

Symmetry breaking, subgroup embeddings and the Weyl group

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Abstract

We present a systematic approach to writing adjoint Higgs vacuum expectation values (vevs), which break a symmetry G to differently embedded isomorphic copies of a subgroup belonging to the chain $G \supset H_1 \supset \dots \supset H_l$, as linear combinations of each other. Given an adjoint Higgs vacuum expectation value h breaking $G \rightarrow H$, a full complement of vevs breaking G to different embeddings of the subgroup H can be generated through the Weyl group orbit of h . An explicit formula for recovering each vev is given. We focus on the case when H stabilizes the highest weight of the lowest dimensional fundamental representation, where the formula is exceedingly simple. We also discuss cases when the Higgs field is not in the adjoint representation and apply these techniques to current research problems, especially in domain-wall brane model building.

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I. INTRODUCTION

Symmetry breaking is a crucial aspect of modern particle physics. In particular the symmetry breaking sectors of theories extending the standard model are studied intensively. Many of the most puzzling problems in generic standard model extensions, such as the gauge hierarchy and parameter proliferation problems, arise because of the use of elementary scalar fields to spontaneously break symmetries. Deeper insights into both the physics and mathematics of symmetry breaking are thus worth having.

The purpose of this paper is to explain the mathematics of differently embedded but isomorphic subgroups, where the latter are obtained from the former through symmetry breaking. This issue has arisen in a number of contexts in the high-energy physics literature, including:

- Grand unified theories (GUTs), where so-called “flipped” models arise whenever there are alternative embeddings of a given GUT inside a larger GUT [1],[2].
- Domain-wall brane scenarios which use the “clash of symmetries” mechanism [4],[3],[6]. This idea was the main motivation for us to pursue the present study.
- The low-energy limit of Yang-Mills theory.
- Whenever there are multiple copies Φ_1, Φ_2, \dots of a given representation of Higgs fields, with vacuum expectation values (vevs) $\langle \Phi_1 \rangle, \langle \Phi_2 \rangle, \dots$ breaking the gauge group to isomorphic but differently embedded subgroups. This is a special case of what is generally termed “vacuum alignment”.

Each of these physical contexts will be reviewed in more detail in the next section. Of course, there may well be other applications for different embeddings of isomorphic subgroups.

In this paper we give results for the adjoint representation of a Lie group G , describing how (and when) the vacuum expectation values (vevs) of a collection of symmetry breaking fields, which cause an internal symmetry G to break along a chain of subgroups $G \supset H_1 \supset \dots \supset H_l$, can be written as a linear combination of an equivalent set of vevs which break G to an isomorphic but differently embedded subgroup chain $G \supset gH_1g^{-1} \supset \dots \supset gH_lg^{-1}$, for those $g \in G$ which map a Cartan subalgebra into itself. These different embeddings are related by conjugation by an element of the Weyl group. We highlight a simple method for constructing

linear combinations relating vevs which pick out differently embedded isomorphic copies of the subgroup H , which stabilizes the highest weight of the lowest dimensional fundamental representation.

More generally we consider an arbitrary representation. We begin with an explicit choice of the Cartan subalgebra h^1, \dots, h^l , transcribed from the branching rules of the representation for $G \supset H_1 \supset \dots \supset H_l$, where each $H_i = H'_i \times U(1)_{H_1} \times \dots \times U(1)_{H_i}$ includes $U(1)_{H_1, \dots, H_i}$ factors generated by h^1, \dots, h^i respectively. For conjugations of the Lie algebra \mathcal{L} by $g \in G$ we give results for how (and when) the Cartan subalgebra $gh^1g^{-1}, \dots, gh^lg^{-1}$ can be written as a linear combination of h^1, \dots, h^l . We cover the special case of linear combinations relating the generator h of the $U(1)_H$ factor in the subgroup, $H = H' \times U(1)_H$, which stabilizes the highest weight of the representation. We also cover how to write the weights of vevs which break the symmetry to these differently embedded subgroups as linear combinations of each other.

In Sec. VI we clearly state the formula for recovering the adjoint Higgs vevs which break G to different embeddings of a subgroup H as linear combinations of vevs breaking G along the chain $G \supset H_1 \supset H_2 \supset \dots \supset H_l$. We also treat the relation between the weights of vevs causing G to break to different embeddings of a subgroup H , for a nonadjoint Higgs field. This key result is preceded, in Sec. II, by a discussion of the four physical contexts listed above where these results may be applied. In Sec. III we canvas the notation we intend to use. Sections IV-V set up the proof and the explanation behind the formulas presented in Sec. VI. The remaining sections contain case studies which physically contextualize the root systems discussion in this paper and explicitly apply the formulas derived in VI, as well as the conclusion.

II. MOTIVATION

We now explain some of the physical contexts for our work in more detail.

A. Flipped grand unification

The simplest example of flipped grand unification is flipped $SU(5)$ [1],[2]. The relationship between standard and flipped $SU(5)$ may be explained using two different embeddings of

$SU(5) \times U(1)$ inside $SO(10)$. Call these two subgroups $SU(5)_s \times U(1)_{X_s}$ and $SU(5)_f \times U(1)_{X_f}$. One of these embeddings has been labelled s for “standard”, and the other f for “flipped”. The selection of one as standard is purely a matter of convention; the important issue is the relationship between the two embeddings. Having decided to call one embedding “standard”, the standard weak hypercharge generator is identified as the Y_s obtained through $SU(5)_s \rightarrow SU(3) \times SU(2) \times U(1)_{Y_s}$. By contrast in the flipped case, the weak hypercharge generator is Y_f , which arises from a second embedding of $SU(5)$ inside $SO(10)$; namely $SO(10) \rightarrow SU(5)_f \times U(1)_{X_f} \rightarrow [SU(3) \times SU(2) \times U(1)_{Y_f}] \times U(1)_{X_f}$, where X_f is a linear combination of Y_s and X_s . $U(1)_{Y_f}$ is not a subgroup of $SU(5)_s$, in fact Y_f is a linear combination of Y_s and X_s which is linearly independent of X_f .

This concept can be extended through E_6 grand unification. The subgroup chain

$$E_6 \rightarrow SO(10) \times U(1)'' \rightarrow SU(5) \times U(1)' \times U(1)'' \quad (\text{II.1})$$

can be shown to contain three possible candidates for weak hypercharge: standard, flipped, and double-flipped. Standard hypercharge is a generator of $SU(5)$. The flipped choice is a linear combination of standard hypercharge and the $U(1)'$ generator, while the double-flipped choice also involves an admixture of the generator of $U(1)''$. Once again, each of these candidate hypercharges is actually a generator of a differently-embedded $SU(5)$ subgroup within E_6 .

B. Domain-wall brane models

This work was primarily motivated by a study of domain wall topological defects created by an adjoint scalar field, \mathcal{X} . In particular we study the case where the Lagrangian is invariant under a discrete symmetry, Z , and a continuous internal symmetry G but along two distinct antipodal directions the asymptotic configuration of the scalar field breaks $Z \times G$ down to differently embedded isomorphic copies of $H \subset G$. This construction has a natural manifestation in grand-unified models with gauge group G and a single infinite extra dimension. Here the adjoint scalar field interpolates between two vacuum configurations preserving subgroups H and $zgHg^{-1}$ (for some $z \in Z$ and $g \in G$) as a function of the

extra dimensional co-ordinate, y .¹ The case $g = 1$ defines what may be called the standard domain wall or kink. In this case, the spontaneous symmetry breaking produces exactly the same unbroken subgroup H on opposite sides of the domain wall. At generic values of y , the configuration is also stabilized by exactly that same H , except for a finite number of points where the unbroken subgroup may be instantaneously larger than H . The interesting fact is that for certain $g \neq 1$, domain wall solutions can also exist. This situation has been termed the “clash of symmetries (CoS)”, because now the unbroken subgroups in the “bulk” on opposite sides of the domain wall are no longer identical, though they are isomorphic [4],[3], [6].

One reason to be interested in CoS domain walls is the dynamical localization of massless gauge fields to the domain wall, thus effecting a dimensional reduction from a $d + 1$ -dimensional gauge theory to a $(d - 1) + 1$ -dimensional gauge theory. The idea, which is an elaboration of an original proposal due to Dvali and Shifman [5], is as follows. We suppose that the non-Abelian factors in the H and gHg^{-1} gauge theories produced on opposite sides of the wall are in confinement phase. The underlying mechanism for this might, for example, be dual superconductivity. On the wall, the unbroken subgroup is $H \cap gHg^{-1}$, which is a subgroup of both H and gHg^{-1} . The idea is that the gauge fields of a certain subgroup of $H \cap gHg^{-1}$ are dynamically localized, due to the mass gap created by the confining dynamics in the bulk. An example of this situation has been provided in [6]. Here, E_6 breaks to differently embedded $SO(10) \times U(1)$ subgroups in the bulk on opposite sides of the domain wall. For appropriately chosen pairs of these subgroups, their intersection is $SU(5) \times U(1) \times U(1)$. By hypothesising that the $SO(10)$ gauge forces lead to confinement, the conclusion is that the $SU(5)$ gauge fields should be dynamically localized on the wall.² This is interesting for model building when $d = 4$, because the dynamically-localized $d = 3$ $SU(5)$ gauge theory could form the basis for a phenomenologically-realistic standard model extension.

¹ The role of the discrete symmetry breaking is to provide disconnected vacua which then serve as the boundary conditions for topologically non-trivial domain wall solutions. Cosmologically, one expects domain wall formation when causally disconnected patches of spacetime acquire different vacuum configurations.

² It has not been definitely established that the Dvali-Shifman mechanism works, but the heuristics are compelling. Note that for $d > 3$, the bulk dynamics is governed by a non-renormalisable gauge theory that must be implicitly defined with an ultraviolet cut-off, beyond which new physics must be invoked to complete the dynamics. Studies of Yang-Mills theory in $4 + 1$ dimensions at finite lattice spacing, which acts as an ultraviolet cut-off, support the existence of a confinement phase when the gauge coupling constant is above a critical value.

To implement the CoS mechanism we must solve the Euler-Lagrange equations for \mathcal{X} for boundary conditions as $y \rightarrow \pm\infty$ breaking $G \times Z$ to H and $zgHg^{-1}$, respectively. Therefore it is necessary to understand how the boundary conditions breaking G to gHg^{-1} can be written as a linear combination of the adjoint scalar field vevs breaking G along the $H_{1,2,3,\dots,l}$ branching direction in the Cartan subalgebra.

Solutions to the Euler-Lagrange equations satisfying different boundary conditions have different energies. A boundary condition preserving a symmetry H can be continuously transformed into a boundary condition preserving any other isomorphic subgroup gHg^{-1} inside G , and for some choices of g solutions interpolating between the H - and $zgHg^{-1}$ -preserving boundary conditions exist. The phenomenology of each of these domain wall solutions is different because each different non-isomorphic intersection $H \cap gHg^{-1}$ will give rise to a different gauge theory localized on the domain wall. Hence an exhaustive search for the lowest energy stable domain wall configuration must be executed. This search must range through all solutions to the Euler-Lagrange equations with different boundary conditions. In this case a systematic method for finding all the different possible configurations must be established. To trap a copy of the standard model gauge group on the domain wall, the grand unified gauge group must have a comparatively high rank, for example E_6 as in [6]. For high rank groups a method for writing one set of boundary conditions in terms of another becomes critical.

To find the vev for the adjoint \mathcal{X} breaking G to a subgroup gHg^{-1} as a linear combination of vevs along the $H_{1,2,3,\dots,l}$ branching direction in the Cartan subalgebra, the authors of [6] wrote down the Casimir operators (invariants) for a general linear combination of the Cartan subalgebra, h^1, \dots, h^l . The coordinates in the Cartan subalgebra space which extremize the Casimir operators correspond to linear combinations which break G to certain subgroups, including H and gHg^{-1} . The physical reason for this is: invariance of the action under the internal symmetry forces the potential to be a polynomial in the Casimir invariants. Therefore extrema of the Casimir operators correspond to degenerate minima in the vacuum manifold associated with spontaneous breaking of the internal symmetry G to various subgroups, including to differently embedded isomorphic copies of a subgroup $H = H' \times U(1)_H$. Hence the coefficients in the linear combination which extremize the Casimir invariants are precisely the components of the adjoint Higgs field in the original Cartan subalgebra basis which combine to give the $U(1)_{gHg^{-1}}$ generator which spontaneously condenses to break

$G \rightarrow gHg^{-1}$. This approach is labor intensive, and the techniques to be explained in this paper will improve upon it.

C. Low-energy limit of Yang-Mills theory

We have found a natural motivation for our work in domain wall formation due to the breaking of a global symmetry on cosmological scales. At the other end of the spectrum, in low energy effective models for SU(3) (and SU(2)) pure Yang-Mills gauge theories, domain walls form due to a breakdown of Weyl group symmetry caused by gluon condensation. This gives rise to a trapping of gauge fields on the domain wall. Galilo and Nedelko [7] work with an effective potential generated by loop order corrections in a low energy effective field theory approach to QCD:

$$U_{\text{eff}} = \frac{1}{12} \text{Tr} \left(C_1 \hat{F}^2 + \frac{4}{3} C_2 \hat{F}^4 - \frac{16}{9} C_3 \hat{F}^6 \right), \quad (\text{II.2})$$

where the potential is confining provided $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, and the non-Abelian gauge field strength tensor, $\hat{F}_{\mu\nu}$, can be written in terms of the SU(3) Lie algebra structure constants f^{abc} as,

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - i f^{abc} G_\mu^b G_\nu^c, \\ \left(\hat{F}_{\mu\nu} \right)_{bc} &= F_{\mu\nu}^a T_{bc}^a, \quad T_{bc}^a = -i f^{abc}. \end{aligned} \quad (\text{II.3})$$

The second order Casimir invariant $\text{Tr}(\hat{F})^2 = -3F_{\mu\nu}^a F_{\mu\nu}^a \leq 0$, causing the minimum of the effective potential to occur at a nonzero gluon field strength.

$$F_{\mu\nu}^a F_{\mu\nu}^a = \frac{4}{9C_3^2} \left(\sqrt{C_2^2 + 3C_1 C_3} - C_2 \right)^2 \Lambda^4 > 0, \quad (\text{II.4})$$

where Λ is the QCD confinement scale.

Galilo and Nedelko [7] look at the effective potential for $\hat{F}_{\mu\nu} = h^\chi B_{\mu\nu}^\chi$, which involves restricting the full SU(3) gauge theory to the $U(1) \times U(1)$ Abelian subspace, where the generators are given as a linear combination of the diagonal Gell-Mann matrices,

$$h^\chi = \chi^1 \lambda_3 + \chi^2 \lambda_8, \quad (\text{II.5})$$

and the associated field strength $B_{\mu,\nu}^\chi$ can be found by using the Abelian subalgebra version of (II.3) on $B_\mu^\chi = \chi^1 G_\mu^3 + \chi^2 G_\mu^8$. The minima of the effective potential are located at:

$$\chi = \left(\text{Cos} \frac{(2n+1)\pi}{6}, \text{Sin} \frac{(2n+1)\pi}{6} \right) \text{ for } n \in \{0, \dots, 5\}. \quad (\text{II.6})$$

They are related by a discrete Weyl group symmetry. The requirement that QCD remains unbroken despite a non-zero background field strength means the background field must be the average of an ensemble of gauge field configurations with a high degree of disorder and spatial variation of the direction χ in colour-space. This causes different vacua to be picked out in different spatial regions. Galilo and Nedelko [7] explain that domain wall configurations are formed by gauge fields interpolating between these vacua. Collectively the h^x describe the vevs of an adjoint Higgs field which break $\text{SU}(3)$ to $\text{U}(1) \times \text{U}(1)$. Here they again form the boundary conditions for the domain wall.

In the pure $\text{SU}(2)$ Yang-Mills theory domain walls form between vacua preserving different embeddings of a $\text{U}(1)_\alpha$ symmetry associated with magnetic charge [8].

In both the above models there is an opportunity to trap gauge fields on the domain wall. This analysis can be generalized to $\text{SU}(n)$ pure Yang-Mills theory where the rank of the algebra will again necessitate a systematic way of identifying all the boundary conditions for the domain walls.

D. Vacuum alignment

Many extensions of the standard model feature multiple copies Φ_1, Φ_2, \dots of Higgs multiplets transforming according to a given representation of the gauge group G . In general, their vevs $\langle \Phi_1 \rangle, \langle \Phi_2 \rangle, \dots$ are not aligned in the internal representation space, so each multiplet breaks G to a different subgroup, with the net unbroken symmetry being the intersection of all of these individual subgroups. These subgroups may or may not be all isomorphic, depending on the model and the context. For the cases where the individual subgroups are indeed isomorphic but differently embedded in the parent group G , then our analysis is relevant.

III. BACKGROUND AND NOTATION

We now clearly outline some of the terminology and notation we use throughout this document. To do this we devote a paragraph to spontaneous symmetry breaking, which

is the overarching framework. A reader who is familiar with the concept of spontaneous symmetry breaking and standard notation in QCD and root systems may choose to skip this section and use it as a reference.

Spontaneous symmetry breaking occurs when the action has a symmetry G , but the lowest energy configuration of the potential (the vacuum) is stabilized by a subgroup H , called the little group of G . Because the potential is invariant under G , the orbit of H in G , $\mathcal{M} = G/H$, forms a manifold of degenerate minima of the potential known as the vacuum manifold. There exists a point P in the vacuum manifold associated with the coset $1H$. All the elements t_l , belonging to the Lie subalgebra of H , \mathcal{L}_H , generate infinitesimal diffeomorphisms which fix P , while Lie algebra elements, m , belonging to the complement of \mathcal{L}_H in the Lie algebra \mathcal{L} induce parallel translation along geodesics at point P in \mathcal{M} .

A parallel translation (induced by m) from P to a neighboring point in the vacuum manifold, Q , is accompanied by a continuous change in the space of diffeomorphisms which fix P to the space of diffeomorphisms which fix Q . The latter is given by gHg^{-1} , where g is the Lie group element generated by m . Physically, members of the complement of \mathcal{L}_H in \mathcal{L} generate symmetries of the action which nevertheless shift the vacuum of the theory associated with point P . If the vacuum at point P is designated $|0\rangle$ and has the property $H|0\rangle = |0\rangle$ then the vacuum at Q is $g|0\rangle$ and is consequently fixed by $gHg^{-1}g|0\rangle = g|0\rangle$.

We establish some notation which will help elucidate the following discussion. We choose to work exclusively with diagonal Cartan subalgebra generators, which can be done without loss of generality because given an arbitrary Cartan subalgebra it is always possible to simultaneously diagonalize each member using a similarity transformation within the Lie algebra. If the Lie algebra has rank l we choose h^i where $i \in \{1, \dots, l\}$ to refer to our basis for the Cartan subalgebra.

Throughout this document we physically contextualize our result using QCD and the weak force as examples. To do this we choose explicit representations. In each case we make use of the adjoint representation and the lowest dimensional fundamental representation, otherwise known as the smallest faithful representation.

In QCD for the 3 representation of $SU(3)$ we use the Gell-Mann matrices $\lambda_1, \dots, \lambda_8$ as generators. We refer to the gluons as a set of 8 Lorentz 4-vector fields G_μ^i where $i \in \{1, \dots, 8\}$ distributed over the Gell-Mann matrices; we write $X_i^\mu = G_\mu^i \lambda_i$ where there is no intended

sum over i . We also make use of the linear combinations of the off diagonal gluons:

$$Z_\mu^1 = \frac{1}{\sqrt{2}}(X_\mu^1 + iX_\mu^2), \quad Z_\mu^2 = \frac{1}{\sqrt{2}}(X_\mu^4 + iX_\mu^5), \quad Z_\mu^3 = \frac{1}{\sqrt{2}}(X_\mu^6 + iX_\mu^7). \quad (\text{III.1})$$

Correspondingly we take linear combinations of the two diagonal gluons, renamed for notational convenience $X_\mu^3 = A_\mu^1$ and $X_\mu^8 = A_\mu^2$,

$$B_\mu^p = A_\mu^i \alpha_i^p, \quad (\text{III.2})$$

where $p \in \{1, 2, 3\}$ and the α_i^p vectors are the three roots $\alpha_i^1 = (1, 0)$, $\alpha_i^2 = (1/2, \sqrt{3}/2)$, $\alpha_i^3 = (-1/2, \sqrt{3}/2)$. In keeping with this notation we use a relabeling of the diagonal Gell-Mann matrices $\lambda_3 = A^1$ and $\lambda_8 = A^2$ to define the SU(3) Lie algebra generators $\kappa = A^i \alpha_i^1$, $\rho = A^i \alpha_i^2$ and $\varepsilon = A^i \alpha_i^3$ associated respectively with B_μ^1, B_μ^2 and B_μ^3 . We also give rather unimaginative names to the Lie algebra generators associated with the valence gluons Z_μ^1, Z_μ^2 and Z_μ^3 :

$$Z^1 = \lambda_1 + i\lambda_2, \quad Z^{-1} = \lambda_1 - i\lambda_2, \quad (\text{III.3})$$

$$Z^2 = \lambda_4 + i\lambda_5, \quad Z^{-2} = \lambda_4 - i\lambda_5, \quad (\text{III.4})$$

$$Z^3 = \lambda_6 + i\lambda_7, \quad Z^{-3} = \lambda_6 - i\lambda_7. \quad (\text{III.5})$$

Notice these are precisely the raising and lowering operators of the SU(3) Lie algebra.

Extending the SU(3) example we will refer to the I-spin, V-spin and U-spin directions in colour space, which describe the three Cartan preserving embeddings of SU(2) inside SU(3). These are the three embeddings which have the Cartan subalgebra generators for SU(2) as a subset of the Cartan generators for SU(3). In terms of the Gell-Mann matrices, the generators of the SU(2) subgroup in each case are

$$\begin{aligned} & \underbrace{\lambda_1, \lambda_2, \kappa, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8}_{2(\text{I-spin})} \\ & \lambda_1, \lambda_2, \lambda_4, \lambda_5, \underbrace{\varepsilon, \lambda_6, \lambda_7, \varepsilon'}_{2(\text{V-spin})} \\ & \lambda_1 \lambda_2, \underbrace{\rho, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \rho'}_{2(\text{U-spin})} \end{aligned} \quad (\text{III.6})$$

where we have chosen to introduce complementary matrices to the ρ and ε , namely $\rho' = -\sqrt{3}/2A^1 + 1/2A^2$ and $\varepsilon' = \sqrt{3}/2A^1 + 1/2A^2$ respectively, so that each set of Lie algebra

generators contains a diagonal Cartan subalgebra, which is orthogonal under the matrix trace.

In our weak force examples we use the Pauli matrices $\{\tau_1, \tau_2, \tau_3\}$ as a vector space basis for the adjoint representation (note τ_3 is the diagonal generator of the weak isospin gauge group, I_2 , in this representation) and the standard notation for the three gauge bosons $W_\mu^1 = w_\mu^1 \tau^1, W_\mu^2 = w_\mu^2 \tau^2, W_\mu^3 = w_\mu^3 \tau^3$.

Analogously to the QCD case we consider linear combinations of the weak force gauge bosons $W_\mu^+ = w_\mu^+ \tau^+ = W_\mu^1 - iW_\mu^2$, $W_\mu^- = w_\mu^- \tau^- = W_\mu^1 + iW_\mu^2$ and $W_\mu^0 = w_\mu^3 \tau^3 = W_\mu^3$, and the corresponding generators $\tau^+ = \tau_1 - i\tau_2$, $\tau^- = \tau_1 + i\tau_2$ and $\tau_0 = \tau_3$. We use this notation because $+1, -1$ and 0 are the respective $U(1)_Q$ quantum numbers or electric charges of these linear combinations.

The adjoint action of the Lie algebra $\text{ad}_{h^i} \cdot E^\alpha$ on itself is defined by $\text{ad}_{h^i} \cdot E^\alpha = [h^i, E^\alpha]$. In the special cases where the E^α are eigenvectors under the adjoint operation for some h^i we write $[h^i, E^\alpha] = \alpha(h^i)E^\alpha$.

We say a linear transformation stabilizes a point if it maps that point back onto itself. For example if $|\lambda\rangle$ is an eigenvector of a Lie algebra generator $t_k \in \mathcal{L}$, so that $t^k \cdot |\lambda\rangle = \lambda(t^k) |\lambda\rangle$, then we say $|\lambda\rangle$ is stabilized by t_k .

IV. ROOT SYSTEMS, THE WEYL GROUP

Our work relies heavily on the concept of roots and weights. Particle physicists often refer to the roots and weights as the quantum numbers of particles belonging to a nontrivial representation space of a non-Abelian gauge group. Consider the $SU(2)$ -weak lepton doublet,

$$l_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \sim (1, 2)(-1), \quad (\text{IV.1})$$

where by $(1, 2)(-1)$ we mean the lepton doublet does not transform under the $SU(3)$ colour symmetry, however it transforms under a two dimensional representation of the $SU(2)$ weak isospin gauge group and $l \rightarrow e^{-i\theta} l$ under the $U(1)$ weak hypercharge symmetry. The $SU(2)$ weights of the two states in this representation are the isospin quantum numbers of the fermions. The electron neutrino, ν_e , has isospin quantum number $+1/2$. This is the highest weight of the representation. The electron, e , has isospin quantum number $-1/2$. This is the lowest weight of the representation.

The roots are the isospin charges of the self-interacting gauge bosons,

$$W_\mu^+ = \begin{pmatrix} 0 & w_\mu^+ \\ 0 & 0 \end{pmatrix}, \quad W_\mu^- = \begin{pmatrix} 0 & 0 \\ w_\mu^- & 0 \end{pmatrix}. \quad (\text{IV.2})$$

The gauge bosons are associated with the SU(2) raising operator τ^+ , and the SU(2) lowering operator τ^- respectively. These are eigenstates of the adjoint action of the weak-isospin generator I_2 , that is $\text{ad}_{I_2} \cdot \tau^\pm = [I_2, \tau^\pm] = \text{ad}_{I_2}(\tau^\pm)\tau^\pm$. The +1 isospin charge of W^+ and -1 isospin charge of W^- follow from:

$$[I_2, W_\mu^+] = w_\mu^+ [I_2, \tau^+] = 1W_\mu^+ \quad [I_2, W_\mu^-] = w_\mu^- [I_2, \tau^-] = -1W_\mu^-. \quad (\text{IV.3})$$

A. Root Systems and the Weyl group

In general, it is possible to represent a semi-simple rank l Lie Algebra using two types of generators:

- a set of l mutually commuting diagonalizable generators, h^1, \dots, h^l , which together with the linear combinations $\Sigma_i a^i h^i$, form a Cartan subalgebra, \mathcal{C}_G and,
- a collection of simultaneous eigenstates E^α of the adjoint action of every Cartan subalgebra generator.

Collectively the generators satisfy the commutation relations of the Lie algebra, \mathcal{L} ,

$$\begin{aligned} [h^i, h^j] &= 0 \\ [h^i, E^\alpha] &= \text{ad}_{h^i} \cdot E^\alpha = \alpha(h^i)E^\alpha \\ [E^\alpha, E^{-\alpha}] &= h^\alpha \\ [E^\alpha, E^\beta] &= N_{\alpha,\beta}E^{\alpha,\beta} \quad \text{if } \alpha \neq -\beta \end{aligned} \quad (\text{IV.4})$$

where h^α is a linear combination of the h^i . We shall call $\alpha(h^i) = \alpha^i$ for convenience. Each eigenstate E^α can be labeled by an l -dimensional vector $\alpha = (\alpha^1, \dots, \alpha^l)$ called a *root*. The root is a list of the l eigenvalues (structure constants) for the commutator, $[h^i, E^\alpha]$, of E^α with each $h^i \in \mathcal{C}_G$.

The length of the roots depends on choosing a consistent normalization scheme for the generators. We fix the normalization of our Lie algebra generators by choosing $\text{Tr}(E^\alpha E^{-\alpha}) =$

$2/(\alpha, \alpha)$, where (a, b) is an invariant inner product: for example if one used an invariant inner product on the Lie algebra generators, such as the Cartan-Killing form or the regular trace and restricted this inner product to the Cartan generators then because the root space is dual to the Cartan subalgebra this induces an invariant inner product on the root space. This is a condition known as the Chevalley-Serre basis. It guarantees the components of the roots are integers.

It follows from equation (IV.4) that for each root α , labeling a generator $E^\alpha \in \mathcal{L}$, there exists $-\alpha$, labeling the hermitian conjugate generator $E^{-\alpha} = E^{\alpha\dagger} \in \mathcal{L}$. We refer to E^α as a raising operator, and $E^{-\alpha}$ as a lowering operator. This leads us to partition the root system into two disjoint sets: the positive and the negative roots. We elect to call a root, α , whose first non-zero component is positive, a “positive root”. The corresponding negated positive root, $-\alpha$, is termed a ”negative root”. Not all of these roots are linearly independent. It is convenient to introduce a basis for the root space.

A rank l Lie algebra has l independent Cartan subalgebra generators and therefore a set of l linearly independent simple roots called $\{\zeta^{(1)}, \dots, \zeta^{(l)}\}$. The simple roots are conventionally chosen to be the l -dimensional subset of the positive roots, with the property that every positive root can be written as a non-negative linear combination of $\{\zeta^{(1)}, \dots, \zeta^{(l)}\}$.

It is clear from (IV.4) that each root α is the pullback of a member of the Cartan subalgebra,

$$h^\alpha = [E^\alpha, E^{-\alpha}]. \quad (\text{IV.5})$$

Multiplying this expression on the left hand side by $h^j \in \{h^1, \dots, h^l\}$ and taking the matrix trace we see $h^\alpha = \alpha_j^\vee h^j$ (sum over $j \in \{1, \dots, l\}$) where $\alpha_j^\vee = g_{ij} 2\alpha^i / (\alpha, \alpha)$ (sum over $i \in \{1, \dots, l\}$), where $g_{ij} = [\text{Tr}(h^i h^j)]^{-1}$ is the inverse of the $l \times l$ matrix whose ij th element is $g^{ij} = [\text{Tr}(h^i h^j)]$. We call $\alpha^\vee = 2\alpha / (\alpha, \alpha)$ a co-root; for example $\zeta^{(i)\vee} = 2\zeta^{(i)} / (\zeta^{(i)}, \zeta^{(i)})$ is a simple co-root, for $\zeta^{(i)} \in \{\zeta^{(1)}, \dots, \zeta^{(l)}\}$.

Linearity of the commutator bracket now allows us to extend our definition of the adjoint action to any h^β acting on the Lie algebra according to

$$\text{ad}_{h^\beta} \cdot E^\alpha = \alpha(h^\beta) E^\alpha = (\alpha, \beta^\vee) E^\alpha. \quad (\text{IV.6})$$

The set of roots for a Lie algebra have the property that they completely characterize the Lie algebra. They also form a *crystallographic root system* Δ [13]. This means:

Definition IV.1. Δ is a *crystallographic root system* if for all $\alpha, \gamma, \beta \in \Delta$,

1. If $\alpha \in \Delta$, then $\chi \alpha \in \Delta$ if and only if $\chi = \pm 1$.
2. The reflection of β in the hyperplane perpendicular to γ : $s^\gamma \cdot \beta = \beta - (\beta, \gamma^\vee)\gamma$ also belongs to Δ .
3. $(\beta, \gamma^\vee) \in \mathbb{Z}$.

Notice that condition (2) implies that $W = \{s^\gamma | \gamma \in \Delta\}$, the subset of symmetries of Δ generated by reflections in the hyperplanes orthogonal to the roots in Δ , forms a reflection group known as the *Weyl group*.

Any element of W can be expressed as a sequence of reflections in the simple roots. This gives rise to a presentation of the Weyl group, called the Coxeter presentation, generated by reflections in the hyperplanes orthogonal to the simple roots, ζ^i . If we refer to the angle between any two simple roots ζ^i and ζ^j as π/m_{ij} , then the *Coxeter presentation* is:

$$W = \left\{ s^{\zeta^i} \mid \left(s^{\zeta^i} s^{\zeta^j} \right)^{m_{ij}} = 1, \left(s^{\zeta^i} \right)^2 = 1 \right\}. \quad (\text{IV.7})$$

The Coxeter presentation expression for each element, $w^\gamma \in W$, is not unique. However if we define the length of an expression to be the number of reflections, s^{ζ^i} , it contains, then the relations can be used to reduce all Coxeter presentations for w^γ to a fixed minimum length. This fixed length is a property of γ relative to the choice of $\{\zeta^1, \dots, \zeta^l\}$.

To understand the relations in equation (IV.7) let $\mathcal{H}^{\zeta^{i\vee}}$ be the $(l-1)$ -dimensional hyperplane orthogonal to ζ^i . Because ζ^1 and ζ^2 are linearly independent, the intersection $\mathcal{H}^{\zeta^{1\vee}} \cap \mathcal{H}^{\zeta^{2\vee}}$ is an $(l-2)$ -dimensional space, the complementary space being the plane spanned by ζ^1 and ζ^2 . A reflection in $\mathcal{H}^{\zeta^{1\vee}}$ followed by a reflection in $\mathcal{H}^{\zeta^{2\vee}}$, $s^{\zeta^1} s^{\zeta^2}$, is the same as a rotation by twice the angle between $\mathcal{H}^{\zeta^{1\vee}}$ and $\mathcal{H}^{\zeta^{2\vee}}$ (that is a rotation by $2\pi/m_{12}$) in the (ζ^1, ζ^2) plane. Therefore the relation $(s^{\zeta^1} s^{\zeta^2})^{m_{12}} = 1$ is equivalent to the statement that m_{12} concatenations of a rotation by an angle $2\pi/m_{12}$ is the identity transformation.

The Weyl group has a natural analogue in the matrix picture [14]. Here conjugation by the operator

$$w^\gamma = \exp(E^\gamma) \exp(E^{-\gamma}) \exp(E^\gamma), \quad (\text{IV.8})$$

acts on the Cartan subalgebra according to

$$w^\gamma \cdot h^\beta = w^\gamma h^\beta w^{-\gamma} = (s^\gamma \cdot \beta^\vee)_i h^i = (s^\gamma \cdot \beta)_i^\vee h^i, \quad (\text{IV.9})$$

where $w^{-\gamma} = (w^\gamma)^{-1}$. We can check that (IV.8) is a matrix representation for the Weyl group, acting as a module on the Cartan subalgebra \mathcal{C}_G , by checking that $w^\gamma \cdot h^\beta = h^{s^\gamma \cdot \beta}$. This follows directly from the action of $w^\gamma \cdot h^\beta$ on E^α :

$$\begin{aligned} [w^\gamma \cdot h^\beta, E^\alpha] &= (s^\gamma \cdot \beta)_i^\vee [h^i, E^\alpha] = \alpha^i (s^\gamma \cdot \beta)_i^\vee E^\alpha \\ &= (\alpha, (s^\gamma \cdot \beta)^\vee) E^\alpha = [h^{s^\gamma \cdot \beta}, E^\alpha]. \end{aligned} \quad (\text{IV.10})$$

Conversely conjugating (IV.10) by $w^{-\gamma}$:

$$\begin{aligned} [h^\beta, w^{-\gamma} E^\alpha w^\gamma] &= (s^\gamma \cdot \beta, \alpha^\vee) w^{-\gamma} E^\alpha w^\gamma \\ &= (\beta, (s^\gamma \cdot \alpha)^\vee) w^{-\gamma} E^\alpha w^\gamma = [h^\beta, E^{s^\gamma \cdot \alpha}], \end{aligned} \quad (\text{IV.11})$$

leads us to conclude $w^{-\gamma} \cdot E^\alpha = E^{s^\gamma \cdot \alpha}$ and therefore (IV.9) also furnishes a matrix representation for the Weyl group acting as a module on the space of generators $\{E^\alpha \mid \alpha \in \Delta\}$. In the root system picture its elements are orthogonal transformations which act to permute the collection of roots belonging to Δ .

In the matrix picture the Cartan subalgebra is an invariant subspace for the Weyl group and the Weyl group permutes the raising and lowering operators E^α .

B. Weights

More generally, the physical significance of being able to simultaneously diagonalize the Cartan subalgebra is that, for any representation space of the Lie group, there exists a basis, \mathcal{B} , of simultaneous eigenvectors, $|\nu\rangle$, of the entire Cartan subalgebra. Each eigenvector $|\nu\rangle \in \mathcal{B}$, can be labeled by the l -dimensional vector, $\nu = (\nu^1, \dots, \nu^l) = (\nu(h^1), \dots, \nu(h^l))$, formed by listing its eigenvalues, $h^i |\nu\rangle = \nu(h^i) |\nu\rangle$, for $h^i = h^1, \dots, h^l$. These l -dimensional vectors are the weights.

In the adjoint representation the weights are the root vectors. If the Lie group representation acts as a module over a vector space of n -dimensional column vectors (analogously to the SU(2)-weak lepton doublet), then the weights are the eigenvalues under left matrix multiplication by the Cartan subalgebra generators. The eigenvector labeled by the highest weight is annihilated by all raising operators.

The Weyl group action on the adjoint representation space eigenbasis, $w^{-\gamma} \cdot E^\alpha$, and weights of the adjoint representation, $s^\gamma \cdot \alpha$, from equations (IV.11) and condition (2) in

definition IV.1, can be generalized. The Weyl group reflection of a weight ν in the hyperplane orthogonal to root κ is:

$$s^\kappa \cdot \nu = \nu - (\nu, \kappa^\vee) \kappa. \quad (\text{IV.12})$$

In direct analogy to the adjoint representation, an arbitrary representation space for the Lie group furnishes a representation space for the Weyl group. This can be seen directly from the action of (IV.8) on $|\nu\rangle \in \mathcal{B}$

$$w^{-\kappa} \cdot |\nu\rangle = |s^\kappa \cdot \nu\rangle. \quad (\text{IV.13})$$

The result follows from analyzing the action of $h^i \in \mathcal{C}_G$ on $w^{-\kappa} |\nu\rangle$:

$$\begin{aligned} h^i w^{-\kappa} |\nu\rangle &= w^{-\kappa} (w^\kappa h^i w^{-\kappa}) |\nu\rangle \\ &= w^{-\kappa} (h^i - \kappa^i h^\kappa) |\nu\rangle \\ &= w^{-\kappa} (h^i - \kappa^i \kappa_j^\vee h^j) |\nu\rangle \\ &= (\nu^i - (\nu, \kappa^\vee) \kappa^i) w^{-\kappa} |\nu\rangle \\ &= [s^\kappa \cdot \nu]^i w^{-\kappa} |\nu\rangle \\ &= h^i |s^\kappa \cdot \nu\rangle. \end{aligned} \quad (\text{IV.14})$$

where to get the second equality we have used $h^j = \sum_i \delta_i^j h^i$ in equation (IV.9). Thus because of (IV.8) and (IV.9) it is not a coincidence that every representation space for the Lie group furnishes a representation space for the Weyl group.

We introduce some terminology we need to talk about weights. The weights belonging to the Weyl group orbit of the highest weight are called extremal weights.

Consider a representation which has highest weight ν , and let $E^\delta \in \mathcal{L}$ be a generic raising operator for this representation. Then it is easy to see that each extremal weight $\mu = s^\kappa \cdot \nu$ where $\kappa \in \Delta$, is also the highest weight with respect to a different choice of positive roots, as $|\mu\rangle$ is eliminated by an equivalent set of raising operators $w^\kappa \cdot E^\delta \in \mathcal{L}$. However the Weyl group permutes the set of Lie algebra generators, so both representations have the same generators. We would like to have a way of distinguishing between these representations and others which have qualitatively different sets of generators.

It is necessary to work with a basis for the weight space $\{\omega^1, \dots, \omega^l\}$ which is dual to the simple roots, that is $\omega^i \zeta^{j\vee} = \delta^{ij}$. We call $\{\omega^1, \dots, \omega^l\}$ fundamental weights. A linear combination of $\{\omega^1, \dots, \omega^l\}$ with non-negative coefficients is called a dominant weight. Every

dominant weight is the highest weight of a representation, and up to conjugation by the Weyl group every highest weight is dominant.

V. LIE SUBALGEBRAS AND EMBEDDINGS

Lie subalgebras $\mathcal{L}_H \subset \mathcal{L}$ have generators labeled by closed subroot systems $\Delta_H \subset \Delta$, where by a closed subroot system [13] we mean

Definition V.1. A *closed subroot system* is a root system, $\Delta_H \subset \Delta$, such that for all $\alpha, \beta \in \Delta_H$ if $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_H$.

We can see \mathcal{L}_H satisfies the Lie algebra commutation relations (IV.4) because whenever $E^\alpha, E^\beta \in \mathcal{L}_H$ and $N_{\alpha, \beta} \neq 0$, we have $[E^\alpha, E^\beta] \in \mathcal{L}_H$ (closure under the Lie bracket).

The Weyl group of the subroot system Δ_H , $W_H = \{s^\alpha | \alpha \in \Delta_H\}$, is the subgroup of W , which permutes the subset of the roots belonging to Δ_H .

For each subroot system Δ_H , or one of its Weyl group conjugates, there is a systematic way of choosing a basis of simple roots consisting of a proper subset $I_H \subset \{\zeta^1, \dots, \zeta^i\} \cup \{-\zeta_0\}$, of the union of the simple roots for Δ and, the negated highest weight of the adjoint representation (negated highest root). The method is given in the Borel-de-Siebenthal theorem (see the appendix X). The Coxeter presentation for W_H is generated by reflections in the hyperplanes H^{ζ^j} , $\zeta^j \in I_H$ orthogonal to the simple roots of Δ_H .

The Weyl group action on the root system is regular (that is for all $\alpha, \beta \in \Delta$, there exists precisely one $s^\gamma \in W$ such that $\beta = s^\gamma \cdot \alpha$). The orbit $W \cdot \Delta_H$ represents all the embeddings of Δ_H inside Δ . However we know that W_{Δ_H} maps Δ_H back onto itself, so each element in the orbit $W_{\Delta_H} \cdot \Delta_H = \Delta_H$ gives rise to the same embedding of \mathcal{L}_H inside \mathcal{L} . Therefore the set $W/W_{\Delta_H} \cdot \Delta_H$ represents all the “qualitatively different” embeddings of Δ_H and \mathcal{L}_H inside Δ and \mathcal{L} respectively. By “qualitatively different” we mean the raising and lowering operators belonging to \mathcal{L}_H and $w^\kappa \mathcal{L}_H w^{-\kappa}$ are distinct subsets of the full complement of raising and lowering operators belonging to \mathcal{L} .

One outcome of this is that we now know the number of embeddings of \mathcal{L}_H inside \mathcal{L} is $|W/W_{\Delta_H}|$. Because the Weyl group is finite we can simplify this expression³ to $|W|/|W_{\Delta_H}|$.

³ This follows from the orbit stabilizer theorem: Suppose that a linear algebraic group G acts on the set X . If G is finite then $|G| = |G \cdot x| \cdot |\text{Stabilizer}(x)|$, that is, the order of the orbit of x , $|G \cdot x|$, divides $|G|$.

VI. STATEMENT OF PROOF

We are looking for a complete set of embeddings of the subgroup chain $G \supset H_1 \supset H_2 \supset \dots \supset H_l$, where the vevs of the adjoint Higgs fields which break G down to these subgroups and define a basis for the Cartan subalgebra h^1, \dots, h^l , can be written as linear combinations of each other.

We have established that the embeddings of H_1 within G arise from conjugation of the Lie algebra \mathcal{L}_{H_1} for H_1 by the Weyl group $W/W_{\Delta_{H_1}}$ where $W_{\Delta_{H_1}}$ is the Weyl group of the maximal subgroup H_1 . Moreover we know conjugation by any Weyl group element $w^\kappa \in W/W_{\Delta_{H_1}}$ acts on the Cartan subalgebra or vevs h^1, \dots, h^l according to

$$w^\kappa \cdot h^j = \sum_i (\delta_{ij} - \sum_n \kappa^n \delta_{nj} \kappa_i^\vee) h^i = h^j - \kappa^j h^\kappa. \quad (\text{VI.1})$$

So after identifying the generators (roots) excluded from the embedding of $H_1 \subset G$ ($\Delta_{H_1} \subset \Delta$) we have a general formula for writing the vevs of the adjoint Higgs field $w^\kappa \cdot h^1, \dots, w^\kappa \cdot h^l$ causing the breaking of $G \supset w^\kappa H_1 w^{-\kappa} \supset w^\kappa H_2 w^{-\kappa} \supset \dots \supset w^\kappa H_l w^{-\kappa}$ as linear combination of h^1, \dots, h^l . If, after choosing an embedding of H_1 within G , identified with $\mathcal{L}_{H_1} \subset \mathcal{L}$, we wish to find all the different embeddings of H_2 within H_1 which have $\mathcal{L}_{H_2} \subset \mathcal{L}_{H_1}$ we simply repeat this procedure for $W_{\Delta_{H_1}}/W_{\Delta_{H_2}}$.

In the case where we are looking for the adjoint Higgs vevs, $w^\kappa \cdot h$, breaking G to different embeddings of the subgroup $H = H' \times \text{U}(1)_H$ which stabilizes (the representation space state labeled by) the highest weight of the lowest dimensional fundamental representation, $|\lambda\rangle$, these linear combination have a remarkably simple formula. We show that the linear combination giving each vev is $\sum_i \mu(h^i) h^i$ for an extremal weight μ of the fundamental representation.

We first prove the adjoint Higgs vev, h , which breaks G to H is given by the linear combination $h = \sum_i \lambda(h^i) h^i$, where the coefficients are the coordinates of highest weight of the fundamental representation. We then explain why other generators breaking G to different embeddings $w^\kappa \cdot H$, $w^\kappa \cdot h = \sum_i \mu(h^i) h^i$, are the linear combinations of h^1, \dots, h^l which have the co-ordinates of the extremal weights, $\mu(h^i)$ as coefficients.

If $\sum_i \lambda(h^i) h^i$ is the adjoint Higgs vev which breaks G to H , then it is the generator of the $\text{U}(1)_H$ factor in $H = H' \times \text{U}(1)_H$. Therefore $\sum_i \lambda(h^i) h^i$ must stabilize $|\lambda\rangle$ (be a generator of H) and it must commute with each generator, $E^\alpha \in \mathcal{L}_G$, if and only if $E^\alpha \in \mathcal{L}_H$.

It is clear that $\Sigma_i \lambda(h^i)h^i$ is a generator of H because $\Sigma_i \lambda(h^i)h^i |\lambda\rangle = (\Sigma_i \lambda(h^i)^2) |\lambda\rangle$.

Furthermore, let $E^\alpha \in \mathcal{L}_H$. Then E^α is a raising or lowering operator and E^α stabilizes $|\lambda\rangle$, therefore we must have $E^\alpha |\lambda\rangle = 0$. If $\alpha \in \Delta_H$ then $-\alpha \in \Delta_H$, and by the same logic $E^{-\alpha} |\lambda\rangle = 0$. Consider the commutator

$$\begin{aligned}
[E^\alpha, \Sigma_i \lambda(h^i)h^i] &= \Sigma_i \lambda(h^i)[E^\alpha, h^i] \\
&= \Sigma_i \lambda(h^i) \alpha^i E^\alpha \\
&= \lambda(\Sigma_i \alpha^i h^i) E^\alpha \\
&= \lambda(h^\alpha) E^\alpha \\
&= \lambda([E^\alpha, E^{-\alpha}]) E^\alpha \\
&= 0.
\end{aligned} \tag{VI.2}$$

Therefore $\Sigma_i \lambda(h^i)h^i$ commutes with all the elements of \mathcal{L}_H .

Assume $\Sigma_i \lambda(h^i)h^i$ commutes with a generator $E^\kappa \notin \mathcal{L}_H$ which does not belong to the Lie algebra of H , then we have

$$\begin{aligned}
\Sigma_i \lambda(h^i)h^i &= w^\kappa \Sigma_i \lambda(h^i)h^i w^{-\kappa} \\
&= \Sigma_i \lambda(h^i)h^i - \Sigma_i \lambda(h^i) \kappa(h^i) h^\kappa \\
&= \Sigma_i \lambda(h^i)h^i - \Sigma_{ij} \lambda(h^i) \kappa(h^i) \kappa^\vee(h^j) h^j \\
&= \Sigma_i \lambda(h^i)h^i - \Sigma_j \frac{2(\lambda, \kappa)}{(\kappa, \kappa)} \kappa(h^j) h^j \\
&= \Sigma_i (\lambda(h^i) - (\lambda, \kappa^\vee) \kappa(h^i)) h^i \\
&= \Sigma_i [s^\kappa \cdot \lambda] (h^i) h^i.
\end{aligned} \tag{VI.3}$$

This creates a contradiction because we are insisting E^κ does not stabilize $|\lambda\rangle$, so $w^\kappa |\lambda\rangle = |s^\kappa \cdot \lambda\rangle \neq |\lambda\rangle$ and the two sets of coefficients (of the linearly independent Cartan subalgebra generators h^1, \dots, h^l) in the above sum must be different. We have proved $\Sigma_i \lambda(h^i)h^i$ is the adjoint Higgs vev, h , which breaks G to H .

Now each embedding $w^\kappa \cdot H = w^\kappa H w^{-\kappa}$ will stabilize a state in the representation labeled by an extremal weight $w^\kappa \cdot |\lambda\rangle = |\mu\rangle$. By the above argument, the center of the subgroup $w^\kappa \cdot H$ which stabilizes $|\mu\rangle$ is generated by $\Sigma_i \mu(h^i)h^i$. We have a remarkably easy formula for reproducing the vevs which break G to all the different embeddings of the subgroup which stabilizes the highest weight of the lowest dimensional fundamental representation, H , as a linear combination of the Cartan subalgebra h^1, \dots, h^l . Notice that w^κ must belong to a

nontrivial coset in W/W_{Δ_H} , because conjugation by w^κ only takes us from one embedding to another when s^κ does not fix the highest weight.

We present a systematic method for determining the subgroup H directly from the extended Dynkin diagram for the Lie group G . Each unmarked node in the extended Dynkin diagrams is labeled by a simple root. The node with a cross in the center is ζ^0 . To find the Dynkin diagram for H we simply need to determine which of the simple roots in Δ are also in Δ_H . We also need to work out if the highest root ζ^0 is in Δ_H . The subset of $\{\zeta^1, \dots, \zeta^l\} \cup \{-\zeta^0\}$ belonging to Δ_H , will be the simple roots for Δ_H .

First we determine which subset of the simple roots $\{\zeta^1, \dots, \zeta^l\}$ belong to Δ_H . Take the highest weight, λ , and write it as a linear combination of the fundamental weights.

$$\lambda = a_1\omega_1 + \dots + a_l\omega_l \quad (\text{VI.4})$$

We assume this highest weight is dominant ($a_1, \dots, a_l \geq 0$), if it is not then it is always possible to replace λ by one of the extremal weights which is dominant. Construct a set $S_\lambda = \{j | a_j = 0\}$. For all $j \in S_\lambda$ we have $(\lambda, \zeta^{j\nu}) = 0$. We claim that $\zeta^j \in \Delta_H$, that is $E^{\pm\zeta^j} |\lambda\rangle = 0$, for all $j \in S_\lambda$. Otherwise if $E^{\pm\zeta^j} |\lambda\rangle \neq 0$ consider the norm $N_{\lambda \pm \zeta^j} = \langle \lambda | E^{\pm\zeta^j \dagger} E^{\pm\zeta^j} | \lambda \rangle$. Because λ is the highest weight of the representation $N_{\lambda + \zeta^j} = 0$ while $N_{\lambda - \zeta^j} = \langle \lambda | [E^{\zeta^j}, E^{-\zeta^j}] | \lambda \rangle = \langle \lambda | \lambda \rangle (\lambda, \zeta^{j\nu}) = 0$. For the remaining simple roots labeled by $k \notin S_\lambda$, we have $s^{\zeta^k} \cdot \lambda \neq \lambda$, therefore $w^{\zeta^k} |\lambda\rangle \neq |\lambda\rangle$ and from (IV.8) we know that one of $E^{\pm\zeta^k}$ does not stabilize λ .

The highest root (negated highest root) $\pm\zeta^0$ does not belong to Δ_H . This follows from the fact that ζ^0 is some linear combination of all the simple roots (with positive coefficients), therefore if the set S_λ is non-empty $(\lambda, \zeta^0) > 0$.

So the Dynkin diagram for H can be reconstructed from the connected components of the Dynkin diagram for G labeled by simple roots $\{\zeta^j | j \in S_\lambda\}$. This uniquely defines the non-Abelian factor H' of H . The full subgroup H which stabilizes the highest weight is a product of H' with one Abelian factor $U(1)$ for each $k \notin S_\lambda$. These extra $U(1)$ factors are generated by the Cartan subalgebra generators h^{ζ^k} , $k \notin S_\lambda$, which (by definition) stabilize λ , even when the associated raising/lowering operators E^{ζ^k} do not.

If the Higgs field does not belong to the adjoint representation then the above analysis generalizes. The Weyl group reflections still give the different embeddings of the subgroup chain $G \supset H_1 \supset \dots \supset H_l$. If there is an associated Cartan subalgebra h^1, \dots, h^l defined

as the generators of $U(1)_{H_i}$ factors appearing in the subgroup chain through $H_i = H'_i \times U(1)_{H_1} \times \cdots \times U(1)_{H_i}$ (where H'_i is some product of non-Abelian Lie groups) then equation (VI.1) gives the linear combinations for the equivalent Cartan subalgebra generator for the $U(1)_{w^\kappa \cdot H_i}$ factors belonging to the differently embedded subgroup chain $G \supset w^\kappa H_1 w^{-\kappa} \supset \cdots \supset w^\kappa H_l w^{-\kappa}$, where $w^\kappa \in W/W_{\Delta_H}$.

If a subgroup $H \subset G$ annihilates a column vector $|\nu\rangle$, labeled by a weight ν , then the differently embedded subgroup $w^\kappa H w^{-\kappa}$ annihilates the column vector $w^\kappa |\nu\rangle$. Hence if $|\nu\rangle$ breaks G to H , then $|s^\kappa \cdot \nu\rangle$ breaks $G \supset w^\kappa H w^{-\kappa}$ and it follows directly from (IV.14) that equation (IV.12) gives the coordinates of the new weights as a linear combination of ν (and κ).

VII. INSIGHT

We wish to firmly ground the above discussion by applying these concepts to physical systems. We physically contextualize the key concepts in Secs. IV and V via the smallest effective example: embeddings of U-spin, I-spin and V-spin within the SU(3) QCD gauge group. We also tackle the nontrivial problem of finding a full complement of domain wall boundary conditions for an adjoint Higgs field which break E_6 to different embeddings of $SO(10) \times U(1)$ to demonstrate the effectiveness of the techniques developed in section (VI).

A. A Quantum Chromodynamics example

Consider the Weyl group conjugations giving rise to differently embedded copies of the subgroups $SU(2) \times U(1)$ inside $SU(3)$. Following [10] we rewrite the $SU(3)$ pure Yang-Mills quantum chromodynamics Lagrangian in terms of the off diagonal gluons Z_μ^p , $p \in \{1, 2, 3\}$ and the dual potentials to the roots B_μ^p , $p \in \{1, 2, 3\}$ defined in Sec. III:

$$\mathcal{L} = -\frac{1}{4} \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} = \Sigma_p \left\{ -\frac{1}{6} (F_{\mu\nu}^p)^2 + \frac{1}{2} |D_{p\mu} Z_\nu^p - D_{p\nu} Z_\mu^p|^2 - ig F_{\mu\nu}^p Z_p^{\mu*} Z_p^\nu - \frac{1}{2} g^2 [(Z_\mu^{p*} Z_\nu^p)^2 + (Z_\mu^{p*})^2 (Z_\nu^p)^2] \right\} \quad (\text{VII.1})$$

where

$$F_{\mu\nu}^p = \partial_\mu B_\nu^p - \partial_\nu B_\mu^p, \quad D_{p\mu} W_\nu^p = (\partial_\mu - ig B_\mu^p) W_\nu^p. \quad (\text{VII.2})$$

The Weyl group permutes the roots $\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ of the $SU(3)$ Lie algebra. Hence the Weyl group action on the above Lagrangian will cause a permutation of the dual potentials B_μ^p , $p \in \{1, 2, 3\}$ spanning the Cartan subalgebra and also a permutation of the raising and lowering operators $Z_\mu^1 \in \text{span}\{Z^1, Z^{-1}\}$, $Z_\mu^2 \in \text{span}\{Z^2, Z^{-2}\}$, $Z_\mu^3 \in \text{span}\{Z^3, Z^{-3}\}$ labeled by the roots. The permutation is concordant with the geometric picture of the Weyl group reflections of their root labels. Therefore the invariance of the above Lagrangian under Weyl group reflections is encapsulated in the sum over the index p .

The clarity of this presentation is a direct consequence of the associated generators ε, ρ and $Z^{\pm p}$ where $p \in \{1, 2, 3\}$ (it is not necessary to include κ in this list, because $SU(3)$ has rank 2, however we can substitute it for either ε or ρ if we wish) forming a useful computational basis for the Lie algebra: the Chevalley basis. Here each of the three subset $\{\kappa, Z^{\pm 1}\}$, $\{\rho, Z^{\pm 2}\}$ and $\{\varepsilon, Z^{\pm 3}\}$ defines an embedding of $SU(2)$ inside $SU(3)$. These correspond to the closed crystallographic root systems $\{\pm\alpha^1\}$ whose Weyl group fixes a point on the hyperplane orthogonal to α^1 , $\{\pm\alpha^2\}$ and $\{\pm\alpha^3\}$, whose Weyl groups fix analogous points. Cross checking this with Sec. III we see these are precisely the I-spin, V-spin and U-spin embeddings. Each embedding commutes with one of the Abelian subgroup generators $\lambda_8 (= \kappa'), \rho'$ or ε' which we now have the tools to write as $\Sigma_i \mu(h^i) h^i$ for any diagonal Cartan subalgebra $\{h^1, h^2\}$ for $SU(3)$ (in Sec. III our Cartan subalgebra was chosen to be λ_3 and λ_8) and the three extremal weights of the lowest dimensional fundamental representation for $SU(3)$.

B. Adjoint Higgs field domain-wall-brane boundary conditions breaking $E_6 \rightarrow SO(10) \times U(1)$

We use the method developed in the previous section to find all the adjoint Higgs vevs which break E_6 to all the different embeddings of $SO(10) \times U(1)$; this example is directly motivated by an extra-dimensional “clash of symmetries” domain wall brane model [6]. Our choice of Cartan subalgebra for E_6 is explained in table I, the entries of this table follow

directly from the branching rules [9]:

$$\begin{aligned}
E_6 &\supset SO(10) \times U(1)_{h^1} \supset SU(5) \times U(1)_{h^1} \times U(1)_{h^2} \\
&\supset SU(3) \times SU(2) \times U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \\
&\supset SU(3) \times U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \times U(1)_{h^4} \\
&\supset SU(2) \times U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \times U(1)_{h^4} \times U(1)_{h^5} \\
&\supset U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \times U(1)_{h^4} \times U(1)_{h^5} \times U(1)_{h^6}. \tag{VII.3}
\end{aligned}$$

As mentioned in Sec. II our primary motivation for studying this problem arose from a co-dimension-1 clash-of-symmetries domain-wall brane. The brane originates from an E_6 adjoint Higgs field \mathcal{X} which condenses spontaneously to break translational invariance along the extra dimension of a 4+1-dimensional space-time manifold.

The Lagrangian for this theory is invariant under a $Z_2 \times E_6$ internal symmetry. It is a linear combination of the invariant kinetic term $\text{Tr}[D^\mu \mathcal{X} D_\mu \mathcal{X}]$ and a potential formed from the E_6 Casimir invariants $I_2 = \text{Tr} \mathcal{X}^2$ and $I_6 = \text{Tr} \mathcal{X}^6$ as well as the powers I_2^2 and I_2^3 . Casimir invariants corresponding to odd powers of \mathcal{X} must be omitted due to the imposed Z_2 , $\mathcal{X} \rightarrow -\mathcal{X}$, symmetry. The potential is truncated at 6th order because the coupling constants of higher order invariants have negative mass dimensions and are therefore suppressed by powers of the putative ultraviolet completion scale (see Sec. II), yet the fourth order invariants exhibit an accidental $O(78)$ symmetry, so we must include a $\text{Tr} \mathcal{X}^6$ term. A subset of the local minima of the Casimir invariants occur at adjoint Higgs vevs which break $Z_2 \times E_6 \rightarrow SO(10) \times U(1)$. If the solution \mathcal{X} to the associated Euler-Lagrange equations interpolates between vacuum expectation values which break $Z_2 \times E_6$ to a specific pair of differently embedded copies of $SO(10) \times U(1)$ then [6] postulates a copy of the standard model particles can be trapped on the 3+1-dimensional domain-wall brane. To find \mathcal{X} it is necessary to write the boundary conditions at the two antipodal extremes of the extra dimension as a linear combination of the adjoint Higgs vevs h^1, \dots, h^6 , the generators of the Abelian subgroup factors given in (VII.3).

Because $SO(10) \times U(1)$ stabilizes the highest weight of the lowest dimensional fundamental representation for E_6 this is now a trivial problem. Each of the possible boundary conditions which break $E_6 \rightarrow SO(10) \times U(1)$ can be written as a linear combination of h^1, \dots, h^6 using $\Sigma_i \mu(h^i) h^i$ where μ is one of the 27 extremal weights of the lowest dimensional fundamental representation for E_6 . Explicitly the 27 different vevs breaking $E_6 \rightarrow SO(10) \times U(1)$ are

	$60h^1$	$60h^2$	$60h^3$	$60h^4$	$60h^5$	$60h^6$
1	20	0	0	0	0	0
2	-10	$2\sqrt{15}$	$3\sqrt{10}$	$-5\sqrt{6}$	0	0
3	-10	$2\sqrt{15}$	$3\sqrt{10}$	$5\sqrt{6}$	0	0
4	-10	$2\sqrt{15}$	$-2\sqrt{10}$	0	$5\sqrt{2}$	$5\sqrt{6}$
5	-10	$2\sqrt{15}$	$-2\sqrt{10}$	0	$5\sqrt{2}$	$-5\sqrt{6}$
6	-10	$2\sqrt{15}$	$-2\sqrt{10}$	0	$-10\sqrt{2}$	0
7	-10	$-2\sqrt{15}$	$-3\sqrt{10}$	$5\sqrt{6}$	0	0
8	-10	$-2\sqrt{15}$	$-3\sqrt{10}$	$-5\sqrt{6}$	0	0
9	-10	$-2\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$
10	-10	$-2\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$5\sqrt{6}$
11	-10	$-2\sqrt{15}$	$2\sqrt{10}$	0	$10\sqrt{2}$	0
12	5	$-5\sqrt{15}$	0	0	0	0
13	5	$3\sqrt{15}$	$-3\sqrt{10}$	$5\sqrt{6}$	0	0
14	5	$3\sqrt{15}$	$-3\sqrt{10}$	$-5\sqrt{6}$	0	0
15	5	$3\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$
16	5	$3\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$5\sqrt{6}$
17	5	$3\sqrt{15}$	$2\sqrt{10}$	0	$10\sqrt{2}$	0
18	5	$-\sqrt{15}$	$\sqrt{10}$	$-5\sqrt{6}$	$5\sqrt{2}$	$5\sqrt{6}$
19	5	$-\sqrt{15}$	$\sqrt{10}$	$-5\sqrt{6}$	$5\sqrt{2}$	$-5\sqrt{6}$
20	5	$-\sqrt{15}$	$\sqrt{10}$	$-5\sqrt{6}$	$-10\sqrt{2}$	0
21	5	$-\sqrt{15}$	$\sqrt{10}$	$5\sqrt{6}$	$5\sqrt{2}$	$5\sqrt{6}$
22	5	$-\sqrt{15}$	$\sqrt{10}$	$5\sqrt{6}$	$5\sqrt{2}$	$-5\sqrt{6}$
23	5	$-\sqrt{15}$	$\sqrt{10}$	$5\sqrt{6}$	$-10\sqrt{2}$	0
24	5	$-\sqrt{15}$	$-4\sqrt{10}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$
25	5	$-\sqrt{15}$	$-4\sqrt{10}$	0	$-5\sqrt{2}$	$5\sqrt{6}$
26	5	$-\sqrt{15}$	$-4\sqrt{10}$	0	$10\sqrt{2}$	0
27	5	$-\sqrt{15}$	$6\sqrt{10}$	0	0	0

TABLE I: The six diagonal generators h^{1-6} of E_6 . The diagonal elements of the generator h^n are found by taking the n^{th} column and multiplying it by $1/60$. Also the rows give the coefficients f_{1-6} of these generators that yield a linear combination that breaks $E_6 \rightarrow \text{SO}(10) \times \text{U}(1)$.

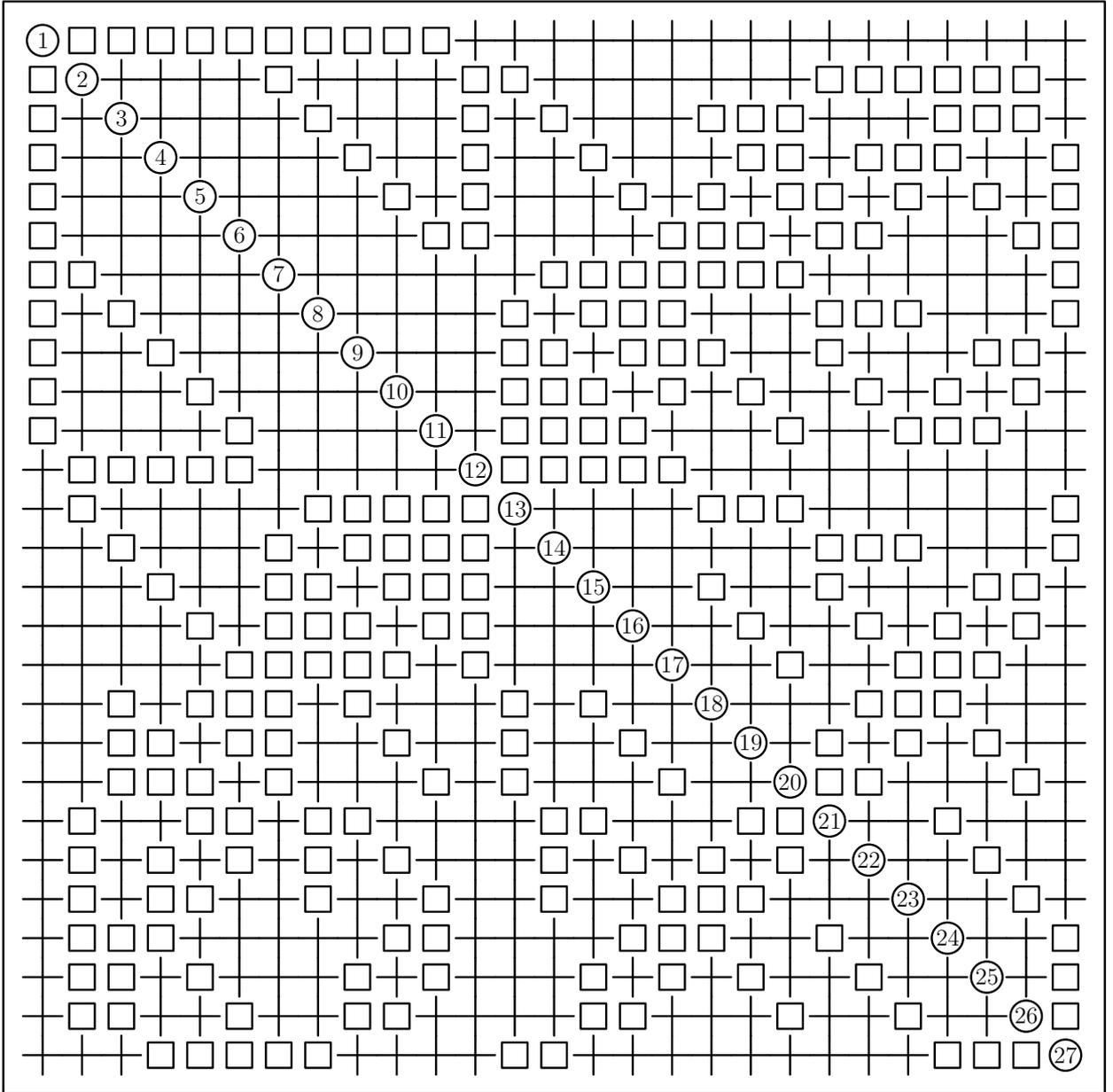


FIG. 1: A pictorial representation of the twenty-seven rearrangements of the diagonal generator h^1 of E_6 . Each rearrangement can be reconstructed from one of the twenty-seven rows (or columns) of symbols in this picture. To find the diagonal entries of the n^{th} rearrangement, read along the n^{th} row and translate the symbols according to: circles \circ correspond to the single $1/3$ entry, squares \square to $-1/6$ and crosses $+$ to $1/12$ (note that adjacent crosses are touching). The number in the centre of each circle tells its row and column number (being the same). Row n of this picture corresponds precisely to row n of Table I in the sense that the linear combination $\sum_{a=1}^6 f_a h^a$, where the f_{1-6} are chosen from row n of Table I, yields the rearranged version of the generator h^1 represented by the symbols of row n in this picture.

$$\langle \mathcal{X} \rangle \propto \sum_{a=1}^6 f_a h^a \quad (\text{VII.4})$$

where the sextuplet $f_{1,\dots,6}$ takes values from one of the rows of the Table I. We include a figure from [12] which graphically identifies the diagonal entries of each of these 27 vevs breaking $E_6 \rightarrow SO(10) \times U(1)$.

VIII. CONCLUSION

In a linear combination of the adjoint Higgs vevs which break $G \supset H_1 \supset H_2 \supset \dots \supset H_l$ we have described how to choose the coefficients so that the resulting vev breaks G to a differently embedded isomorphic subgroup belonging to the chain $G \supset gH_1g^{-1} \supset gH_2g^{-1} \supset \dots \supset gH_lg^{-1}$, for some $g \in G$. We have highlighted the simple case when the subgroup we are breaking to stabilizes the highest weight of the lowest dimensional fundamental representation for G and complemented our discussion with physical examples.

We also covered the more general case when the Higgs field is not in the adjoint representation. Here we discussed the relationship between the weights of vevs breaking G to differently embedded copies of a particular subgroups. In addition we canvassed the relationship between the Cartan subalgebra generators h^1, \dots, h^l which generate the Abelian subgroups in a chain $G \supset H_1 \supset H_2 \supset \dots \supset H_l$ where each $H_i = H'_i \times U(1)_{H_1} \times \dots \times U(1)_{H_i}$ and the Cartan subalgebra generators for the conjugated chain $G \supset gH_1g^{-1} \supset gH_2g^{-1} \supset \dots \supset gH_lg^{-1}$.

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X. APPENDIX

The Borel-de-Siebenthal theorem gives a systematic way of identifying the maximal Lie subgroups of G directly from the extended Dynkin diagram for G [13]. It does this by identifying which subset of the simple roots belonging to Δ , the root system for the Lie algebra \mathcal{L} of G , are also simple roots for Δ_H , the subroot system of the Lie algebra \mathcal{L}_H of maximal subgroup $H \subset G$. The nodes of the extended Dynkin diagram of G which are labeled by simple roots ζ^i which **do not** belong to Δ_H are then removed, along with all their adjacent edges. The remaining graph is the Dynkin diagram for H . If the subgroup $H \subset K \subset G$, is not maximal then this procedure can be iterated to determine $K \subset G$ and $H \subset K$.

Theorem X.1. (Borel-de-Siebenthal) *Let Δ be an irreducible crystallographic root system. Let $\{\zeta^1, \dots, \zeta^l\}$ be the simple roots for Δ . Let ζ^0 be the highest root of Δ with respect to $\{\zeta^1, \dots, \zeta^l\}$. Expand:*

$$\zeta^0 = \sum_i c_i \zeta^i \tag{X.1}$$

Then the maximal closed subroot systems of Δ (up to Weyl group reflections) are those with fundamental systems

- $\{\zeta^1, \zeta^2, \dots, \hat{\zeta}^i, \dots, \zeta^l\}$ where $c_i = 1$;
- $\{-\zeta^0, \zeta^1, \dots, \hat{\zeta}^i, \dots, \zeta^l\}$ where $c_i = p$ (prime)

Where “ $\hat{\zeta}^i$ ” is being used to denote elimination.