Generalized Schubert Calculus

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Dedicated to C.S. Seshadri on the occasion of his 80th birthday

Abstract

In this paper we study the T-equivariant generalized cohomology of flag varieties using two models, the Borel model and the moment graph model. We study the differences between the Schubert classes and the Bott-Samelson classes. After setup of the general framework we compute, for classes of Schubert varieties of complex dimension $\leqslant 3$ in rank 2 (including A_2, B_2, G_2 and $A_1^{(1)}$), moment graph representatives, Pieri-Chevalley formulas and products of Schubert classes. These computations generalize the computations in equivariant K-theory for rank 2 cases which are given in Griffeth-Ram [GR].

1 Introduction

This paper is a study of the generalized equivariant cohomology of flag varieties. We set up a general framework for working with the generalized (equivariant) Schubert calculus which allows for detailed study without the need for knowledge of cobordism or generalized cohomology theories. Working in the context of a complex reductive algebraic group G, the (generalized) flag variety is G/B, where B is a Borel subgroup containing the maximal torus T. The equivariant generalized cohomology theory h_T comes with a (formal) group which is used to combinatorially construct the ring $S = h_T(\mathrm{pt})$. The Borel model presents $h_T(G/B)$ as a 'coinvariant ring' $S \otimes_{S^{W_0}} S$ and the moment graph model presents $h_T(G/B)$ via the image of the inclusions of the T-fixed points of G/B. Special cases of generalized equivariant cohomology theories are 'ordinary' cohomology (corresponding to the additive group) and K-theory (corresponding to the multiplicative group). The universal formal group law corresponds to complex cobordism.

Our work follows papers of Bressler-Evens [BE1, BE2], Calmès-Petrov-Zainoulline [CPZ], Harada-Holm-Henriques [HHH], Hornbostel-Kiritchenko [HK], and Kiritchenko-Krishna [KiKr], which have laid important foundations. Combining these tools we study the equivariant cohomology of the flag varieties, partial flag varieties, and Schubert varieties via the algebraic and combinatorial study of the rings which appear in the Borel model and the moment graph model. In Sections 2.3 and 3 we review the setup for these models and the connection to the (generalized) nil affine Hecke algebra and the BGG-Demazure operators (see also [HLSZ] and [BE1, BE2]).

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One of the main points of our work is to shift the focus from Bott-Samelson classes to Schubert classes. In ordinary equivariant cohomology and equivariant K-theory these agree, but in generalized cohomology the Schubert classes and the Bott-Samelson classes usually differ. Since the Schubert varieties are not, in general, smooth it is not even clear how the Schubert classes (the fundamental classes of the Schubert varieties) should be defined. In Section 5 we give explicit examples of "naive pushforwards" and Bott-Samelson classes and explain why neither of these can possibly be the Schubert classes in general. There are several directions to explore in searching for a good way to define Schubert classes:

(a) One can take the lead of Borisov-Libgober [BL] (see also [To]), and define the Schubert class $[X_w]$ as a 'corrected' version of the Bott-Samelson class $[Z_{\vec{w}}]$ which, in the end, does not depend on the reduced word $\vec{w}$ chosen for w. Borisov-Libgober [BL, Definition 3.1] obtain a correction factor for the elliptic genus from the discrepancies of the components of the exceptional divisor of a resolution of singularities of a variety with at worst log terminal singularities. Recent papers of Anderson-Stapledon [AS] and Kumar-Schwede [KS] explain that Schubert varieties have Kawamata log terminal singularities and analyze the exceptional divisor in the Bott-Samelson resolution. In Section 5 we compute a possible equivariant algebraic cobordism correction factor for the smallest singular (complex dimension 3) Schubert variety in all rank 2 cases. Though the approach of Borisov-Libgober was a motivation for our computations we have not yet understood how to make our computation of the correction factor for equivariant algebraic cobordism relate to the correction suggested by Borisov-Libgober for the elliptic genus.

(b) One can try to define the Schubert classes as classes determined, hopefully uniquely, by positivity properties under multiplication. We have not yet managed to make a definition that is satisfying but our computations of Schubert products do display remarkable positivity features.

(c) One can try to use the theory of Soergel bimodules (see [Soe]) to pick out particular generators (as (S,S)-bimodules) of the generalized cohomologies of Schubert varieties which serve as Schubert classes. Though we have not had space to exhibit our computations of the algebraic cobordism case of Soergel bimodules in this paper, our preliminary computations show that generalizing the Soergel bimodule theory to the ring S which appears in Theorem 3.1 is useful for obtaining better understanding of the equivariant generalized cohomology of Schubert varieties.

In Section 7 we provide explicit computations of Schubert classes, and products with Schubert classes in the rank 2 cases. Our computations hold for all rank two cases, but we have only given specific results for Schubert classes of Schubert varieties in G/B of (complex) dimension $\leqslant 3$. In partiuclar, this provides complete results for types A_2 and B_2 and partial results for G_2 and $A_1^{(1)}$.

To some extent this paper is a sequel to [GR]. That paper considers the case of equivariant K-theory. In retrospect, [GR] did not capitalize on the full power of the moment graph model, in particular, that the map Φ in Theorem 3.1 is a *ring homomorphism.* This key point is the feature which we exploit in this paper to execute computations similar to those in [GR], but with greater ease and in greater generality.

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this beautiful way of working with T-equivariant cohomology theories. We thank Alex Ghitza, Matthew Ando, Megumi Harada, Dave Anderson and Michel Brion for helpful conversations. We also thank Craig Westerland for answering many many questions of all shapes and sizes all along the way. It is a pleasure to dedicate this paper to C.S. Seshadri who, for so many years, has provided so much Schubert calculus support and inspiration.

2 The Schubert calculus framework

2.1 Flag and Schubert varieties

The basic data is

$$\begin{array}{ll} G & \text{a connected complex reductive algebraic group} \\ \cup\!| & \\ B & \text{a Borel subgroup} \\ \cup\!| & \\ T & \text{a maximal torus.} \end{array} \tag{2.1}$$

The *Weyl group*, the *character lattice* and *cocharacter lattice* are, respectively,

$$W_0 = N(T)/T, \qquad \mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times) \qquad \text{and} \qquad \mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T), \tag{2.2}$$

where $\mathrm{Hom}(H, K)$ is the abelian group of algebraic group homomorphisms from H to K with product given by pointwise multiplication, $(\phi\psi)(h) = \phi(h)\psi(h)$. Since the Weyl group acts on T, it also acts on $\mathfrak{h}_{\mathbb{Z}}^*$ and on $\mathfrak{h}_{\mathbb{Z}}$.

A *standard parabolic subgroup* of G is a subgroup $P_J \supseteq B$ such that G/P_J is a projective variety. A *parabolic subgroup* of G is a conjugate of a standard parabolic subgroup.

$$\text{The } \textit{flag variety} \text{ is } G/B \qquad \text{and} \qquad G/P_J \text{ are the } \textit{partial flag varieties.} \tag{2.3}$$

These are studied via the *Bruhat decomposition*

$$G = \bigsqcup_{w \in W_0} BwB \qquad \text{and} \qquad G = \bigsqcup_{u \in W^J} BuP_J \tag{2.4}$$

where $W_J = \{v \in W_0 \mid vT \subseteq P_J\}$ and

$$W^J = \{\text{coset representatives } u \text{ of cosets in } W_0/W_J\}. \tag{2.5}$$

The *Schubert varieties* are

$$X_w = \overline{BwB} \quad \text{in } G/B \qquad \text{and} \qquad X_u^J = \overline{BuP_J} \quad \text{in } G/P_J, \tag{2.6}$$

and the *Bruhat orders* are the partial orders on W_0 and W_J given by

$$X_w = \overline{BwB} = \bigsqcup_{v \leqslant w} BvB \qquad \text{and} \qquad X_u^J = \overline{BuP_J} = \bigsqcup_{z \leqslant u} BzP_J. \tag{2.7}$$

The T-fixed points

$$\text{in } G/B \text{ are } \{wB \mid w \in W_0\} \quad \text{and} \qquad \text{in } G/P_J \text{ are } \{uP_J \mid u \in W^J\}. \tag{2.8}$$

Let $P_1, \ldots, P_n$ be the minimal parabolic subgroups $P_i \neq B$. Then

$$W_i = W_{\{i\}} = \{1, s_i\} \qquad \text{and} \qquad s_1, \ldots, s_n \text{ are the \textit{simple reflections} in } W_0. \tag{2.9}$$

With respect to the action of W_0 on $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}$, the s_i are reflections in the hyperplanes $(\mathfrak{h}^*)^{s_i} = \{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid s_i\mu = \mu\}$. An alternative description of the standard parabolic subgroups is to let $J \subseteq \{1, 2, \ldots, n\}$ and let

$$W_J = \langle s_j \mid j \in J \rangle. \qquad \text{Then} \qquad P_J = \bigsqcup_{v \in W_J} BvB. \tag{2.10}$$

In particular, $P_i = P_{\{i\}} = B \sqcup Bs_iB$, for $i = 1, 2, \ldots, n$.

Theorem 2.1. *(Coxeter) The group W_0 is generated by $s_1, \ldots, s_n$ with relations*

$$s_i^2 = 1 \qquad \textit{and} \qquad \underbrace{s_is_js_i\cdots}_{m_{ij}\ \text{factors}} = \underbrace{s_js_is_j\cdots}_{m_{ij}\ \text{factors}}$$

where $\pi/m_{ij} = (\mathfrak{h}^*)^{s_i} \angle (\mathfrak{h}^*)^{s_j}$ is the angle between $(\mathfrak{h}^*)^{s_i}$ and $(\mathfrak{h}^*)^{s_j}$.

The definitions in (2.3), (2.8) and (2.6) provide T-equivariant maps

$$\begin{array}{rccc} p_J\colon & G/B & \longrightarrow & G/P_J \\ & gB & \longmapsto & gP_J \end{array} \qquad \begin{array}{rccc} \iota_w\colon & \mathrm{pt} & \hookrightarrow & G/B \\ & \mathrm{pt} & \longmapsto & wB \end{array} \qquad \begin{array}{rccc} \sigma_w\colon & X_w & \hookrightarrow & G/B \\ & gB & \longmapsto & gB \end{array} \tag{2.11}$$

and

$$\begin{array}{rccc} \iota^J_u\colon & \mathrm{pt} & \hookrightarrow & G/P_J \\ & \mathrm{pt} & \longmapsto & uP_J \end{array} \qquad \begin{array}{rccc} \sigma^J_u\colon & X^J_u & \hookrightarrow & G/P_J \\ & gP_J & \longmapsto & gP_J \end{array} \tag{2.12}$$

for $J \subseteq \{1, 2, \ldots, \ell\}$, $w \in W_0$, and $u \in W^J$.

For example, in type $G = GL_3$, with T and B the subgroups given by

$$T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \qquad \text{and} \qquad B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

then $W_0 = \langle s_1, s_2 \mid s_i^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle$, where

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{tikzcd} & X_{w_0} = G/B & \\ X_{s_1s_2} \arrow[ur, hook] & & X_{s_2s_1} \arrow[ul, hook] \\ X_{s_1} \arrow[u, hook] \arrow[urr, hook] & & X_{s_2} \arrow[u, hook] \arrow[ull, hook] \\ & \mathrm{pt} = X_1 \arrow[ul, hook] \arrow[ur, hook] \arrow[uuu, hook] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & G/B \arrow[dl] \arrow[dr] & \\ G/P_1 \arrow[dr] & & G/P_2 \arrow[dl] \\ & G/G = \mathrm{pt} & \end{tikzcd}$$

where P_1 and P_2 are the subgroups of $G = GL_3(\mathbb{C})$ given by

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} = B \sqcup Bs_1B \qquad \text{and} \qquad P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} = B \sqcup Bs_2B.$$

2.2 Generalized cohomology theories

Schubert calculus is the study of the cohomology of flag and Schubert varieties. Although the home for our computations is the particular ring $S = \mathbb{L}[[y_\lambda]]$ of (3.4) the motivation comes from the formalism of generalized cohomology theories h. Model examples are: ordinary cohomology H, K-theory K, elliptic cohomology (see [MR, GKV, Gr, An, Lu]) and complex and algebraic cobordism Ω (see [LM]). Key to our point of view is that if $f\colon X \to Y$ is a morphism of spaces, the contravariance of the cohomology theory provides

$$\text{a pullback} \quad f^*\colon h(Y) \to h(X), \qquad \text{and} \quad \text{a pushforward} \quad f_!\colon h(X) \to h(Y)$$

exists if the morphism f is nice enough. Our true interest is in the morphisms in (2.11) and (2.12) and (4.5). Sometimes we will try to consider, by combinatorial gadgetry, pushforwards across these morphisms even in cases where we are not sure that, for any given cohomology theory, the pushforward properly exists.

As in [CPZ, §8.2], the important property for the analysis of Schubert calculus is that an oriented cohomology theory h comes with a formal group law F over the coefficient ring $h(pt)$ such that

$$F(c_1^h(\mathcal{L}_1), c_1^h(\mathcal{L}_2)) = c_1^h(\mathcal{L}_1 \otimes \mathcal{L}_2),$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on X and c_1^h denotes the first Chern class in the cohomology theory h (see [LM, Cor. 4.1.8]). The *Lazard ring* $\mathbb{L}$ is generated by symbols a_{ij}, for $i, j \in \mathbb{Z}_{>0}$, which satisfy the relations given by the equations

$$F(x, F(y, z)) = F(F(x, y), z), \qquad F(x, y) = F(y, x), \qquad F(x, 0) = x, \tag{2.13}$$

where

$$F(x, y) = x + y + a_{11}xy + a_{12}xy^2 + a_{21}x^2y + \cdots$$

The ring $\mathbb{L}$ is the universal coefficient ring for a formal group law F. This ring is one of the ingredients for the construction of the ring S where we do our computations.

A equivariant cohomology theory h_T is a functor from T-spaces (some appropriate class of topological or geometric objects with T-action) to some class of algebraic objects (in most of our model examples, $h_T(\text{pt})$-algebras). Important features and properties of the theory include:

(0) Normalization: specification of $h_T(\text{pt})$,

(1) nice behaviour under products, smashes, suspensions: such as axioms for computing $h_{G\times K}(M \times N)$,

(2) functoriality/pullbacks: if $f\colon X \to Y$ then we have $f^*\colon h_T(Y) \to h_T(X)$

(3) Thom isomorphism/orientability/pushforwards: For certain classes of maps $f\colon X \to Y$ there exists a pushforward $f_!\colon h_T(X) \to h_T(Y)$,

(4) Change of groups: For certain classes of groups G and K and group homomorphisms $\varphi\colon G \to K$ there exist $\chi_\varphi\colon h_G \to h_K$ and $\chi^\varphi\colon h_K \to h_G$.

The art of choosing appropriate categories of input "T-spaces", of output "algebraic objects" and widening the classes of maps on which pushforwards and/or change of groups homomorphisms are defined is a beautiful chapter in algebraic topology and geometry. The challenge of extending a nonequivariant generalized cohomology theory to the equivariant case can be considerable. For such a genuinely equivariant theory the formal groups above will be replaced by actual groups but we do not emphasize this point of view here. For a small selection of references we refer the reader to [Ad, p. 37-29] for a discussion of the connection to formal group laws and spectra, [Ma, Chapt. XIII] and [Oko] for a discussion of equivariant orientable theories as Mackey functors and [GKV, (1.5)] for discussion of axioms for equivariant elliptic cohomology.

In order to specify a home for our computations in Schubert calculus in equivariant cohomology theories we follow [HHH]. They restrict their class of spaces to *GKM spaces*: stratified T-spaces

$$X = \bigcup_{i \in \mathbb{Z}_{>0}} X_i, \qquad X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots,$$

where the successive quotients X_i/X_{i-1} are homeomorphic to the Thom spaces $Th(V_i)$ of some h-orientable T-vector bundles $V_i \to F_i$ (see [HHH, (2.1)]). As pointed out in [HHH, Remark 3.3], for the case of flag and Schubert varieties that are the focus of this paper, the F_i are points and the V_i are one dimensional representations of T. In particular, the assumptions of [HHH, §3] hold for these cases.

2.3 The Borel model for $h_T(G/B)$

The general combinatorial Schubert calculus uses $\mathfrak{h}_{\mathbb{Z}}^*$ and $\mathfrak{h}_{\mathbb{Z}}$ to build a C-algebra R with an action of W_0 on R by C-algebra automorphisms (in favorite examples C may be $\mathbb{Z}$, or the ring $\widetilde{Th}_0$ of holomorphic functions on the upper half plane, or the Lazard ring $\mathbb{L}$, see the examples below). If

$$R^{W_0} = \{f \in R \mid wf = f \text{ for } w \in W_0\} \qquad \text{is the } \textit{invariant ring},$$

then, conceptually,

$$R = h_T(\mathrm{pt}) \qquad \text{and} \qquad R^{W_0} = h_G(\mathrm{pt}), \tag{2.14}$$

for the equivariant cohomology theory h_T under analysis. By definition, the *coinvariant ring* is

$$R \otimes_{R^{W_0}} R = \frac{R \otimes_C R}{\langle f \otimes 1 - 1 \otimes f \mid f \in R^{W_0} \rangle}, \tag{2.15}$$

where the terminology is chosen to be representative of the classical terminology in the study of the cohomology of G/B, not to reflect a notion of coinvariants with respect to a group action. Then (see [Bo, Proposition 26.1], [KL, Proposition 1.6], [KiKr, Theorem 4.7]) the ring

$$R \otimes_{R^{W_0}} R \qquad \text{is a good combinatorial model for} \quad h_T(G/B), \tag{2.16}$$

where the product on $R \otimes_{R^{W_0}} R$ is given by $(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1 f_2 \otimes g_1 g_2$.

There are four favorite examples:

Cohomology: $h_T = H_T$. Here

$$H_T(\mathrm{pt}) = S(\mathfrak{h}_{\mathbb{Z}}^*) = \mathbb{C}[x_1, \ldots, x_n] \qquad \text{and} \qquad H_G(\mathrm{pt}) = H_T(\mathrm{pt})^{W_0} = \mathbb{C}[x_1, \ldots, x_n]^{W_0},$$

where $x_i = x_{\omega_i}$, where $\omega_1, \ldots, \omega_n$ is a $\mathbb{Z}$-basis of $\mathfrak{h}_{\mathbb{Z}}^*$. Alternatively, $H_T(\mathrm{pt})$ is the ring

$$\mathbb{C}[x_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*] \quad \text{with} \quad x_{\lambda+\mu} = x_\lambda + x_\mu,$$

for $\lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}^*$ and with $wx_\lambda = x_{w\lambda}$ for $w \in W_0$ and $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$. Then

$$H_T(G/B) = H_T(\mathrm{pt}) \otimes_{H_G(\mathrm{pt})} H_T(\mathrm{pt}) = \frac{\mathbb{C}[y_1, \ldots, y_n, x_1, \ldots, x_n]}{\langle f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) \mid f \in \mathbb{C}[x_1, \ldots, x_n]^{W_0} \rangle}.$$

K-theory: $h_T = K_T$. Here

$$K_T(\mathrm{pt}) = \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \qquad \text{and} \qquad K_G(\mathrm{pt}) = K_T(\mathrm{pt})^{W_0} = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{W_0},$$

where $X_i = e^{\omega_i}$, where $\omega_1, \ldots, \omega_n$ is a $\mathbb{Z}$-basis of $\mathfrak{h}_{\mathbb{Z}}^*$. Alternatively, $K_T(\mathrm{pt})$ is the ring

$$\mathbb{C}[e^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*] \quad \text{with} \quad e^{\lambda+\mu} = e^\lambda e^\mu,$$

for $\lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}^*$ and with $we^\lambda = e^{w\lambda}$ for $w \in W_0$ and $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$. Then

$$K_T(G/B) = K_T(\mathrm{pt}) \otimes_{K_G(\mathrm{pt})} K_T(\mathrm{pt}) = \frac{\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}]}{\langle f(X_1, \ldots, X_n) - f(Y_1, \ldots, Y_n) \mid f \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{W_0} \rangle}.$$

Elliptic cohomology: $h_T = Ell_T$. Here $Ell_T(\mathrm{pt})$ is the structure sheaf of the abelian variety $A_\tau = \mathfrak{h}_{\mathbb{C}}^*/(\mathfrak{h}_{\mathbb{Z}}^* + \tau\mathfrak{h}_{\mathbb{Z}}^*)$. The homogeneous coordinate ring

$$\text{for } A_\tau \text{ is} \qquad \widetilde{Th} = \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \widetilde{Th}_m, \qquad \text{and} \qquad \text{for } A_\tau/W_0 \text{ is} \qquad \widetilde{Th}^{W_0}.$$

Then the graded $\widetilde{Th}$-module corresponding to

$$\text{the sheaf } Ell_T(G/B) \text{ on } A_\tau \text{ is} \qquad \widetilde{Th} \otimes_{\widetilde{Th}^{W_0}} \widetilde{Th}.$$

Complex or algebraic cobordism: $h_T = \Omega_T$. Algebraic cobordism is treated in the book of Levine-Morel [LM] and T-equivariant algebraic cobordism Ω_T is treated in [Kr] and [KiKr]. The following summary of our setting is made precise by Theorem 3.1 below.

The *Lazard ring* $\mathbb{L}$ is the coefficient ring for the universal formal group law F so that $\mathbb{L}$ is given by generators a_{ij} with relations given by setting

$$F(x,y) = x + y + \sum_{i,j \in \mathbb{Z}_{>0}} a_{ij} x^i y^j \qquad \text{in } \mathbb{L}[[x,y]],$$

and requiring

$$F(x,0) = F(0,x) = x, \quad F(x,y) = F(y,x), \quad F(x, F(y,z)) = F(F(x,y),z).$$

Then

$$\Omega_T(\mathrm{pt}) = \mathbb{L}[[x_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]] \quad \text{with} \quad x_{\lambda+\mu} = x_\lambda +_F x_\mu = F(x_\lambda, x_\mu),$$

for $\lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}^*$. Then

$$\Omega_G(\mathrm{pt}) = \Omega_T(\mathrm{pt})^{W_0} = \mathbb{L}[[x_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]]^{W_0}, \qquad \text{where} \quad wx_\lambda = x_{w\lambda},$$

for $w \in W_0$ and $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, and

$$\Omega_T(G/B) = \Omega_T(\mathrm{pt}) \otimes_{\Omega_G(\mathrm{pt})} \Omega_T(\mathrm{pt}) = \frac{\mathbb{L}[[y_\lambda, x_\mu \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]]}{\langle f(x) - f(y) \mid f \in \mathbb{L}[[x_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]]^{W_0} \rangle}.$$

Sample references for such identities are [KK1] for the case of $H_T(G/B)$, [KK2, KL, CG] for $K_T(G/B)$, [KP, Gr, GKV, An, Ga] for $Ell_T(G/B)$ and [HHH, CPZ, HK, KiKr] for $\Omega_T(G/B)$.

The cobordism case specializes to the cases of cohomology H_T and K-theory K_T by setting

$$F(x,y) = \begin{cases} x+y, & \text{in } H_T, \\ x+y-xy, & \text{in } K_T, \end{cases} \qquad \text{and} \qquad x_\lambda = \begin{cases} x_\lambda, & \text{in } H_T, \\ 1 - e^\lambda, & \text{in } K_T. \end{cases}$$

3 The moment graph model

3.1 T-fixed points and the map Φ

Following Goresky-Kottwitz-MacPherson [GKM, Theorem 1.2.2] a powerful way to think about this theory is via the *moment graph model*. This means that for a T-variety X where the imbeddings of the T-fixed points of X into X are

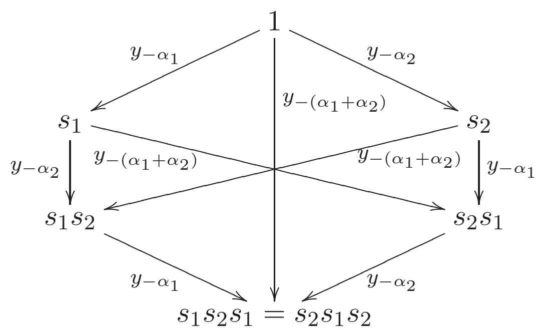
$$\begin{array}{rccc} \iota_w\colon & \mathrm{pt} & \to & X \\ & * & \mapsto & w \end{array} \qquad \text{consider} \qquad \iota^* = \textstyle\bigoplus_{w\in W} \iota_w^*\colon \Omega_T^*(X) \longrightarrow \displaystyle\bigoplus_{w\in W} \Omega_T(\mathrm{pt}), \tag{3.1}$$

where the sums are over an index set W for the T-fixed points in X. When X is a "GKM-space" (see [GKM, Theorem 14] for several equivalent characterization of a GKM space for equivariant ordinary cohomology and [**?**, HHH]or equivariant generalized cohomology theories) the ring homomorphism ι^* is injective with image

$$\operatorname{im}\iota^* = \left\{ (g_w)_{w\in W_0} \in \bigoplus_{w\in W_0} \Omega_T(\mathrm{pt}), \;\middle|\; \begin{array}{c} g_w - g_{w'} \in y_\alpha \Omega_T(\mathrm{pt}) \text{ if there is a} \\ \text{1-dimensional } T\text{-orbit containing } w \text{ and } w' \end{array} \right\},$$

where y_α is the T-equivariant Chern class of the tangent along the 1-dimensional orbit connecting w and w'.

Computations are facilitated by encoding the information of $\operatorname{im}\iota^*$ with a *moment graph*, which has vertices corresponding to the T-fixed points of X and labeled edges $w \xrightarrow{\alpha} w'$ corresponding to 1-dimensional T-orbits in X. For example, for G/B for type GL_3 the graph is



(3.2)

A *moment graph section* is a tuple $(g_w)_{w\in W}$ of elements of $\Omega_T(\mathrm{pt})$ which is an element of $\operatorname{im}\iota^*$.

A morphism of GKM-spaces is a morphism of T-spaces

$$f\colon X \to Y \qquad \text{which provides, by restriction,} \qquad f\colon W \to V$$

from the set W of T-fixed points of X to the set V of T-fixed points of Y. Viewing elements of $H_T(X)$ and $H_T(Y)$ as moment graph sections the maps

$$f^*\colon H_T(Y) \to H_T(X) \qquad \text{and} \qquad f_!\colon H_T(X) \to H_T(Y)$$

are given by

$$(f^*(c))_w = c_{f(w)}, \qquad \text{and} \qquad (f_!(\gamma))_v = \sum_{w\in f^{-1}(v)} \gamma_w \frac{1}{e(f)_{wv}}, \tag{3.3}$$

where the *Euler class* of f from v to w is

$$e(f)_{wv} = \left(\prod_{\substack{\text{edges of } W \\ \text{adjacent to } w}} y_\beta \right) \left(\prod_{\substack{\text{edges of } V \\ \text{adjacent to } v}} y_\beta \right)^{-1}.$$

The second formula in (3.3) is a form of the familiar formula for push forwards by "localization at the T-fixed points" as found, for example, in [AB, (3.8)]. The Euler class of f from v to w is the contribution measured by the difference between the tangent space at the T-fixed point w in X to the tangent space to the T-fixed point $v = f(w)$ in Y.

The Borel model and the moment graph model for G/B for equivariant algebraic cobordism $\Omega_T(G/B)$ are summarized in the following Theorem, which is a combination of [KiKr, Theorem 4.7] and [HHH, Theorem 3.1]. The ring S which takes the role of $\Omega_T(\mathrm{pt})$ is as in [CPZ, §2.4]. For comparison to the K-theory case see [KK2, Theorem 3.13] and [LSS, Theorem 3.1].

Theorem 3.1. ([HHH, Theorem 3.1], [KiKr, Theorem 4.7] and [CPZ, §2.4] combined) *Let $G \supseteq B \supseteq T$ be a reductive group datum as in* (2.1) *and let W_0 and $\mathfrak{h}_{\mathbb{Z}}^*$ be the Weyl group and the weight lattice $\mathfrak{h}_{\mathbb{Z}}^*$ as in* (2.2)*. Let $\mathbb{L}$ be the Lazard ring generated by a_{ij} as in* (2.13) *and let S be the $\mathbb{L}$-algebra*

$$S = \mathbb{L}[[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]], \quad \textit{with} \quad y_{\lambda+\mu} = y_\lambda + y_\mu + a_{11}y_\lambda y_\mu + a_{12}y_\lambda y_\mu^2 + a_{21}y_\lambda^2 y_\mu + \cdots. \tag{3.4}$$

The Weyl group

$$W_0 \quad \textit{acts } \mathbb{L}\textit{-linearly on } S \textit{ by} \qquad wy_\lambda = y_{w\lambda},$$

for $w \in W_0$, $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$. Define a product on $\bigoplus_{w\in W_0} S$ pointwise,

$$(f_w)_{w\in W_0} \cdot (g_w)_{w\in W_0} = (f_w g_w)_{w\in W_0}, \tag{3.5}$$

and let $S \otimes_{S^{W_0}} S$ be the coinvariant ring as defined in (2.15)*. The S-algebra homomorphism*

$$\begin{array}{cccc} \Phi\colon S \otimes_{S^{W_0}} S \xrightarrow{\sim} \Omega_T(G/B) \xrightarrow{\sim} \operatorname{im}\Phi & \hookrightarrow & \bigoplus_{w\in W_0} S \\ f \otimes g & \longmapsto & \left(f \cdot (w^{-1}g)\right)_{w\in W_0} \end{array} \tag{3.6}$$

is well defined and injective with

$$\operatorname{im}\Phi = \left\{ (g_w)_{w\in W_0} \in \bigoplus_{w\in W_0} S \;\middle|\; g_w - g_{ws_\alpha} \in y_{-\alpha}S \text{ for } \alpha \in R^+ \text{ and } w \in W_0 \right\},$$

where R^+ is the set of positive roots corresponding to B and $s_\alpha \in W_0$ denotes the reflection corresponding to α.

To provide a feel for the ring S of (3.4), let us provide some formulas which will be useful for computations later. To recapitulate and summarize previous definitions,

$$S = \mathbb{L}[[y_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]] \qquad \text{with} \qquad y_{\lambda+\mu} = y_\lambda + y_\mu - p(y_\lambda, y_\mu)y_\lambda y_\mu, \tag{3.7}$$

where $p(y_\lambda, y_\mu) \in \mathbb{L}[[y_\lambda, y_\mu]]$ is a power series

$$p(y_\lambda, y_\mu) = -a_{11} - a_{12}y_\mu - a_{21}y_\lambda - a_{31}y_\lambda^2 - a_{22}y_\lambda y_\mu - a_{13}y_\mu y_\lambda - \cdots, \tag{3.8}$$

with $a_{ij} \in \mathbb{L}$ satisfying relations such that

$$y_{-\lambda+\lambda} = y_0 = 0, \qquad y_{\lambda+\mu} = y_{\mu+\lambda}, \qquad y_{(\lambda+\mu)+\nu} = y_{\lambda+(\mu+\nu)}. \tag{3.9}$$

Then

$$y_\alpha = \frac{-y_{-\alpha}}{1-p(y_\alpha, y_{-\alpha})y_{-\alpha}}, \qquad \frac{1}{y_{-\alpha}} + \frac{1}{y_\alpha} = p(y_\alpha, y_{-\alpha}), \tag{3.10}$$

and the formula

$$\frac{y_{-\ell\alpha}}{y_{-\alpha}} = \ell - \sum_{j=1}^{\ell-1} p(y_{-\alpha}, y_{-j\alpha})y_{-j\alpha} = 1 + \sum_{j=1}^{\ell-1}(1-p(y_{-\alpha}, y_{-j\alpha})y_{-j\alpha}), \qquad \text{for } \ell \in \mathbb{Z}_{>0}, \tag{3.11}$$

is proved by induction on ℓ. Using (3.11) and the formula $s_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ for the action of a simple reflection on $\mathfrak{h}^*$ produces

$$\frac{y_{s_i\lambda} - y_\lambda}{y_{-\alpha_i}} = (1 - p(y_\lambda, y_{-\langle\lambda,\alpha_i^\vee\rangle\alpha_i})y_\lambda)\left(1 + \sum_{j=1}^{\langle\lambda,\alpha^\vee\rangle-1}(1-p(y_{-\alpha_i}, y_{-j\alpha_i})y_{-j\alpha_i})\right), \tag{3.12}$$

for $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$. Formula (3.12) generalizes one of the favorite formulas for the action of a Demazure operator (see [Ku2, Lemma 8.2.8]). This cobordism case specializes to H_T and K_T by setting

$$p(y_\lambda, y_\mu) = \begin{cases} 0, & \text{in } H_T, \\ 1, & \text{in } K_T, \end{cases} \qquad \text{and} \qquad y_\lambda = \begin{cases} y_\lambda, & \text{in } H_T, \\ 1-e^\lambda, & \text{in } K_T. \end{cases} \tag{3.13}$$

3.2 The nil affine Hecke algebra

Let S be as in (3.4) and (3.7). The point of view of [GR] is that the homomorphism Φ of (3.6) arises naturally from the nil affine Hecke algebra.

The *nil affine Hecke algebra H* is

$$\begin{aligned} H &= (S \otimes_{\mathbb{L}} S) \ltimes \mathbb{L}[W_0] \\ &= S\text{-span}\{gt_w \mid g \in S, w \in W_0\} = \mathbb{L}\text{-span}\{(f\otimes g)t_w \mid f, g \in S, w \in W_0\} \end{aligned}$$

with

$$t_ut_v = t_{uv} \quad \text{and} \quad t_w(f\otimes g) = (f \otimes (wg))t_w, \tag{3.14}$$

for $u, v, w \in W_0$ and $f, g \in S$. The nil affine Hecke algebra H acts on $S \otimes_{\mathbb{L}} S$ and on $S \otimes_{S^{W_0}} S$ by

$$t_w(f\otimes g) = f \otimes wg \qquad \text{and} \qquad (h\otimes p)(f \otimes g) = hf \otimes pg, \tag{3.15}$$

for $h, p, f, g \in S$ and $w \in W_0$. These actions arise from the realization of $S \otimes_{S^{W_0}} S$ as an induced up H-module in (3.16) below.

Let b_1 be a symbol and let Sb_1 be the $S \otimes_{\mathbb{L}} S$ module (a rank 1 free S-module with basis $\{b_1\}$) corresponding to the ring homomorphism

$$\begin{array}{rccc} \varepsilon\colon & S\otimes_{\mathbb{L}} S & \longrightarrow & S \\ & f\otimes g & \longmapsto & fg \end{array} \qquad \text{so that the } S \otimes_{\mathbb{L}} S \text{ action on } Sb_1 \text{ is given by } (f\otimes g)b_1 = fgb_1,$$

for $f, g \in S$. The induced module

$$Hb_1 = \mathrm{Ind}_{S\otimes_{\mathbb{L}}S}^H(Sb_1) \qquad \text{has } S\text{-basis} \qquad \{b_w \mid w \in W_0\}, \quad \text{where } b_w = t_wb_1.$$

Let $\mathbf{1}_0 = \sum_{w\in W_0} t_w$. With the definition of the H action on $S\otimes_{\mathbb{L}} S$ as in (3.15), the sequence of maps (see [GR, Theorem 2.12])

$$\begin{array}{ccccccccc} S\otimes_{\mathbb{L}} S & \longrightarrow & H\mathbf{1}_0 & \hookrightarrow & H & \longrightarrow & Hb_1 & \cong \bigoplus_{w\in W_0} S \\ (f\otimes g) & \longmapsto & (f\otimes g)\mathbf{1}_0 & & & & & \\ & & & & h & \longmapsto & hb_1 & \end{array} \tag{3.16}$$

is a homomorphism of H-modules (with kernel generated by $\{f\otimes 1 - 1\otimes f \mid f\in S^{W_0}\}$). The maps in (3.16) allow for the expansion of any element of $S\otimes_{\mathbb{L}} S$ in terms of the basis $\{b_w \mid w\in W_0\}$ of Hb_1, giving

$$\begin{aligned}(f\otimes g)\mathbf{1}_0 b_1 &= (f\otimes g)\big(\sum_{w\in W_0} t_w\big)b_1 = \sum_{w\in W_0} t_w(f\otimes (w^{-1}g))b_1 \\ &= \sum_{w\in W_0} t_w(f\cdot (w^{-1}g))b_1 = \sum_{w\in W_0} (f\cdot (w^{-1}g))b_w.\end{aligned}$$

This formula illustrates that computing $\Phi(f\otimes g)$ in (3.6) is equivalent to expanding $(f\otimes g)b_1$ in terms of the b_w. Because of this we use (3.6) and (3.16) to

$$\text{identify}\quad \Omega_T(G/B) = Hb_1 = S\text{-span}\{b_w \mid w\in W_0\} \cong \bigoplus_{w\in W_0} S$$

and write elements

$$f\in \Omega_T(G/B) \qquad \text{as} \qquad f = \sum_{w\in W_0} f_w b_w. \tag{3.17}$$

The product in $\Omega_T(G/B)$ is then given by (3.5). To more easily keep track of the left and right factors in $S\otimes_{\mathbb{L}} S$ use the notation

$$x_\mu = 1\otimes y_\mu \qquad \text{and} \qquad y_\mu = y_\mu\otimes 1. \tag{3.18}$$

Then the formulas

$$x_\lambda\cdot 1 = x_\lambda \sum_{w\in W_0} t_w b_1 = \sum_{w\in W_0} t_w x_{w^{-1}\lambda} b_1 = \sum_{w\in W_0} y_{w^{-1}\lambda} b_w, \quad \text{and} \tag{3.19}$$

$$t_v \sum_{w\in W_0} f_w b_w = \sum_{w\in W_0} f_w t_v b_w = \sum_{w\in W_0} f_w b_{vw} = \sum_{z\in W_0} f_{v^{-1}z} b_z, \tag{3.20}$$

provide the formulas for action of the nil affine Hecke algebra in terms of moment graph sections (see (3.15)). We often view the values f_w as labels on the vertices of the moment graph so that, for exmaple, in type GL_3 where the moment graph is as in (3.2), (3.19) can be written

$$x_\lambda \quad = \quad \begin{array}{ccc} & y_\lambda & \\ y_{s_1\lambda} & & y_{s_2\lambda} \\ y_{s_2s_1\lambda} & & y_{s_1s_2\lambda} \\ & y_{s_1s_2s_1\lambda} & \end{array}$$

4 Partial flag varieties and Bott-Samelson classes $[Z_{\vec{w}}]$

In this section we review the formulas for the Bott-Samelson classes as established in, for example, [HK, CPZ, BE1, BE2]. Though some of these references are not considering the equivariant case, the same machinery applies to define these classes in $\Omega_T(G/B)$. In particular, this is the place in the theory where the BGG/Demazure operators are derived from the geometry. These operators play a fundamental role in the combinatorial study of $\Omega_T(G/B)$.

4.1 Pushforwards to partial flag varieties: BGG/Demazure operators

Using the notation for parabolic subgroups and partial flag varieties as in (2.3), if $J \subseteq \{1, 2, \ldots, n\}$ and

$$\begin{array}{cccc} \pi_J\colon & G/B & \to & G/P_J \\ & gB & \mapsto & gP_J \end{array} \qquad \text{then} \qquad \pi_J(wB) = uP_J, \quad \text{where } wW_J = uP_J.$$

Then, in the setting of Theorem 3.1,

$$S \otimes_{S_0^W} S^{W_J} \cong \Omega_T(G/P_J),$$

and $\pi_J^*\colon \Omega_T(G/P_J) \to \Omega_T(G/B)$ and $(\pi_J)_!\colon \Omega_T(G/B) \to \Omega_T(G/P_J)$ correspond to

$$\pi_J^*\colon S \otimes_{S^{W_0}} S^{W_J} \hookrightarrow S \otimes_{S^{W_0}} S \qquad \text{and} \qquad (\pi_J)_!\colon S \otimes_{S^{W_0}} S \longrightarrow S \otimes_{S^{W_0}} S^{W_J} \tag{4.1}$$

where $(\pi_J)_!$ is given by the operator in the nil affine Hecke algebra given by

$$(\pi_J)_! = \left(\sum_{v \in W_J} t_v\right) \frac{1}{x_J}, \qquad \text{where} \quad x_J = \prod_{\alpha \in R_J^+} x_{-\alpha}.$$

with R_J^+ the set of positive roots for $P_J \supseteq B \supseteq T$. A special case is when $J = \{i\}$, for which

$$W_J = \{1, s_i\} \quad \text{and} \quad \pi_i^*(\pi_i)_! = A_i = (1 + t_{s_i}) \frac{1}{x_{-\alpha_i}}, \tag{4.2}$$

is the *BGG-Demazure operator* (see [BE1, Cor.-Def. 1.9]). The calculus of the operators A_i is controlled via the identities in Section 8.

4.2 Bott-Samelson classes

For a sequence $\vec{w} = (i_1, \ldots, i_\ell)$ with $1 \leqslant i_1, \ldots, i_\ell \leqslant n$ define the *Bott-Samelson class*

$$[Z_{\vec{w}}] = [Z_{i_1 i_2 \cdots i_\ell}] = A_{i_1} A_{i_2} \cdots A_{i_\ell} [Z_{\mathrm{pt}}], \tag{4.3}$$

where, in the notation of (3.17),

$$[Z_{\mathrm{pt}}]_v = \begin{cases} \prod_{\alpha \in R^+} y_{-\alpha}, & \text{if } v = 1, \\ 0, & \text{if } v \neq 1. \end{cases} \tag{4.4}$$

Theorem 4.1. ([BE2, Prop. 1], [HK, Prop. 3.1], [KK2, Lemma 3.15], see also [HHH, Proposition 4.1]) *The generalized cohomology*

$$h_T(G/B) \text{ has } h_T(\mathrm{pt})\text{-basis} \qquad \{[Z_{\vec{w}}] = [\gamma_{\vec{w}}\colon \Gamma_{\vec{w}} \to G/B] \mid w \in W_0\},$$

where, for each $w \in W_0$, $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a fixed reduced word for w.

Let us explain where this comes from. Let X be a T-variety. Following [Fu, Example 1.9.1], or [CG, §5.5], a *cellular decomposition* of X is a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_d = X$$

by closed subvarieties such that $X_i = X_{i-1}$ are isomorphic to a disjoint union of affine spaces $\mathbb{A}^{\ell_i}$ for $i = 1, 2, \ldots, d$. The "cells" of X are the $X_i - X_{i-1}$.

Theorem 4.2. (see [G, Prop. 7]; [Fu, Example 1.9.1] who refers to [Ch]; [CG, Lemma 5.5.1]; [BE2, Proposition 1]; [HK, Theorem 2.5]) *Let X be a T-variety with a cellular decomposition. Then $h_T(X)$ has an $h_T(\mathrm{pt})$-basis given by resolutions of cell closures (choose one resolution for each cell).*

For $X = G/B$, the Bruhat decomposition

$$G = \bigsqcup_{w\in W_0} BwB \qquad \text{provides the desired cell decomposition}$$

and the *Schubert varieties* $X_w = \overline{BwB}$ are the closures of the Schubert cells. Let $P_1, \ldots, P_n$ be the minimal parabolics of G (with $P_i \supseteq B$ and $P_i \neq B$) and let $s_1, \ldots, s_n$ be the corresponding *simple reflections* in W_0. The group W_0 is generated by $s_1, \ldots, s_n$. Let $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w. Then the Bott-Samelson variety $\Gamma_{i_1,\ldots,i_\ell} = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell}/B$ provides a resolution of X_w,

$$\begin{array}{rcccc} \gamma_{i_1,\ldots,i_\ell}\colon & P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell} \times_B \mathrm{pt} & \longrightarrow & X_w & \hookrightarrow G/B \\ & [g_1, \ldots, g_\ell] & \longmapsto & g_1 \cdots g_\ell B & \end{array} \tag{4.5}$$

Then following, for example, the proof of [BE2, Prop. 2], since the diagram

$$\begin{tikzcd} P_{i_1} \times_B \cdots \times_B P_{i_\ell} \times_B P_{i_{\ell+1}} \times_B \mathrm{pt} \arrow[rr, "\gamma_{i_1\ldots i_{\ell+1}}"] \arrow[d, "\tau"'] & & G/B \arrow[d, "\pi_{i_{\ell+1}}"] \\ P_{i_1} \times_B \cdots \times_B P_{i_\ell} \times_B \mathrm{pt} \arrow[r, "\gamma_{i_1\ldots i_\ell}"'] & G/B \arrow[r, "\pi_{i_{\ell+1}}"'] & G/P_{i_{\ell+1}} \end{tikzcd} \tag{4.6}$$

(a) commutes, and

(b) has both vertical maps fibrations with fibre $P_{i_{\ell+1}}/B$,

it is a pullback square. Thus

$$\begin{aligned} (\gamma_{i_1\ldots i_{\ell+1}})_!(\iota^*(1)) &= \pi^*_{i_{\ell+1}}(\pi_{i_{\ell+1}} \circ \gamma_{i_1\ldots i_\ell})_!(1) \\ &= \pi^*_{i_{\ell+1}}(\pi_{i_{\ell+1}})_!(\gamma_{i_1\ldots i_\ell})_!(1) = A_{i_{\ell+1}}(\gamma_{i_1\ldots i_\ell})_!(1). \end{aligned} \tag{4.7}$$

The following result then follows by induction.

Theorem 4.3. ([HK, Theorem 3.2], [BE2, Proposition 2]) *If $I = (i_1, \ldots, i_\ell)$ is a sequence in $\{1, \ldots, n\}$ and $\gamma_{i_1\ldots i_\ell}$ is as in (4.5) then*

$$[Z_{i_1\cdots i_\ell}] = [(\gamma_{i_1\ldots i_\ell})_!(1)] = A_{i_1} \cdots A_{i_\ell}[Z_{\mathrm{pt}}], \qquad \textit{where } [Z_{\mathrm{pt}}] \textit{ is the class of a point.}$$

Theorem 4.3 says that the values on the vertices of the element $[Z_{i_1\cdots i_\ell}]$ on the moment graph of $\Gamma_{i_1,\ldots,i_\ell}$ are exactly the coefficients of the 2^ℓ terms in the expansion of

$$A_{i_1} \cdots A_{i_\ell} = (1 + t_{s_{i_1}})\frac{1}{x_{-\alpha_{i_1}}} \cdots (1 + t_{s_{i_\ell}})\frac{1}{x_{-\alpha_{i_\ell}}}.$$

For example, in type GL_3,

$$[Z_{121}] = \begin{pmatrix} \frac{y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_1}} 1 \cdot 1 \cdot 1 \\ + \frac{y_{-\alpha_2}}{y_{\alpha_1}} t_{s_1} \cdot 1 \cdot 1 + 1 \cdot t_{s_2} \cdot 1 + \frac{y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_1}} 1 \cdot 1 \cdot t_{s_1} \\ + t_{s_1} \cdot t_{s_2} \cdot 1 + \frac{y_{-\alpha_2}}{y_{\alpha_1}} t_{s_1} \cdot 1 \cdot t_{s_1} + 1 \cdot t_{s_2} \cdot t_{s_1} \\ + t_{s_1} \cdot t_{s_2} \cdot t_{s_1} \end{pmatrix} b_1$$

provides the expansion of $[Z_{121}] = (1 + t_{s_1})\frac{1}{x_{-\alpha_1}}(1 + t_{s_1})\frac{1}{x_{-\alpha_1}}(1 + t_{s_1})\frac{1}{x_{-\alpha_1}} y_{R^-} b_1$ in the basis $\{b_w \mid w \in W_0\}$. An example of the pushpull in (4.6) in the case of type GL_3

$$\begin{tikzcd}
P_1 \times_B P_2 \times_B P_1 \times_B \mathrm{pt} \arrow[rr, "\gamma_{121}"] \arrow[d, "\tau"'] & & GL_3/B \arrow[d, "\pi_1"] \\
P_1 \times_B P_2 \times_B \mathrm{pt} \arrow[r, "\gamma_{12}"'] & GL_3/B \arrow[r, "\pi_1"'] & GL_3/P_1
\end{tikzcd} \tag{4.8}$$

has moment graphs as in Figure 1, and the computation in (4.7) for this example is

$$\begin{array}{ccccc}
\begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ & 1 & \end{array} & & \xrightarrow{(\gamma_{121})_!} & & \begin{array}{ccc} & \Delta_{121} & \\ \Delta_{121} & & 1 \\ 1 & & 1 \\ & 1 & \end{array} \\
\uparrow{\scriptstyle \tau^*} & & & & \uparrow{\scriptstyle \pi_1^*} \\
\begin{array}{ccc} & 1 & \\ 1 & 1 & \\ 1 & & \end{array} & \xrightarrow{(\gamma_{12})_!} & \begin{array}{ccc} & y_{-(\alpha_1+\alpha_2)} & \\ y_{-\alpha_2} & & y_{-(\alpha_1+\alpha_2)} \\ 0 & & y_{-\alpha_2} \\ & 0 & \end{array} & \xrightarrow{(\pi_1)_!} & \begin{array}{cc} \Delta_{121} & \\ & 1 \\ & 1 \end{array}
\end{array}$$

where $\Delta_{121} = \frac{y_{-(\alpha_1+\alpha_2)}}{y_{-\alpha_1}} + \frac{y_{-\alpha_2}}{y_{\alpha_1}}$.

4.3 Change of groups morphisms across $\iota\colon B \hookrightarrow P_J$

In the same way that Theorem 3.1 provides $S \otimes_S^{W_0} S \cong \Omega_T(G/B)$ one can obtain

$$S^{W_J} \otimes_{S_0^W} S \cong \Omega_{P_J}(G/B),$$

and, if $\iota\colon B \hookrightarrow P_J$ is the inclusion then the change of group homomorphisms

$$\iota^J : \Omega_{P_J}(G/B) \to \Omega_T(G/B) \qquad \text{and} \qquad \iota_J : \Omega_T(G/B) \to \Omega_{P_J}(G/B)$$

are given, combinatorially, by

$$\iota^J : S^{W_J} \otimes_{S_0^W} S \hookrightarrow S \otimes_{S_0^W} S \qquad \text{and} \qquad \iota_J : S \otimes_{S_0^W} S \longrightarrow S^{W_J} \otimes_{S_0^W} S,$$

with

$$\iota^J(f \otimes g) = \sum_{w \in W_J} w\left(\frac{1}{y_J} f\right) \otimes g, \qquad \text{where} \quad y_J = \prod_{\alpha \in R_J^+} y_{-\alpha},$$

with R_J^+ the set of positive roots for $P_J \supseteq B \supseteq T$. The pushforward ι^J is similar to the pushforward operator $(\pi_J)_!$ appearing in (4.1) except acting on the left factor of $S \otimes_{S^{W_0}} S$ (see, for example, the definition of δ_i in [Ka, §7]).

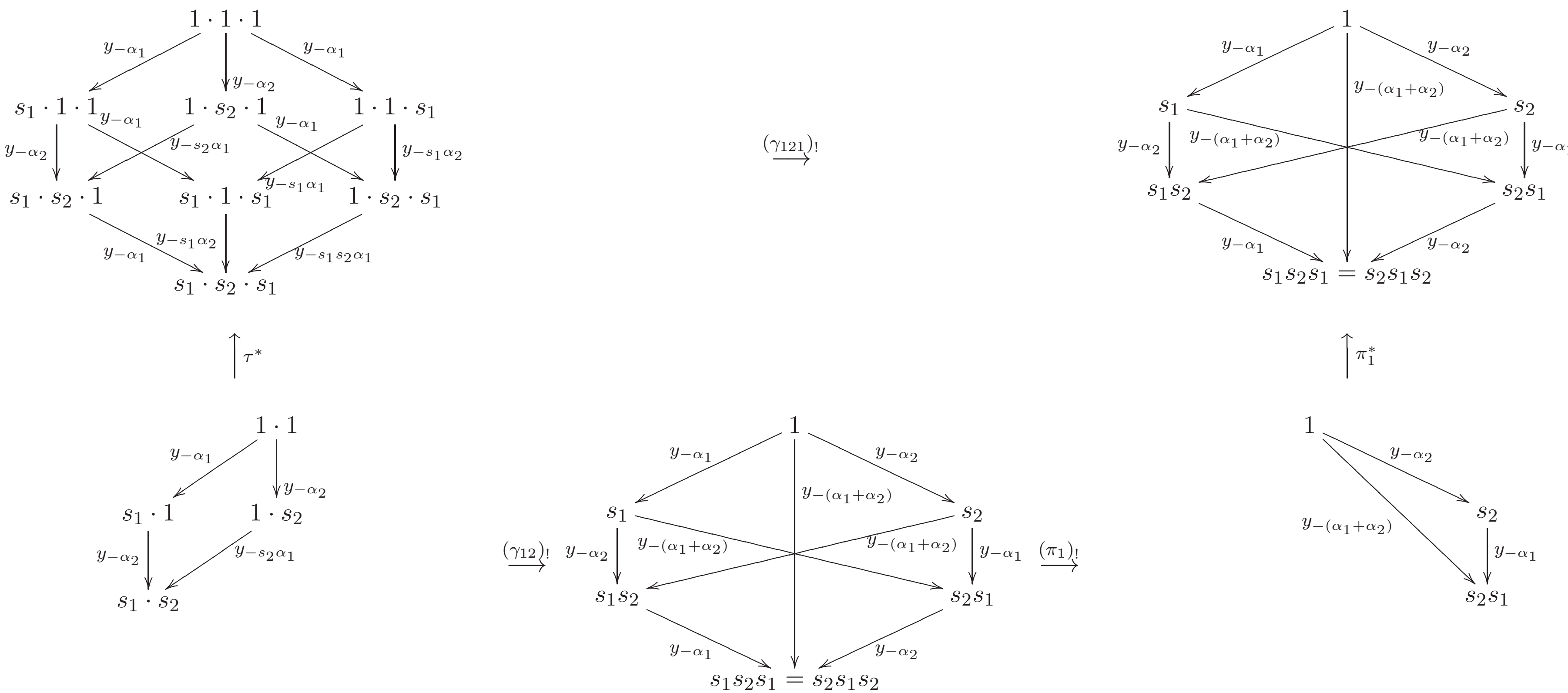


Figure 1: An example of the moment graphs for the diagram (4.8)

5 Schubert classes $[X_w]$

Now we consider the inclusions $\sigma_w\colon X_w \longrightarrow G/B$ of the Schubert varieties into the flag variety. For $w \in W_0$, define the *Schubert classes*

$$[X_w] = (\sigma_w)_!(1), \qquad \text{where} \quad (\sigma_w)_!\colon \Omega_T(X_w) \to \Omega_T(G/B). \tag{5.1}$$

If X_w is not smooth then, as discussed further below, it is not clear that $(\sigma_w)_!$ is well defined. Though we consider various approaches to the analysis of $[X_w] = (\sigma_w)_!(1)$ below, we have not yet found a definition of $(\sigma_w)_!$ which is fully satisfying (at least to us) in the singular case.

In *generalized* cohomology

$$\text{the Schubert class } [X_w] \text{ is } \textit{not always} \text{ equal to } [Z_{\vec{w}}]$$

for a reduced word $\vec{w}$ of w, although, in equivariant cohomology and equivariant K-theory, $[X_w] = [Z_{\vec{w}}]$ if $\vec{w}$ is a reduced word for w. We consider various approaches to the analysis of $[X_w] = (\sigma_w)_!(1)$:

(a) Defining $(\sigma_w)_!(1)$ by (3.3);

(b) Comparing $[X_w] = (\sigma_w)_!(1)$ and the Bott-Samelson class $[Z_{\vec{w}}]$ via the diagram

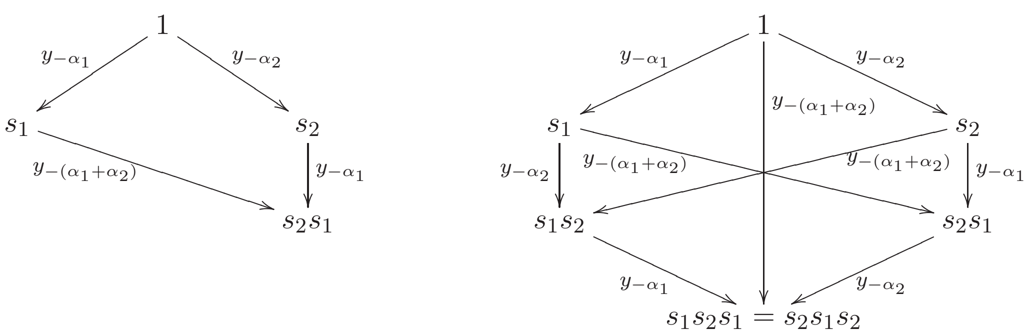
$$\begin{tikzcd} \Omega_T(\Gamma_{\vec{w}}) \arrow[d, "(\tilde{\gamma}_{\vec{w}})_!"'] \arrow[dr, "(\gamma_{\vec{w}})_!"] & \\ \Omega_T(X_w) \arrow[r, "(\sigma_w)_!"] & \Omega_T(G/B) \end{tikzcd} \tag{5.2}$$

(c) Combinatorial forcing by support conditions, normalization and/or (S,S)-bimodule structure of the cohomology.

(a) Is $(\sigma_w)_!(1)$ given by (3.3)? As pointed out in [Ty, Proposition 2.7], since X_w is filtered by Schubert cells BvB with $v \leqslant w$ and $BvB \cong \mathbb{C}^{\ell(v)}$ has even real dimension, the Schubert variety X_w has no odd-dimensional cohomology, and thus, by [GKM, Theorem 14], the Schubert variety X_w is 'equivariantly formal' (i.e., is a GKM-space) and the moment graph theory applies. The moment graph of X_w is the subgraph of the moment graph of G/B with vertices $\{v \in W_0 \mid v \leqslant w\}$. If X_w is smooth then there are no challenges in defining the pushforward $(\sigma_w)_!$ and the pushforward formula in (3.3) gives that

$$\text{if } X_w \text{ is smooth,} \qquad \text{then} \quad [X_w]_v = \frac{y_{R^-}}{\displaystyle\prod_{\substack{\beta\in R^+ \\ vs_\beta \leqslant w}} y_{-\beta}}, \quad \text{for } v \in W_0 \text{ such that } v \leqslant w, \tag{5.3}$$

as found, for example, in [BiLa, Theorem 7.2.1] (the notation $f = \sum_{w\in W_0} f_w b_w$ for elements of $\Omega_T(G/B)$ is as (3.17)). For example, the inclusion $\sigma_{s_2s_1}\colon X_{s_2s_1} \to G/B$ for $G = GL_3$ corresponds to the inclusion of moment graphs



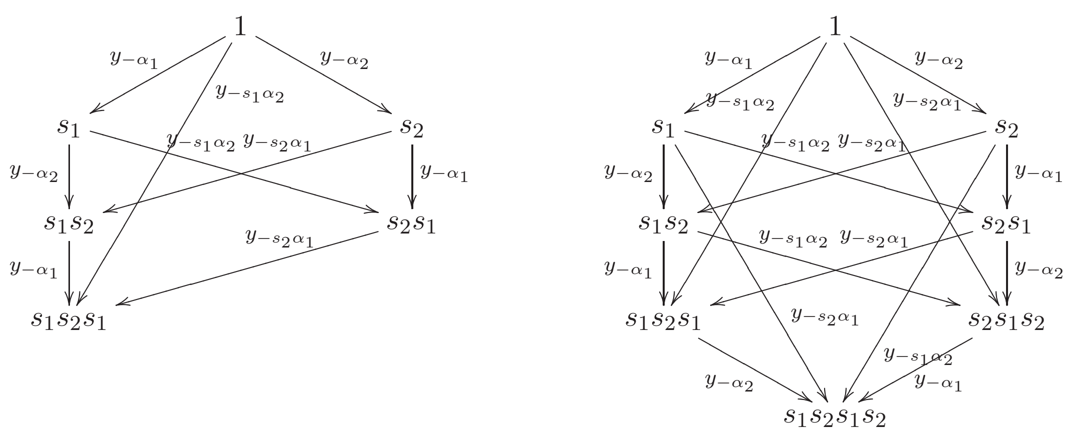
so that

$$[X_{s_2s_1}] \quad = \quad \begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}} \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}} \\ & 0 & \end{array}$$

The following example illustrates that this procedure does not work well when X_w is not smooth. From [Ku, Prop. 6.1], the singular Schubert varieties for G of rank 2 are

Type	Singular	Locus
B_2	$X_{s_1s_2s_1}$	X_{s_1}
G_2	$X_{s_1s_2s_1}$	X_{s_1}
G_2	$X_{s_1s_2s_1s_2}$	$X_{s_1s_2}$
G_2	$X_{s_2s_1s_2s_1}$	$X_{s_2s_1}$
G_2	$X_{s_1s_2s_1s_2s_1}$	$X_{s_1s_2s_1}$
G_2	$X_{s_2s_1s_2s_1s_2}$	X_{s_2}

The inclusion $\sigma_{s_1s_2s_1}\colon X_{s_1s_2s_1} \to G/B$ for $G = Sp_4$ (Type B_2) corresponds to the inclusion of moment graphs



but the direct "naive" application of the pushforward formula in (3.3) produces

$$[X_{s_1s_2s_1}]?=? \qquad \begin{array}{ccc} & y_{-s_2\alpha_1} & \\ y_{-s_2\alpha_1} & & y_{-s_1\alpha_2} \\ y_{-s_1\alpha_2} & & y_{-\alpha_2} \\ y_{-\alpha_2} & & 0 \\ & 0 & \end{array} = \begin{array}{ccc} & y_{-(\alpha_1+\alpha_2)} & \\ y_{-(\alpha_1+\alpha_2)} & & y_{-(2\alpha_1+\alpha_2)} \\ y_{-(2\alpha_1+\alpha_2)} & & y_{-\alpha_2} \\ y_{-\alpha_2} & & 0 \\ & 0 & \end{array} \tag{5.4}$$

which cannot be correct for $[X_{s_1s_2s_1}]$ since the right hand side does not satisfy the condition to be in $\operatorname{im}\Phi$ (the difference across the edge $1 \to s_2$ is not divisible by $y_{-\alpha_2}$). This answer needs to be corrected by finding N so that

$$[X_{s_1s_2s_1}] \; = \; \begin{array}{ccc} & Ny_{-(\alpha_1+\alpha_2)} & \\ Ny_{-(\alpha_1+\alpha_2)} & & y_{-(2\alpha_1+\alpha_2)} \\ y_{-(2\alpha_1+\alpha_2)} & & y_{-\alpha_2} \\ y_{-\alpha_2} & & 0 \\ & 0 & \end{array}$$

where the correction factor N appears on vertices corresponding to the singular locus.

In the example in (5.4) we see that the moment graph knows that $X_{s_1s_2s_1}$ is not smooth! It is interesting to contrast (5.4) with the same analysis for $\sigma_{s_2s_1s_2}\colon X_{s_2s_1s_2} \to G/B$, where the pushforward formula gives

$$[X_{s_2s_1s_2}] = \begin{array}{ccc} & y_{-s_1\alpha_2} & \\ y_{-s_2\alpha_1} & & y_{-s_1\alpha_2} \\ y_{-\alpha_1} & & y_{-s_2\alpha_1} \\ 0 & & y_{-\alpha_1} \\ & 0 & \end{array} = \begin{array}{ccc} & y_{-(2\alpha_1+\alpha_2)} & \\ y_{-(\alpha_1+\alpha_2)} & & y_{-(2\alpha_1+\alpha_2)} \\ y_{-\alpha_1} & & y_{-(\alpha_1+\alpha_2)} \\ 0 & & y_{-\alpha_1} \\ & 0 & \end{array}$$

which is in $\operatorname{im}\Phi$ (this case works out well since $X_{s_2s_1s_2}$ is smooth).

(b) Using (5.2) **to compare** $[X_w]$ **and** $[Z_{\vec{w}}]$**.** Working in rank 2, use notations y_R^-, Δ_{121} and Δ_{212} as in (7.2), so that (see (4.8) and Figure 1)

$$[Z_{212}] \quad = \quad \begin{array}{ccc} & \Delta_{212} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} & & \Delta_{212} \\ \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} \\ & 0 & \end{array}$$

Since $X_{s_1s_2s_1}$ is smooth it is reasonable to apply the pushforward formula in (3.3) which gives

$$[X_{s_2s_1s_2}] \quad = \quad \begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} \\ & 0 & \end{array}$$

Using these and computing with the formulas (3.10)-(3.12) gives the formula

$$\begin{aligned}
[Z_{212}] &= [X_{s_2s_1s_2}] + \left(\Delta_{212} - \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}\right)\frac{y_{-\alpha_2}}{y_{R^-}}[X_{s_2}] \\
&= [X_{s_2s_1s_2}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}\left(\frac{y_{-s_2\alpha_1} - y_{-\alpha_1}}{y_{-\alpha_2}} + p(y_{\alpha_2}, y_{-\alpha_2})y_{-\alpha_1} - 1\right)\frac{y_{-\alpha_2}}{y_{R^-}}[X_{s_2}] \\
&= [X_{s_2s_1s_2}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}\left((1 - p(y_{-\alpha_1}, y_{-\alpha_2})y_{-\alpha_1} + p(y_{\alpha_2}, y_{-\alpha_2})y_{-\alpha_1} - 1\right)\frac{y_{-\alpha_2}}{y_{R^-}}[X_{s_2}] \\
&= [X_{s_2s_1s_2}] + \frac{1}{y_{-s_2\alpha_1}}\big(p(y_{\alpha_2}, y_{-\alpha_2}) - p(y_{-\alpha_1}, y_{-\alpha_2})\big)y_{-\alpha_1}\frac{y_{-\alpha_2}}{y_{R^-}}[X_{s_2}]
\end{aligned}$$

which is reflected in [CPZ, 17.3, first equation] and [HK, §5.2]. Similarly, with our conjectured correction factor N as in (7.3), we get a formula which would provide $[Z_{121}] - [X_{s_1s_2s_1}] = 0$ in cohomology and K-theory but have $[Z_{121}] - [X_{s_1s_2s_1}] \neq 0$ in complex or algebraic cobordism:

$$\begin{aligned}
[Z_{121}] - [X_{s_1s_2s_1}] &= \left(\Delta_{121} - \frac{Ny_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\right)\frac{y_{-\alpha_1}}{y_{R^-}}[X_{s_1}] \\
&= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\left(\frac{y_{-s_1\alpha_2} - y_{-\alpha_2}}{y_{-\alpha_1}} + p(y_{\alpha_1}, y_{-\alpha_1})y_{-\alpha_2} - N\right)\frac{y_{-\alpha_1}}{y_{R^-}}[X_{s_1}] \\
&= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\left(\begin{array}{l} (1 - p(y_{-\alpha_2}, y_{-j\alpha_1})y_{-\alpha_2})\big(1 + \sum_{k=1}^{j-1}(1 - p(y_{-\alpha_1}, y_{-k\alpha_1})y_{-k\alpha_1}\big) \\ \quad +p(y_{\alpha_1}, y_{-\alpha_1})y_{-\alpha_2} - N \end{array}\right)\frac{y_{-\alpha_1}}{y_{R^-}}[X_{s_1}] \\
&= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\big(p(y_{\alpha_1}, y_{-\alpha_1}) - p(y_{-\alpha_2}, y_{-j\alpha_1})\big)y_{-\alpha_2}\frac{y_{-\alpha_1}}{y_{R^-}}[X_{s_1}] \\
&= \frac{1}{y_{-s_1\alpha_2}}\big(p(y_{\alpha_1}, y_{-\alpha_1}) - p(y_{-\alpha_2}, y_{-j\alpha_1})\big)[X_{s_1}].
\end{aligned}$$

(c) Combinatorial forcing: The Schubert classes satisfy

(a) (normalization) $[X_w]_w = \prod_{\alpha\in R(w)} y_{-\alpha}$, where $R(w) = \{\alpha \in R^+ \mid w\alpha \notin R^+\}$.

(b) If $W_J u = W_J z$ then $[X_{w_J u}]_v = [X_{w_J u}]_z$,

(c) (support) $[X_w]_v = 0$ unless $v \leqslant w$.

These properties do not characterize the Schubert classes; the Bott-Samelson classes also satisfy these properties. As observed, for example, in [HHH, Proposition 4.3], in equivariant cohomology a degree condition can be imposed to get uniqueness. It is not clear to us how to generalize the degree condition to equivariant K-theory and/or equivariant cobordism. It seems plausible that in generalized equivariant cohomology the Schubert classes might be characterized by positivity properties, or by using the (S, S)-bimodule structure of $\Omega_T(X_w)$ and $\Omega_T(Z_{\vec{w}})$ as in the theory of Soergel bimodules (see [Soe] and [EW]).

6 Products with Schubert classes

For $w \in W_0$ define Schubert classes $[X_w]$ by $[X_w] = (\sigma_w)_!(1)$ as in (5.1). Continue to use notations $f = \sum_{w\in W_0} f_w b_w$ for elements of $\Omega_T(G/B)$, as in (3.17).

The Schubert product problem: *Find a combinatorial description of the $c^w_{uv} \in R$ given by*

$$[X_u][X_v] = \sum_{w\in W_0} c^w_{uv}[X_w]. \tag{6.1}$$

As is visible from the formula (6.3) below and the formulas at the end of this section, if $v \leqslant u$ in Bruhat order then

$$[X_u][X_v] = [X_u]_v[X_v] + \sum_{w<v} c^w_{uv}[X_w], \tag{6.2}$$

and so the determination of the moment graph values $[X_u]_v$ is a subproblem of the Schubert product problem. The other coefficients c^w_{uv} are determined by the $[X_u]_v$ in an intricate but, perhaps, controllable fashion. Furthermore, our computations of products in the rank two cases display a certain amount of positivity, indicating that there may be a positivity statement for equivariant cobordism analogous to that which holds for equivariant cohomology and equivariant K-theory (see [Gra] and [AGM]).

Properties (a) and (c) are already enough to provide an algorithm for expanding an element $f = \sum_{w\in W_0} f_w b_w$ in terms of Schubert classes. If f has support on w with $\ell(w)\leqslant k$ then

$$f - \sum_{\ell(w)=k} f_w \frac{1}{[X_w]_w}[X_w] = \sum_{\ell(v)\leqslant k-1}\left(f_v - \sum_{\ell(w)=k} f_w \frac{[X_w]_v}{[X_w]_w}\right) b_v$$

has support on v with $\ell(v)\leqslant k-1$. Then

$$\begin{aligned}
f - &\sum_{\ell(w)=k} f_w \frac{1}{[X_w]_w}[X_w] - \sum_{\ell(v)=k-1}\left(f_v - \sum_{\substack{\ell(v)=k-1\\ \ell(w)=k}} f_w\frac{[X_w]_v}{[X_w]_w}\right)\frac{1}{[X_v]_v}[X_v] \\
&= \sum_{\ell(z)\leqslant k-2}\left(f_z - \sum_{\ell(w)=k} f_w\frac{[X_w]_z}{[X_w]_w} - \sum_{\ell(v)=k-1} f_v\frac{[X_v]_z}{[X_v]_v} + \sum_{\substack{\ell(v)=k-1\\ \ell(w)=k}} f_w\frac{[X_w]_v}{[X_w]_w}\frac{[X_v]_z}{[X_v]_v}\right) b_z
\end{aligned}$$

and induction gives that

$$f = \sum_{z\in W_0}\left(\sum_{k=1}^{\ell(w_0)}\sum_{w_1>\cdots>w_k=z}(-1)^{k-1} f_{w_1}\frac{[X_{w_1}]_{w_2}}{[X_{w_1}]_{w_1}}\frac{[X_{w_2}]_{w_3}}{[X_{w_2}]_{w_2}}\cdots\frac{[X_{w_{k-1}}]_{w_k}}{[X_{w_{k-1}}]_{w_{k-1}}}\frac{1}{[X_{w_k}]_{w_k}}\right)[X_z] \tag{6.3}$$

with the terms in the sum naturally indexed by chains in the Bruhat order (compare to, for example, [BS]).

For example, in rank 2 using notations as in Section 7, if $f = \sum_{w\leqslant s_1s_2s_1s_2} f_w b_w$ then

$$\begin{aligned}
f &= f_{s_1s_2s_1s_2}\frac{1}{[X_{s_1s_2s_1s_2}]_{s_1s_2s_1s_2}}[X_{s_1s_2s_1s_2}] \\
&\quad + (f_{s_1s_2s_1} - f_{s_1s_2s_1s_2})\frac{1}{[X_{s_1s_2s_1}]_{s_1s_2s_1}}[X_{s_1s_2s_1}] + (f_{s_2s_1s_2} - f_{s_1s_2s_1s_2})\frac{1}{[X_{s_2s_1s_2}]_{s_2s_1s_2}}[X_{s_2s_1s_2}] \\
&\quad + \left((f_{s_1s_2} - f_{s_2s_1s_2}) + (f_{s_1s_2s_1s_2} - f_{s_1s_2s_1})\frac{[X_{s_1s_2s_1}]_{s_1s_2}}{[X_{s_1s_2s_1}]_{s_1s_2s_1}}\right)\frac{1}{[X_{s_1s_2}]_{s_1s_2}}[X_{s_1s_2}] \\
&\quad + \left((f_{s_2s_1} - f_{s_1s_2s_1}) + (f_{s_1s_2s_1s_2} - f_{s_2s_1s_2})\frac{[X_{s_2s_1s_2}]_{s_2s_1}}{[X_{s_2s_1s_2}]_{s_2s_1s_2}}\right)\frac{1}{[X_{s_2s_1}]_{s_2s_1}}[X_{s_2s_1}] \\
&\quad + \left(\begin{array}{l}(f_{s_1} - f_{s_2s_1}) + (f_{s_2s_1s_2} - f_{s_1s_2})\frac{[X_{s_1s_2}]_{s_1}}{[X_{s_1s_2}]_{s_1s_2}} \\ \quad +(f_{s_1s_2s_1s_2} - f_{s_1s_2s_1})\left(\frac{[X_{s_1s_2s_1}]_{s_1}}{[X_{s_1s_2s_1}]_{s_1s_2s_1}} - \frac{[X_{s_1s_2s_1}]_{s_1s_2}}{[X_{s_1s_2s_1}]_{s_1s_2s_1}}\frac{[X_{s_1s_2}]_{s_1}}{[X_{s_1s_2}]_{s_1s_2}} - 1\right)\end{array}\right)\frac{1}{[X_{s_1}]_{s_1}}[X_{s_1}] \\
&\quad + \left(\begin{array}{l}(f_{s_2} - f_{s_1s_2}) + (f_{s_1s_2s_1} - f_{s_2s_1})\frac{[X_{s_2s_1}]_{s_2}}{[X_{s_2s_1}]_{s_2s_1}} \\ \quad +(f_{s_2s_1s_2s_1} - f_{s_2s_1s_2})\left(\frac{[X_{s_2s_1s_2}]_{s_2}}{[X_{s_2s_1s_2}]_{s_2s_1s_2}} - \frac{[X_{s_2s_1s_2}]_{s_2s_1}}{[X_{s_2s_1s_2}]_{s_2s_1s_2}}\frac{[X_{s_2s_1}]_{s_2}}{[X_{s_2s_1}]_{s_2s_1}} - 1\right)\end{array}\right)\frac{1}{[X_{s_2}]_{s_2}}[X_{s_2}] \\
&\quad + (f_1 - f_{s_1} - f_{s_2} + f_{s_1s_2} + f_{s_2s_1} - f_{s_1s_2s_1} - f_{s_2s_1s_2} + f_{s_1s_2s_1s_2})\frac{1}{[X_1]_1}[X_1]
\end{aligned}$$

and we may use the explicit values of $[X_w]_v$ given in Figure 2 to derive

$$
\begin{aligned}
f = {} & f_{s_1s_2s_1s_2}\frac{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_1s_1\alpha_2}}{y_{R^-}}[X_{s_1s_2s_1s_2}] \\
& + (f_{s_1s_2s_1} - f_{s_1s_2s_1s_2})\frac{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}{y_{R^-}}[X_{s_1s_2s_1}] + (f_{s_2s_1s_2} - f_{s_1s_2s_1s_2})\frac{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}{y_{R^-}}[X_{s_2s_1s_2}] \\
& + \left((f_{s_1s_2} - f_{s_2s_1s_2}) + (f_{s_1s_2s_1s_2} - f_{s_1s_2s_1})\frac{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}\right)\frac{y_{-\alpha_2}y_{-s_2\alpha_1}}{y_{R^-}}[X_{s_1s_2}] \\
& + \left((f_{s_2s_1} - f_{s_1s_2s_1}) + (f_{s_1s_2s_1s_2} - f_{s_2s_1s_2})\frac{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\right)\frac{y_{-\alpha_1}y_{-s_1\alpha_2}}{y_{R^-}}[X_{s_2s_1}] \\
& + \left(\begin{array}{l} (f_{s_1} - f_{s_2s_1}) + (f_{s_2s_1s_2} - f_{s_1s_2})\frac{y_{-\alpha_2}y_{-s_2\alpha_1}}{y_{-\alpha_1}y_{-\alpha_2}} \\ \quad +(f_{s_1s_2s_1s_2} - f_{s_1s_2s_1})\left(\frac{Ny_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} - \frac{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}\frac{y_{-\alpha_2}y_{-s_2\alpha_1}}{y_{-\alpha_1}y_{-\alpha_2}} - 1\right) \end{array}\right)\frac{y_{-\alpha_1}}{y_{R^-}}[X_{s_1}] \\
& + \left(\begin{array}{l} (f_{s_2} - f_{s_1s_2}) + (f_{s_1s_2s_1} - f_{s_2s_1})\frac{y_{-\alpha_1}y_{-s_1\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}} \\ \quad +(f_{s_2s_1s_2s_1} - f_{s_2s_1s_2})\left(\frac{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} - \frac{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\frac{y_{-\alpha_1}y_{-s_1\alpha_2}}{y_{-\alpha_1}y_{-\alpha_2}} - 1\right) \end{array}\right)\frac{y_{-\alpha_2}}{y_{R^-}}[X_{s_2}] \\
& + (f_1 - f_{s_1} - f_{s_2} + f_{s_1s_2} + f_{s_2s_1} - f_{s_1s_2s_1} - f_{s_2s_1s_2} + f_{s_1s_2s_1s_2})\frac{1}{y_{R^-}}[X_1]
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
y_{R^-}f = {} & f_{s_1s_2s_1s_2}y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}y_{-s_1s_1s_1\alpha_2}[X_{s_1s_2s_1s_2}] \\
& + (f_{s_1s_2s_1} - f_{s_1s_2s_1s_2})y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}[X_{s_1s_2s_1}] \\
& + (f_{s_2s_1s_2} - f_{s_1s_2s_1s_2})y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}[X_{s_2s_1s_2}] \\
& + ((f_{s_1s_2} - f_{s_2s_1s_2})y_{-\alpha_2}y_{-s_2\alpha_1} + (f_{s_1s_2s_1s_2} - f_{s_1s_2s_1})y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1})[X_{s_1s_2}] \\
& + ((f_{s_2s_1} - f_{s_1s_2s_1})y_{-\alpha_1}y_{-s_1\alpha_2} + (f_{s_1s_2s_1s_2} - f_{s_2s_1s_2})y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2})[X_{s_2s_1}] \\
& + \left(\begin{array}{l} (f_{s_1} - f_{s_2s_1})y_{-\alpha_1} + (f_{s_2s_1s_2} - f_{s_1s_2})y_{-s_2\alpha_1} \\ \quad +(f_{s_1s_2s_1s_2} - f_{s_1s_2s_1})\left(\frac{Ny_{-s_1s_2\alpha_1}y_{-\alpha_1}}{y_{-\alpha_2}} - \frac{y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}{y_{-\alpha_2}} - y_{-\alpha_1}\right) \end{array}\right)[X_{s_1}] \\
& + \left(\begin{array}{l} (f_{s_2} - f_{s_1s_2})y_{-\alpha_2} + (f_{s_1s_2s_1} - f_{s_2s_1})y_{-s_1\alpha_2} \\ \quad +(f_{s_2s_1s_2s_1} - f_{s_2s_1s_2})\left(\frac{y_{-s_2s_1\alpha_2}y_{-\alpha_2}}{y_{-\alpha_1}} - \frac{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}{y_{-\alpha_1}} - y_{-\alpha_2}\right) \end{array}\right)[X_{s_2}] \\
& + (f_1 - f_{s_1} - f_{s_2} + f_{s_1s_2} + f_{s_2s_1} - f_{s_1s_2s_1} - f_{s_2s_1s_2} + f_{s_1s_2s_1s_2})[X_1]
\end{aligned}
$$

This last formula allows for quick computation of products with Schubert classes in rank 2 for low dimensional Schubert varieties. In particular, for $g = \sum_{w\in W_0} g_w b_w$ in $\Omega_T(G/B)$,

$$
\begin{aligned}
g[X_1] &= g_1[X_1], \\
g[X_{s_1}] &= g_{s_1}[X_{s_1}] + g_{1,s_1}[X_1], \qquad \text{where} \quad g_{1,s_1} = \frac{g_1 - g_{s_1}}{y_{-\alpha_1}}, \\
g[X_{s_2}] &= g_{s_2}[X_{s_2}] + g_{1,s_2}[X_1], \qquad \text{where} \quad g_{1,s_2} = \frac{g_1 - g_{s_2}}{y_{-\alpha_2}}, \\
g[X_{s_1s_2}] &= g_{s_1s_2}[X_{s_1s_2}] + g_{s_1,s_1s_2}[X_{s_1}] + g_{s_2,s_1s_2}[X_{s_2}] + \frac{g_{1,s_1} - g_{s_2,s_1s_2}}{y_{-\alpha_2}}[X_1], \\
g[X_{s_2s_1}] &= g_{s_2s_1}[X_{s_2s_1}] + g_{s_1,s_2s_1}[X_{s_1}] + g_{s_2,s_2s_1}[X_{s_2}] + \frac{g_{1,s_2} - g_{s_1,s_2s_1}}{y_{-\alpha_1}}[X_1],
\end{aligned}
$$

where

$$g_{s_1,s_1s_2} = \frac{g_{s_1} - g_{s_1s_2}}{y_{-\alpha_2}}, \quad g_{s_2,s_1s_2} = \frac{g_{s_2} - g_{s_1s_2}}{y_{-s_2\alpha_1}}, \quad g_{s_1,s_2s_1} = \frac{g_{s_1} - g_{s_2s_1}}{y_{-s_1\alpha_2}}, \quad g_{s_2,s_2s_1} = \frac{g_{s_2} - g_{s_2s_1}}{y_{-\alpha_1}}.$$

Using (3.19), Pieri-Chevalley rules giving the expansions of products $x_\lambda[X_w]$ in terms of Schubert classes are directly determined from these formulas.

7 Schubert classes and products in rank 2

In rank 2, W_0 is a dihedral group generated by s_1 and s_2 with $s_i^2 = 1$, $s_1\alpha_1 = -\alpha_1$, $s_2\alpha_2 = -\alpha_2$,

$$\begin{array}{ll} s_1\alpha_1 = -\alpha_1, & s_1\alpha_2 = j\alpha_1 + \alpha_2, \\ s_2\alpha_1 = \alpha_1 + \alpha_2, & s_2\alpha_2 = -\alpha_2, \end{array}$$

with

$$j = \begin{cases} 1, \text{in Type } A_2, \\ 2, \text{in Type } B_2, \\ 3, \text{in Type } G_2, \end{cases}$$

and

$$\begin{array}{ccc} & b_1 & \\ b_{s_1} & & b_{s_2} \\ b_{s_1s_2} & & b_{s_2s_1} \\ b_{s_1s_2s_1} & & b_{s_2s_1s_2} \\ b_{s_1s_2s_1s_2} & & b_{s_2s_1s_2s_1} \\ b_{s_1s_2s_1s_2s_1} & & b_{s_2s_1s_2s_1s_2} \\ \vdots & & \vdots \end{array}$$

b_w basis

$$\begin{array}{ccc} & y_{-\alpha_1} & \\ y_{\alpha_1} & & y_{-s_2\alpha_1} \\ y_{s_2\alpha_1} & & y_{-s_1s_2\alpha_1} \\ y_{s_1s_2\alpha_1} & & y_{-s_2s_1s_2\alpha_1} \\ y_{s_2s_1s_2\alpha_1} & & y_{-s_1s_2s_1s_2\alpha_1} \\ y_{s_1s_2s_1s_2\alpha_1} & & y_{-s_2s_1s_2s_1s_2\alpha_1} \\ \vdots & & \vdots \end{array} \qquad \begin{array}{ccc} & y_{-\alpha_2} & \\ y_{-s_1\alpha_2} & & y_{\alpha_2} \\ y_{-s_2s_1\alpha_2} & & y_{s_1\alpha_2} \\ y_{-s_1s_2s_1\alpha_2} & & y_{s_2s_1\alpha_2} \\ y_{-s_2s_1s_2s_1\alpha_2} & & y_{s_1s_2s_1\alpha_2} \\ y_{-s_1s_2s_1s_2s_1\alpha_2} & & y_{s_2s_1s_2s_1\alpha_2} \\ \vdots & & \vdots \end{array}$$

$x_{-\alpha_1}$ $\qquad$ $x_{-\alpha_2}$

Let

$$y_{R^-} = \prod_{\alpha \in R^+} y_{-\alpha}, \tag{7.1}$$

$$\begin{aligned} \Delta_{121} &= y_{R^-}\left(\frac{1}{y_{-\alpha_2}y_{-\alpha_1}y_{-\alpha_2}} + \frac{1}{y_{-s_2\alpha_2}y_{-s_2\alpha_1}y_{-\alpha_2}}\right) \\ &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\left(\frac{y_{-s_1\alpha_2} - y_{-\alpha_2}}{y_{-\alpha_1}} + p(y_{\alpha_1}, y_{-\alpha_1})y_{-\alpha_2}\right) \\ \Delta_{212} &= \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_2\alpha_1}}\left(\frac{y_{-s_2\alpha_1} - y_{-\alpha_1}}{y_{-\alpha_2}} + p(y_{\alpha_2}, y_{-\alpha_2})y_{-\alpha_1}\right), \quad \text{and} \end{aligned} \tag{7.2}$$

$$N = 1 + (1 - p(y_{-\alpha_2}, y_{-j\alpha_1})y_{-\alpha_2})\Big(\sum_{k=1}^{j-1}(1 - p(y_{-\alpha_1}, y_{-k\alpha_1})y_{-k\alpha_1}\Big). \tag{7.3}$$

We note that, *for ordinary cohomology H_T and K-theory K_T*,

$$N = \begin{cases} 1 + (j-1), & \text{in } H_T, \\ 1 + e^{-\alpha_2}(e^{-\alpha_1} + \cdots + e^{-(j-1)\alpha_1}), & \text{in } K_T, \end{cases} \qquad \text{and} \qquad \Delta_{121} = \frac{N y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}.$$

The Schubert and Bott-Samelson cycles for rank 2 and length $\leqslant 1$ are given

$$\begin{array}{ccc} & y_{R^-} & \\ 0 & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ & & \\ & [X_1]=[Z_{\mathrm{pt}}] & \end{array} \qquad \begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_1}} & \\ \frac{y_{R^-}}{y_{-\alpha_1}} & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ & & \\ & [X_{s_1}]=[Z_1] & \end{array} \qquad \begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_2}} & \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_2}} \\ 0 & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ & & \\ & [X_{s_2}]=[Z_2] & \end{array}$$

The remaining Schubert and Bott-Samelson cycles for rank 2 and length $\leqslant 3$ are given in Figure 2.

7.1 Schubert products in rank 2

Using the explicit moment graph representations of the Schubert classes, the formulas for products $g[X_w]$ given at the end of Section 6 allow for quick computations of the products of Schubert classes in rank 2 for Weyl group elements up to length 3. It is straightforward to check that these generalise the corresponding computations for equivariant cohomology and equivariant K-theory which were given in [GR, §5]. Since $[X_{s_1s_2s_1s_2}] = [X_{s_2s_1s_2s_1}] = 1$ in Type B_2, these calculations completely determine all Schubert products generalized equivariant Schubert products for Types A_2 and B_2.

The Schubert products for low dimensional Schubert varieties are as follows.

$$[X_1]^2 = y_{R^-}[X_1], \qquad [X_1][X_{s_1}] = \frac{y_{R^-}}{y_{-\alpha_1}}[X_1], \qquad [X_1][X_{s_2}] = \frac{y_{R^-}}{y_{-\alpha_2}}[X_1],$$

$$[X_1][X_{s_1s_2}] = \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}[X_1], \qquad [X_1][X_{s_2s_1}] = \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}}[X_1],$$

$$[X_1][X_{s_1s_2s_1}] = \frac{Ny_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}[X_1], \qquad [X_1][X_{s_2s_1s_2}] = \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_2\alpha_1}}[X_1],$$

$$[X_{s_1}]^2 = \frac{y_{R^-}}{y_{-\alpha_1}}[X_{s_1}], \qquad [X_{s_1}][X_{s_1s_2}] = \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}[X_{s_1}], \qquad [X_{s_1}][X_{s_1s_2s_1}] = \frac{Ny_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}[X_{s_1}],$$

$$[X_{s_1}][X_{s_2}] = \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}[X_1],$$

$$[X_{s_1}][X_{s_2s_1}] = \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}}[X_{s_1}] + \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_1\alpha_2}}\left(\frac{y_{-s_1\alpha_2} - y_{-\alpha_2}}{y_{-\alpha_1}}\right)[X_1],$$

$$[X_{s_1}][X_{s_2s_1s_2}] = \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_1\alpha_2}}[X_{s_1}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}y_{-s_2\alpha_1}}\left(\frac{y_{-s_1\alpha_2} - y_{-s_2\alpha_1}}{y_{-\alpha_1}}\right)[X_1],$$

$$[X_{s_2}]^2 = \frac{y_{R^-}}{y_{-\alpha_2}}[X_{s_2}], \quad [X_{s_2}][X_{s_2s_1}] = \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}}[X_{s_2}], \quad [X_{s_2}][X_{s_2s_1s_2}] = \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_2\alpha_1}}[X_{s_2}],$$

$$[X_{s_2}][X_{s_1s_2}] = \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}}[X_{s_2}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}\left(\frac{y_{-s_2\alpha_1} - y_{-\alpha_1}}{y_{-\alpha_2}}\right)[X_1],$$

$$[X_{s_2}][X_{s_1s_2s_1}] = \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}[X_{s_2}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}y_{-s_2\alpha_1}}\left(\frac{Ny_{-s_2\alpha_1} - y_{-s_1\alpha_2}}{y_{-\alpha_2}}\right)[X_1],$$

$$\begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}} & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \end{array} \qquad \begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}} \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}} \\ 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \end{array}$$

$$[X_{s_1s_2}] = [Z_{12}] \qquad\qquad [X_{s_2s_1}] = [Z_{21}]$$

$$\begin{array}{ccc} & \frac{Ny_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} & \\ \frac{Ny_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \end{array} \qquad \begin{array}{ccc} & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} \\ 0 & & 0 \\ \vdots & & \vdots \end{array}$$

$$[X_{s_1s_2s_1}] \qquad\qquad [X_{s_2s_1s_2}]$$

$$\begin{array}{ccc} & \Delta_{121} & \\ \Delta_{121} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \end{array} \qquad \begin{array}{ccc} & \Delta_{212} & \\ \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} & & \Delta_{212} \\ \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} & & \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} \\ 0 & & \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} \\ 0 & & 0 \\ \vdots & & \vdots \end{array}$$

$$[Z_{121}] \qquad\qquad [Z_{212}]$$

Figure 2: Schubert and Bott-Samelson cycles for rank 2 and length $\leqslant 3$.

$$
\begin{aligned}
[X_{s_1s_2}]^2 &= \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}}[X_{s_1s_2}] + \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_2\alpha_1}}\left(\frac{y_{-s_2\alpha_1}-y_{-\alpha_1}}{y_{-\alpha_2}}\right)[X_{s_1}],\\
[X_{s_1s_2}][X_{s_2s_1}] &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}[X_{s_1}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}[X_{s_2}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}y_{-s_2\alpha_1}}\left(\left(\frac{y_{-s_2\alpha_1}-y_{-\alpha_1}}{y_{-\alpha_2}}\right)\left(\frac{y_{-s_1\alpha_2}-y_{-\alpha_2}}{y_{-\alpha_1}}\right)-1\right)[X_1],\\
[X_{s_1s_2}][X_{s_1s_2s_1}] &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}}[X_{s_1s_2}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}y_{-s_2\alpha_1}}\left(\frac{Ny_{-s_2\alpha_1}-y_{-s_1\alpha_2}}{y_{-\alpha_2}}\right)[X_{s_1}],\\
[X_{s_1s_2}][X_{s_2s_1s_2}] &= \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}[X_{s_1s_2}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_1\alpha_2}y_{-s_2s_1\alpha_2}}\left(\frac{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}-y_{-\alpha_1}y_{-s_1\alpha_2}}{y_{-\alpha_2}}\right)[X_{s_1}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}\left(\frac{y_{-s_2s_1\alpha_2}-y_{-\alpha_1}}{y_{-s_2\alpha_1}}\right)[X_{s_2}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_2}^2}\left(\frac{1}{y_{-\alpha_1}^2y_{-s_2\alpha_1}} - \frac{1}{y_{-s_2\alpha_1}^2y_{-\alpha_1}} - \frac{1}{y_{-\alpha_1}^2y_{-s_1\alpha_2}} + \frac{1}{y_{-s_2\alpha_1}^2y_{-s_2s_1\alpha_2}}\right)[X_1],
\end{aligned}
$$

$$
\begin{aligned}
[X_{s_2s_1}]^2 &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}}[X_{s_2s_1}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}}\left(\frac{y_{-s_1\alpha_2}-y_{-\alpha_2}}{y_{-\alpha_1}}\right)[X_{s_2}],\\
[X_{s_2s_1}][X_{s_1s_2s_1}] &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}[X_{s_2s_1}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}^2}\left(\frac{N}{y_{-\alpha_2}} - \frac{1}{y_{-s_1s_2\alpha_1}}\right)[X_{s_1}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}^2}\left(\frac{1}{y_{-s_2\alpha_1}y_{-\alpha_2}} - \frac{1}{y_{-s_1s_2\alpha_1}y_{-s_1\alpha_2}}\right)[X_{s_2}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}^2}\left(\frac{N}{y_{-\alpha_2}^2y_{-s_1\alpha_2}} - \frac{N}{y_{-\alpha_2}y_{-s_1\alpha_2}^2} - \frac{1}{y_{-\alpha_2}^2y_{-s_2\alpha_1}} + \frac{1}{y_{-s_1\alpha_2}^2y_{-s_1s_2\alpha_1}}\right)[X_1],\\
[X_{s_2s_1}][X_{s_2s_1s_2}] &= \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}y_{-s_1\alpha_2}}[X_{s_2s_1}] + \frac{y_{R^-}}{y_{-\alpha_2}y_{-\alpha_1}^2}\left(\frac{1}{y_{-s_2\alpha_1}} - \frac{1}{y_{-s_1\alpha_2}}\right)[X_{s_2}],
\end{aligned}
$$

$$
\begin{aligned}
[X_{s_1s_2s_1}]^2 &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}[X_{s_1s_2s_1}] + \frac{y_{R^-}}{y_{-\alpha_1^2}}\left(\frac{1}{y_{-\alpha_2}y_{-s_2\alpha_1}} - \frac{1}{y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}\right)[X_{s_1s_2}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}\left(\frac{N^2}{y_{-\alpha_2}y_{-s_1\alpha_2}^2} - \frac{N}{y_{-s_1\alpha_2}^2y_{-s_1s_2\alpha_1}} - \frac{1}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}} + \frac{1}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}\right)[X_1],\\
[X_{s_1s_2s_1}][X_{s_2s_1s_2}] &= \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}[X_{s_1s_2}] + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}}[X_{s_2s_1}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}\left(\frac{N}{y_{-\alpha_2}y_{-s_1\alpha_2}^2} - \frac{1}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}} - \frac{1}{y_{-s_1\alpha_2}^2y_{-s_1s_2\alpha_1}}\right)[X_{s_1}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}\left(\frac{1}{y_{-\alpha_1}y_{-s_2\alpha_1}^2} - \frac{1}{y_{-\alpha_1}y_{-s_1\alpha_2}y_{-s_1s_2\alpha_1}} - \frac{1}{y_{-s_2\alpha_1}^2y_{-s_2s_1\alpha_2}}\right)[X_{s_2}],\\
[X_{s_2s_1s_2}]^2 &= \frac{y_{R^-}}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}[X_{s_2s_1s_2}] + \frac{y_{R^-}}{y_{-\alpha_2^2}}\left(\frac{1}{y_{-\alpha_1}y_{-s_1\alpha_2}} - \frac{1}{y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}\right)[X_{s_2s_1}]\\
&\quad + \frac{y_{R^-}}{y_{-\alpha_1}y_{-\alpha_2}}\left(\frac{1}{y_{-\alpha_1}y_{-s_2\alpha_1}^2} - \frac{1}{y_{-s_2\alpha_1}^2y_{-s_2s_1\alpha_2}} - \frac{1}{y_{-\alpha_1}y_{-\alpha_2}y_{-s_1\alpha_2}} + \frac{1}{y_{-\alpha_2}y_{-s_2\alpha_1}y_{-s_2s_1\alpha_2}}\right)[X_1],
\end{aligned}
$$

8 The calculus of BGG operators

The *nil affine Hecke algebra* is the algebra over $\mathbb{L}$ with generators x_λ, y_λ, t_w, with $\lambda, \mu \in \mathfrak{h}_\mathbb{Z}^*$ and $w \in W_0$, with relations

$$x_{\lambda+\mu} = x_\lambda + x_\mu - p(x_\lambda, x_\mu)x_\lambda x_\mu, \qquad y_{\lambda+\mu} = y_\lambda + y_\mu - p(y_\lambda, y_\mu)y_\lambda y_\mu, \qquad x_\lambda y_\mu = y_\mu x_\lambda,$$

and

$$t_v t_w = t_{vw}, \qquad t_w y_\lambda = y_\lambda t_w, \qquad t_w x_\lambda = x_{w\lambda} t_w, \qquad \text{for } v, w \in W_0,\ \lambda \in \mathfrak{h}_\mathbb{Z}^*.$$

Recall from (4.2) that the *pushpull operators*, or *BGG-Demazure operators* are given by

$$A_i = (1 + t_{s_i})\frac{1}{x_{-\alpha_i}}, \qquad \text{for } i = 1, 2, \ldots, n. \tag{8.1}$$

In general,

$$\begin{aligned} A_i &= (1 + t_{s_i})\frac{1}{x_{-\alpha_i}} = \frac{1}{x_{-\alpha_i}} + \frac{1}{x_{\alpha_i}} t_{s_i} = \frac{1}{x_{-\alpha_i}} - \frac{1 - p(x_{\alpha_i}, x_{-\alpha_i})x_{-\alpha_i}}{x_{-\alpha_i}} t_{s_i} \\ &= \frac{1}{x_{-\alpha_i}}\left(1 - (1 - p(x_{\alpha_i}, x_{-\alpha_i})x_{-\alpha_i})t_{s_i}\right) = \frac{1}{x_{-\alpha_i}}(1 - t_{s_i}) + p(x_{\alpha_i}, x_{-\alpha_i})t_{s_i}. \end{aligned} \tag{8.2}$$

so that A_i is a divided difference operator plus an extra term. As in [BE1, Prop. 3.1],

$$\begin{aligned} A_i^2 &= (1 + t_{s_i})\frac{1}{x_{-\alpha_i}}(1 + t_{s_i})\frac{1}{x_{-\alpha_i}} = \left(\frac{1}{x_{-\alpha_i}} + \frac{1}{x_{\alpha_i}} t_{s_i}\right)(1 + t_{s_i})\frac{1}{x_{-\alpha_i}} \\ &= \left(\frac{1}{x_{-\alpha_i}} + \frac{1}{x_{\alpha_i}}\right)(1 + t_{s_i})\frac{1}{x_{-\alpha_i}} = \left(\frac{1}{x_{-\alpha_i}} + \frac{1}{x_{\alpha_i}}\right)A_i, \end{aligned}$$

so that

$$A_i^2 = \left(\frac{1}{x_{-\alpha_i}} + \frac{1}{x_{\alpha_i}}\right)A_i = A_i\left(\frac{1}{x_{-\alpha_i}} + \frac{1}{x_{\alpha_i}}\right) = A_i p(x_{\alpha_i}, x_{-\alpha_i}). \tag{8.3}$$

Note also that

$$t_{s_i} A_i = t_{s_i}(1 + t_{s_i})\frac{1}{x_{-\alpha_i}} = A_i \quad \text{and} \tag{8.4}$$

$$A_i t_{s_i} = (1 + t_{s_i})\frac{1}{x_{-\alpha_i}} t_{s_i} = (1 + t_{s_i})\frac{1}{x_{\alpha_i}} = A_i \frac{x_{-\alpha_i}}{x_{\alpha_i}}. \tag{8.5}$$

If $f \in \mathbb{L}[[x_\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*]]$ then

$$\begin{aligned} fA_i &= f(1 + t_{s_i})\frac{1}{x_{-\alpha_i}} = f\frac{1}{x_{-\alpha_i}} + f t_{s_i}\frac{1}{x_{-\alpha_i}} \quad \text{and} \\ A_i(s_i f) &= (1 + t_{s_i})\frac{s_i f}{x_{-\alpha_i}} = (s_i f + f t_{s_i})\frac{1}{x_{-\alpha_i}}, \end{aligned}$$

so that

$$fA_i = A_i(s_i f) + \left(\frac{f - s_i f}{x_{-\alpha_i}}\right). \tag{8.6}$$

The relation (8.6) is the analogue, for this setting, of a key relation in the definition of the classical nil-affine Hecke algebra (see [CG, Lemma 7.1.10] or [GR, (1.3)]).

Next are useful, expansions of products of t_{s_i} in terms of products of A_i with xs on the left,

$$
\begin{aligned}
t_{s_1} &= x_{\alpha_1} A_1 - \frac{x_{\alpha_1}}{x_{-\alpha_1}}, \\
t_{s_2} t_{s_1} &= x_{s_2\alpha_1} x_{\alpha_2} A_2 A_1 - x_{s_2\alpha_1} \frac{x_{\alpha_2}}{x_{-\alpha_2}} A_1 - \frac{x_{s_2\alpha_1}}{x_{-s_2\alpha_1}} x_{\alpha_2} A_2 + \frac{x_{s_2\alpha_1}}{x_{-s_2\alpha_1}} \frac{x_{\alpha_2}}{x_{-\alpha_2}} \\
t_{s_1} t_{s_2} t_{s_1} &= x_{s_1 s_2\alpha_1} x_{s_1\alpha_2} x_{\alpha_1} A_1 A_2 A_1 - x_{s_1 s_2\alpha_1} x_{s_1\alpha_2} \frac{x_{\alpha_1}}{x_{-\alpha_1}} A_2 A_1 - \frac{x_{s_1 s_2\alpha_1}}{x_{-s_1 s_2\alpha_1}} x_{s_1\alpha_2} x_{\alpha_1} A_1 A_2 \\
&\quad + \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} x_{s_2\alpha_1} \frac{x_{\alpha_2}}{x_{-\alpha_2}} A_1 + \frac{x_{s_1 s_2\alpha_1}}{x_{-s_1 s_2\alpha_1}} x_{s_1\alpha_2} \frac{x_{\alpha_1}}{x_{-\alpha_1}} A_2 - \frac{x_{s_1 s_2\alpha_1}}{x_{-s_1 s_2\alpha_1}} \frac{x_{s_1\alpha_2}}{x_{-s_1\alpha_2}} \frac{x_{\alpha_1}}{x_{-\alpha_1}} \\
&\quad + \left(\frac{x_{s_1\alpha_2}}{x_{-s_1\alpha_2}} \frac{x_{s_1 s_2\alpha_1}}{x_{-s_1 s_2\alpha_1}} x_{\alpha_1} - \frac{x_{s_1\alpha_2}}{x_{-s_1\alpha_2}} x_{s_1 s_2\alpha_1} - \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} x_{s_2\alpha_1} \frac{x_{\alpha_2}}{x_{-\alpha_2}} \right) A_1 \\
t_{s_1} t_{s_2} t_{s_1} t_{s_2} &= x_{s_2 s_1 s_2\alpha_1} x_{s_2 s_1\alpha_2} x_{s_2\alpha_1} x_{\alpha_2} A_2 A_1 A_2 A_1 \\
&\quad - x_{s_2 s_1 s_2\alpha_1} x_{s_2 s_1\alpha_2} x_{s_2\alpha_1} \frac{x_{\alpha_2}}{x_{-\alpha_2}} A_1 A_2 A_1 - \frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} x_{s_2 s_1\alpha_2} x_{s_2\alpha_1} x_{\alpha_2} A_2 A_1 A_2 \\
&\quad + \frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} x_{s_2 s_1\alpha_2} x_{s_2\alpha_1} \frac{x_{\alpha_2}}{x_{-\alpha_2}} A_1 A_2 \\
&\quad + \left(\frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} x_{s_2\alpha_1} x_{\alpha_2} - x_{s_2 s_1 s_2\alpha_1} \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} x_{\alpha_2} - x_{s_2 s_1 s_2\alpha_1} x_{s_2 s_1\alpha_2} \frac{x_{s_2\alpha_1}}{x_{-s_2\alpha_1}} \right) A_2 A_1 \\
&\quad - \left(\frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} x_{s_2\alpha_1} - x_{s_2 s_1 s_2\alpha_1} \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} \right) \frac{x_{\alpha_2}}{x_{-\alpha_2}} A_1 \\
&\quad + \left(\frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} x_{s_2 s_1\alpha_2} \frac{x_{s_2\alpha_1}}{x_{-s_2\alpha_1}} - \frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} \frac{x_{s_2\alpha_1}}{x_{-s_2\alpha_1}} x_{\alpha_2} \right) A_2 \\
&\quad + \frac{x_{s_2 s_1 s_2\alpha_1}}{x_{-s_2 s_1 s_2\alpha_1}} \frac{x_{s_2 s_1\alpha_2}}{x_{-s_2 s_1\alpha_2}} \frac{x_{s_2\alpha_1}}{x_{-s_2\alpha_1}} \frac{x_{\alpha_2}}{x_{-\alpha_2}},
\end{aligned}
$$

and expansions of products of t_{s_i} in terms of products of A_i with xs on the right,

$$
\begin{aligned}
t_{s_1} &= A_1 x_{-\alpha_1} - 1, \\
t_{s_1} t_{s_2} &= A_1 A_2 x_{-\alpha_2} x_{-s_2\alpha_1} - A_1 x_{-s_2\alpha_1} - A_2 x_{-\alpha_2} + 1, \\
t_{s_1} t_{s_2} t_{s_1} &= A_1 A_2 A_1 x_{-\alpha_1} x_{-s_1\alpha_2} x_{-s_1 s_2\alpha_1} - A_1 A_2 x_{-s_1\alpha_2} x_{-s_1 s_2\alpha_1} - A_2 A_1 x_{-\alpha_1} x_{-s_1\alpha_2} \\
&\quad + A_1 x_{-s_2\alpha_1} + A_2 x_{-s_1\alpha_2} - 1 + A_1 \left(x_{-\alpha_1} - x_{-s_2\alpha_1} - \frac{x_{-\alpha_1}}{x_{\alpha_1}} x_{-s_1 s_2\alpha_1} \right), \\
t_{s_1} t_{s_2} t_{s_1} t_{s_2} &= A_1 A_2 A_1 A_2 x_{-\alpha_2} x_{-s_2\alpha_1} x_{-s_2 s_1\alpha_2} x_{-s_2 s_1 s_2\alpha_1} \\
&\quad - A_1 A_2 A_1 x_{-s_2\alpha_1} x_{-s_2 s_1\alpha_2} x_{-s_2 s_1 s_2\alpha_1} - A_2 A_1 A_2 x_{-\alpha_2} x_{-s_2\alpha_1} x_{-s_2 s_1\alpha_2} \\
&\quad + A_1 A_2 \left(-\frac{x_{-\alpha_2}}{x_{\alpha_2}} x_{-s_2 s_1\alpha_2} x_{-s_2 s_1 s_2\alpha_1} - x_{-\alpha_2} \frac{x_{-s_2\alpha_1}}{x_{s_2\alpha_1}} x_{-s_2 s_1 s_2\alpha_1} + x_{-\alpha_2} x_{-s_2\alpha_1} \right) \\
&\quad + A_2 A_1 x_{-s_2\alpha_1} x_{-s_2 s_1\alpha_2} \\
&\quad - A_1 \left(x_{-s_2\alpha_1} - \frac{x_{-s_2\alpha_1}}{x_{s_2\alpha_1}} x_{-s_2 s_1 s_2\alpha_1} \right) - A_2 \left(x_{-\alpha_2} - \frac{x_{-\alpha_2}}{x_{\alpha_2}} x_{-s_2 s_1\alpha_2} \right) + 1.
\end{aligned}
$$

Finally, there are expansions of products of A_i in terms of products of t_{s_i}:

$$A_1 = (t_{s_1}+1)\frac{1}{x_{-\alpha_1}},$$

$$A_1A_2 = (t_{s_1}+1)\left(t_{s_2}\frac{1}{x_{-\alpha_2}x_{-s_2\alpha_1}} + \frac{1}{x_{-\alpha_1}x_{-\alpha_2}}\right),$$

$$A_1A_2A_1 = (t_{s_1}+1)\begin{pmatrix} t_{s_2}t_{s_1}\frac{1}{x_{-\alpha_1}x_{-s_1\alpha_2}x_{-s_1s_2\alpha_1}} + t_{s_2}\frac{1}{x_{-\alpha_1}x_{-\alpha_2}x_{-s_2\alpha_1}} \\ +\frac{1}{x_{-\alpha_1}}\left(\frac{1}{x_{-\alpha_1}x_{-\alpha_2}} + \frac{1}{x_{-s_1\alpha_1}x_{-s_2\alpha_1}}\right)\end{pmatrix},$$

$$A_1A_2A_1A_2 = (t_{s_1}+1)\begin{pmatrix} t_{s_2}t_{s_1}t_{s_2}\frac{1}{x_{-\alpha_2}x_{-s_2\alpha_1}x_{-s_2s_1\alpha_2}x_{-s_2s_1s_2\alpha_1}} \\ +t_{s_2}t_{s_1}\frac{1}{x_{-\alpha_2}x_{-\alpha_1}x_{-s_1\alpha_2}x_{-s_1s_2\alpha_1}} \\ +t_{s_2}\frac{1}{x_{-\alpha_2}x_{-s_2\alpha_1}}\left(\frac{1}{x_{-\alpha_2}x_{-\alpha_1}} + \frac{1}{x_{-s_2\alpha_1}x_{-s_2\alpha_2}} + \frac{1}{x_{-s_2s_1\alpha_2}x_{-s_2s_1\alpha_1}}\right) \\ +\frac{1}{x_{-\alpha_1}x_{-\alpha_2}}\left(\frac{1}{x_{-\alpha_2}x_{-\alpha_1}} + \frac{1}{x_{-s_2\alpha_1}x_{-s_2\alpha_2}} + \frac{1}{x_{-s_1\alpha_2}x_{-s_1\alpha_1}}\right)\end{pmatrix}.$$

These formulas arranged so that products beginning with t_{s_2} and A_2 are obtained from the above formulas by switching 1s and 2s. In particular, the "braid relations" for the operators A_i are the equations given by, for example, in the case that $s_1s_2s_1 = s_2s_1s_2$ so that $s_1\alpha_2 = s_2\alpha_1 = \alpha_1+\alpha_2$ then

$$0 = t_{s_1}t_{s_2}t_{s_1} - t_{s_2}t_{s_1}t_{s_2}$$

is equivalent to

$$\begin{aligned} A_2A_1A_2 &- \left(\frac{1}{x_{-\alpha_2}x_{-\alpha_1}} - \frac{1}{x_{-\alpha_1}x_{-\alpha_3}} + \frac{1}{x_{\alpha_2}x_{-\alpha_3}}\right)A_2 \\ &= A_1A_2A_1 - \left(\frac{1}{x_{-\alpha_1}x_{-\alpha_2}} - \frac{1}{x_{-\alpha_2}x_{-\alpha_3}} + \frac{1}{x_{\alpha_1}x_{-\alpha_3}}\right)A_1, \end{aligned}$$

as indicated in [HLSZ, Proposition 5.7].

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