_\ell}B = Bs_{i_1}\cdots s_{i_\ell}B.$$

By Step 2 this is impossible and so $c_1'=c_1$. Then, by induction, $c_i'=c_i$ for $i=1,2,\ldots,\ell$.

Step 4: From the definition of $R(w)$ it follows that if $\alpha,\beta\in R(w)$ and $\alpha+\beta\in R_{\mathrm{re}}$ then $\alpha+\beta\in R(w)$ and if $\alpha,\beta\in R(w)$ then α,β form a prenilpotent pair. Thus, by [St, Lemma 17], any total order on the set $R(w)$ can be taken in the statement of the theorem. □

Remark 4.2. Suppose that $\lambda\in\mathfrak{h}^*$ is dominant integral and $M(\lambda)$ is an (integrable) highest weight representation of G generated by a highest weight vector v_λ^+. Then the set $BwBv_\lambda^+$ contains the vector wv_λ^+ and is contained in the sum $\bigoplus_{\nu\geq w\lambda}M(\lambda)_\nu$ of the weight spaces with weights $\geq w\lambda$. This is another way to show that if $w\neq v$ then $BwB\neq BvB$ and accomplish Step 2 in the proof of Theorem 4.1.

5 Loop Lie algebras and their extensions

This section gives a presentation of the theory of loop Lie algebras. The main lines of the theory are exactly as in the classical case (see, for example, [Mac2, §4] and [Kac, ch. 7]) but, following recent trends (see [Ga]. [GK], [GR] and [Rou]) we treat the more general setting of the loop Lie algebra of a Kac-Moody Lie algebra.

Let $\mathfrak{g}_0$ be a symmetrizable Kac-Moody Lie algebra with bracket $[,]_0\colon \mathfrak{g}_0\otimes\mathfrak{g}_0 \to \mathfrak{g}_0$ and invariant form $\langle,\rangle_0\colon \mathfrak{g}_0\times\mathfrak{g}_0\to\mathbb{C}$. The *loop Lie algebra* is

$$\mathfrak{g}_0[t,t^{-1}] = \mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}}\mathfrak{g}_0 \qquad \text{with bracket} \qquad [t^mx,t^ny]_0 = t^{m+n}[x,y]_0,$$

for $x,y\in\mathfrak{g}_0$. Let

$$\mathfrak{g} = \mathfrak{g}_0[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d, \qquad \mathfrak{g}' = \mathfrak{g}_0[t,t^{-1}]\oplus\mathbb{C}c, \qquad \bar{\mathfrak{g}}' = \mathfrak{g}_0[t,t^{-1}] = \frac{\mathfrak{g}'}{\mathbb{C}c}$$

where the bracket on $\mathfrak{g}$ is given by

$$[t^mx,t^ny] = t^{m+n}[x,y]_0+\delta_{m+n,0}m\langle x,y\rangle_0 c, \qquad c\in Z(\mathfrak{g}), \qquad [d,t^mx]=mt^mx. \tag{5.1}$$

By [Kac, Ex. 7.8], $\mathfrak{g}'$ is the universal central extension of $\bar{\mathfrak{g}}'$. An invariant symmetric form on $\mathfrak{g}$ is given by

$$\langle c,d\rangle = 1, \qquad \langle c,t^my\rangle = \langle d,t^my\rangle = 0, \qquad \langle c,c\rangle = \langle d,d\rangle = 0, \tag{5.2}$$

and

$$\langle t^mx,t^ny\rangle = \begin{cases}\langle x,y\rangle_0, & \text{if } m+n=0,\\ 0, & \text{otherwise,}\end{cases} \tag{5.3}$$

for $x,y\in\mathfrak{g}_0$, $m,n\in\mathbb{Z}$.

Fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and let

$$\mathfrak{h} = \mathfrak{h}_0\oplus\mathbb{C}c\oplus\mathbb{C}d, \qquad \mathfrak{h}' = \mathfrak{h}_0\oplus\mathbb{C}c, \qquad \bar{\mathfrak{h}}' = \mathfrak{h}_0. \tag{5.4}$$

As in (2.2), let $h_1,\ldots,h_n,d_1,\ldots,d_\ell$ be a basis of $\mathfrak{h}_0$ and let

$$\begin{matrix}\{h_1,\ldots,h_n,d_1,\ldots,d_\ell,c,d\} \text{ be a basis of } \mathfrak{h} \text{ and}\\ \{\omega_1,\ldots,\omega_n,\delta_1,\ldots,\delta_\ell,\Lambda_0,\delta\} \text{ the dual basis in } \mathfrak{h}^*\end{matrix} \tag{5.5}$$

so that

$$\delta(\mathfrak{h}_0)=0,\quad \delta(c)=0,\quad \delta(d)=1, \qquad \text{and} \qquad \Lambda_0(\mathfrak{h}_0)=0,\quad \Lambda_0(c)=1,\quad \Lambda_0(d)=0. \tag{5.6}$$

Let R be as in (2.9). As an $\mathfrak{h}$-module

$$\mathfrak{g} = \Bigg(\bigoplus_{\substack{\alpha\in R\\ k\in\mathbb{Z}}}\mathfrak{g}_{\alpha+k\delta}\Bigg)\oplus\Bigg(\bigoplus_{k\in\mathbb{Z}_{\neq 0}}\mathfrak{g}_{k\delta}\Bigg)\oplus\mathfrak{h}, \quad \text{where} \quad \mathfrak{h}=\mathfrak{h}_0\oplus\mathbb{C}c\oplus\mathbb{C}d, \tag{5.7}$$

$$\mathfrak{g}_{\alpha+k\delta} = t^k\mathfrak{g}_\alpha, \qquad \mathfrak{g}_{k\delta} = t^k\mathfrak{h}_0, \qquad \text{and} \qquad \tilde{R} = (R+\mathbb{Z}\delta)\cup\mathbb{Z}_{\neq 0}\delta \tag{5.8}$$

is the set of *roots* of $\mathfrak{g}$.

Let $\alpha\in R_{\mathrm{re}}$ with $\alpha = w\alpha_i$ and fix a choice of e_α, f_α and h_α in (2.18) (choose $\tilde{w}$). Then

$$e_{-\alpha+k\delta} = t^kf_\alpha, \qquad f_{-\alpha+k\delta} = t^{-k}e_\alpha, \qquad h_{-\alpha+k\delta} = -h_\alpha + k\langle e_\alpha,f_\alpha\rangle_0 c, \tag{5.9}$$

span a subalgebra isomorphic to $\mathfrak{sl}_2$. If $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is the decomposition in (2.10) and

$\mathfrak{n}^+$ is the subalgebra generated by $\mathfrak{n}_0^+$ and $e_{-\alpha+k\delta}$ for $\alpha \in R_{\mathrm{re}}$, $k \in \mathbb{Z}_{>0}$, and
$\mathfrak{n}^-$ is the subalgebra generated by $\mathfrak{n}_0^-$ and $f_{-\alpha+k\delta}$ for $\alpha \in R_{\mathrm{re}}$, $k \in \mathbb{Z}_{>0}$,

then

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{with} \quad \mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \left(\bigoplus_{\substack{\alpha \in R \cup \{0\} \\ k \in \mathbb{Z}_{>0}}} \mathfrak{g}_{\alpha+k\delta} \right) \quad \text{and} \quad \mathfrak{n}^- = \mathfrak{n}_0^- \oplus \left(\bigoplus_{\substack{\alpha \in R \cup \{0\} \\ k \in \mathbb{Z}_{<0}}} \mathfrak{g}_{\alpha+k\delta} \right).$$

The elements $e_{-\alpha+k\delta}$ and $f_{-\alpha+k\delta}$ in (5.9) act locally nilpotently on $\mathfrak{g}$ because f_α and e_α act locally nilpotently on $\mathfrak{g}_0$. Thus

$$\tilde{s}_{-\alpha+k\delta} = \exp(\operatorname{ad} t^k f_\alpha) \exp(-\operatorname{ad} t^{-k} e_\alpha) \exp(\operatorname{ad} t^k f_\alpha) \tag{5.10}$$

is a well defined automorphism of $\mathfrak{g}$ and

$$\tilde{s}_{-\alpha+k\delta}\mathfrak{g}_\beta = \mathfrak{g}_{s_{-\alpha+k\delta}\beta} \qquad \text{and} \qquad \tilde{s}_{-\alpha+k\delta}h = s_{-\alpha+k\delta}h, \tag{5.11}$$

for $h \in \mathfrak{h}$ and $\beta \in \tilde{R}$, where $s_{-\alpha+k\delta}\colon \mathfrak{h}^* \to \mathfrak{h}^*$ and $s_{-\alpha+k\delta}\colon \mathfrak{h} \to \mathfrak{h}$ are given by

$$s_{-\alpha+k\delta}\lambda = \lambda - \lambda(h_{-\alpha+k\delta})(-\alpha+k\delta) \quad \text{and} \quad s_{-\alpha+k\delta}h = h - (-\alpha+k\delta)(h)h_{-\alpha+k\delta}, \tag{5.12}$$

for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. The *Weyl group* of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h}^*)$ (or $GL(\mathfrak{h})$) generated by the reflections $s_{-\alpha+k\delta}$,

$$W_{\mathrm{aff}} = \langle s_{-\alpha+k\delta} \mid \alpha \in R_{\mathrm{re}}, k \in \mathbb{Z} \rangle. \tag{5.13}$$

Noting that $\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ and $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$, use (5.12) to compute

$$\begin{array}{ll}
s_{-\alpha+k\delta}(\bar{\lambda}) = \bar{\lambda} + \bar{\lambda}(h_\alpha)(-\alpha+k\delta), & s_{-\alpha+k\delta}(\bar{h}) = \bar{h} + \alpha(\bar{h})(-h_\alpha + k\langle e_\alpha, f_\alpha\rangle_0 c), \\
s_{-\alpha+k\delta}(\ell\Lambda_0) = \ell\Lambda_0 - k\ell\langle e_\alpha, f_\alpha\rangle_0(-\alpha+k\delta), & s_{-\alpha+k\delta}(mc) = mc, \\
s_{-\alpha+k\delta}(m\delta) = m\delta, & s_{-\alpha+k\delta}(\ell d) = \ell d - k\ell(-h_\alpha + k\langle e_\alpha, f_\alpha\rangle_0 c).
\end{array}$$

for $\bar{\lambda} \in \mathfrak{h}_0^*$, $\bar{h} \in \mathfrak{h}_0$, $m, \ell \in \mathbb{C}$. For $\alpha \in R_{\mathrm{re}}$ and $k \in \mathbb{Z}$

$$\text{define } t_{k\alpha^\vee} \in W_{\mathrm{aff}} \text{ by} \qquad s_{-\alpha+k\delta} = t_{k\alpha^\vee} s_{-\alpha}, \tag{5.14}$$

and use (2.26) and (2.27) to compute

$$\begin{array}{ll}
t_{k\alpha^\vee}(\bar{\lambda}) = \bar{\lambda} - \bar{\lambda}(kh_\alpha)\delta, & t_{k\alpha^\vee}(\bar{h}) = \bar{h} - k\alpha^\vee(\bar{h})c, \\
t_{k\alpha^\vee}(\ell\Lambda_0) = \ell\Lambda_0 + \ell k\alpha^\vee - \ell\frac{1}{2}\langle kh_\alpha, kh_\alpha\rangle_0\delta, & t_{k\alpha^\vee}(mc) = mc, \\
t_{k\alpha^\vee}(m\delta) = m\delta, & t_{k\alpha^\vee}(\ell d) = \ell d + \ell kh_\alpha - \ell\frac{1}{2}\langle kh_\alpha, kh_\alpha\rangle_0 c.
\end{array}$$

Then $t_{k\alpha^\vee}t_{j\beta^\vee}(\bar{\lambda}) = t_{kh_\alpha}(\bar{\lambda} - \bar{\lambda}(jh_\beta)\delta) = \bar{\lambda} - \bar{\lambda}(kh_\alpha + jh_\beta)\delta$, and

$$\begin{aligned}
t_{k\alpha^\vee}t_{j\beta^\vee}(\ell\Lambda_0) &= t_{k\alpha^\vee}\left(\ell\Lambda_0 + \ell j\beta^\vee - \ell\tfrac{1}{2}\langle jh_\beta, jh_\beta\rangle_0\delta\right) \\
&= \ell\Lambda_0 + \ell k\alpha^\vee - \ell\tfrac{1}{2}\langle kh_\alpha, kh_\alpha\rangle_0\delta + \ell j\beta^\vee - \ell j\beta^\vee(kh_\alpha)\delta - \ell\tfrac{1}{2}\langle jh_\beta, jh_\beta\rangle_0\delta \\
&= \ell\Lambda_0 + \ell(k\alpha^\vee + j\beta^\vee) - \ell\tfrac{1}{2}\langle kh_\alpha + jh_\beta, kh_\alpha + jh_\beta\rangle_0\delta.
\end{aligned}$$

This computation shows that $t_{k\alpha^\vee}t_{j\beta^\vee} = t_{j\alpha^\vee+k\beta^\vee}$. Thus, if W_0 is the Weyl group of $\mathfrak{g}_0$ and $Q^* = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ then

$$W_{\mathrm{aff}} = \{t_{\lambda^\vee}w \mid \lambda^\vee \in Q^*, w \in W_0\} \qquad \text{with} \qquad t_{\lambda^\vee}t_{\mu^\vee} = t_{\lambda^\vee+\mu^\vee} \quad \text{and} \quad wt_{\lambda^\vee} = t_{w\lambda^\vee}w, \tag{5.15}$$

for $w \in W_0$, $\lambda^\vee, \mu^\vee \in Q^*$.

Since $\mathbb{C}\delta$ is $W_{\rm aff}$-invariant, the group $W_{\rm aff}$ acts on $\mathfrak{h}^*/\mathbb{C}\delta$ and $W_{\rm aff}$ acts on the set

$$\begin{array}{ccc} (\mathfrak{h}_0^* + \Lambda_0 + \mathbb{C}\delta)/\mathbb{C}\delta & \xrightarrow{\sim} & \mathfrak{h}_0^* \\ \bar\lambda + \Lambda_0 + \mathbb{C}\delta & \longmapsto & \bar\lambda \end{array} \tag{5.16}$$

and the $W_{\rm aff}$-action on the right hand side is given by

$$s_\alpha(\bar\lambda) = \bar\lambda - \bar\lambda(h_\alpha)\alpha \qquad \text{and} \qquad t_{k\alpha^\vee}(\bar\lambda) = \bar\lambda + k\alpha^\vee, \qquad \text{for } \bar\lambda \in \mathfrak{h}_0. \tag{5.17}$$

Here $\mathfrak{h}_0^*$ is a set with a $W_{\rm aff}$-action, the action of $W_{\rm aff}$ is *not linear*.

6 Loop groups and the affine flag variety G/I

This section gives a short treatment of loop groups following [St, Ch. 8] and [Mac1, §2.5 and 2.6]. This theory is currently a subject of intense research as evidenced by the work in [Ga], [GK], [Rem], [Rou], [GR].

Let $\mathfrak{g}_0$ be a symmetrizable Kac-Moody Lie algebra and let $\mathfrak{h}_\mathbb{Z}$ be a $\mathbb{Z}$-lattice in $\mathfrak{h}_0$ that contains $Q^\vee = \mathbb{Z}\text{-span}\{h_1, \ldots, h_n\}$.

$$\text{The } \textit{loop group} \text{ is the Tits group } G = G_0(\mathbb{C}((t))) \tag{6.1}$$

over the field $\mathbb{F} = \mathbb{C}((t))$. Let $K = G_0(\mathbb{C}[[t]])$ and $G_0(\mathbb{C})$ be the Tits groups of $\mathfrak{g}_0$ and $\mathfrak{h}_\mathbb{Z}$ over the rings $\mathbb{C}[[t]]$ and $\mathbb{C}$, respectively, and let $B(\mathbb{C})$ be the standard *Borel subgroup* of $G_0(\mathbb{C})$ as defined in (4.2). Let

$$U^- \text{ be the subgroup of } G \text{ generated by } x_{-\alpha}(f) \text{ for } \alpha \in R^+_{\rm re} \text{ and } f \in \mathbb{C}((t)), \tag{6.2}$$

and define the standard *Iwahori subgroup* I of G by

$$\begin{array}{ccccc} G & = & G_0(\mathbb{C}((t))) & & \\ \cup\!\vert & & \cup\!\vert & & \\ K & = & G_0(\mathbb{C}[[t]]) & \stackrel{{\rm ev}_{t=0}}{\longrightarrow} & G_0(\mathbb{C}) \\ \cup\!\vert & & \cup\!\vert & & \cup\!\vert \\ I & = & {\rm ev}_{t=0}^{-1}(B(\mathbb{C})) & \stackrel{{\rm ev}_{t=0}}{\longrightarrow} & B(\mathbb{C}). \end{array} \tag{6.3}$$

The *affine flag variety* is G/I.

For $\alpha + j\delta \in R_{\rm re} + \mathbb{Z}\delta$ and $c \in \mathbb{C}$, define

$$x_{\alpha+j\delta}(c) = x_\alpha(ct^j) \qquad \text{and} \qquad t_{\lambda^\vee} = h_{\lambda^\vee}(t^{-1}), \tag{6.4}$$

and, for $c \in \mathbb{C}^\times$, define

$$n_{\alpha+j\delta}(c) = x_{\alpha+j\delta}(c)x_{-\alpha-j\delta}(-c^{-1})x_{\alpha+j\delta}(c), \tag{6.5}$$

$$n_{\alpha+j\delta} = n_{\alpha+j\delta}(1), \quad \text{and} \quad h_{(\alpha+j\delta)^\vee}(c) = n_{\alpha+j\delta}(c)n_{\alpha+j\delta}^{-1} \tag{6.6}$$

analogous to (3.3).

The group

$$\widetilde{W} = \{t_{\lambda^\vee} w \mid \lambda^\vee \in \mathfrak{h}_\mathbb{Z}, w \in W_0\} \qquad \text{with} \quad t_{\lambda^\vee}t_{\mu^\vee} = t_{\lambda^\vee+\mu^\vee} \quad \text{and} \quad wt_{\lambda^\vee} = t_{w\lambda^\vee}w, \tag{6.7}$$

acts on $\mathfrak{h}_0^* \oplus \mathbb{C}\delta$ by

$$v(\mu + k\delta) = v\mu + k\delta \qquad \text{and} \qquad t_{\lambda^\vee}(\mu + k\delta) = \mu + (k - \langle \lambda^\vee, \mu\rangle)\delta \tag{6.8}$$

for $v \in W_0$, $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$, $\mu \in \mathfrak{h}_\mathbb{Z}^*$, and $k \in \mathbb{Z}$. Then $n_{\alpha+j\delta}(c) = t_{-j\alpha^\vee} n_\alpha(c) = n_\alpha(ct^j)$,

$$n_\alpha x_{\beta+k\delta}(c) n_\alpha^{-1} = n_\alpha x_\beta(ct^k) n_\alpha^{-1} = x_{s_\alpha\beta}(\epsilon_{\alpha,\beta} ct^k) = x_{s_\alpha(\beta+k\delta)}(\epsilon_{\alpha,\beta} c)$$

for $\alpha \in R_{\mathrm{re}}$, and, for $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$,

$$t_{\lambda^\vee} x_{\beta+k\delta}(c) t_{\lambda^\vee}^{-1} = x_{\beta+k\delta}(t^{-\langle\lambda^\vee,\beta\rangle} c) = x_{t_{\lambda^\vee}(\beta+k\delta)}(c).$$

Thus the *root subgroups*

$$\mathcal{X}_{\alpha+j\delta} = \{x_{\alpha+j\delta}(c) \mid c \in \mathbb{C}\} \qquad \text{satisfy} \qquad w\mathcal{X}_{\alpha+j\beta} w^{-1} = \mathcal{X}_{w(\alpha+j\delta)} \tag{6.9}$$

for $w \in \widetilde{W}$ and $\alpha + j\delta \in R_{\mathrm{re}} + \mathbb{Z}\delta$. These relations are a reflection of the symmetry of the group G under the group defined in (3.8):

$$\widetilde{N} = N(\mathbb{C}((t))) \quad \text{generated by } n_\alpha(g),\ h_{\lambda^\vee}(g), \text{ for } g \in \mathbb{C}((t))^\times, \tag{6.10}$$

$\alpha \in R_{\mathrm{re}}$, and $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$. The homomorphism $\widetilde{N} \to W_0$ from (3.9) lifts to a surjective homomorphism (see [Mac1, p. 26 and p. 28])

$$\begin{array}{ccc} \widetilde{N} & \longrightarrow & \widetilde{W} \\ n_{\alpha+j\delta} & \longmapsto & t_{-j\alpha^\vee} s_\alpha \\ t_{\lambda^\vee} & \longmapsto & t_\lambda^\vee \end{array} \qquad \text{with kernel } H \text{ generated by } h_\lambda(d),\ d \in \mathbb{C}[[t]]^\times.$$

Define

$$\tilde{R}_{\mathrm{re}}^I = (R_{\mathrm{re}}^+ + \mathbb{Z}_{\geq 0}\delta) \sqcup (-R_{\mathrm{re}}^+ + \mathbb{Z}_{>0}\delta) \quad \text{and} \quad \tilde{R}_{\mathrm{re}}^U = -R_{\mathrm{re}}^+ + \mathbb{Z}\delta \tag{6.11}$$

so that

$$\begin{array}{llll} \mathcal{X}_{\alpha+j\delta} \subseteq I & \text{if and only if} & \alpha + j\delta \in \tilde{R}_{\mathrm{re}}^I & \text{and} \\ \mathcal{X}_{\alpha+j\delta} \subseteq U^- & \text{if and only if} & \alpha + j\delta \in \tilde{R}_{\mathrm{re}}^U. & \end{array} \tag{6.12}$$

Note that $\tilde{R}_{\mathrm{re}}^I \sqcup (-\tilde{R}_{\mathrm{re}}^I) = \tilde{R}_{\mathrm{re}}^U \sqcup (-\tilde{R}_{\mathrm{re}}^U) = R_{\mathrm{re}} + \mathbb{Z}\delta$.

7 The folding algorithm and the intersections $U^- vI \cap IwI$

In this section we prove our main theorem, which gives a precise connection between the alcove walks in [Ra] and the points in the affine flag variety. The algorithm here is essentially that which is found in [BD] and, with our setup from the earlier sections, it is the 'obvious one'. The same method has, of course, been used in other contexts, see, for example, [C].

A special situation in the loop group theory is when $\mathfrak{g}_0$ is finite dimensional. In this case, the extended loop Lie algebra $\mathfrak{g}$ defined in (5.1) is also a Kac-Moody Lie algebra. If G_0 is the Tits group of $\mathfrak{g}_0$ and $G = G_0(\mathbb{C}((t)))$ is the corresponding loop group then the subgroup I defined in (6.3) differs from the Borel subgroup of the Kac-Moody group G_{KM} for $\mathfrak{g}$ only by elements of T, and the affine flag variety of G coincides with the flag variety of G_{KM}. Thus, in this case, Theorem 4.1 provides a labeling of the points of the affine flag variety.

Suppose that $\mathfrak{g}_0$ is a finite dimensional complex semisimple Lie algebra presented as a Kac-Moody Lie algebra with generators $e_1, \ldots, e_n, f_1, \ldots, f_n, h_1, \ldots, h_n$ and Cartan matrix $A =$

$(\alpha_i(h_j))_{1\le i,j,\le n}$. Let φ be the highest root of R (the highest weight of the adjoint representation), fix

$$e_\varphi \in \mathfrak{g}_\varphi, \quad f_\varphi \in \mathfrak{g}_{-\varphi} \quad \text{such that} \quad \langle e_\varphi, f_\varphi\rangle_0 = 1,$$

and let

$$e_0 = e_{-\varphi+\delta} = tf_\varphi, \quad f_0 = f_{-\varphi+\delta} = t^{-1}e_\varphi, \qquad h_0 = [e_0, f_0] = [tx_{-\varphi}, t^{-1}x_\varphi] = -h_\varphi + c,$$

as in (5.9). The magical fact is that, in this case, $\mathfrak{g} = \mathfrak{g}_0[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Kac-Moody Lie algebra with generators $e_0, \dots, e_n$, $f_0, \dots, f_n$, $h_0, \dots, h_n, d$ and Cartan matrix

$$A^{(1)} = (\alpha_i(h_j))_{0\le i,j\le n}, \qquad \text{where} \quad \alpha_0 = -\varphi + \delta \quad \text{and} \quad h_0 = -h_\varphi + c, \tag{7.1}$$

where δ is as in (5.6) (see [Kac, Thm. 7.4]).

The *alcoves* are the open connected components of

$$\mathfrak{h}_\mathbb{R} \backslash \bigcup_{-\alpha+j\delta\in \tilde{R}^I_{\mathrm{re}}} H_{-\alpha+j\delta}, \qquad \text{where} \quad H_{-\alpha+j\delta} = \{x^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle x^\vee, \alpha\rangle = j\}.$$

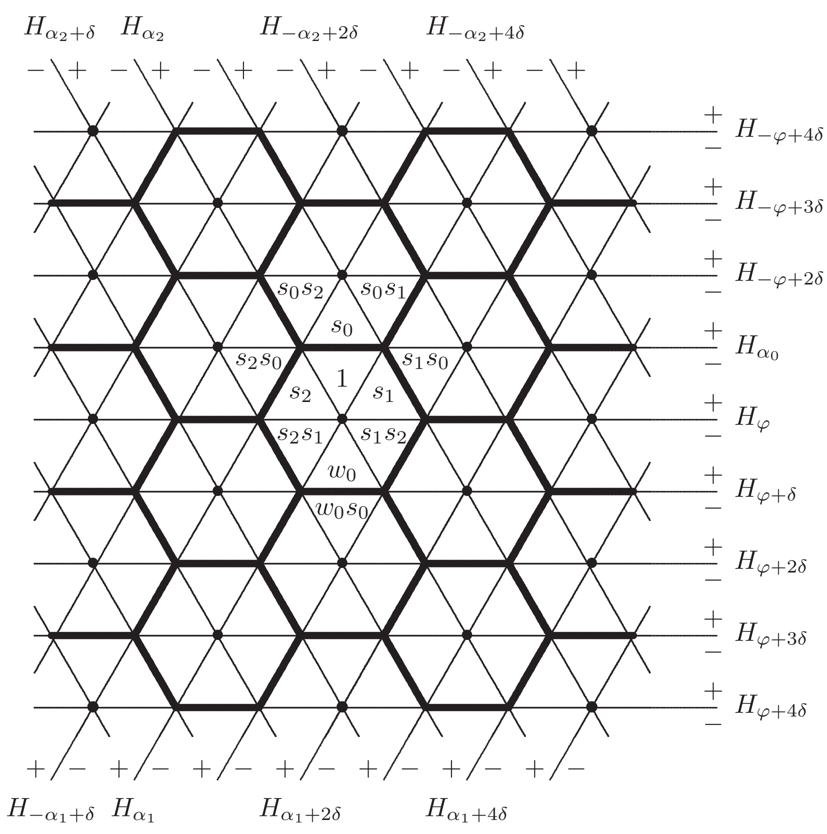
Under the map in (5.16) the chambers wC of the Tits cone X (see (2.20) and (2.21)) become the alcoves. Each alcove is a fundamental region for the action of W_{aff} on $\mathfrak{h}_\mathbb{R}$ given by (5.17) and W_{aff} acts simply transitively on the set of alcoves (see [Kac, Prop. 6.6]). Identify $1 \in W_{\mathrm{aff}}$ with the *fundamental alcove*

$$A_0 = \{x^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle x^\vee, \alpha_i\rangle > 0 \text{ for all } 0 \le i \le n\}$$

to make a bijection

$$W_{\mathrm{aff}} \longleftrightarrow \{\text{alcoves}\}.$$

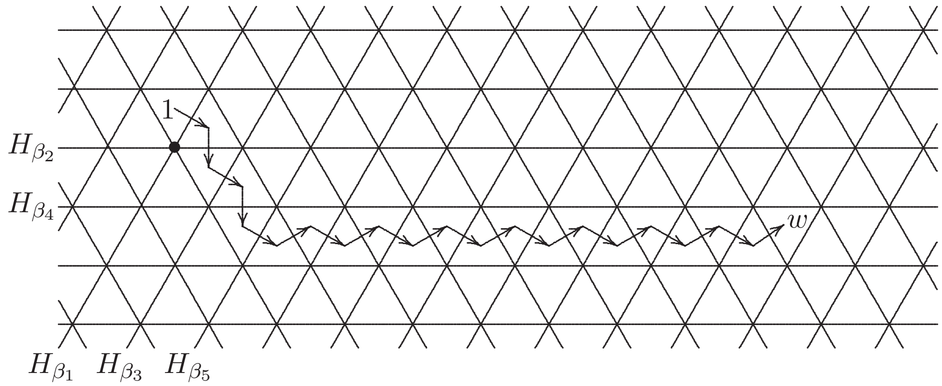
For example, when $\mathfrak{g}_0 = \mathfrak{sl}_3$,



(7.2)

The alcoves are the triangles and the (centres of) hexagons are the elements of $Q^\vee$.

Let $w \in W_{\mathrm{aff}}$. Following the discussion in (4.4)-(4.6), a reduced expression $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a *walk* starting at 1 and ending at w,

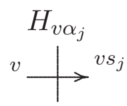


and the points of

$$IwI = \{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}I \mid c_1,\ldots,c_\ell \in \mathbb{C}\} \tag{7.3}$$

are in bijection with labelings of the edges of the walk by complex numbers $c_1,\ldots,c_\ell$. The elements of $R(w) = \{\beta_1,\ldots,\beta_\ell\}$ are the elements of $\tilde{R}^I_{\mathrm{re}}$ corresponding to the sequence of hyperplanes crossed by the walk.

The labeling of the hyperplanes in (7.2) is such that neighboring alcoves have



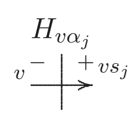
with $v\alpha_j \in \tilde{R}^I_{\mathrm{re}}$ if v is closer to 1 than vs_j. (7.4)

The *periodic orientation* (illustrated in (7.2)) is the orientation of the hyperplanes $H_{\alpha+k\delta}$ such that

(a) 1 is on the positive side of H_α for $\alpha \in R^+_{\mathrm{re}}$,

(b) $H_{\alpha+k\delta}$ and H_α have parallel orientiations.

This orientation is such that

$v\alpha_j \in \tilde{R}^U_{\mathrm{re}}$ if and only if


. (7.5)

Together, (7.4) and (7.5) provide a powerful combinatorics for analyzing the intersections $U^-vI \cap IwI$. We shall use the first identity in (3.3), in the form

$$x_\alpha(c)n_\alpha^{-1} = x_{-\alpha}(c^{-1})x_\alpha(-c)h_{\alpha^\vee}(c) \qquad \text{(main folding law)}, \tag{7.6}$$

to rewrite the points of IwI given in (7.3) as elements of U^-vI. Suppose that

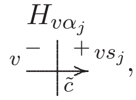
$$x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} = x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_v b, \qquad \text{where } b \in I, \tag{7.7}$$

$v \in W_{\mathrm{aff}}$ and $n_v = n_{j_1}^{-1}\cdots n_{j_k}^{-1}$ if $v = s_{i_1}\cdots s_{i_k}$ is a reduced word, and $\gamma_1,\ldots,\gamma_\ell \in \tilde{R}^U_{\mathrm{re}}$ so that $x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell') \in U^-$. Then the procedure described in (7.8)-(7.10) will compute $c_{\ell+1}' \in \mathbb{C}$, $b' \in I$, $v' \in W_{\mathrm{aff}}$ and $\gamma_{\ell+1} \in \tilde{R}^U_{\mathrm{re}}$ so that

$$x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}x_j(c)n_j^{-1} = x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')x_{\gamma_{\ell+1}}(c_{\ell+1})n_{v'}b'.$$

Keep the notations in (7.7). Since $bx_j(c)n_j^{-1} \in Is_jI$ there are unique $\tilde{c} \in \mathbb{C}$ and $b' \in I$ such that $bx_j(c)n_j^{-1} = x_j(\tilde{c})n_j^{-1}b'$ and

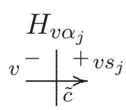
$$\begin{aligned} x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} x_j(c)n_j^{-1} &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v b x_j(c) n_j^{-1} \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_j(\tilde{c}) n_j^{-1} b'. \end{aligned}$$

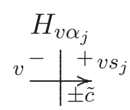
Case 1: If $v\alpha_j \in \tilde{R}^U_{\mathrm{re}}$, 

, then $x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_j(\tilde{c}) n_j^{-1} b'$ is equal to

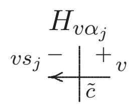
$$x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{v\alpha_j}(\pm\tilde{c}) n_{vs_j} b' \in U^- vs_jI \cap Iws_jI.$$

In this case, $\gamma_{\ell+1} = v\alpha_j$, $v' = vs_j$, and



becomes


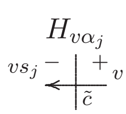
. (7.8)

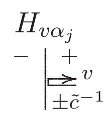
Case 2: If $v\alpha_j \notin \tilde{R}^U_{\mathrm{re}}$ and $\tilde{c} \neq 0$, 

, then

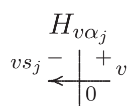
$$\begin{aligned} x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{\alpha_j}(\tilde{c}) n_j^{-1} b' &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{-\alpha_j}(\tilde{c}^{-1}) x_{\alpha_j}(-\tilde{c}) h_{\alpha_j^\vee}(\tilde{c}) b' \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{-\alpha_j}(\tilde{c}^{-1}) b'' \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{\gamma_{\ell+1}}(\pm\tilde{c}^{-1}) n_v b'' \in U^- vI \cap Iws_jI, \end{aligned}$$

where $\gamma_{\ell+1} = -v\alpha_j$ and $b'' = x_{\alpha_j}(-\tilde{c})h_{\alpha_j^\vee}(\tilde{c})b'$. So



becomes


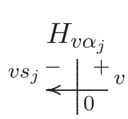
(7.9)

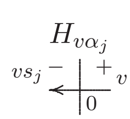
Case 3: If $v\alpha_j \notin \tilde{R}^U_{\mathrm{re}}$ and $\tilde{c} = 0$, 

, then

$$\begin{aligned} x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{\alpha_j}(0) n_j^{-1} b' &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{-\alpha_j}(0) n_j^{-1} b' \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{\gamma_{\ell+1}}(0) n_{vs_j} b' \in U^- vs_jI \cap Iws_jI, \end{aligned}$$

where $\gamma_{\ell+1} = -v\alpha_j$. So



becomes


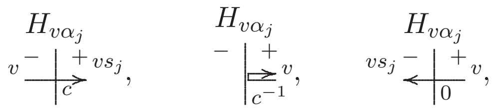
(7.10)

We have proved the following theorem.

Theorem 7.1. *If $w \in W_{\mathrm{aff}}$ and $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a minimal length walk to w define*

$$\mathcal{P}(\vec{w})_v = \left\{ \begin{matrix} \text{labeled folded paths } p \text{ of type } \vec{w} \\ \text{which end in } v \end{matrix} \right\} \qquad \text{for } v \in W_{\mathrm{aff}},$$

where a labeled folded path of type $\vec{w}$ *is a sequence of steps of the form*



where the kth step has $j = i_k$.

Viewing $U^- vI \cap IwI$ *as a subset of* G/I*, there is a bijection*

$$\mathcal{P}(\vec{w})_v \longleftrightarrow U^- vI \cap IwI.$$

8 An example

For the group $G = SL_3(\mathbb{C}((t)))$,

$$x_{\alpha_1}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad h_{\alpha_1^\vee}(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$x_{\alpha_2}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad h_{\alpha_2^\vee}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$x_{\alpha_0}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ct & 0 & 1 \end{pmatrix} \quad h_{\alpha_0^\vee}(c) = \begin{pmatrix} c^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}, \quad n_0 = \begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & 1 & 0 \\ t & 0 & 0 \end{pmatrix}.$$

Let $w = s_2 s_1 s_0 s_2 s_0 s_1 s_0 s_2 s_0$ and $v = s_2 s_1 s_0 s_2 s_1 s_2 s_0$ so that

$$w = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-2} & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 & 0 \\ t^2 & 0 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}.$$

We shall use Theorem 7.1 to show that the points of $IwI \cap U^- vI$ are

$$x_2(c_1)n_2^{-1}x_1(c_2)n_1^{-1}x_0(c_3)n_0^{-1}x_2(c_4)n_2^{-1}x_0(c_5)n_0^{-1}x_1(c_6)n_1^{-1}x_0(c_7)n_0^{-1}x_2(c_8)n_2^{-1}x_0(c_9)n_0^{-1}I,$$

with $c_1, \ldots, c_9 \in \mathbb{C}$ such that

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad c_5 \neq 0, \quad c_6 = 0, \quad c_7 \neq 0, \quad c_9 = c_7^{-1} c_8. \tag{8.1}$$

Precisely,

$$x_2(0)n_2^{-1}x_1(0)n_1^{-1}x_0(0)n_0^{-1}x_2(0)n_2^{-1}x_0(c_5)n_0^{-1}x_1(0)n_1^{-1}x_0(c_7)n_0^{-1}x_2(c_8)n_2^{-1}x_0(c_7^{-1}c_8)n_0^{-1}$$

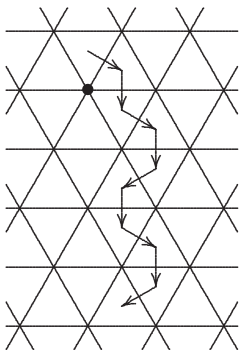
is equal to $u_9 v_9 b_9$, with $u_9 \in U^-$, $v_9 \in N$, $b_9 \in I$ given by

$$\begin{gathered} u_9 = \begin{pmatrix} 1 & 0 & 0 \\ c_5^{-1} - c_5^{-2} c_7^{-1} c_8 t & 1 & 0 \\ c_5^{-1} c_7^{-1} t^{-2} & 0 & 1 \end{pmatrix}, \qquad v_9 = \begin{pmatrix} 0 & 1 & 0 \\ -t^2 & 0 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \\ b_9 = \begin{pmatrix} c_5^{-1} - c_5^{-2} c_7^{-1} c_8 t & -c_5^{-2} c_7^{-1} c_8^2 & c_5^{-2} c_7^{-2} c_8^2 \\ -t^2 & c_5 c_7 + c_8 t & -c_5 - c_7^{-1} c_8 t \\ -c_5^{-1} c_7^{-1} t^2 & -c_5^{-1} c_7^{-1} c_8 t & c_7^{-1} + c_5^{-1} c_7^{-2} c_8 t \end{pmatrix}, \end{gathered} \tag{8.2}$$

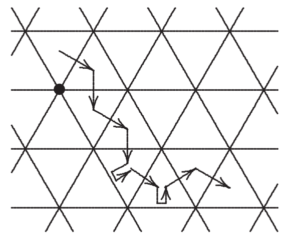
so that $u_9 = x_{-\alpha_2}(d_1)x_{-\varphi}(d_2)x_{-\alpha_2-\delta}(d_3)x_{-\varphi-\delta}(d_4)x_{-\alpha_1}(d_5)x_{-\alpha_2-2\delta}(d_6)x_{-\varphi-3\delta}(d_7)x_{-\alpha_1+\delta}(d_8)$ $\cdot x_{-\alpha_2-3\delta}(d_9)$ with

$$d_1 = d_2 = d_3 = d_4 = 0, \quad d_5 = c_5^{-1}, \quad d_6 = 0, \quad d_7 = c_5^{-1}c_7^{-1}, \quad d_8 = -c_5^{-2}c_7^{-1}c_8, \quad d_9 = 0. \tag{8.3}$$

Pictorially, the walk with labels $c_1, \ldots, c_9$



becomes



,

the labeled folded path with labels $d_1, \ldots, d_9$.

The step by step computation is as follows:

Step 1: If $c_1 = 0$ then

$$x_2(c_1)n_2^{-1} = x_{-\alpha_2}(0)n_2^{-1} = u_1v_1b_1, \qquad \text{with}$$

$$u_1 = x_{-\alpha_2}(0), \qquad v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad b_1 = 1.$$

Step 2: If $c_2 = 0$ then, since $v_1x_1(c_2)v_1^{-1} = x_\varphi(c_2)$,

$$u_1v_1b_1x_1(c_2)n_1^{-1} = u_1x_\varphi(c_2)v_1n_1^{-1}b_1 = u_1x_{-\varphi}(0)v_1n_1^{-1}b_1 = u_2v_2b_2, \qquad \text{with}$$

$$u_2 = u_1x_{-\varphi}(0), \qquad v_2 = v_1n_1^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad b_2 = 1.$$

Step 3: If $c_3 = 0$ then, since $v_2x_0(c_3)v_2^{-1} = x_{\alpha_2+\delta}(-c_3)$,

$$u_2v_2b_2x_0(c_3)n_0^{-1} = u_2x_{\alpha_2+\delta}(-c_3)v_2n_0^{-1}b_2 = u_2x_{-\alpha_2-\delta}(0)v_2n_0^{-1}b_2 = u_3v_3b_3, \qquad \text{with}$$

$$u_3 = u_2x_{-\alpha_2-\delta}(0), \qquad v_3 = v_2n_0^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ t & 0 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}, \qquad \text{and} \qquad b_3 = 1.$$

Step 4: If $c_4 = 0$ then, since $v_3x_2(c_4)v_3^{-1} = x_{\varphi+\delta}(-c_4)$,

$$u_3v_3b_3x_2(c_4)n_2^{-1} = u_3x_{\varphi+\delta}(-c_4)v_3n_2^{-1}b_3 = u_3x_{-\varphi-\delta}(0)v_3n_2^{-1}b_3 = u_4v_4b_4, \qquad \text{with}$$

$$u_4 = u_3x_{-\varphi-\delta}(0), \qquad v_4 = v_3n_2^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ t & 0 & 0 \\ 0 & t^{-1} & 0 \end{pmatrix} \qquad \text{and} \qquad b_4 = 1.$$

Step 5: If $c_5 \neq 0$ then by the folding law and the fact that $v_4 x_{-\alpha_0}(c_5^{-1}) v_4^{-1} = x_{-\alpha_1}(c_5^{-1})$,

$$u_4 v_4 b_4 x_0(c_5) n_0^{-1} = u_4 v_4 x_{-\alpha_0}(c_5^{-1}) x_{\alpha_0}(-c_5) h_{\alpha_0^\vee}(c_5) b_4 = u_4 x_{-\alpha_1}(c_5^{-1}) v_4 b_5 = u_5 v_5 b_5,$$

where

$$u_5 = u_4 x_{-\alpha_1}(c_5^{-1}), \qquad v_5 = v_4, \qquad \text{and} \qquad b_5 = x_{\alpha_0}(-c_5) h_{\alpha_0^\vee}(c_5) b_4 = \begin{pmatrix} c_5^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & c_5 \end{pmatrix}.$$

Step 6: If $c_5^{-1} c_6 = 0$ (so $c_6 = 0$) then

$$u_5 v_5 b_5 x_1(c_6) n_1^{-1} = u_5 v_5 x_1(c_5^{-1} c_6) n_1^{-1} b_5' = u_5 x_{-\alpha_2 - 2\delta}(0) v_5 n_1^{-1} b_5' = u_6 v_6 b_6,$$

with

$$u_6 = u_5 x_{-\alpha_2 - 2\delta}(0), \quad v_6 = v_5 n_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -t & 0 \\ t^{-1} & 0 & 0 \end{pmatrix} \quad \text{and} \quad b_6 = b_5' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_5^{-1} & 0 \\ -c_6 t & t & c_5 \end{pmatrix}$$

so that $b_5 x_1(c_6) n_1^{-1} = x_1(c_5^{-1} c_6) n_1^{-1} b_5'$.
Step 7: If $c_5 c_7 \neq 0$ then, since $v_6 x_{-\alpha_0}(c) v_6^{-1} = x_{-\varphi - 2\delta}(c)$,

$$\begin{aligned} u_6 v_6 b_6 x_0(c_7) n_0^{-1} &= u_6 v_6 x_0(c_5 c_7) n_0^{-1} b_6' = u_6 v_6 x_{-\alpha_0}(c_5^{-1} c_7^{-1}) x_{\alpha_0}(-c_5 c_7) h_{\alpha_0^\vee}(c_5 c_7) b_6' \\ &= u_6 x_{-\varphi - 2\delta}(c_5^{-1} c_7^{-1}) v_6 b_7 = u_7 v_7 b_7, \end{aligned}$$

where

$$u_7 = u_6 x_{-\varphi - 2\delta}(c_5^{-1} c_7^{-1}), \qquad v_7 = v_6, \qquad \text{and}$$

$$b_6' = \begin{pmatrix} c_5 & -1 & 0 \\ 0 & c_5^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b_7 = x_{\alpha_0}(-c_5 c_7) h_{\alpha_0^\vee}(c_5 c_7) b_6' = \begin{pmatrix} c_7^{-1} & -c_5^{-1} c_7^{-1} & 0 \\ 0 & c_5^{-1} & 0 \\ -c_5 t & t & c_5 c_7 \end{pmatrix},$$

so that $b_6 x_0(c_7) n_0^{-1} = x_0(c_5 c_7) n_0^{-1} b_6'$.
Step 8: No restrictions on $c_5^{-2} c_7^{-1} c_8$. Since $v_7 x_{\alpha_2}(c) v_7^{-1} = x_{-\alpha_1 + \delta}(-c)$,

$$u_7 v_7 b_7 x_2(c_8) n_2^{-1} = u_7 v_7 x_2(c_5^{-2} c_7^{-1} c_8) n_2^{-1} b_7' = u_7 x_{-\alpha_1 + \delta}(-c_5^{-2} c_7^{-1} c_8) v_7 n_2^{-1} b_7' = u_8 v_8 b_8,$$

with

$$u_8 = u_7 x_{-\alpha_1 + \delta}(-c_5^{-2} c_7^{-1} c_8), \qquad v_8 = v_7 n_2^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & t \\ t^{-1} & 0 & 0 \end{pmatrix}, \qquad \text{and}$$

$$b_8 = b_7' = \begin{pmatrix} c_7^{-1} & -c_5^{-1} c_7^{-1} c_8 & c_5^{-1} c_7^{-1} \\ -c_5 t & c_5 c_7 + c_8 t & -t \\ -c_5^{-1} c_7^{-1} c_8 t & c_5^{-2} c_7^{-1} c_8^2 t & c_5^{-1} - c_5^{-2} c_7^{-1} c_8 t \end{pmatrix},$$

so that $b_7 x_2(c_8) n_2^{-1} = x_2(c_5^{-2} c_7^{-1} c_8) n_2^{-1} b_7'$.
Step 9: If $c_5^{-1} c_7 c_9 - c_5^{-1} c_8 = 0$ (so $c_9 = c_7^{-1} c_8$) then

$$u_8 v_8 b_8 x_0(c_9) n_0^{-1} = u_8 v_8 x_0(c_5^{-1} c_7 c_9 - c_5^{-1} c_8) n_0^{-1} b_8' = u_8 x_{-\alpha_2 - 3\delta}(0) v_8 n_0^{-1} b_8' = u_9 v_9 b_9$$

with u_9, v_9 and b_9 as in (8.2).

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