

Two boundary Hecke Algebras and combinatorics of type C

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Abstract

This paper gives a Schur-Weyl duality approach to the representation theory of the affine Hecke algebras of type C with unequal parameters. The first step is to realize the affine braid group of type C_k as the group of braids on k strands with two poles. Generalizing familiar methods from the one pole (type A) case, this provides commuting actions of the quantum group $U_q\mathfrak{g}$ and the affine braid group of type C_k on a tensor space $M \otimes N \otimes V^{\otimes k}$. Special cases provide Schur-Weyl pairings between the affine Hecke algebra of type C_k and the quantum group of type \mathfrak{gl}_n , resulting in natural labelings of many representations of the affine Hecke algebras of type C by partitions. Following an analysis of the structure of weights of affine Hecke algebra representations (extending the one parameter case to the three parameter case necessary for affine Hecke algebras of type C), we provide an explicit identification of the affine Hecke algebra representations that appear in tensor space (essentially by identifying their Langlands parameters).

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1 Introduction

This paper explores a Schur-Weyl duality approach to the representations of the affine Hecke algebras of type C with unequal parameters. Following Kazhdan-Lusztig [KL], the irreducible representations of the affine Hecke algebra are usually constructed via the K-theory of generalized Springer fibers. This method works well when an algebraic group is available, which is only for special cases of the three parameters t, t_0, t_k of the affine Hecke algebras of type C.

G. Lusztig gave a general approach to the unequal parameter case using Kazhdan-Lusztig bases and cells. In [Lu2], the challenges for pushing this method through in type C are outlined in a set of conjectures, many of which have now been settled in work of Geck, Bonnafé, and others (see [Ge, Bo, Gu] and references there). Another analytic approach, closer to the original classification and construction of Kazhdan-Lusztig, is given by Opdam and Solleveld (see [OS] and [So] and the references there). In the type C case, Kato [Kt] explained that the “exotic nilpotent cone” can be used to replace the Kazhdan-Lusztig geometry and obtain a complete geometric classification of the irreducible representations of affine Hecke algebras (with mild restrictions on parameters).

In the type A case, there is a powerful alternative to the geometric method via Schur-Weyl duality (see for example [AS, OR, VV]). In this paper we provide an analogue of this Schur-Weyl duality approach for the type C case, with unequal parameters. This is a generalization of the degenerate case studied by Daugherty [Da].

The method is the following: Let $U_q\mathfrak{gl}_n$ be the Drinfeld-Jimbo quantum group corresponding to the general linear Lie algebra, and let $V = \mathbb{C}^n$ be the standard representation of $U_q\mathfrak{gl}_n$. Write $L(\lambda)$ for the irreducible polynomial representation of $U_q\mathfrak{gl}_n$ indexed by the partition λ , let $M = L((a^c))$ and $N = L((b^d))$ be irreducible representations of $U_q\mathfrak{gl}_n$ indexed by $a \times c$ and $b \times d$ rectangles. There is an action of an extension of the affine Hecke algebra of type C_k , denoted H_k^{ext} , with parameters

$$t^{\frac{1}{2}} = q, \quad t_0^{\frac{1}{2}} = -iq^{b+d}, \quad \text{and} \quad t_k^{\frac{1}{2}} = -iq^{a+c} \quad (\text{where } i = \sqrt{-1}),$$

such that

$$M \otimes N \otimes V^{\otimes k} \quad \text{is a } (U_q\mathfrak{gl}_n, H_k^{\text{ext}})\text{-bimodule.}$$

We show that the commuting actions of $U_q\mathfrak{gl}_n$ and H_k^{ext} provide a Schur-Weyl duality, which can be used to derive the representation theory of H_k^{ext} from the quantum group $U_q\mathfrak{gl}_n$. We work out the combinatorics of this correspondence, relating the natural indexing of H_k^{ext} -modules coming from the Schur-Weyl duality to the other indexings, by describing the weights for the action of the polynomial part (generated by Bernstein generators) on each irreducible module.

A significant portion of the work in identifying the centralizer of the $U_q\mathfrak{gl}_n$ action on $M \otimes N \otimes V^{\otimes k}$ as an extended affine Hecke algebra of type C is in relating Coxeter and Bernstein presentations, and putting the parameter conversions into focus. The relationships between these presentations are given in Theorem 2.1 for the affine braid group of type C, and in Theorem 2.2 for the affine

Hecke algebra of type C. Sections 3, 4 and 5 could, perhaps have stood as papers on their own. In Section 3, we give the combinatorics of local regions and standard tableaux for the case of type C with unequal parameters (following the equal parameter case done in [Ra2]). The main result of Section 3, Theorem 3.3, provides a classification and a construction of all irreducible calibrated H_k^{ext} -modules. As in [Ra2], this classification is via *skew local regions*, whose precise definition of skew local regions depends on the careful analysis of the structure of the irreducible representations of rank two affine Hecke algebras. This analysis was done in the single parameter case in [Ra1]. Since the corresponding analysis for *three distinct parameters in the type C_2 case* is, to our knowledge, not available in the literature, we have provided it in Section 4. This will ensure that our classification of calibrated irreducible representations for H_k^{ext} with distinct parameters, as given in Theorem 3.3, is on firm footing. The construction of the action of H_k^{ext} on tensor space is completed in Theorems 5.1 and 5.4. Finally, armed with these tools we prove the main result, Theorem 5.5, which determines exactly which representations of H_k^{ext} appear in tensor space, comparing the natural indexing from the highest weight theory for \mathfrak{gl}_n to the combinatorics of the weights of the action of the polynomial part of H_k^{ext} .

Following the schematic from [OR], one would like to generalize the analysis in this paper by replacing finite-dimensional M and N with, for example, other modules from category \mathcal{O} . In the finite-dimensional case, the key is that R -matrices for $M \otimes V$ and $N \otimes V$ have only two eigenvalues. This strongly restricts the choices for M and N . Non-finite-dimensional choices of modules M and N that satisfy these conditions exist in category \mathcal{O} , but additional work toward understanding the combinatorics of $M \otimes N \otimes V^{\otimes k}$ in these cases is needed. This understanding would yield an interesting generalization of the work in this paper.

The seeds of the idea for this paper were sown during conversations of A. Ram with P. Pyatov and V. Rittenberg in Bonn in 2005. They suggested that one should analyze two boundary spin chains by R -matrices, thus implying the possibility for Schur-Weyl duality approach to representations of affine braid groups of type C. This idea was completed in the degenerate case in [Da], and significant information was obtained in the Temperley-Lieb case in [GN] (see also references there). In [DR] we shall complete the connection to the statistical mechanics by using the results of this paper to identify the representations of the two boundary Temperley-Lieb algebra given, in a diagrammatic form, by de Gier and Nichols in [GN].

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2 The two boundary Hecke algebra

In this section we define the two boundary braid group and Hecke algebras and establish multiple presentations of each. The conversion between presentations is important for matching the algebraic approach to the representation theory with the Schur-Weyl duality approach that we give in Section 5.

For generators g_i, g_j , encode relations graphically by

$$\begin{aligned}
\begin{array}{c} g_i \\ \circ \end{array} & \begin{array}{c} g_j \\ \circ \end{array} & \text{means } g_i g_j = g_j g_i, \\
\begin{array}{c} g_i \\ \circ \end{array} & \text{---} & \begin{array}{c} g_j \\ \circ \end{array} & \text{means } g_i g_j g_i = g_j g_i g_j, \text{ and} \\
\begin{array}{c} g_i \\ \circ \end{array} & \text{=} & \begin{array}{c} g_j \\ \circ \end{array} & \text{means } g_i g_j g_i g_j = g_j g_i g_j g_i.
\end{aligned} \tag{2.1}$$

For example, the group of signed permutations,

$$\mathcal{W}_0 = \left\{ \begin{array}{l} \text{bijections } w: \{-k, \dots, -1, 1, \dots, k\} \rightarrow \{-k, \dots, -1, 1, \dots, k\} \\ \text{such that } w(-i) = -w(i) \text{ for } i = 1, \dots, k \end{array} \right\}, \tag{2.2}$$

has a presentation by generators s_0, s_1, \dots, s_{k-1} , with relations

$$\begin{array}{c} s_0 \quad s_1 \quad s_2 \quad \dots \quad s_{k-2} \quad s_{k-1} \\ \circ \text{=} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad \text{and} \quad s_i^2 = 1 \text{ for } i = 0, 1, 2, \dots, k-1. \tag{2.3}$$

2.1 The two boundary braid group

The *two boundary braid group* is the group \mathcal{B}_k generated by $\bar{T}_0, \bar{T}_1, \dots, \bar{T}_k$, with relations

$$\begin{array}{c} \bar{T}_0 \quad \bar{T}_1 \quad \bar{T}_2 \quad \dots \quad \bar{T}_{k-2} \quad \bar{T}_{k-1} \quad \bar{T}_k \\ \circ \text{=} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \text{=} \circ \end{array} . \tag{2.4}$$

Pictorially, the generators of \mathcal{B}_k are identified with the braid diagrams

$$\begin{aligned}
\bar{T}_k &= \left[\begin{array}{c} \text{Diagram with } k \text{ vertical strands, } k \text{ caps on the left, and a crossing between strands } k-1 \text{ and } k \end{array} \right], & \bar{T}_0 &= \left[\begin{array}{c} \text{Diagram with } k \text{ vertical strands, } k \text{ caps on the right, and a crossing between strands } 1 \text{ and } 2 \end{array} \right], & \text{and} \\
\bar{T}_i &= \left[\begin{array}{c} \text{Diagram with } k \text{ vertical strands, } k \text{ caps on the left, and a crossing between strands } i \text{ and } i+1 \end{array} \right] & \text{for } i = 1, \dots, k-1, & \tag{2.5}
\end{aligned}$$

and the multiplication of braid diagrams is given by placing one diagram on top of another.

To make explicit the Schur-Weyl duality approach to representations of \mathcal{B}_k appearing in Section 5, it is useful to move the rightmost pole to the left by conjugating by the diagram

$$\sigma = \left[\begin{array}{c} \text{Diagram with } k \text{ vertical strands, } k \text{ caps on the left, and a crossing between strands } k-1 \text{ and } k \end{array} \right]. \tag{2.6}$$

Define

$$T_i = \sigma \bar{T}_i \sigma^{-1} = \left[\begin{array}{c} \text{Diagram with } k \text{ vertical strands, } k \text{ caps on the right, and a crossing between strands } i \text{ and } i+1 \end{array} \right], \quad Y_1 = \sigma \bar{T}_0 \sigma^{-1} = \left[\begin{array}{c} \text{Diagram with } k \text{ vertical strands, } k \text{ caps on the left, and a crossing between strands } 1 \text{ and } 2 \end{array} \right], \tag{2.7}$$

and

$$X_1 = T_1^{-1}T_2^{-1}\cdots T_{k-1}^{-1}\sigma\bar{T}_k\sigma^{-1}T_{k-1}\cdots T_1 = \begin{array}{c} \text{Diagram of } X_1 \end{array}. \quad (2.8)$$

Define

$$Z_1 = X_1Y_1 \quad \text{and} \quad Z_i = T_{i-1}T_{i-2}\cdots T_1X_1Y_1T_1\cdots T_{i-1} = \begin{array}{c} \text{Diagram of } Z_i \end{array}, \quad (2.9)$$

for $i = 2, \dots, k$.

Theorem 2.1. *The two boundary braid group \mathcal{B}_k is presented in the following three ways, using the notation defined in (2.1).*

(a) \mathcal{B}_k is presented by generators $X_1, Y_1, Z_1, T_1, \dots, T_{k-1}$ and relations

$$X_1 \text{---} T_1 \text{---} T_2 \text{---} \cdots \text{---} T_{k-2} \text{---} T_{k-1} \quad (a1)$$

$$Y_1 \text{---} T_1 \text{---} T_2 \text{---} \cdots \text{---} T_{k-2} \text{---} T_{k-1} \quad (a2)$$

$$Z_1 \text{---} T_1 \text{---} T_2 \text{---} \cdots \text{---} T_{k-2} \text{---} T_{k-1} \quad (a3)$$

and

$$Z_1 = X_1Y_1. \quad (a4)$$

(b) \mathcal{B}_k is presented by generators $X_1, Y_1, T_1, \dots, T_{k-1}$ and relations (a1), (a2), and

$$(T_1X_1T_1^{-1})Y_1 = Y_1(T_1X_1T_1^{-1}). \quad (b3)$$

(c) \mathcal{B}_k is presented by generators $Z_1, \dots, Z_k, Y_1, T_1, \dots, T_{k-1}$, and relations (a2),

$$Z_iZ_j = Z_jZ_i \quad \text{for } i, j = 1, \dots, k, \quad (c1)$$

$$Y_1Z_i = Z_iY_1 \quad \text{for } i = 2, \dots, k, \text{ and} \quad (c2)$$

$$T_iZ_j = Z_jT_i \quad \text{for } j \neq i, i+1, \text{ with } i = 1, \dots, k-1, \text{ and } j = 1, \dots, k, \quad (c3)$$

and

$$Z_{i+1} = T_iZ_iT_i \quad \text{for } i = 1, \dots, k-1. \quad (c4)$$

Proof. With σ as in (2.6) let $T_k = \sigma\bar{T}_k\sigma^{-1}$, so that the original generators are the σ -conjugates of

$$T_0, T_1, \dots, T_k. \quad (o)$$

Conjugate the relations in (2.4) by σ to rewrite them in the form

$$\begin{array}{c} Y_1 \quad T_1 \quad T_2 \quad \quad \quad T_{k-2} \quad T_{k-1} \\ \circ \text{---} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \end{array}, \quad T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k, \quad (\text{o1})$$

$$T_k Y_1 = Y_1 T_k, \quad \text{and} \quad T_k T_i = T_i T_k, \quad \text{for } i = 1, \dots, k-2. \quad (\text{o2})$$

The conversions between the generators in presentations (a), (b), and (c) are given in (2.7), (2.8), and (2.9). For generators (a) and (b) in terms of generators (o), the key relations are

$$Y_1 = T_0, \quad X_1 = T_1^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1 \quad \text{and} \quad T_k = T_{k-1} \cdots T_1 X_1 T_1^{-1} \cdots T_{k-1}^{-1}.$$

Relations (a) from relations (b): Relation (a4) is the conversion from generators (b) to generators (a). The relations in (a3) then follow from

$$T_i Z_1 = T_i X_1 Y_1 = X_1 T_i Y_1 = X_1 Y_1 T_i = Z_1 T_i, \quad \text{for } i = 2, \dots, k-1,$$

and

$$\begin{aligned} T_1 Z_1 T_1 Z_1 &= T_1 X_1 Y_1 T_1 X_1 Y_1 = T_1 X_1 (Y_1 T_1 X_1 T_1^{-1}) T_1 Y_1 = T_1 X_1 (T_1 X_1 T_1^{-1} Y_1) T_1 Y_1 \\ &= X_1 T_1 X_1 T_1 T_1^{-1} Y_1 T_1 Y_1 = X_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 Y_1 = X_1 T_1 X_1 T_1^{-1} Y_1 T_1 Y_1 T_1 \\ &= X_1 Y_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 = Z_1 T_1 Z_1 T_1. \end{aligned}$$

Relations (b) from relations (a): Multiplying

$$\begin{aligned} T_1 X_1 (T_1 X_1 T_1^{-1} Y_1) T_1 Y_1 &= X_1 T_1 X_1 T_1 T_1^{-1} Y_1 T_1 Y_1 = X_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 Y_1 \\ &= X_1 T_1 X_1 T_1^{-1} Y_1 T_1 Y_1 T_1 = X_1 Y_1 T_1 X_1 T_1^{-1} T_1 Y_1 T_1 = Z_1 T_1 Z_1 T_1 \\ &= T_1 Z_1 T_1 Z_1 = T_1 X_1 Y_1 T_1 X_1 Y_1 = T_1 X_1 (Y_1 T_1 X_1 T_1^{-1}) T_1 Y_1 \end{aligned}$$

on the left by $(T_1 X_1)^{-1}$ and on the right by $(T_1 Y_1)^{-1}$ gives $T_1 X_1 T_1^{-1} Y_1 = Y_1 T_1 X_1 T_1^{-1}$, establishing (b3).

Relations (b) from relations (o): The pictorial computations

and

show that $X_1 T_i = T_i X_1$ for $i = 1, 2, \dots, k-1$, $Y_1 T_1 X_1 T_1^{-1} = T_1 X_1 T_1^{-1} Y_1$, and $X_1 T_1 X_1 T_1 = T_1 X_1 T_1 X_1$. Hence the relations (a1) and (a2) follow from the relations in (o1) and (o2).

Relations (o) from relations (b): The first set of relations in (o1) are the same as the relations in (a2). Let $A = T_{k-1} \cdots T_1$ and $B = T_{k-1} \cdots T_2$. Since X_1 commutes with T_i for $i = 2, \dots, k-1$, then $BX_1B^{-1} = X_1$ so that

$$ABX_1B^{-1}A^{-1} = \left(\begin{array}{c} \text{Diagram: } \text{Two vertical lines on the left, followed by a crossing, then } k-1 \text{ strands crossing each other in a braid-like pattern.} \end{array} \right) = \left(\begin{array}{c} \text{Diagram: } \text{Two vertical lines on the left, followed by } k-1 \text{ vertical lines, then a crossing.} \end{array} \right) = T_k,$$

and

$$ABT_1B^{-1}A^{-1} = \left(\begin{array}{c} \text{Diagram: } \text{Two vertical lines on the left, followed by a crossing, then } k-1 \text{ strands crossing each other in a braid-like pattern.} \end{array} \right) = \left(\begin{array}{c} \text{Diagram: } \text{Two vertical lines on the left, followed by } k-1 \text{ vertical lines, then a crossing.} \end{array} \right) = T_{k-1}.$$

Thus, by conjugation by AB , the relation $X_1T_1X_1T_1 = T_1X_1T_1X_1$ becomes $T_kT_{k-1}T_kT_{k-1} = T_{k-1}T_kT_{k-1}T_k$, establishing the second relation in (o1). For $i = 1, \dots, k-2$,

$$\begin{aligned} T_iT_k &= T_iT_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_k^{-1} = T_{k-1} \cdots T_{i+2}T_iT_{i+1}T_i \cdots T_1X_1T_1^{-1} \cdots T_k^{-1} \\ &= T_{k-1} \cdots T_{i+2}T_{i+1}T_iT_{i+1}T_{i-1} \cdots T_1X_1T_1^{-1} \cdots T_k^{-1} \\ &= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_{i-1}^{-1}T_{i+1}T_i^{-1}T_{i+1}^{-1} \cdots T_k^{-1} \\ &= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_{i-1}^{-1}T_i^{-1}T_{i+1}^{-1}T_iT_{i+2} \cdots T_k^{-1} \\ &= T_{k-1} \cdots T_1X_1T_1^{-1} \cdots T_k^{-1}T_i = T_kT_i. \end{aligned}$$

Similarly, (b3) gives

$$\begin{aligned} Y_1T_k &= Y_1T_{k-1} \cdots T_2T_1X_1T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1} = T_{k-1} \cdots T_2(Y_1T_1X_1T_1^{-1})T_2^{-1} \cdots T_{k-1}^{-1} \\ &= T_{k-1} \cdots T_2(T_1X_1T_1^{-1}Y_1)T_2^{-1} \cdots T_{k-1}^{-1} = T_{k-1} \cdots T_2T_1X_1T_1^{-1}T_2^{-1} \cdots T_{k-1}^{-1}Y_1 = T_kY_1, \end{aligned}$$

giving the relations in (o2).

Relations (c) from relations (o): The first set of relations in (o1) are the same as the relations in (a2). Relations (c4) are exactly the definitions in the second part of (2.9). The pictorial computation

$$Z_jZ_i = \left(\begin{array}{c} \text{Diagram: } \text{Strands } i \text{ and } j \text{ crossing, with } i \text{ strands on the left and } j \text{ strands on the right.} \end{array} \right) = \left(\begin{array}{c} \text{Diagram: } \text{Strands } i \text{ and } j \text{ crossing, with } i \text{ strands on the left and } j \text{ strands on the right.} \end{array} \right) = Z_iZ_j$$

give relations (c1). Similarly, pictorial computations readily show that $Y_1Z_i = Z_iY_1$ for $i > 1$ and $T_iZ_j = Z_jT_i$ for $i \neq j, j+1$, proving relations (c2) and (c3).

Generators (o) from generators (c): The key formula for the generator T_k is

$$\begin{aligned} T_k &= T_{k-1} \cdots T_1 (T_1^{-1} \cdots T_{k-1}^{-1} T_k T_{k-1} \cdots T_1) Y_1 (T_1 \cdots T_{k-1}) (T_{k-1}^{-1} \cdots T_1^{-1}) Y_1^{-1} (T_1^{-1} \cdots T_{k-1}^{-1}) \\ &= (T_{k-1} \cdots T_1) X_1 Y_1 (T_1 \cdots T_{k-1}) T_{s_\varphi} = Z_k T_{s_\varphi}^{-1}, \end{aligned}$$

where

$$T_{s_\varphi} = T_{k-1} T_{k-2} \cdots T_1 Y_1 T_1 \cdots T_{k-2} T_{k-1} = \left\| \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \right\|.$$

Relations (o) from relations (c): The first set of relations in (o1) are the same as the relations in (a2). The relations

$$T_{s_\varphi} Y_1 = Y_1 T_{s_\varphi} \quad \text{and} \quad T_{s_\varphi} T_i = T_i T_{s_\varphi}, \quad \text{for } i = 1, \dots, k-2, \quad (2.10)$$

are verified pictorially by

$$\left\| \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \right\| = \left\| \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \right\| \quad \text{and} \quad \left\| \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \right\| = \left\| \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \right\|.$$

or by direct computation using the relations in (a2).

By (2.8) and (2.9), $Z_k = T_k T_{s_\varphi}$ and, by (c3) and (c2) respectively,

$$\begin{aligned} T_k T_i &= Z_k T_{s_\varphi}^{-1} T_i = Z_k T_i T_{s_\varphi}^{-1} = T_i Z_k T_{s_\varphi}^{-1} = T_i T_k, \quad \text{for } i = 1, \dots, k-2, \text{ and} \\ T_k Y_1 &= Z_k T_{s_\varphi}^{-1} Y_1 = Z_k Y_1 T_{s_\varphi}^{-1} = Y_1 Z_k T_{s_\varphi}^{-1} = Y_1 T_k, \end{aligned} \quad (2.11)$$

which proves the relations in (o2).

By the relations in (2.11) and the second set of relations in (2.10),

$$(T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) T_k = T_k (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) \quad \text{and} \quad (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) T_{s_\varphi} = T_{s_\varphi} (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}),$$

so that $(T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) (T_k T_{s_\varphi}) = (T_k T_{s_\varphi}) (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1})$. Using these and the equality

$$T_{k-1} Z_k Z_{k-1} = T_{k-1} Z_{k-1} Z_k = Z_k T_{k-1}^{-1} Z_k = Z_k Z_{k-1} T_{k-1},$$

we have

$$\begin{aligned} T_{k-1} Z_k Z_{k-1} &= T_{k-1} Z_k (T_{k-1}^{-1} Z_k T_{k-1}^{-1}) = T_{k-1} (T_k T_{s_\varphi}) T_{k-1}^{-1} (T_k T_{s_\varphi}) T_{k-1}^{-1} \\ &= T_{k-1} T_k T_{k-1} (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) T_k T_{s_\varphi} T_{k-1}^{-1} = (T_{k-1} T_k T_{k-1} T_k) (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) \\ &= Z_k Z_{k-1} T_{k-1} = Z_k (T_{k-1}^{-1} Z_k T_{k-1}^{-1}) T_{k-1} = (T_k T_{s_\varphi}) T_{k-1}^{-1} (T_k T_{s_\varphi}) = T_k T_{k-1} (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) (T_k T_{s_\varphi}) \\ &= T_k T_{k-1} (T_k T_{s_\varphi}) (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}) = (T_k T_{k-1} T_k T_{k-1}) (T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1}). \end{aligned}$$

Multiplying on the right by $(T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1} T_{s_\varphi} T_{k-1}^{-1})^{-1}$ gives $T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k$, establishing the last relation in (o1). \square

If

$$P^{1/2} = \begin{array}{c} \text{X} \\ | \\ | \\ | \\ | \\ | \\ | \end{array} \quad (2.12)$$

then

$$P^{1/2}Y_1P^{-1/2} = \begin{array}{c} \text{X} \\ | \\ | \\ | \\ | \\ | \\ | \end{array} = \begin{array}{c} \text{X} \\ | \\ | \\ | \\ | \\ | \\ | \end{array} = Y_1^{-1}X_1Y_1 \quad (2.13)$$

and

$$P^{1/2}X_1P^{-1/2} = \begin{array}{c} \text{X} \\ | \\ | \\ | \\ | \\ | \\ | \end{array} = \begin{array}{c} \text{X} \\ | \\ | \\ | \\ | \\ | \\ | \end{array} = Y_1 \quad (2.14)$$

Following these pictorial computations, the *extended affine braid group* is the group $\mathcal{B}_k^{\text{ext}}$ generated by \mathcal{B}_k and P with the additional relations

$$PX_1P^{-1} = Z_1^{-1}X_1Z_1, \quad PY_1P^{-1} = Z_1^{-1}Y_1Z_1, \quad (2.15)$$

$$PZ_1P^{-1} = Z_1, \quad \text{and} \quad PT_iP^{-1} = T_i \text{ for } i = 1, \dots, k-1. \quad (2.16)$$

Note that the element

$$Z_0 = PZ_1 \cdots Z_k \text{ is central in } \mathcal{B}_k^{\text{ext}} \quad (\text{c0})$$

since the group $\mathcal{B}_k^{\text{ext}}$ is a subgroup of the braid group on $k+2$ strands, and Z_0 is the generator of the center of the braid group on $k+2$ strands (see [GM, Theorem 4.2]). So

$$\text{if } \mathcal{D} = \{Z_0^j \mid j \in \mathbb{Z}\} \quad \text{then} \quad \mathcal{B}_k^{\text{ext}} = \mathcal{D} \times \mathcal{B}_k, \quad \text{with } \mathcal{D} \cong \mathbb{Z}.$$

2.2 The two boundary Hecke algebra H_k^{ext}

In this subsection we define the two boundary Hecke algebras and relate it to the presentation of the affine Hecke algebra of type C that is found, for example, in [Lu1, Proposition 3.6] and [Mac2, (4.2.4)].

Fix $a_1, a_2, b_1, b_2, t^{\frac{1}{2}} \in \mathbb{C}^\times$. The *extended two boundary Hecke algebra* H_k^{ext} is the quotient of $\mathcal{B}_k^{\text{ext}}$ by the relations

$$(X_1 - a_1)(X_1 - a_2) = 0, \quad (Y_1 - b_1)(Y_1 - b_2) = 0, \quad \text{and} \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0, \quad (\text{h})$$

for $i = 1, \dots, k-1$. Let

$$t^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}} \quad \text{and} \quad t^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}}. \quad (2.17)$$

With $Z_i \in H_k^{\text{ext}}$ as in (2.9), define

$$T_0 = b_1^{-\frac{1}{2}}(-b_2)^{-\frac{1}{2}}Y_1, \quad W_i = -(a_1a_2b_1b_2)^{-\frac{1}{2}}Z_i \text{ for } i = 1, \dots, k, \text{ and} \quad (2.18)$$

$$W_0 = PW_1 \cdots W_k = (-1)^k(a_1a_2b_1b_2)^{-\frac{k}{2}}PZ_1 \cdots Z_k = (-1)^k(a_1a_2b_1b_2)^{-\frac{k}{2}}Z_0. \quad (2.19)$$

Then

$$X_1 = Z_1Y_1^{-1} = a_1^{\frac{1}{2}}(-a_2)^{\frac{1}{2}}W_1T_0^{-1}. \quad (2.20)$$

Theorem 2.2. Fix $t_0, t_k, t \in \mathbb{C}^\times$ and use notations for relations as defined in (2.1). The extended affine Hecke algebra H_k^{ext} defined in (h) is presented by generators, $T_0, T_1, \dots, T_{k-1}, W_0, W_1, \dots, W_k$ and relations

$$W_0 \in Z(H_k^{\text{ext}}), \quad \begin{array}{ccccccc} T_0 & T_1 & T_2 & & T_{k-2} & T_{k-1} & ; \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array} \quad (\text{B1})$$

$$W_i W_j = W_j W_i, \quad \text{for } i, j = 0, 1, \dots, k; \quad (\text{B2})$$

$$T_0 W_j = W_j T_0, \quad \text{for } j \neq 1; \quad (\text{B3})$$

$$T_i W_j = W_j T_i \text{ for } i = 1, \dots, k-1 \text{ and } j = 1, \dots, k \text{ with } j \neq i, i+1; \quad (\text{B4})$$

$$(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0, \quad \text{and} \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0 \text{ for } i = 1, \dots, k-1. \quad (\text{H})$$

For $i = 1, \dots, k-1$,

$$T_i W_i = W_{i+1} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_i - W_{i+1}}{1 - W_i W_{i+1}^{-1}}, \quad T_i W_{i+1} = W_i T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_{i+1} - W_i}{1 - W_i W_{i+1}^{-1}}, \quad (\text{C1})$$

$$\text{and} \quad T_0 W_1 = W_1^{-1} T_0 + \left((t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) W_1^{-1} \right) \frac{W_1 - W_1^{-1}}{1 - W_1^{-2}}. \quad (\text{C2})$$

Proof. The conversion between the different sets of generators of H_k^{ext} is provided by (2.18).

Equivalence between (c0–c4) and the second and third relations of (h) with the relations (B1–B4) and (H). Since T_0 and Y_1 differ by a constant, and W_i and Z_i differ by a constant, the relations in (c0–c4) are equivalent to the relations in (B1–B4), respectively. Since

$$\begin{aligned} 0 &= (Y_1 - b_1)(Y_1 - b_2) = b_1^{\frac{1}{2}}(-b_2)^{\frac{1}{2}}(T_0 - b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}})b_1^{\frac{1}{2}}(-b_2)^{\frac{1}{2}}(T_0 + b_1^{-\frac{1}{2}}(-b_2)^{\frac{1}{2}}) \\ &= -b_1 b_2 (T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}), \end{aligned}$$

the relations (H) are equivalent to the second and third relations in (h).

Relations (C1–C2) from relations (c0–c4) and (h): From (2.9) and (2.18), $W_{i+1} = T_i W_i T_i$, and by the last relation in (h), $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. So

$$\begin{aligned} T_i W_i &= W_{i+1} T_i^{-1} = W_{i+1} (T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})) = W_{i+1} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_i - W_{i+1}}{1 - W_i W_{i+1}^{-1}} \quad \text{and} \\ T_i W_{i+1} &= T_i^2 W_i T_i = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) W_{i+1} + W_i T_i = W_i T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_{i+1} - W_i}{1 - W_i W_{i+1}^{-1}}, \end{aligned}$$

which establishes the relations in (C1).

By the first relation in (h), $X_1^{-1} = -a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1})$. Since $W_1 = a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}X_1T_0$ and $T_0 - T_0^{-1} = t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}$,

$$\begin{aligned} T_0W_1 - W_1^{-1}T_0 &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 - a_1(-a_2)T_0^{-1}X_1^{-1}T_0) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 + a_1a_2T_0^{-1}(-a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1}))T_0) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}((T_0 - T_0^{-1})X_1T_0 + (a_1 - (-a_2))) \\ &= (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})W_1 + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}), \end{aligned}$$

which establishes (C2).

The first relation in (h) from the relations (B1–B4), (H) and (C1–C2). By (C2),

$$\begin{aligned} a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 - a_1(-a_2)T_0^{-1}X_1^{-1}T_0) &= T_0W_1 - W_1^{-1}T_0 = (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})W_1 + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}((T_0 - T_0^{-1})X_1T_0 + (a_1 - (-a_2))) \\ &= a_1^{-\frac{1}{2}}(-a_2)^{-\frac{1}{2}}(T_0X_1T_0 + a_1a_2T_0^{-1}(-a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1}))T_0), \end{aligned}$$

giving $X_1^{-1} = -a_1^{-1}a_2^{-1}X_1 + (a_1^{-1} + a_2^{-1})$, which establishes the first relation in (h). \square

As vector spaces,

$$H_k^{\text{ext}} = \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}] \otimes H_k^{\text{fin}}, \quad (2.21)$$

where H_k^{fin} is the subalgebra of H_k^{ext} generated by T_0, T_1, \dots, T_{k-1} . The algebra H_k^{fin} is the Iwahori-Hecke algebra of finite type C_k . If s_0, s_1, \dots, s_{k-1} are the generators of \mathcal{W}_0 as given in (2.3), write $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ for a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, so that

$$\{T_w \mid w \in \mathcal{W}_0\} \quad \text{is a } \mathbb{C}\text{-basis of } H_k^{\text{fin}}.$$

Thus (2.21) means that any element $h \in H_k^{\text{ext}}$ can be written uniquely as

$$h = \sum_{w \in \mathcal{W}_0} h_w T_w, \quad \text{with } h_w \in \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}].$$

Let

$$W^\lambda = W_0^{\lambda_0} W_1^{\lambda_1} W_2^{\lambda_2} \cdots W_k^{\lambda_k} \quad \text{for } \lambda = (\lambda_0, \lambda_1, \dots, \lambda_k) \in \mathbb{Z}^{k+1}. \quad (2.22)$$

Relations (C1) and (C2) produce an action of \mathcal{W}_0 on

$$\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}] = \text{span}_{\mathbb{C}}\{W^\lambda \mid \lambda = (\lambda_0, \lambda_1, \dots, \lambda_k) \in \mathbb{Z}^{k+1}\}.$$

Namely, for $w \in \mathcal{W}_0$ and $\lambda \in \mathbb{Z}^{k+1}$,

$$\begin{aligned} wW^\lambda &= W^{w\lambda}, \quad \text{where } s_0\lambda = s_0(\lambda_0, \lambda_1, \dots, \lambda_k) = (\lambda_0, -\lambda_1, \dots, \lambda_k), \quad \text{and} \\ s_i\lambda &= s_i(\lambda_0, \lambda_1, \dots, \lambda_k) = (\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \dots, \lambda_k), \end{aligned} \quad (2.23)$$

for $i = 1, 2, \dots, k-1$ (see [Ra2, (1.12)]). With this notation, for $\lambda \in \mathbb{Z}^{k+1}$, the relations (C1) and (C2) give

$$T_i W^\lambda = W^{s_i \lambda} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W^\lambda - W^{s_i \lambda}}{1 - W_i W_{i+1}^{-1}} \quad \text{and} \quad (2.24)$$

$$T_0 W^\lambda = W^{s_0 \lambda} T_0 + \left((t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) W_1^{-1} \right) \frac{W^\lambda - W^{s_0 \lambda}}{1 - W_1^{-2}}, \quad (2.25)$$

and, replacing $s_i \lambda$ by μ ,

$$W^\mu T_i = T_i W^{s_i \mu} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W^\mu - W^{s_i \mu}}{1 - W_i W_{i+1}^{-1}} \quad \text{and} \quad (2.26)$$

$$W^\mu T_0 = T_0 W^{s_0 \mu} + \left((t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) W_1^{-1} \right) \frac{W^\mu - W^{s_0 \mu}}{1 - W_1^{-2}}, \quad \text{for } \mu \in \mathbb{Z}^{k+1}. \quad (2.27)$$

The subalgebra $H_k \subseteq H_k^{\text{ext}}$ generated by W_1, \dots, W_k and T_0, \dots, T_{k-1} is the affine Hecke algebra of type C considered, for example, in [Lu1]. The following theorem determines the center of H_k^{ext} and shows that, as algebras, H_k^{ext} is a tensor product of H_k by the algebra of Laurent polynomials in one variable. It follows that the irreducible representations of H_k^{ext} are indexed by $\mathbb{C}^\times \times \hat{H}_k$, where \hat{H}_k is an indexing set for the irreducible representations of H_k .

Theorem 2.3. *Let H_k be the subalgebra of H_k^{ext} generated by W_1, \dots, W_k and T_0, \dots, T_{k-1} . As algebras,*

$$H_k^{\text{ext}} \cong \mathbb{C}[W_0^{\pm 1}] \otimes H_k, \quad (2.28)$$

The center of H_k^{ext} is

$$Z(H_k^{\text{ext}}) = \mathbb{C}[W_0^{\pm 1}] \otimes \mathbb{C}[W_1^{\pm 1}, \dots, W_k^{\pm 1}]^{\mathcal{W}_0},$$

and H_k^{ext} is a free module of rank $\text{Card}(\mathcal{W}_0)^2 = 2^{2k}(k!)^2$ over $Z(H_k^{\text{ext}})$.

Proof. As observed in (c0), Z_0 is central in $\mathcal{B}_k^{\text{ext}}$, and therefore $W_0 = (-1)^k (a_1 a_2 b_1 b_2)^{k/2} Z_0$ is central in H_k^{ext} . Thus

$$H_k^{\text{ext}} = \mathbb{C}[W_0^{\pm 1}] \otimes H_k. \quad (2.29)$$

By the formulas in (2.23), the Laurent polynomial ring $\mathbb{C}[W_1^{\pm 1}, \dots, W_k^{\pm 1}]$ is a \mathcal{W}_0 -submodule of $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}]$, and

$$\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}]^{\mathcal{W}_0} = \mathbb{C}[W_1^{\pm 1}, \dots, W_k^{\pm 1}]^{\mathcal{W}_0} \otimes \mathbb{C}[W_0^{\pm 1}]. \quad (2.30)$$

The proof that $Z(H_k^{\text{ext}}) = \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}]^{\mathcal{W}_0}$ is exactly as in [RR, Thm. 4.12]. The fact that H_k^{ext} is a free module of rank $\text{Card}(\mathcal{W}_0)^2$ over $\mathbb{C}[\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}]^{\mathcal{W}_0}]$ follows from (2.21) and [Ra2, Theorem1.17]. \square

2.3 Weights of representations and intertwiners

Let $t^{\frac{1}{2}} \in \mathbb{C}^\times$ be such that $(t^{\frac{1}{2}})^\ell \neq 1$ for $\ell \in \mathbb{Z}$. All irreducible complex representations γ of the algebra $\mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}]$ are one-dimensional. Identify the sets

$$\begin{aligned} \mathcal{C} &= \{ \text{irreducible representations } \gamma \text{ of } \mathbb{C}[W_0^{\pm 1}, W_1^{\pm 1}, \dots, W_k^{\pm 1}] \} & (2.31) \\ &\leftrightarrow \{ \text{sequences } (z, \gamma_1, \dots, \gamma_k) \in (\mathbb{C}^\times)^{k+1} \} \\ &\leftrightarrow \{ \text{sequences } (\zeta, c_1, \dots, c_k) \in \mathbb{C}^{k+1} \} \end{aligned}$$

via

$$\gamma(W_0) = z = (-1)^k t^\zeta \quad \text{and} \quad \gamma(W_i) = \gamma_i = -t^{c_i} \text{ for } i = 1, \dots, k \quad (2.32)$$

(the choice of sign in the last equation is an artifact of equations (5.32) and (5.33) and an effort to make the combinatorics of contents of boxes Section 5 optimally helpful). The action of \mathcal{W}_0 from (2.23) induces an action of \mathcal{W}_0 on \mathcal{C} by

$$(w\gamma)(W^\lambda) = \gamma(W^{w^{-1}\lambda}), \quad \text{for } w \in \mathcal{W}_0 \text{ and } \lambda \in \mathbb{Z}^{k+1}. \quad (2.33)$$

Equivalently, on sequences (ζ, c_1, \dots, c_k) , this action is given by

$$w(\zeta, c_1, \dots, c_k) = (\zeta, c_{w^{-1}(1)}, \dots, c_{w^{-1}(k)}), \quad \text{for } w \in \mathcal{W}_0. \quad (2.34)$$

Let \tilde{H}_k^{ext} be the extensions of H_k^{ext} by the rational functions in W_1, \dots, W_k :

$$\tilde{H}_k^{\text{ext}} = \mathbb{C}[W_0^{\pm 1}] \otimes \mathbb{C}(W_1, \dots, W_k) \otimes H_k^{\text{fn}},$$

where H_k^{fn} is the subalgebra of H_k^{ext} generated by T_0, T_1, \dots, T_{k-1} . The *intertwining operators* for \tilde{H}_k^{ext} are

$$\tau_0 = T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \quad \text{and} \quad \tau_i = T_i - \frac{t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}} \quad (2.35)$$

for $i = 1, 2, \dots, k-1$. Proposition 2.4 shows that these operators satisfy $\tau_0 W^\lambda = W^{s_0 \lambda} \tau_0$ and $\tau_i W^\lambda = W^{s_i \lambda} \tau_i$ so that, for $w \in \mathcal{W}_0$ and $\lambda = (\lambda_0, \dots, \lambda_k) \in \mathbb{Z}^{k+1}$,

$$\tau_w W^\lambda = W^{w\lambda} \tau_w, \quad \text{where } \tau_w = \tau_{i_1} \dots \tau_{i_\ell} \quad (2.36)$$

for a reduced expression $w = s_{i_1} \dots s_{i_\ell}$.

Each H_k^{ext} -module M can be written as $M = \bigoplus_{\gamma \in \mathcal{C}} M_\gamma^{\text{gen}}$, where for each $\gamma = (z, \gamma_1, \dots, \gamma_k) \in \mathcal{C}$,

$$M_\gamma^{\text{gen}} = \left\{ m \in M \mid \begin{array}{l} \text{there exists } N \in \mathbb{Z}_{>0} \text{ such that } (W_0 - z)^N m = 0 \\ \text{and } (W_i - \gamma_i)^N m = 0 \text{ for } i = 1, \dots, k \end{array} \right\} \quad (2.37)$$

is the *generalized weight space* associated to γ . The intertwiners (2.35) define vector space homomorphisms

$$\tau_0: M_\gamma^{\text{gen}} \mapsto M_{s_0 \gamma}^{\text{gen}} \quad \text{and} \quad \tau_i: M_\gamma^{\text{gen}} \mapsto M_{s_i \gamma}^{\text{gen}} \text{ for } i = 1, \dots, k-1, \quad (2.38)$$

where

τ_0 is defined only when $\gamma_1 \neq 1$, so that $(1 - W_1^{-1})^{-1}$ is well-defined on M_γ^{gen} and τ_i is defined only when $\gamma_i \neq \gamma_{i+1}$, so that $(1 - W_i W_{i+1}^{-1})^{-1}$ is well-defined on M_γ^{gen}

for $i = 1, \dots, k-1$.

Proposition 2.4. (Intertwiner presentation) *The algebra \tilde{H}_k^{ext} is generated by $\tau_0, \dots, \tau_k, W_0$, and $\mathbb{C}(W_1, \dots, W_k)$ with relations*

$$\begin{array}{ccccccc} \tau_0 & \tau_1 & \tau_2 & & \tau_{k-2} & \tau_{k-1} & \\ \circ & \text{---} \circ & \text{---} \circ & \text{---} \text{---} \text{---} & \text{---} \circ & \text{---} \circ & \end{array} \quad (2.39)$$

in the notation of (2.1);

$$\tau_0 W_1 = W_1^{-1} \tau_0 \quad \text{and} \quad \tau_0 W_j = W_j \tau_0 \quad \text{for } j \neq 1; \quad (2.40)$$

for $i = 1, \dots, k-1$,

$$\tau_i W_i = W_{i+1} \tau_i \quad \text{and} \quad \tau_i W_{i+1} = W_i \tau_i \quad \text{for } i > 0, \quad \text{and} \quad \tau_i W_j = W_j \tau_i \quad \text{for } j \neq i, i+1; \quad (2.41)$$

$$\tau_0^2 = \frac{(1 - t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} W_1^{-1}) (1 + t_0^{\frac{1}{2}} t_k^{-\frac{1}{2}} W_1^{-1}) (1 + t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}} W_1^{-1}) (1 - t_0^{-\frac{1}{2}} t_k^{-\frac{1}{2}} W_1^{-1})}{1 - W_1^{-1} \quad 1 + W_1^{-1} \quad 1 + W_1^{-1} \quad 1 - W_1^{-1}}; \quad (2.42)$$

$$\text{and} \quad \tau_i^2 = \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}} W_i^{-1} W_{i+1})(t^{\frac{1}{2}} - t^{-\frac{1}{2}} W_{i+1}^{-1} W_i)}{(1 - W_i^{-1} W_{i+1})(1 - W_{i+1}^{-1} W_i)} \quad \text{for } i = 1, \dots, k-1. \quad (2.43)$$

Proof. The proof of the relations in (2.39) is accomplished exactly as in the proof of [Ra2, Proposition 2.14(e)]; relation (2.43) is [Ra2, Proposition 2.14(c)]. Let us check the relations in (2.41) and (2.42).

Using (C1),

$$\begin{aligned} \tau_i W_i &= \left(T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}} \right) W_i = W_{i+1} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_i - W_{i+1}}{1 - W_i W_{i+1}^{-1}} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_i}{1 - W_i W_{i+1}^{-1}} \\ &= W_{i+1} \left(T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}} \right) = W_{i+1} \tau_i. \end{aligned}$$

Similarly, using (C2),

$$\begin{aligned} \tau_0 W_1 &= \left(T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) W_1^{-1}}{1 - W_1^{-2}} \right) W_1 \\ &= W_1^{-1} T_0 + (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) W_1 + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) W_1 + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})}{1 - W_1^{-2}} \\ &= W_1^{-1} \left(T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) W_1^{-1}}{1 - W_1^{-2}} \right) = W_1^{-1} \tau_0. \end{aligned}$$

For $i = 0, \dots, k-1$ and $j \neq i, i+1$, τ_i and W_j commute by the second set of relations in (C1). These computations establish the relations in (2.40) and (2.41).

By the first relation in (H), $T_0 = T_0^{-1} + (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})$, so that

$$\begin{aligned}\tau_0 &= T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} = T_0^{-1} + (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})W_1^2 + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \\ &= T_0^{-1} + \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2}.\end{aligned}$$

Then

$$\begin{aligned}\tau_0^2 &= \tau_0 \left(T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) = \tau_0 T_0 - \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) \tau_0 \\ &= \left(T_0^{-1} + \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) T_0 \\ &\quad - \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) \left(T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) \\ &= 1 + \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1}{1 - W_1^2} \right) \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) \\ &= 1 - \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})W_1^{-2} + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) \left(\frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \right) \\ &= \frac{\left(\begin{array}{l} 1 - 2W_1^{-2} + W_1^{-4} - ((t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})^2 + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})^2)W_1^{-2} \\ -(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1} - (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})(t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-3} \end{array} \right)}{(1 - W_1^{-2})^2} \\ &= \frac{(1 - t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}W_1^{-1})(1 + t_0^{\frac{1}{2}}t_k^{-\frac{1}{2}}W_1^{-1})(1 + t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}}W_1^{-1})(1 - t_0^{-\frac{1}{2}}t_k^{-\frac{1}{2}}W_1^{-1})}{1 + W_1^{-1} \quad 1 - W_1^{-1} \quad 1 - W_1^{-1} \quad 1 + W_1^{-1}},\end{aligned}$$

establishing (2.42). □

3 Calibrated representations of H_k^{ext}

A *calibrated H_k^{ext} -module* is an H_k^{ext} -module M such that W_0, W_1, \dots, W_k are simultaneously diagonalizable as operators on M . In the context of (2.37), M is calibrated if

$$M = \bigoplus_{\gamma \in \mathcal{C}} M_\gamma, \quad \text{where } M_\gamma = \{m \in M \mid W_0 m = z m \text{ and } W_i m = \gamma_i m \text{ for } i = 1, \dots, k\} \quad (3.1)$$

for $\gamma = (z, \gamma_1, \dots, \gamma_k) \in \mathcal{C}$. Another formulation is that M is calibrated if M has a basis of simultaneous eigenvectors for W_0, \dots, W_k . This section follows the framework of [Ra2] in developing

combinatorial tools for describing the structure and the classification of irreducible calibrated H_k^{ext} -modules. In Section 5 we will use this combinatorics to analyze and classify the H_k^{ext} -modules arising in the Schur-Weyl duality settings.

With notations as in the definition of \mathcal{W}_0 in (2.2), the *reflection representation* of \mathcal{W}_0 is the action of \mathcal{W}_0 on $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}^k$ given by

$$w(c_1, \dots, c_k) = (c_{w^{-1}(1)}, \dots, c_{w^{-1}(k)}), \quad \text{where } c_{-i} = -c_i \text{ for } i = 1, 2, \dots, k.$$

The dual space $\mathfrak{h}_{\mathbb{R}}^*$ has basis $\varepsilon_1, \dots, \varepsilon_k$, where $\varepsilon_i: \mathfrak{h}_{\mathbb{R}} \rightarrow \mathbb{R}$ is the \mathbb{R} -linear map given by $\varepsilon_i(\gamma_1, \dots, \gamma_k) = \gamma_i$. With $\varepsilon_{-i} = -\varepsilon_i$, the action of \mathcal{W}_0 on \mathbb{R}^k produces an action on $\mathfrak{h}_{\mathbb{R}}^*$ given by $w\varepsilon_i = \varepsilon_{w^{-1}(i)}$.

Let

$$\begin{aligned} R^+ &= \{\varepsilon_1, \dots, \varepsilon_k\} \sqcup \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \mid 1 \leq i < j \leq k\} \\ &= \{\varepsilon_1, \dots, \varepsilon_k\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 1 \leq i < j \leq k\} \sqcup \{\varepsilon_j - \varepsilon_{-i} \mid 1 \leq i < j \leq k\} \\ &= \{\varepsilon_1, \dots, \varepsilon_k\} \sqcup \{\varepsilon_j - \varepsilon_i \mid i, j \in \{-k, \dots, -1, 1, \dots, k\}, i < j, i \neq -j\}. \end{aligned}$$

If $w \in \mathcal{W}_0$, the *inversion set* of w is

$$R(w) = \{\alpha \in R^+ \mid w\alpha \notin R^+\} \tag{3.2}$$

$$\begin{aligned} &= \{\varepsilon_i \mid \text{if } i > 0 \text{ and } w(i) < 0\} \sqcup \{\varepsilon_j - \varepsilon_i \mid \text{if } 0 < i < j \text{ and } w(i) > w(j)\} \\ &\quad \sqcup \{\varepsilon_j + \varepsilon_i \mid \text{if } 0 < i < j \text{ and } -w(i) > w(j)\}. \end{aligned} \tag{3.3}$$

The chambers are the connected components of $\mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha \in R^+} \mathfrak{h}^\alpha$, where $\mathfrak{h}^\alpha = \{\gamma \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(\gamma) = 0\}$. The fundamental chamber in $\mathfrak{h}_{\mathbb{R}}$ is

$$C = \{\mathbf{c} \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(\gamma) \in \mathbb{R}_{>0} \text{ for } \alpha \in R^+\} = \{(c_1, \dots, c_k) \in \mathbb{R}^k \mid 0 < c_1 < c_2 < \dots < c_k\},$$

and the group \mathcal{W}_0 can be identified with the set of chambers via the bijection

$$\begin{array}{ccc} \mathcal{W}_0 & \longleftrightarrow & \{\text{chambers}\} \\ w & \longmapsto & w^{-1}C \end{array}. \quad \text{Since } w^{-1}C = \left\{ \mathbf{c} \in \mathfrak{h}_{\mathbb{R}} \mid \begin{array}{l} \alpha(\mathbf{c}) \in \mathbb{R}_{<0} \text{ if } \alpha \in R(w) \text{ and} \\ \alpha(\mathbf{c}) \in \mathbb{R}_{>0} \text{ if } \alpha \in R^+ \setminus R(w) \end{array} \right\},$$

the set $R(w)$ determines w .

3.1 Local regions

For $\gamma = (\gamma_1, \dots, \gamma_k) \in (\mathbb{C}^\times)^k$, define

$$\begin{aligned} Z(\gamma) &= \{\varepsilon_i \mid \gamma_i = \pm 1\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, \gamma_i \gamma_j^{-1} = 1\} \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, \gamma_i \gamma_j = 1\}, \\ P(\gamma) &= \{\varepsilon_i \mid \gamma_i \in \{(t_0^{\frac{1}{2}} t_k^{\frac{1}{2}})^{\pm 1}, (-t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}})^{\pm 1}\}\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, \gamma_i \gamma_j^{-1} = t^{\pm 1}\} \\ &\quad \sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, \gamma_i \gamma_j = t^{\pm 1}\}. \end{aligned} \tag{3.4}$$

Using the conversion from γ_i to c_i as in (2.32), let

$$\gamma_i = -t^{c_i}, \quad \text{and set } -t^{r_1} = -t_k^{\frac{1}{2}} t_0^{-\frac{1}{2}} \quad \text{and} \quad -t^{r_2} = t_k^{\frac{1}{2}} t_0^{\frac{1}{2}}, \tag{3.5}$$

so that $-t^{\pm r_1}$ and $-t^{\pm r_2}$ are the eigenvalues of W_1 that cause τ_0^2 to have a nonzero kernel (see (2.42)). Then, for $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{C}^k$ let $c_{-i} = -c_i$ and define

$$\begin{aligned} Z(\mathbf{c}) = \{ \varepsilon_i \mid c_i = 0 \} \sqcup \{ \varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = 0 \} \\ \sqcup \{ \varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = 0 \}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} P(\mathbf{c}) = \{ \varepsilon_i \mid c_i \in \{ \pm r_1, \pm r_2 \} \} \sqcup \{ \varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = \pm 1 \} \\ \sqcup \{ \varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = \pm 1 \}. \end{aligned} \quad (3.7)$$

A *local region* is a pair (\mathbf{c}, J) with $\mathbf{c} \in \mathbb{C}^k$ and $J \subseteq P(\mathbf{c})$. The set of *standard tableaux* of shape (\mathbf{c}, J) is

$$\mathcal{F}^{(\mathbf{c}, J)} = \{ w \in \mathcal{W}_0 \mid R(w) \cap Z(\mathbf{c}) = \emptyset, R(w) \cap P(\mathbf{c}) = J \}. \quad (3.8)$$

As in [Ra2, §5 and §8] the local regions (\mathbf{c}, J) and standard tableaux $w \in \mathcal{F}^{(\mathbf{c}, J)}$ can be converted to configurations of boxes κ and standard tableaux S of shape κ similar to those that are familiar in the literature on irreducible representations of Weyl groups of classical types. As explained in [Ra2, §5.11], the definitions of $Z(\mathbf{c})$ and $P(\mathbf{c})$ make it possible to view the general case $\mathbf{c} \in \mathbb{C}^k$ as pieced together from the cases $\mathbf{c} \in (\mathbb{Z} + \beta)^k$ where β runs over a set of representatives of the \mathbb{Z} -cosets in \mathbb{C} . Below we make the conversion between local regions and configurations of boxes explicit for the cases when $\mathbf{c} \in \mathbb{Z}^k$ and $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$. These are the cases that appear in the Schur-Weyl duality approach to the representations of H_k^{ext} explored in Section 5. As in [Ra2, §8], it is also true that these cases are sufficient to determine the general $\mathbf{c} \in (\mathbb{Z} + \beta)^k$ setting.

Let (\mathbf{c}, J) be a local region with $\mathbf{c} = (c_1, \dots, c_k)$,

$$\mathbf{c} \in \mathbb{Z}^k \quad \text{or} \quad \mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k, \quad \text{and} \quad 0 \leq c_1 \leq \dots \leq c_k. \quad (3.9)$$

Start with an infinite arrangement of NW to SE diagonals, numbered consecutively from \mathbb{Z} or $\mathbb{Z} + \frac{1}{2}$, increasing southwest to northeast (see Example 1). The *configuration* κ of boxes corresponding to the local region (\mathbf{c}, J) has $2k$ boxes (labeled $\text{box}_{-k}, \dots, \text{box}_{-1}, \text{box}_1, \dots, \text{box}_k$) with the following conditions.

($\kappa 1$) Location: box_i is on diagonal c_i , where $c_{-i} = -c_i$ for $i \in \{-k, \dots, -1\}$.

($\kappa 2$) Same diagonals: box_i is NW of box_j if $i < j$ and box_i and box_j are on the same diagonal.

($\kappa 3$) Adjacent diagonals:

If $\varepsilon_j - \varepsilon_i \in J$, then box_j is NW (strictly north and weakly west) of box_i : $\begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array}$

If $\varepsilon_j - \varepsilon_i \in P(\mathbf{c}) - J$, then box_j is SE (weakly south and strictly east) of box_i : $\begin{array}{|c|c|} \hline i & j \\ \hline \end{array}$

($\kappa 4$) Markings: There is a marking on each of the diagonals $r_1, -r_1, r_2$ and $-r_2$.

If $\varepsilon_i \in J$, box_i is NW of the marking on diagonal c_i : $\begin{array}{|c|} \hline i \\ \hline \end{array} \bullet$

If $\varepsilon_i \in P(\mathbf{c}) - J$, then box_i is SE of the marking in diagonal c_i : $\bullet \begin{array}{|c|} \hline i \\ \hline \end{array}$

Condition ($\kappa 1$) enables the values $(c_{-k}, \dots, c_{-1}, c_1, \dots, c_k)$ to be read off of configuration κ . The

sets $Z(\mathbf{c})$, $P(\mathbf{c})$, and J can also be determined from the configuration κ since

$$\begin{aligned}
Z(\mathbf{c}) &= \{\varepsilon_i \mid 0 < i \text{ and } \text{box}_i \text{ is in diagonal } 0\} \\
&\sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } \text{box}_i \text{ and } \text{box}_j \text{ are in the same diagonal}\} \\
&\sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } \text{box}_i \text{ and } \text{box}_j \text{ are both in diagonal } 0\}, \\
P(\mathbf{c}) &= \{\varepsilon_i \mid 0 < i \text{ and } \text{box}_i \text{ is in diagonal } r_1 \text{ or } r_2\}, \\
&\sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } \text{box}_i \text{ and } \text{box}_j \text{ are in adjacent diagonals}\} \\
&\sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } \text{box}_{-i} \text{ and } \text{box}_j \text{ are in adjacent diagonals}\}, \quad \text{and} \\
J &= \{\varepsilon_i \in P(\mathbf{c}) \mid \text{box}_i \text{ is NW of the marking}\} \\
&\sqcup \{\varepsilon_j - \varepsilon_i \in P(\mathbf{c}) \mid \text{box}_j \text{ is northwest of } \text{box}_i\} \\
&\sqcup \{\varepsilon_j + \varepsilon_i \in P(\mathbf{c}) \mid \text{box}_j \text{ is northwest of } \text{box}_{-i}\}.
\end{aligned}$$

A *standard filling* of the boxes of κ is a bijective function $S: \kappa \rightarrow \{-k, \dots, -1, 1, \dots, k\}$ such that

(S1) Symmetry: $S(\text{box}_{-i}) = -S(\text{box}_i)$.

(S2) Same diagonals:

If $0 < i < j$ and box_i and box_j are on the same diagonal then $S(\text{box}_i) < S(\text{box}_j)$.

(S3) Adjacent diagonals:

If $0 < i < j$, box_i and box_j are on adjacent diagonals, and box_j is NW of box_i , then $S(\text{box}_j) < S(\text{box}_i)$.

If $0 < i < j$, box_i and box_j are on adjacent diagonals, and box_j is SE of box_i , then $S(\text{box}_j) > S(\text{box}_i)$.

(S4) Markings:

If box_i is on a marked diagonal and is SE of the marking, then $S(\text{box}_i) > 0$.

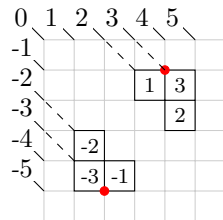
If box_i is on a marked diagonal and is NW of the marking, then $S(\text{box}_i) < 0$.

The *identity filling* of a configuration κ is the filling F of the boxes of κ given by $F(\text{box}_i) = i$, for $i = -k, \dots, -1, 1, \dots, k$. The identity filling of κ is usually not a standard filling of κ (see Example 1).

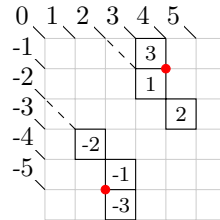
Example 1. Let $k = 4$, $r_1 = 1$, and $r_2 = 3$. Consider $\mathbf{c} = (-3, -2, -2, 2, 2, 3)$. Then

$$Z(\mathbf{c}) = \{\varepsilon_2 - \varepsilon_1\} \quad \text{and} \quad P(\mathbf{c}) = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.$$

The box configurations corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ and $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$ (filled with their identity fillings) are

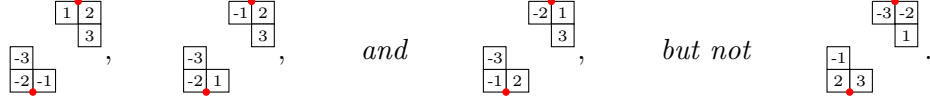


$$J = \{\varepsilon_3 - \varepsilon_2\}$$



$$J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$$

For both configurations, the identity filling is not a standard filling. Examples of standard fillings of the configuration corresponding to $J = \{\varepsilon_3 - \varepsilon_2\}$ include



The proof of the following proposition is a straightforward, though slightly tedious, check that the conditions $R(w) \cap Z(\mathbf{c}) = \emptyset$ and $R(w) \cap P(\mathbf{c}) = J$ from (3.8) convert to the conditions (S2), (S3), (S4) on standard fillings of shape κ . The proof is similar to the proof of [Ra2, Thm. 5.9].

Proposition 3.1. *Let κ be a configuration of boxes corresponding to a local region (\mathbf{c}, J) with $\mathbf{c} \in \mathbb{Z}^k$ or $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$. For $w \in \mathcal{W}_0$ let S_w be the filling of the boxes of κ given by*

$$S_w(\text{box}_i) = w(i), \quad \text{for } i = -k, \dots, -1, 1, \dots, k.$$

The map

$$\begin{array}{ccc} \mathcal{F}^{(\mathbf{c}, J)} & \longrightarrow & \{\text{standard fillings } S \text{ of the boxes of } \kappa\} \\ w & \longmapsto & S_w \end{array} \quad \text{is a bijection.}$$

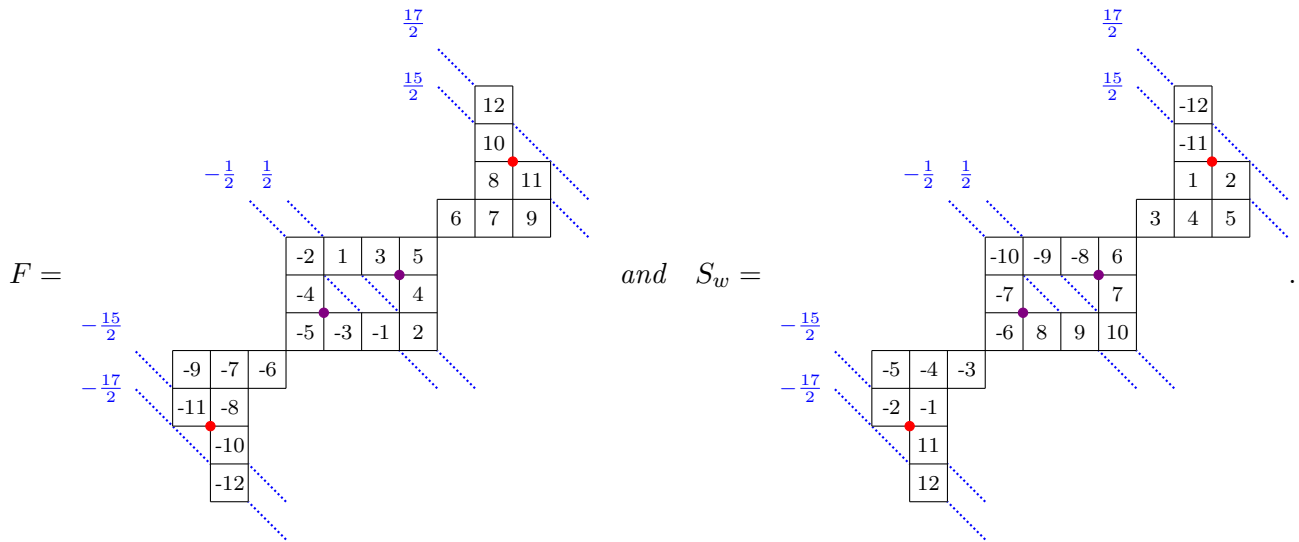
Example 2. Let $k = 12$, $r_1 = \frac{3}{2}$, $r_2 = \frac{15}{2}$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{13}{2}, \frac{15}{2}, \frac{15}{2}, \frac{17}{2})$ and

$$J = \left\{ \begin{array}{l} \varepsilon_3, \varepsilon_{10}, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_8 - \varepsilon_7, \\ \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{10}, \varepsilon_{12} - \varepsilon_{11} \end{array} \right\}$$

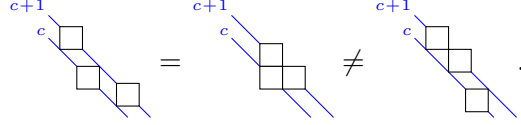
Let

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ -9 & 10 & -8 & 7 & 6 & 3 & 4 & 1 & 5 & -11 & 2 & -12 \end{pmatrix} \in \mathcal{F}^{(\mathbf{c}, J)}.$$

Then, for the corresponding configuration of boxes κ , the identity filling F , and the standard filling S_w corresponding to w are

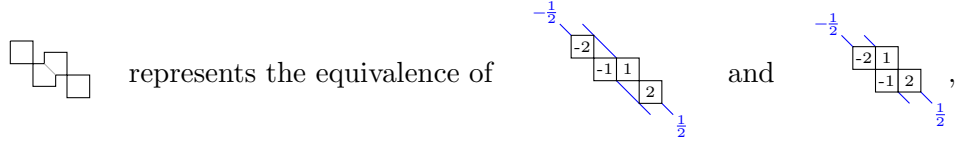


Remark 3.2. Borrowing a physical intuition, configurations are invariant under sliding boxes along diagonals like beads on an abacus, so long as boxes that run into each other are not allowed to exchange places, i.e. for most $c \in \mathbb{Z}$,

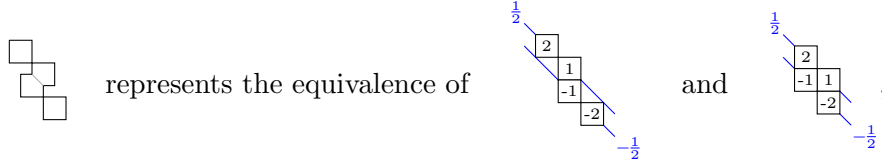


Then by arranging configurations so that the boxes are packed together, standard fillings of configurations are exactly analogous to standard tableaux for partitions.

The only exception to this physical intuition is for boxes on the diagonals $\pm \frac{1}{2}$. Note that if $c_i = \frac{1}{2}$, then box_i and box_{-i} are on adjacent diagonals. However, since $2\varepsilon_i = \varepsilon_i - \varepsilon_{-i} \notin R^+$ and therefore never in $P(\mathbf{c})$, the relative positions of box_i and box_{-i} will never be recorded in the set J . For example, in Figure 2, the point where $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$ has two configurations, each with two boxes overlapping in indication that box_i and box_{-i} may “slide past each other”. The drawing



(with boxes filled in the identity filling) where box_1 and box_{-1} can move freely past each other, and



where box_2 and box_{-2} can move freely past each other. In these two examples $\varepsilon_1 - \varepsilon_{-2} \in P(\mathbf{c})$ and $\varepsilon_2 - \varepsilon_{-1} \in P(\mathbf{c})$ and so the relative orientation of box_2 and box_{-1} and the relative orientation of box_1 and box_{-2} are recorded in J . Each configuration has exactly two standard fillings.

3.2 Classifying and constructing calibrated representations

Theorem 3.3 below provides an indexing of the calibrated irreducible H_k^{ext} -modules by skew local regions. A *skew local region* is a local region (\mathbf{c}, J) , $\mathbf{c} = (c_1, \dots, c_k)$, such that

if $w \in \mathcal{F}(\mathbf{c}, J)$ then $w\mathbf{c} = ((w\mathbf{c})_1, \dots, (w\mathbf{c})_n)$ satisfies

$$(w\mathbf{c})_1 \neq 0, \quad (w\mathbf{c})_2 \neq 0, \quad (w\mathbf{c})_1 \neq -(w\mathbf{c})_2, \tag{3.10}$$

$$(w\mathbf{c})_i \neq (w\mathbf{c})_{i+1} \text{ for } i = 1, \dots, k-1, \quad \text{and} \quad (w\mathbf{c})_i \neq (w\mathbf{c})_{i+2} \text{ for } i = 1, \dots, k-2.$$

Theorem 3.3 is completely analogous to the same theorem for the case $t^{\frac{1}{2}} = t_0^{\frac{1}{2}} = t_k^{\frac{1}{2}}$ in [Ra2, Theorem 3.5]. As explained in the discussion and remarks before [Ra2, Lemma 3.1] in [Ra2, §3], getting exactly the right definition of skew local region for the purpose of Theorem 3.3 is accomplished by a detailed computation of the irreducible representations in rank two cases. More specifically, for

$I \subseteq \{0, \dots, k\}$, let H_I be the subalgebra of H_k^{ext} generated by $\{T_i\}_{i \in I}$ and $\mathbb{C}[W_1^{\pm 1}, \dots, W_k^{\pm 1}]$. Then the conditions in (3.10) guarantee that for $w \in \mathcal{F}^{(\mathbf{c}, J)}$ and $i, j \in \{0, 1, \dots, k-1\}$,

there exists a calibrated $H_{\{i, j\}}$ -module M with $M_{w\mathbf{c}}^{\text{gen}} \neq 0$.

The cases where $H_{\{i, j\}}$ is of type $A \times A_1$ or of type A_2 are checked in [Ra1]. However, when $H_{\{i, j\}}$ is of type C_2 and there are *three distinct parameters*, we do not know a reference for this. So in the effort to provide a more complete presentation, we have done the appropriate analysis in Section 4 for all generic choices of the three parameters $t^{\frac{1}{2}}$, $t_0^{\frac{1}{2}}$, and $t_k^{\frac{1}{2}}$, as given in the following theorem (see also (4.1))

Theorem 3.3. *Assume $t^{\frac{1}{2}}$, $t_0^{\frac{1}{2}}$, and $t_k^{\frac{1}{2}}$ are invertible, $t^{\frac{1}{2}}$ is not a root of unity, and*

$$t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}, -t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}} \notin \{1, -1, t^{\pm \frac{1}{2}}, -t^{\pm \frac{1}{2}}, t^{\pm 1}, -t^{\pm 1}\} \quad \text{and} \quad t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} \neq (-t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}})^{\pm 1}.$$

(a) *Let (\mathbf{c}, J) be a skew local region and let $z \in \mathbb{C}^\times$. Define*

$$H_k^{(z, \mathbf{c}, J)} = \text{span}_{\mathbb{C}}\{v_w \mid w \in \mathcal{F}^{(\mathbf{c}, J)}\}, \quad (3.11)$$

so that the symbols v_w are a labeled basis of the vector space $H_k^{(z, \mathbf{c}, J)}$. Let

$$\gamma_i = -t^{c_i} \quad \text{for } i = 1, 2, \dots, k, \quad \text{and} \quad \gamma_0 = z \gamma_{w^{-1}(1)}^{-1} \cdots \gamma_{w^{-1}(k)}^{-1}.$$

Then the following formulas make $H_k^{(z, \mathbf{c}, J)}$ into an irreducible H_k^{ext} -module:

$$PW_1 \cdots W_k v_w = z v_w, \quad Pv_w = \gamma_0 v_w, \quad W_i v_w = \gamma_{w^{-1}(i)} v_w, \quad (3.12)$$

$$T_i v_w = [T_i]_{ww} v_w + \sqrt{-([T_i]_{ww} - t^{\frac{1}{2}})([T_i]_{ww} + t^{-\frac{1}{2}})} v_{s_i w}, \quad \text{for } i = 1, \dots, k-1, \quad (3.13)$$

$$T_0 v_w = [T_0]_{ww} v_w + \sqrt{-([T_0]_{ww} - t_0^{\frac{1}{2}})([T_0]_{ww} + t_0^{-\frac{1}{2}})} v_{s_0 w}, \quad (3.14)$$

where $v_{s_i w} = 0$ if $s_i w \notin \mathcal{F}^{(\mathbf{c}, J)}$, and

$$[T_i]_{ww} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)} \gamma_{w^{-1}(i+1)}^{-1}} \quad \text{and} \quad [T_0]_{ww} = \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) \gamma_{w^{-1}(1)}^{-1}}{1 - \gamma_{w^{-1}(1)}^{-2}}. \quad (3.15)$$

(b) *The map*

$$\begin{array}{ccc} \mathbb{C}^\times \times \{\text{skew local regions } (\mathbf{c}, J)\} & \longleftrightarrow & \{\text{irreducible calibrated } H_k^{\text{ext}}\text{-modules}\} \\ (z, \mathbf{c}, J) & \longmapsto & H_k^{(z, \mathbf{c}, J)} \end{array}$$

is a bijection.

Proof. This result follows from [Ra2, Theorems 3.2 and 3.5]. It is only necessary to establish that the formulas in (3.12), (3.13), and (3.14) are correct. These are derived in a similar manner to [Ra2,

Proposition 3.3] as follows. As in [Ra2, Theorem 3.2], if M is an irreducible calibrated H_k^{ext} -module then

$$M = \bigoplus_{w \in \mathcal{W}_0} M_{w\gamma}^{\text{gen}}, \quad \text{with } \dim(M_{w\gamma}^{\text{gen}}) = 1 \text{ if } M_{w\gamma}^{\text{gen}} \neq 0.$$

For $w \in \mathcal{W}_0$, if $M_{w\gamma}^{\text{gen}} \neq 0$, let v_w be a nonzero vector in $M_{w\gamma}^{\text{gen}}$; otherwise if $M_{w\gamma}^{\text{gen}} = 0$, let $v_w = 0$. By (2.38), $\tau_i v_w = [T_i]_{s_i w, w} v_{s_i w}$ for some constant $[T_i]_{s_i w, w}$ and the definition of τ_i in (2.35) gives that

$$T_i v_w = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)} \gamma_{w^{-1}(i+1)}^{-1}} v_w + [T_i]_{s_i w, w} v_{s_i w} \quad \text{for } i = 1, \dots, k, \quad (3.16)$$

and

$$T_0 v_\gamma = \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) \gamma_{w^{-1}(1)}^{-1}}{1 - \gamma_{w^{-1}(1)}^{-2}} v_w + [T_0]_{s_0 w, w} v_{s_0 w}. \quad (3.17)$$

Thus T_0 is an operator on the subspace $\text{span}_{\mathbb{C}}\{v_w, v_{s_0 w}\}$ satisfying $(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0$ by (H). Restricting to the action on $\text{span}_{\mathbb{C}}\{v_w, v_{s_0 w}\}$, the formulas in (3.14) now follow from the following argument about general 2×2 matrices.

If a 2×2 matrix $[T_0]$ has eigenvalues α_1 and α_2 ,

$$[T_0] = \begin{pmatrix} [T_0]_{ww} & [T_0]_{w, s_0 w} \\ [T_0]_{s_0 w, w} & [T_0]_{s_0 w, s_0 w} \end{pmatrix}, \quad \text{then} \quad ([T_0] - \alpha_1)([T_0] - \alpha_2) = 0$$

is the characteristic polynomial for $[T_0]$, and it follows that

$$\begin{aligned} \text{Tr}([T_0]) &= [T_0]_{ww} + [T_0]_{s_0 w, s_0 w} = \alpha_1 + \alpha_2, & \text{and} \\ \det([T_0]) &= [T_0]_{ww} [T_0]_{s_0 w, s_0 w} - [T_0]_{w, s_0 w} [T_0]_{s_0 w, w} = \alpha_1 \alpha_2. \end{aligned}$$

Thus

$$\begin{aligned} -[T_0]_{w, s_0 w} [T_0]_{s_0 w, w} &= \alpha_1 \alpha_2 - [T_0]_{ww} [T_0]_{s_0 w, s_0 w} = \alpha_1 \alpha_2 - [T_0]_{ww} ((\alpha_1 + \alpha_2) - [T_0]_{ww}) \\ &= \alpha_1 \alpha_2 - (\alpha_1 + \alpha_2) [T_0]_{ww} + ([T_0]_{ww})^2 = ([T_0]_{ww} - \alpha_1)([T_0]_{ww} - \alpha_2). \end{aligned}$$

Choosing a normalization of $v_{s_0 w}$ so that the matrix of $[T_0]$ is symmetric, we have $[T_0]_{w, s_0 w} = [T_0]_{s_0 w, w}$ and

$$[T_0]_{s_0 w, w} = \sqrt{([T_0]_{s_0 w, w})^2} = \sqrt{[T_0]_{w, s_0 w} [T_0]_{s_0 w, w}} = \sqrt{-([T_0]_{ww} - \alpha_1)([T_0]_{ww} - \alpha_2)}.$$

□

4 Classification of irreducible representations of H_2

In this section we do a complete classification of the irreducible representations of the algebra H_2^{ext} . An important reason for doing this classification of H_2^{ext} representations is to provide a sound basis for the definition of a skew local region (see the remarks immediately after the definition of skew local region in (3.10)). The classification and construction of calibrated representations of

H_k^{ext} in terms of skew local regions in Theorem 3.3 is important for determining the irreducible representations of H_k^{ext} that arise in the Schur-Weyl duality framework (see Theorem 5.5). We will do the classification of irreducible H_2^{ext} representations under **genericity assumptions on the parameters**: $t^{\frac{1}{2}}$ is not a root of unity and

$$t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}, -t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}} \notin \{1, -1, t^{\pm\frac{1}{2}}, -t^{\pm\frac{1}{2}}, t^{\pm 1}, -t^{\pm 1}\} \quad \text{and} \quad t_0^{\frac{1}{2}}t_k^{\frac{1}{2}} \neq (-t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}})^{\pm 1}. \quad (4.1)$$

More specifically, this condition is used for the (rank 2) computation in equation (4.4). Similar methods apply to the nongeneric cases but the final classification needs to be stated differently and we will not treat the nongeneric cases here. The nongeneric case $t_0^{\frac{1}{2}} = t_k^{\frac{1}{2}} = t^{\frac{1}{2}}$ is done in [Ra1, Ra2] and [Re]; the case where $t_0^{\frac{1}{2}} = t_k^{\frac{1}{2}} \neq t^{\frac{1}{2}}$ appears in [En] (see also [KR]).

The algebra H_2 is generated by $W_1^{\pm 1}, W_2^{\pm 1}, T_0$, and T_1 , and the Weyl group \mathcal{W}_0 is generated by s_0 and s_1 with relations $s_i^2 = 1$ and $s_0s_1s_0s_1 = s_1s_0s_1s_0$. By (2.29),

$$H_2^{\text{ext}} = \mathbb{C}[W_0^{\pm}] \otimes H_2 \quad \text{as algebras,}$$

and therefore it is sufficient to do the classification of irreducible representations of H_2 . This is because all irreducible representations of $\mathbb{C}[W_0^{\pm 1}]$ are one dimensional and determined by the image of W_0 ; and all irreducible representations of H_2^{ext} are the tensor product of an irreducible representation of $\mathbb{C}[W_0^{\pm 1}]$ and an irreducible representation of H_2 .

The group \mathcal{W}_0 acts on $(\mathbb{C}^\times)^2$ by

$$s_0(\gamma_1, \gamma_2) = (\gamma_1^{-1}, \gamma_2) \quad \text{and} \quad s_1(\gamma_1, \gamma_2) = (\gamma_2, \gamma_1). \quad (4.2)$$

By (2.35), the intertwiners are

$$\tau_0 = T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}} \quad \text{and} \quad \tau_1 = T_1 - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - W_1W_2^{-1}}.$$

4.1 Classification of central characters

Following [Ra1, §5], the classification of irreducible H_k^{ext} -modules begins with a classification of possible pairs $(Z(\mathbf{c}), P(\mathbf{c})) = (Z(\gamma), P(\gamma))$ (where γ and \mathbf{c} are related as in (2.32)). It is straightforward (though slightly tedious) to enumerate all the possibilities by taking note of the following:

- (0) Since $(Z(w\gamma), P(w\gamma)) = (wZ(\gamma), wP(\gamma))$, it is sufficient to do the analysis for a single representative γ of each \mathcal{W}_0 -orbit on $(\mathbb{C}^\times)^k$.
- (1) The \mathcal{W}_0 -orbits of roots are $\{\pm\varepsilon_1, \pm\varepsilon_2\}$ and $\{\pm(\varepsilon_2 \pm \varepsilon_1)\}$, and our preferred representative of the \mathcal{W}_0 -orbit will have ε_1 or $\varepsilon_2 - \varepsilon_1$ in $Z(\gamma)$ if $Z(\gamma) \neq \emptyset$.
- (2) If $Z(\gamma) = \emptyset$ and $P(\gamma) \neq \emptyset$ then our preferred representative of the \mathcal{W}_0 -orbit will have ε_1 or $\varepsilon_2 - \varepsilon_1$ in $Z(\gamma)$.

With these preferences, the classification of $(Z(\gamma), P(\gamma))$ is accomplished by noting that

- (a) if $\gamma \in \{(1, 1), (-1, -1)\}$ then $(Z(\gamma), P(\gamma)) = (\{\varepsilon_1, \varepsilon_2, \varepsilon_2 \pm \varepsilon_1\}, \emptyset)$;

- (b) if $\gamma \in \{(1, -1), (-1, 1)\}$ then $(Z(\gamma), P(\gamma)) = (\{\varepsilon_1, \varepsilon_2\}, \emptyset)$;
- (c) $\varepsilon_2 - \varepsilon_1 \in Z(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_1)$;
- (d) $\varepsilon_2 + \varepsilon_1 \in Z(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_1^{-1})$;
- (e) $\varepsilon_1 \in Z(\gamma)$ if and only if $\gamma = (1, \gamma_2)$ or $\gamma = (-1, \gamma_2)$;
- (f) $\varepsilon_2 \in Z(\gamma)$ if and only if $\gamma = (\gamma_1, 1)$ or $\gamma = (\gamma_1, -1)$;
- (g) $\varepsilon_1 \in P(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 \in \{t_0^{\frac{1}{2}}t_k^{\frac{1}{2}}, -t_0^{-\frac{1}{2}}t_k^{\frac{1}{2}}, -t_0^{\frac{1}{2}}t_k^{-\frac{1}{2}}, t_0^{-\frac{1}{2}}t_k^{-\frac{1}{2}}\}$;
- (h) $\varepsilon_2 - \varepsilon_1 \in P(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_2 = \gamma_1 t^{\pm 1}$;
- (i) $\varepsilon_2 + \varepsilon_1 \in P(\gamma)$ if and only if $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 \gamma_2 = t^{\pm 1}$.

We shall freely use the conversion between $\gamma = (\gamma_1, \gamma_2)$ and $\mathbf{c} = (c_1, c_2)$ given by (2.32),

$$\gamma_1 = -t^{c_1}, \quad \gamma_2 = -t^{c_2}, \quad \text{and write } (Z(\mathbf{c}), P(\mathbf{c})) = (Z(\gamma), P(\gamma)).$$

Representatives of the 12 possible $(Z(\mathbf{c}), P(\mathbf{c}))$ with $Z(\mathbf{c}) = \emptyset$ are displayed in Figure 1. Representatives of the 9 possible $(Z(\mathbf{c}), P(\mathbf{c}))$ with $Z(\mathbf{c}) \neq \emptyset$ are displayed in Figure 2. It works out that, in each case, the pair $(Z(\mathbf{c}), P(\mathbf{c}))$ is attained by an element \mathbf{c} that has real coordinates (the one complex character in the equal parameter case that behaves differently from the real characters, namely the point t_b in [Ra1, Figure 5.1], does not appear in the generic unequal parameter case assumed in (4.1)).

With notation as at the beginning of Section 3, in Figures 1 and 2, the fundamental region C is the shaded area, the solid lines are the hyperplanes \mathfrak{h}^α for $\alpha \in R^+$, and the dotted hyperplanes are labeled by the equation that defines them. If $\mathbf{c} = (c_1, c_2) \in C$, so that $0 \leq c_1 \leq c_2$, then

$$Z(\mathbf{c}) = \{\text{solid hyperplanes through } \mathbf{c}\} \quad \text{and} \quad P(\mathbf{c}) = \{\text{dotted hyperplanes through } \mathbf{c}\}.$$

The bijection

$$\begin{array}{ccc} \mathcal{W}_0 & \leftrightarrow & \{\text{chambers}\} \\ w & \mapsto & w^{-1}C \end{array} \quad \text{identifies each } \mathcal{F}^{(\mathbf{c}, J)} \text{ with a set of chambers,} \quad (4.3)$$

a *local region* in $\mathfrak{h}_{\mathbb{R}}^*$. As illustrated by the example at the bottom right of Figures 1 and 2, $\mathcal{F}^{(\mathbf{c}, J)}$ is identified with the set of chambers that are on the negative side of the hyperplanes in J and on the positive side of the hyperplanes in $P(\mathbf{c}) - J$. For each (\mathbf{c}, J) the corresponding configuration of boxes κ is displayed in the local region of chambers corresponding to the elements of $\mathcal{F}^{(\mathbf{c}, J)}$ by (4.3). In Figure 1, only the boxes on positive diagonals are shown, since they determine the entire doubled configuration when $Z(\mathbf{c}) = \emptyset$. The diagram at the bottom right of each figure gives an example of the correspondence between chambers corresponding to $\mathcal{F}^{(\mathbf{c}, J)}$, the elements of $\mathcal{F}^{(\mathbf{c}, J)}$, and the standard fillings of the corresponding configuration of boxes κ : the point $\mathbf{c} = (r_1 - 1, r_1)$ in the bottom right of Figure 1, and the point $\mathbf{c} = (0, 1)$ in the bottom right of Figure 2.

In Figure 2, the small graphs nearby each marked $\mathbf{c} = (c_1, c_2)$ indicate the structure (generalized weight spaces and intertwiner maps) of the irreducible modules M of central character \mathbf{c} . This structure is determined below in Section 4.2. There is a vertex in the chamber $w^{-1}C$ for each element of a basis of $M_{w\mathbf{c}}^{\text{gen}}$ and there is an edge if the matrix of τ_i (or T_i if τ_i is not defined on $M_{w\mathbf{c}}^{\text{gen}}$) is nonzero in the entry corresponding to the two vertices that are connected.

Figure 1: Regular central characters in rank 2. See the description in Section 4.1

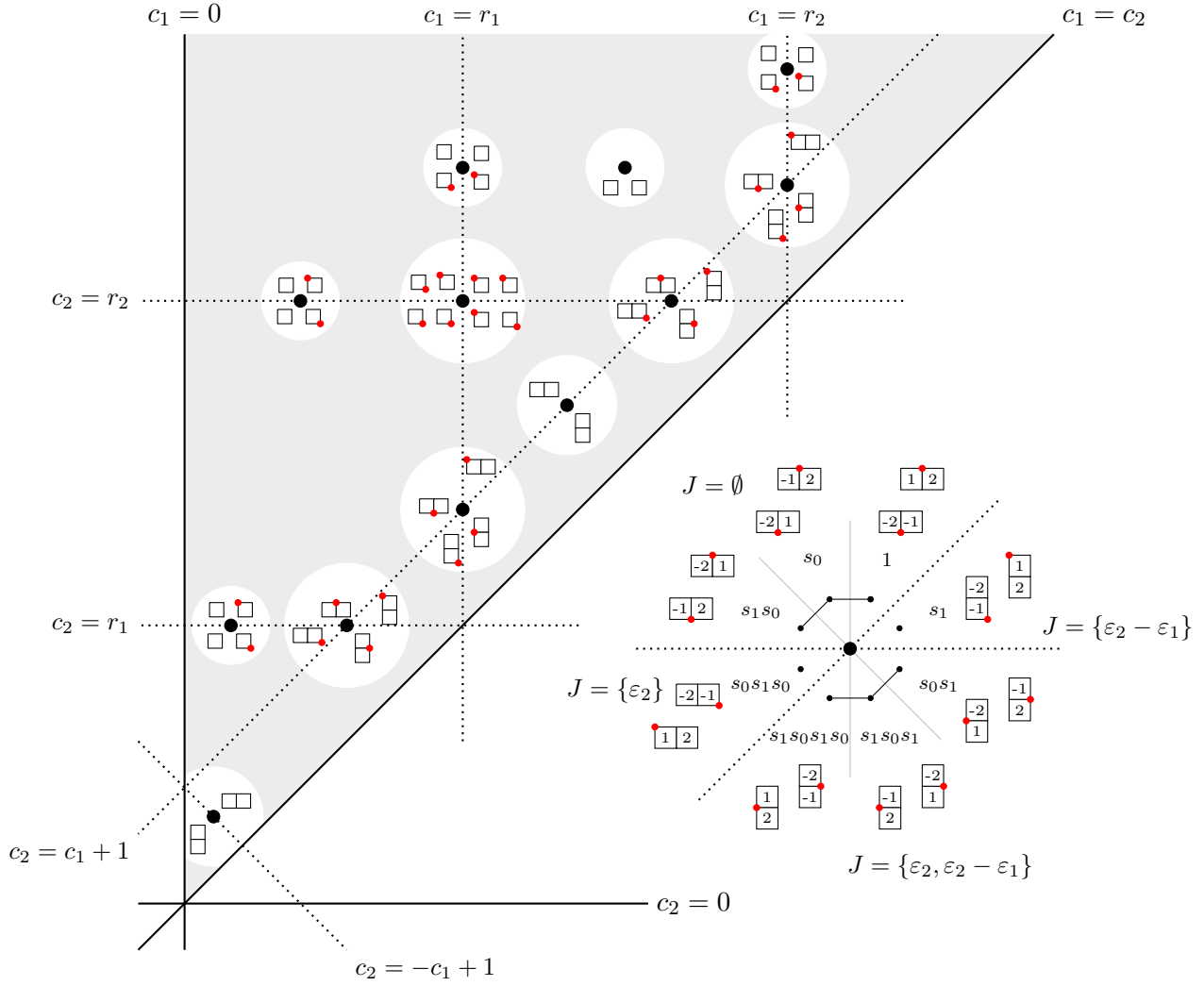
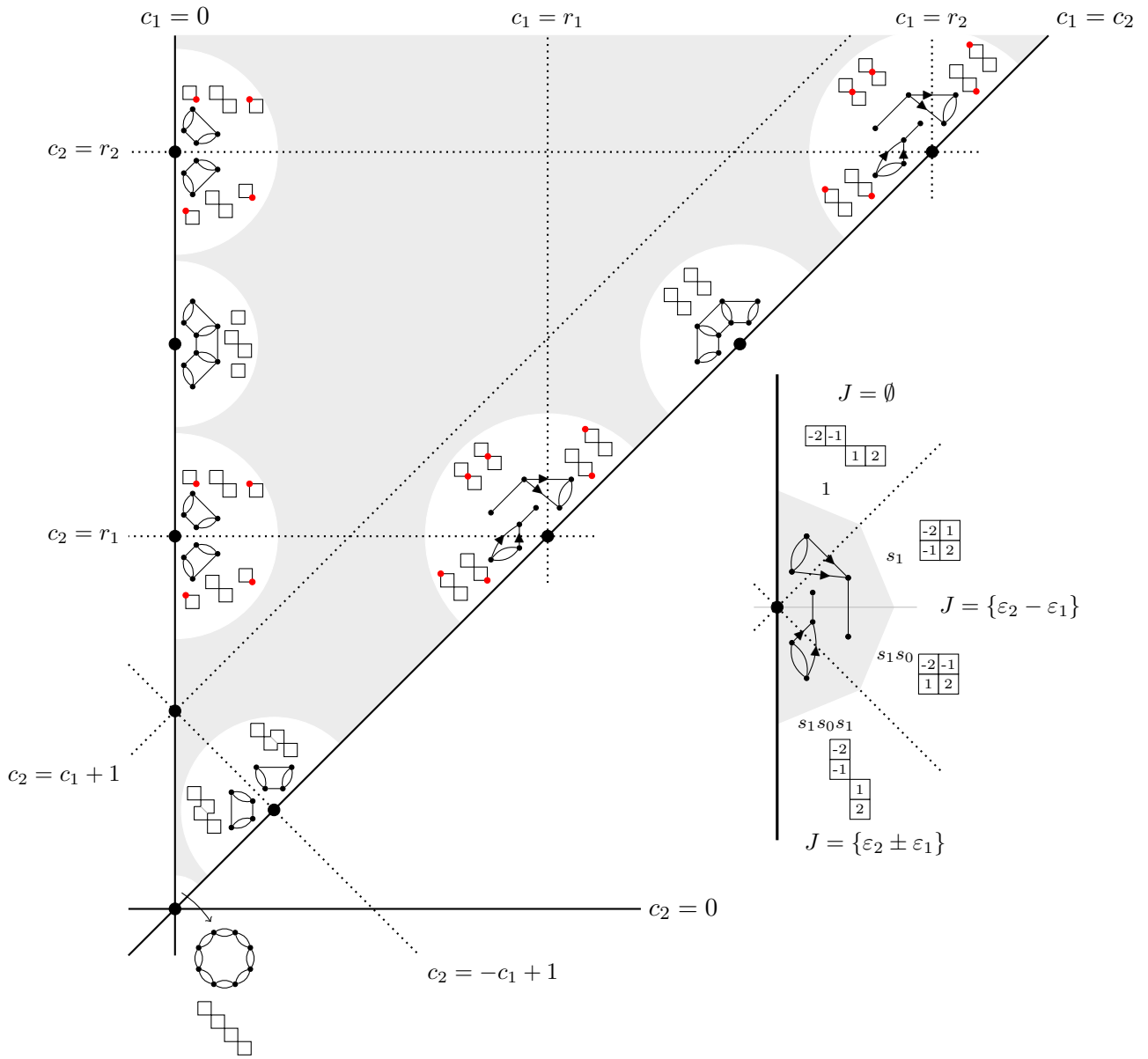


Figure 2: Non-regular points



4.2 Construction of the irreducible H_2 -modules

The group \mathcal{W}_0 acts on $(\mathbb{C}^\times)^2$ as in (4.2) and the *central characters* are the \mathcal{W}_0 -orbits on $(\mathbb{C}^\times)^2$. The *regular central characters* are the \mathcal{W}_0 -orbits of $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{C}^\times)^2$ that have $Z(\gamma) = \emptyset$, i.e. where the intertwining operators in (2.38) are defined. Let $\mathbb{C}[W] = \mathbb{C}[W_1^{\pm 1}, W_2^{\pm 1}] \subseteq H_2$. By Kato's criterion (see [Ra2, Proposition 2.11b]), for central characters $\gamma = (\gamma_1, \gamma_2)$ with $P(\gamma) = \emptyset$ there is a single irreducible module of dimension eight given by

$$L_{(\gamma_1, \gamma_2)} = \text{Ind}_{\mathbb{C}[W]}^H(\mathbb{C}_{\gamma_1, \gamma_2}), \quad \text{where } \mathbb{C}_{\gamma_1, \gamma_2} = \mathbb{C}v \text{ with } W_1v = \gamma_1v \text{ and } W_2v = \gamma_2v.$$

All irreducible modules with $Z(\gamma) = \emptyset$ are calibrated and can be constructed as in Theorem 3.3.

Representatives of the \mathcal{W}_0 -orbits of $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{C}^\times)^2$ that have $Z(\gamma) \neq \emptyset$ and $P(\gamma) \neq \emptyset$ are as follows:

$\gamma = (\gamma_1, \gamma_2)$	$Z(\gamma)$	$P(\gamma)$
$(t^{\frac{1}{2}}, t^{\frac{1}{2}}), (-t^{\frac{1}{2}}, -t^{\frac{1}{2}})$	$\{\varepsilon_2 - \varepsilon_1\}$	$\{\varepsilon_2 + \varepsilon_1\}$
$(t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}, t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}), (-t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}}, -t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}})$	$\{\varepsilon_2 - \varepsilon_1\}$	$\{\varepsilon_1, \varepsilon_2\}$
$(1, t), (-1, -t)$	$\{\varepsilon_1\}$	$\{\varepsilon_2 - \varepsilon_1, \varepsilon_2 + \varepsilon_1\}$
$(\pm 1, t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}), (\pm 1, -t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}})$	$\{\varepsilon_1\}$	$\{\varepsilon_2\}$

(4.4)

This classification is valid under the genericity assumption on the parameters (4.1), which guarantees that none of these representatives are in the \mathcal{W}_0 -orbit of another.

The following analysis of modules of central character $\gamma = (\gamma_1, \gamma_2)$ in (4.4) shows that no irreducible calibrated H_2 -modules appear at these central characters. As in (3.5), the values r_1 and r_2 are defined by

$$-t^{r_1} = -t^{\frac{1}{2}} t_0^{-\frac{1}{2}} \quad \text{and} \quad -t^{r_2} = t^{\frac{1}{2}} t_0^{\frac{1}{2}}.$$

Case $(\gamma_1, \gamma_2) = (-1, -t^{r_i})$ for $i = 1$ or 2 : Let $H_{\{0\}}$ be the subalgebra of H_2 generated by $T_0, W_1^{\pm 1}, W_2^{\pm 1}$. For each of $i = 1$ and $i = 2$, there are two irreducible modules of central character $\mathbf{c} = (0, r_i)$:

$$L_{(0, r_i)}^+ = \text{Ind}_{H_{\{0\}}}^{H_2}(\mathbb{C}_{(r_i, 0)}), \quad \text{where } \mathbb{C}_{(r_i, 0)} = \mathbb{C}v \text{ with } \begin{aligned} W_1v &= -t^{r_i}v, \\ W_2v &= -v, \\ T_0v &= t_0^{\frac{1}{2}}v, \end{aligned}$$

and

$$L_{(0, r_i)}^- = \text{Ind}_{H_{\{0\}}}^{H_2}(\mathbb{C}_{(-r_i, 0)}), \quad \text{where } \mathbb{C}_{(-r_i, 0)} = \mathbb{C}v \text{ with } \begin{aligned} W_1v &= -t^{-r_i}v, \\ W_2v &= -v, \\ T_0v &= -t_0^{-\frac{1}{2}}v. \end{aligned}$$

With $M = L_{(0, r_i)}^+$, the generalized weight space decomposition is

$$M = M_{(r_i, 0)}^{\text{gen}} \oplus M_{(0, r_i)}^{\text{gen}}, \quad \text{with } \dim(M_{(r_i, 0)}^{\text{gen}}) = \dim(M_{(0, r_i)}^{\text{gen}}) = 2. \quad (4.5)$$

The element $W_1W_2^{-1}$ acts on $M_{(r_i, 0)}^{\text{gen}}$ with eigenvalues t^{r_i} . Since the parameters are generic (see (4.1)), $t^{r_i} \neq t^{\pm 1}$ and thus, by (2.43), τ_1^2 has no kernel. Thus the intertwiner $\tau_1: M_{(r_i, 0)}^{\text{gen}} \rightarrow M_{(0, r_i)}^{\text{gen}}$ is

invertible and $M = L_{(0,r_i)}^+$ is irreducible. Replacing r_i with $-r_i$ in (4.5) yields the decomposition of $M = L_{(0,r_i)}^-$ analogously.

Case $(\gamma_1, \gamma_2) = (-t^{\frac{1}{2}}, -t^{\frac{1}{2}})$: Let $H_{\{1\}}$ be the subalgebra of H_2 generated by $T_1, W_1^{\pm 1}, W_2^{\pm 1}$. There are two irreducible modules of central character $\mathbf{c} = (\frac{1}{2}, \frac{1}{2})$:

$$L_{(\frac{1}{2}, \frac{1}{2})}^+ = \text{Ind}_{H_{\{1\}}}^{H_2} (\mathbb{C}_{(-\frac{1}{2}, \frac{1}{2})}), \quad \text{where } \mathbb{C}_{(-\frac{1}{2}, \frac{1}{2})} = \mathbb{C}v \text{ with} \quad \begin{aligned} W_1 v &= -t^{-\frac{1}{2}}v, \\ W_2 v &= -t^{\frac{1}{2}}v, \\ T_1 v &= t^{\frac{1}{2}}v, \end{aligned}$$

and

$$L_{(\frac{1}{2}, \frac{1}{2})}^- = \text{Ind}_{H_{\{1\}}}^{H_2} (\mathbb{C}_{(\frac{1}{2}, -\frac{1}{2})}), \quad \text{where } \mathbb{C}_{(\frac{1}{2}, -\frac{1}{2})} = \mathbb{C}v \text{ with} \quad \begin{aligned} W_1 v &= -t^{\frac{1}{2}}v, \\ W_2 v &= -t^{-\frac{1}{2}}v, \\ T_1 v &= -t^{-\frac{1}{2}}v. \end{aligned}$$

With $M = L_{(\frac{1}{2}, \frac{1}{2})}^+$, the generalized weight space decomposition is

$$M = M_{(\frac{1}{2}, \frac{1}{2})}^{\text{gen}} \oplus M_{(-\frac{1}{2}, \frac{1}{2})}^{\text{gen}}, \quad \text{with } \dim(M_{(\frac{1}{2}, \frac{1}{2})}^{\text{gen}}) = \dim(M_{(-\frac{1}{2}, \frac{1}{2})}^{\text{gen}}) = 2. \quad (4.6)$$

The element W_1^{-1} acts on $M_{(\frac{1}{2}, \frac{1}{2})}^{\text{gen}}$ with eigenvalues $-t^{\frac{1}{2}}$. Since the parameters are generic (see (4.1)), $-t^{\frac{1}{2}} \notin \{-t^{\pm r_1}, -t^{\pm r_2}\}$ and thus, by (2.42), τ_0^2 has no kernel. Thus the intertwiner $\tau_0: M_{(\frac{1}{2}, -\frac{1}{2})}^{\text{gen}} \rightarrow M_{(-\frac{1}{2}, -\frac{1}{2})}^{\text{gen}}$ is invertible and $M = L_{(\frac{1}{2}, \frac{1}{2})}^+$ is irreducible. Similarly, the structure of $M = L_{(\frac{1}{2}, \frac{1}{2})}^-$ is given by swapping $\frac{1}{2}$ and $-\frac{1}{2}$ in (4.6).

Case $(\gamma_1, \gamma_2) = (-t^{r_i}, -t^{r_i})$ **for** $i = 1$ **or** 2 : Let $H_{\{0\}}$ be the subalgebra of H_2 generated by $T_0, W_1^{\pm 1}, W_2^{\pm 1}$. For each of $i = 1$ and $i = 2$, there are two irreducible modules of central character $\mathbf{c} = (r_i, r_i)$:

$$L_{(r_i, r_i)}^+ = \text{Ind}_{H_{\{0\}}}^{H_2} (\mathbb{C}_{(r_i, -r_i)}), \quad \text{where } \mathbb{C}_{(r_i, -r_i)} = \mathbb{C}v \text{ with} \quad \begin{aligned} W_1 v &= -t^{r_i}v, \\ W_2 v &= -t^{-r_i}v, \\ T_0 v &= t_0^{\frac{1}{2}}v, \end{aligned}$$

and

$$L_{(r_i, r_i)}^- = \text{Ind}_{H_{\{0\}}}^{H_2} (\mathbb{C}_{(-r_i, r_i)}), \quad \text{where } \mathbb{C}_{(-r_i, r_i)} = \mathbb{C}v \text{ with} \quad \begin{aligned} W_1 v &= -t^{-r_i}v, \\ W_2 v &= -t^{r_i}v, \\ T_0 v &= -t_0^{-\frac{1}{2}}v. \end{aligned}$$

The irreducibility of $L_{(r_i, r_i)}^+$ and $L_{(r_i, r_i)}^-$ is not immediate. We will show that $M = L_{(r_i, r_i)}^+$ is irreducible; the irreducibility of $L_{(r_i, r_i)}^-$ is proved analogously.

The generalized weight space decomposition of $M = L_{(r_i, r_i)}^+$ is

$$M = M_{(r_i, -r_i)}^{\text{gen}} \oplus M_{(-r_i, r_i)}^{\text{gen}} \oplus M_{(r_i, r_i)}^{\text{gen}} \quad \text{with} \quad \begin{aligned} \dim(M_{(r_i, -r_i)}^{\text{gen}}) &= \dim(M_{(-r_i, r_i)}^{\text{gen}}) = 1, \\ \dim(M_{(r_i, r_i)}^{\text{gen}}) &= 2. \end{aligned}$$

The element $W_1 W_2^{-1}$ acts on $M_{(r_i, -r_i)}^{\text{gen}}$ with eigenvalue $t^{r_i - (-r_i)}$. Since the parameters are generic (see (4.1)), $t^{2r_i} \neq t^{\pm 1}$ and thus, by (2.43), τ_1^2 has no kernel. Thus the intertwiner $\tau_1: M_{(r_i, -r_i)}^{\text{gen}} \rightarrow M_{(-r_i, r_i)}^{\text{gen}}$ is invertible. As a $H_{\{0\}}$ -module, $M_{(r_i, r_i)}^{\text{gen}}$ is irreducible (2-dimensional). So either $N = M_{(r_i, r_i)}^{\text{gen}}$ is an H_2 -submodule or M is irreducible.

For the purpose of deriving a contradiction, assume that $N = M_{(r_i, r_i)}^{\text{gen}}$ is an H_2 -submodule of M . The space N has a basis

$$\{n_\gamma, T_1 n_\gamma\} \quad \text{with} \quad W_1 n_\gamma = -t^{r_i} n_\gamma, \quad \text{and} \quad W_2 n_\gamma = -t^{r_i} n_\gamma.$$

By (2.24), $W_1^{-1} T_1 n_\gamma = T_1 W_2^{-1} n_\gamma + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) W_1^{-1} n_\gamma = T_1 (-t^{-r_i}) n_\gamma + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) (-t^{-r_i}) n_\gamma$ and the action of W_1^{-1} and W_1^{-2} on the basis $\{n_\gamma, T_1 n_\gamma\}$ are given by the matrices

$$\rho(W_1^{-1}) = (-t^{-r_i}) \begin{pmatrix} 1 & (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(W_1^{-2}) = \rho(W_1^{-1})^2 = t^{-2r_i} \begin{pmatrix} 1 & 2(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \rho(1 - W_1^{-2}) &= (1 - t^{-2r_i}) \begin{pmatrix} 1 & \frac{-2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i}}{1 - t^{-2r_i}} \\ 0 & 1 \end{pmatrix} \quad \text{and} \\ \rho(1 - W_1^{-2})^{-1} &= \frac{1}{(1 - t^{-2r_i})} \begin{pmatrix} 1 & \frac{2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i}}{1 - t^{-2r_i}} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since N is a submodule of M , we have $0 = \tau_0 = T_0 - \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})W_1^{-1}}{1 - W_1^{-2}}$ (see (2.35)

for the formula for τ_0), and so

$$\begin{aligned} \rho(T_0) &= ((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})W_1^{-1})(1 - W_1^{-2})^{-1} \\ &= \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})}{1 - t^{-2r_i}} \begin{pmatrix} 1 & \frac{(t_k^{1/2} - t_k^{-1/2})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})(-t^{-r_i})}{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{-2r_i}}{1 - t^{-2r_i}} \\ 0 & 1 \end{pmatrix} \\ &= \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})}{1 - t^{-2r_i}} \begin{pmatrix} 1 & \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{-t^{r_i}} \left(\frac{2(-t^{-r_i})}{1 - t^{-2r_i}} + \frac{(t_k^{1/2} - t_k^{-1/2})}{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})} \right) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Recall, from (3.5), that $-t^{r_i} = \pm t_k^{\pm \frac{1}{2}} t_0^{\frac{1}{2}}$, so that

$$\frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})}{1 - t^{-2r_i}} = \frac{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(\pm t_k^{\mp \frac{1}{2}} t_0^{-\frac{1}{2}})}{1 - t_k^{\mp 1} t_0^{-1}} = t_0^{\frac{1}{2}}.$$

The eigenvalues of $\rho(T_0)$ are $t_0^{\frac{1}{2}}$ and, since $(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0$, the Jordan blocks of $\rho(T_0)$ are of size 1, forcing

$$\begin{aligned} 0 &= \frac{2(-t^{-r_i})}{1 - t^{-2r_i}} + \frac{(t_k^{1/2} - t_k^{-1/2})}{(t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-t^{-r_i})} = \frac{2(-t^{-r_i})}{1 - t^{-2r_i}} + \frac{(t_k^{1/2} - t_k^{-1/2})}{(1 - t^{-2r_i})t_0^{1/2}} \\ &= \frac{2(-t^{-r_i})t_0^{\frac{1}{2}} + (t_k^{1/2} - t_k^{-1/2})}{(1 - t^{-2r_i})t_0^{1/2}} = \frac{2(\pm t_k^{\mp \frac{1}{2}} t_0^{-\frac{1}{2}})t_0^{\frac{1}{2}} + (t_k^{1/2} - t_k^{-1/2})}{(1 - t^{-2r_i})t_0^{1/2}} = \frac{\pm(t_k^{1/2} + t_k^{-1/2})}{(1 - t^{-2r_i})t_0^{1/2}}. \end{aligned}$$

This is a contradiction since, by the generic condition on parameters in (4.1), $1 \neq (-t^{r_1})(-t^{r_2}) = (-t_k^{\frac{1}{2}}t_0^{-\frac{1}{2}})(t_k^{\frac{1}{2}}t_0^{\frac{1}{2}}) = -(t_k^{\frac{1}{2}})^2$. Thus N is not a submodule of M , and so M is irreducible.

Case $(\gamma_1, \gamma_2) = (-1, -t)$: Let $H_{\{1\}}$ be the subalgebra of H_2 generated by $T_1, W_1^{\pm 1}, W_2^{\pm 1}$. There are two irreducible modules of central character $\mathbf{c} = (0, 1)$:

$$L_{(0,1)}^+ = \text{Ind}_{H_{\{1\}}}^{H_2} (\mathbb{C}_{(-1,0)}), \quad \text{where } \mathbb{C}_{(-1,0)} = \mathbb{C}v \text{ with } \begin{array}{l} W_1 v = -t^{-1}v, \\ W_2 v = -v, \\ T_1 v = t^{\frac{1}{2}}v, \end{array}$$

and

$$L_{(0,1)}^- = \text{Ind}_{H_{\{1\}}}^{H_2} (\mathbb{C}_{(1,0)}), \quad \text{where } \mathbb{C}_{(1,0)} = \mathbb{C}v \text{ with } \begin{array}{l} W_1 v = -tv, \\ W_2 v = -v, \\ T_1 v = -t^{-\frac{1}{2}}v. \end{array}$$

The irreducibility of $L_{(0,1)}^+$ and $L_{(0,1)}^-$ is not immediate. We will show that $M = L_{(0,1)}^+$ is irreducible; the irreducibility of $L_{(0,1)}^-$ is proved analogously.

The generalized weight space decomposition of $M = L_{(0,1)}^+$ is

$$M = M_{(-1,0)}^{\text{gen}} \oplus M_{(1,0)}^{\text{gen}} \oplus M_{(0,1)}^{\text{gen}} \quad \text{with} \quad \begin{array}{l} \dim(M_{(-1,0)}^{\text{gen}}) = \dim(M_{(1,0)}^{\text{gen}}) = 1, \\ \dim(M_{(0,1)}^{\text{gen}}) = 2. \end{array}$$

The element W_1^{-1} acts on $M_{(-1,0)}^{\text{gen}}$ with eigenvalue $-t$. Since the parameters are generic (see (4.1)), $-t \notin \{-t^{\pm r_1}, -t^{\pm r_2}\}$ and thus, by (2.42), τ_0^2 has no kernel. Thus the intertwiner $\tau_0: M_{(-1,0)}^{\text{gen}} \rightarrow M_{(1,0)}^{\text{gen}}$ is invertible. Since $M_{(0,1)}^{\text{gen}}$ is irreducible as a $H_{\{0\}}$ -module, we have either $N = M_{(0,1)}^{\text{gen}}$ is an H_2 -submodule or M is irreducible.

For the purpose of deriving a contradiction, assume that $N = M_{(0,1)}^{\text{gen}}$ is an H_2 -submodule of M . The space N has a basis

$$\{n_\gamma, T_0 n_\gamma\} \quad \text{with} \quad W_1 n_\gamma = -n_\gamma, \quad \text{and} \quad W_2 n_\gamma = -t n_\gamma.$$

By (C2) and (B3),

$$\begin{aligned} W_1 W_2^{-1} T_0 n_\gamma &= T_0 W_1^{-1} W_2^{-1} n_\gamma + ((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2}) W_1^{-1}) \frac{W_1 - W_1^{-1}}{1 - W_1^{-2}} W_2^{-1} n_\gamma \\ &= T_0 t^{-1} n_\gamma + ((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1)) t^{-1} n_\gamma, \end{aligned}$$

and the action of $W_1 W_2^{-1}$ on the basis $\{n_\gamma, T_0 n_\gamma\}$ is given by the matrix

$$\rho(W_1 W_2^{-1}) = \begin{pmatrix} t^{-1} & ((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1)) t^{-1} \\ 0 & t^{-1} \end{pmatrix}.$$

Thus

$$\begin{aligned} \rho(1 - W_1 W_2^{-1}) &= \begin{pmatrix} 1 - t^{-1} & -((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1)) t^{-1} \\ 0 & 1 - t^{-1} \end{pmatrix} \\ &= (1 - t^{-1}) \begin{pmatrix} 1 & -\frac{((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1)) t^{-1}}{1 - t^{-1}} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$\rho(1 - W_1 W_2^{-1})^{-1} = \frac{1}{(1 - t^{-1})} \begin{pmatrix} 1 & \frac{((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1))t^{-1}}{1 - t^{-1}} \\ 0 & 1 \end{pmatrix}.$$

If N is a submodule of M then $0 = \tau_1 = T_1 - \frac{t^{\frac{1}{2}} + t^{-\frac{1}{2}}}{1 - W_1 W_2^{-1}}$ (see (2.35) for the formula for τ_1). Thus

$$\rho(T_1) = t^{\frac{1}{2}} \begin{pmatrix} 1 & \frac{((t_0^{1/2} - t_0^{-1/2}) + (t_k^{1/2} - t_k^{-1/2})(-1))t^{-1}}{1 - t^{-1}} \\ 0 & 1 \end{pmatrix}.$$

Since $(T_1 - t^{\frac{1}{2}})(T_1 + t^{-\frac{1}{2}}) = 0$ the Jordan blocks of $\rho(T_1)$ are of size one, forcing

$$0 = (t_0^{1/2} - t_0^{-1/2}) - (t_k^{1/2} - t_k^{-1/2}) = t_0^{-\frac{1}{2}}(t_0^{1/2} + t_k^{-1/2})(t_0^{1/2} - t_k^{1/2}).$$

This is a quadratic equation in $t_0^{\frac{1}{2}}$ with two solutions, $t_0^{\frac{1}{2}} = t_k^{\frac{1}{2}}$ and $t_0^{\frac{1}{2}} = -t_k^{-\frac{1}{2}}$. This is a contradiction since, by the generic condition on parameters in (4.1), $-t^{-r_1} = -t_0^{\frac{1}{2}} t_k^{-\frac{1}{2}} \neq -1$ and $-t^{r_2} = t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} \neq -1$. Thus N is not a submodule of M , and so M is irreducible.

5 Representations of $\mathcal{B}_k^{\text{ext}}$ in tensor space

In this section we give a Schur-Weyl duality approach to the representations of the two boundary Hecke algebras H_k^{ext} . More generally, in Theorem 5.1 we show that, for a quantum group or quasitriangular Hopf algebra $U_q \mathfrak{g}$ and three $U_q \mathfrak{g}$ -modules M , N and V , there is an action of the two boundary braid group $\mathcal{B}_k^{\text{ext}}$ on tensor space $M \otimes N \otimes V^{\otimes k}$ that commutes with the $U_q \mathfrak{g}$ -action. This means that there is a weak Schur-Weyl duality pairing between $U_q \mathfrak{g}$ -modules and $\mathcal{B}_k^{\text{ext}}$ -modules, so that if $M \otimes N \otimes V^{\otimes k}$ is completely reducible as a $U_q \mathfrak{g}$ -module then

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda} L(\lambda) \otimes B_k^{\lambda} \quad \text{as } (U_q \mathfrak{g}, \mathcal{B}_k^{\text{ext}})\text{-modules,}$$

where $L(\lambda)$ are irreducible $U_q \mathfrak{g}$ -modules and B_k^{λ} are $\mathcal{B}_k^{\text{ext}}$ -modules. In Section 5.4 we will explain that when $\mathfrak{g} = \mathfrak{gl}_n$ and M and N and V are appropriately chosen, the $\mathcal{B}_k^{\text{ext}}$ -action provides an action of the two boundary Hecke algebra H_k^{ext} (where the parameters depend on the choice of M and N). Our main theorem, Theorem 5.5, proves that the H_k^{ext} -modules B_k^{λ} that appear in tensor space $M \otimes N \otimes V^{\otimes k}$ are irreducible, and identifies them in terms of the classification of irreducible calibrated H_k^{ext} -modules which is given in Theorem 3.3.

5.1 Quantum groups and R -matrices

Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form, and let $U_q \mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to \mathfrak{g} . The quantum group $U_q \mathfrak{g}$ is a ribbon Hopf algebra with invertible R -matrix

$$R = \sum_R R_1 \otimes R_2 \quad \text{in } U_q \mathfrak{g} \otimes U_q \mathfrak{g}, \quad \text{and ribbon element } v = q^{-2\rho} u,$$

where $u = \sum_R S(R_2)R_1$ and ρ is the staircase weight (see [LR, Corollary (2.15)]). For $U_q\mathfrak{g}$ -modules M and N , the map

$$\check{R}_{MN}: \begin{array}{ccc} N \otimes M & \longrightarrow & M \otimes N \\ n \otimes m & \longmapsto & \sum_R R_2 m \otimes R_1 n \end{array} \quad \begin{array}{c} M \otimes N \\ \curvearrowright \\ N \otimes M \end{array} \quad (5.1)$$

is a $U_q\mathfrak{g}$ -module isomorphism. The quasitriangularity of a ribbon Hopf algebra provides the relations (see, for example, [OR, (2.9), (2.10), and (2.12)]),

$$\begin{array}{ccc} \begin{array}{c} M \otimes N \\ \square \varphi \\ \curvearrowright \\ N \otimes M \end{array} & = & \begin{array}{c} M \otimes N \\ \curvearrowright \\ N \otimes M \\ \square \varphi \end{array} \end{array} \quad \text{for any isomorphism } \varphi : M \rightarrow M, \\ (\varphi \otimes \text{id}_N)\check{R}_{MN} = \check{R}_{MN}(\text{id}_N \otimes \varphi),$$

$$\begin{array}{ccc} \begin{array}{c} M \otimes N \otimes V \\ \curvearrowright \\ V \otimes N \otimes M \end{array} & = & \begin{array}{c} M \otimes N \otimes V \\ \curvearrowright \\ V \otimes N \otimes M \end{array} \end{array} \\ (\check{R}_{MN} \otimes \text{id}_V)(\text{id}_N \otimes \check{R}_{MV})(\check{R}_{NV} \otimes \text{id}_M) = (\text{id}_M \otimes \check{R}_{NV})(\check{R}_{MV} \otimes \text{id}_N)(\text{id}_V \otimes \check{R}_{MN}),$$

$$\begin{array}{ccc} \begin{array}{c} M \otimes (N \otimes V) \\ \curvearrowright \\ (N \otimes V) \otimes M \end{array} & = & \begin{array}{c} M \otimes N \otimes V \\ \curvearrowright \\ N \otimes V \otimes M \end{array} & \quad & \begin{array}{c} (M \otimes N) \otimes V \\ \curvearrowright \\ V \otimes (M \otimes N) \end{array} & = & \begin{array}{c} M \otimes N \otimes V \\ \curvearrowright \\ V \otimes M \otimes N \end{array} \\ (\check{R}_{M \otimes N, V}) & = & (\text{id}_M \otimes \check{R}_{NV})(\check{R}_{MV} \otimes \text{id}_N) & \quad & (\check{R}_{M \otimes N, V}) & = & (\text{id}_M \otimes \check{R}_{NV})(\check{R}_{MV} \otimes \text{id}_N). \end{array} \quad (5.2)$$

For a $U_q\mathfrak{g}$ -module M define

$$C_M: \begin{array}{ccc} M & \longrightarrow & M \\ m & \longmapsto & vm \end{array} \quad \text{so that} \quad C_{M \otimes N} = (\check{R}_{MN}\check{R}_{NM})^{-1}(C_M \otimes C_N) \quad (5.3)$$

(see [Dr, Prop. 3.2]). Let $L(\lambda)$ denote the simple $U_q\mathfrak{g}$ -module generated by a highest weight vector v_λ^+ of weight λ . Then

$$C_{L(\lambda)} = q^{-\langle \lambda, \lambda + 2\rho \rangle} \text{id}_{L(\lambda)} \quad (5.4)$$

(see [LR, Prop. 2.14] or [Dr, Prop. 5.1]). From (5.4) and (5.3), it follows that if $M = L(\mu)$ and $N = L(\nu)$ are finite-dimensional irreducible $U_q\mathfrak{g}$ -modules of highest weights μ and ν respectively, then $\check{R}_{MN}\check{R}_{NM}$ acts on the $L(\lambda)$ -isotypic component $L(\lambda)^{\oplus c_{\mu\nu}^\lambda}$ of the decomposition

$$L(\mu) \otimes L(\nu) = \bigoplus_{\lambda} L(\lambda)^{\oplus c_{\mu\nu}^\lambda} \quad \text{by the scalar} \quad q^{\langle \lambda, \lambda+2\rho \rangle - \langle \mu, \mu+2\rho \rangle - \langle \nu, \nu+2\rho \rangle}. \quad (5.5)$$

Proposition 5.1. *Let \mathfrak{g} be a finite-dimensional complex Lie algebra with a symmetric nondegenerate ad-invariant bilinear form, let $U_q\mathfrak{g}$ be the corresponding Drinfeld-Jimbo quantum group, and let $\mathcal{Z} = Z(U_q\mathfrak{g})$ be the center of $U_q\mathfrak{g}$. Let M, N , and V be $U_q\mathfrak{g}$ -modules. Then $M \otimes N \otimes V^{\otimes k}$ is a $\mathcal{Z}\mathcal{B}_k^{\text{ext}}$ -module with action given by*

$$\begin{aligned} \Phi: \quad \mathcal{Z}\mathcal{B}_k^{\text{ext}} &\longrightarrow \text{End}_{U_q\mathfrak{g}}(M \otimes N \otimes V^{\otimes k}) \\ T_i &\longmapsto \check{R}_i, & \text{for } i = 1, \dots, k-1, \\ X_1 &\longmapsto \check{R}_M^2, \\ Y_1 &\longmapsto \check{R}_N^2, \\ Z_1 &\longmapsto \check{R}_0^2, \\ P &\longmapsto (\check{R}_{MN}\check{R}_{NM}) \otimes \text{id}_V^{\otimes(k)}, \end{aligned} \quad (5.6)$$

where

$$\check{R}_0^2 = (\check{R}_{(M \otimes N)V}\check{R}_{V(M \otimes N)}) \otimes \text{id}_V^{\otimes(k-1)}, \quad \check{R}_i = \text{id}_M \otimes \text{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \text{id}_V^{\otimes(k-i-1)}$$

for $i = 1, \dots, k-1$,

$$\check{R}_M^2 = ((\text{id}_M \otimes \check{R}_{NV})(\check{R}_{MV}\check{R}_{VM}) \otimes \text{id}_N)(\text{id}_M \otimes \check{R}_{NV}^{-1}) \otimes \text{id}_V^{\otimes(k-1)}, \quad \text{and}$$

$$\check{R}_N^2 = \text{id}_M \otimes (\check{R}_{NV}\check{R}_{VN}) \otimes \text{id}_V^{\otimes(k-1)},$$

with \check{R}_{MV} as in (5.1). Moreover, this $\mathcal{Z}\mathcal{B}_k^{\text{ext}}$ action commutes with the $U_q\mathfrak{g}$ -action on $M \otimes N \otimes V^{\otimes k}$.

Proof. This proof follows the proof of [OR, Prop. 3.1], checking that the images of the generators T_i, X_1, Y_1 , and Z_1 under the map Φ satisfy the relations of presentation (a) of the two boundary braid group in Theorem 2.1, as well as relations (2.15) and (2.16) for the extended two boundary braid group. For $i \in \{1, \dots, k-2\}$,

$$\Phi(T_i)\Phi(T_{i+1})\Phi(T_i) = \check{R}_i\check{R}_{i+1}\check{R}_i = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \end{array} = \check{R}_{i+1}\check{R}_i\check{R}_{i+1} = \Phi(T_{i+1})\Phi(T_i)\Phi(T_{i+1}).$$

Using the notation $\check{R}_{M \otimes N}$ for the endomorphism \check{R}_0 , we have that, for $L = M, N$, or $M \otimes N$,

$$\check{R}_L^2\check{R}_1\check{R}_L^2\check{R}_1 = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \text{---} \end{array} = \check{R}_1\check{R}_L^2\check{R}_1\check{R}_L^2,$$

which establishes

$$\Phi(A)\Phi(T_1)\Phi(A)\Phi(T_1) = \Phi(T_1)\Phi(A)\Phi(T_1)\Phi(A) \quad \text{for } A = X_1, Y_1, \text{ and } Z_1, \text{ respectively.}$$

The formula

$$\Phi(Z_1) = \check{R}_0^2 = \check{R}_M^2 \check{R}_N^2 = \Phi(X_1)\Phi(Y_1)$$

is a consequence of the third set of relations (cabling relations) in (5.2). Finally, the relations

$$\Phi(P)\Phi(Y_1)\Phi(P) = \Phi(Z_1^{-1})\Phi(Y_1)\Phi(Z_1) \quad \text{and} \quad \Phi(P)\Phi(X_1)\Phi(P) = \Phi(Z_1^{-1})\Phi(X_1)\Phi(Z_1)$$

follow from the first and second sets of relations for \check{R} -matrices in (5.2) by the same braid computation by which the identities (2.13) were derived. The remainder of the relations (commuting generators) follow directly from the definitions of $\Phi(T_i)$, $\Phi(X_1)$, $\Phi(Y_1)$, $\Phi(Z_1)$, and $\Phi(P)$. \square

5.2 The $\mathcal{B}_k^{\text{ext}}$ -modules B_k^λ

Assume that M , N , and V are finite-dimensional $U_q\mathfrak{g}$ -modules and that ω is the highest weight of V so that

$$V = L(\omega) \quad \text{is irreducible of highest weight } \omega.$$

Let $\mathcal{P}^{(j)}$ be an index set for the irreducible $U_q\mathfrak{g}$ -modules that appear in $M \otimes N \otimes V^{\otimes j}$ and let $\mathcal{P}^{(-1)}$ be an index set for the irreducible $U_q\mathfrak{g}$ -modules in M . The *Bratteli diagram* for the sequence of $U_q\mathfrak{g}$ -modules

$$M, \quad M \otimes N, \quad M \otimes N \otimes V, \quad M \otimes N \otimes V \otimes V, \quad \dots \quad (5.7)$$

is the graph with

vertices on level j labeled by $\mu \in \mathcal{P}^{(j)}$, for $j \in \mathbb{Z}_{\geq -1}$,

$m_{\mu\lambda}$ edges $\mu \rightarrow \lambda$ for $\mu \in \mathcal{P}^{(j)}$ and $\lambda \in \mathcal{P}^{(j+1)}$, and where $L(\mu) \otimes V \cong \bigoplus_{\lambda \in \mathcal{P}^{(j+1)}} L(\lambda)^{\oplus m_{\mu\lambda}}$,

each edge $\mu \rightarrow \lambda$ labeled with $\frac{1}{2}(\langle \lambda, \lambda + 2\rho \rangle - \langle \omega, \omega + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle)$.

A specific example in the case where $\mathfrak{g} = \mathfrak{gl}_n$ is given in Figure 3.

If M and N are finite-dimensional then $M \otimes N \otimes V^{\otimes k}$ is completely decomposable as a $U_q\mathfrak{g}$ -module. If B_k^λ is the space of highest weight vectors of weight λ in $M \otimes N \otimes V^{\otimes k}$, then

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^\lambda, \quad \text{as } (U_q\mathfrak{g}, \mathcal{B}_k^{\text{ext}})\text{-bimodules.} \quad (5.8)$$

The $\mathcal{B}_k^{\text{ext}}$ -modules B_k^λ are not necessarily irreducible and not necessarily nonisomorphic, though they will be in the (mostly rare but very important) settings where $\Phi(\mathcal{CB}_k^{\text{ext}}) = \text{End}_{U_q\mathfrak{g}}(M \otimes N \otimes V^{\otimes k})$.

Recall from (2.9) that

$$Z_i = T_{i-1} \cdots T_1 Z_1 T_1 \cdots T_{i-1} \quad \text{for } i = 1, \dots, k.$$

The following proposition shows that, as operators on B_k^λ , the Z_i are simultaneously diagonalizable and have eigenvalues determined by the edges on the Bratteli diagram. The proof follows the same schematic that is used, for example, in the proof of [OR, Proposition 3.2].

Proposition 5.2. *Assume M , N , and V are finite-dimensional $U_q\mathfrak{g}$ modules with V irreducible. For $\lambda \in \mathcal{P}^{(k)}$, let B_k^λ be the $\mathcal{B}_k^{\text{ext}}$ -module in (5.8) and let*

$$\mathcal{T}_k^\lambda = \{\text{paths } S = (S^{(-1)} \xrightarrow{e_0} S^{(0)} \xrightarrow{e_1} \dots \xrightarrow{e_k} S^{(k)} = \lambda) \text{ in the Bratteli diagram}\}.$$

Then

$$B_k^\lambda \text{ has a basis } \{v_S \mid S \in \mathcal{T}_k^\lambda\}$$

of simultaneous eigenvectors for the action of P, Z_1, \dots, Z_k , with

$$Pv_S = q^{2e_0}v_S \quad \text{and} \quad Z_i v_S = q^{2e_i}v_S, \quad \text{for } i = 1, \dots, k,$$

so that the eigenvalues of P and Z_1, \dots, Z_k on v_S are determined by the labels on the edges of the path S .

Proof. The basis $\{v_S \mid S \in \mathcal{T}_k^\lambda\}$ is constructed inductively. For the initial case, choose any basis \hat{B}_{-1} of the highest weight vectors in M , and let \hat{B}_{-1}^ν be the set of basis elements in \hat{B}_{-1} of weight ν . For the inductive step, assume that $\hat{B}_{k-1}^\mu = \{v_T \mid T \in \mathcal{T}_{k-1}^\mu\}$ has been constructed so that

$$M \otimes N \otimes V^{\otimes(k-1)} = \bigoplus_{\mu \in \mathcal{P}^{(k-1)}} L(\mu) \otimes B_{k-1}^\mu = \bigoplus_{\mu \in \mathcal{P}^{(k-1)}} L(\mu) \otimes \left(\sum_{T \in \mathcal{T}_{k-1}^\mu} \mathbb{C}v_T \right),$$

The set $\hat{B}_{k-1}^\mu = \{v_T \mid T \in \mathcal{T}_{k-1}^\mu\}$ is a basis of the vector space of highest weight vectors of weight μ in $M \otimes N \otimes V^{\otimes(k-1)}$ that is indexed by the paths $T = (T^{(-1)} \rightarrow \dots \rightarrow T^{(k-1)} = \mu)$ of length k in the Bratteli diagram that end at μ . In this form $L(\mu) \otimes \mathbb{C}v_T$ denotes the irreducible $U_q\mathfrak{g}$ -submodule of $M \otimes N \otimes V^{\otimes(k-1)}$ with highest weight vector v_T of weight μ .

Then, for each $T = (T^{(-1)} \rightarrow \dots \rightarrow T^{(k-1)} = \mu)$ in \mathcal{T}_{k-1}^μ , choose a basis

$$\hat{B}_k^{T \rightarrow \lambda} = \{v_S \mid S = (T^{(-1)} \rightarrow \dots \rightarrow T^{(k-1)} = \mu \rightarrow \lambda)\}$$

of highest weight vectors in the submodule of $M \otimes N \otimes V^{\otimes k}$ given by

$$(L(\mu) \otimes \mathbb{C}v_T) \otimes V = L(\mu) \otimes V \otimes \mathbb{C}v_T = \sum_{\mu \rightarrow \lambda} L(\lambda) \otimes \mathbb{C}v_S.$$

The basis $\hat{B}_k^{T \rightarrow \lambda}$ is indexed by the edges in the Bratteli diagram from μ to a partition λ on level k . Then

$$\hat{B}_k^\lambda = \bigsqcup_{\mu} \bigsqcup_{T \in \mathcal{T}_{k-1}^\mu} \mathcal{T}_k^{T \rightarrow \lambda} \quad \text{is a basis of } B_k^\lambda.$$

The central element $q^{-2\rho}u$ in $U_q\mathfrak{g}$ acts on the submodule $L(\mu) \otimes \mathbb{C}v_T$ of $M \otimes N \otimes V^{\otimes(k-1)}$ by the constant $q^{-\langle \mu, \mu + 2\rho \rangle}$. From (5.2), (5.3), and (5.4) it follows that Z_i acts on $M \otimes N \otimes V^{\otimes k}$ by

$$\begin{aligned} \Phi(Z_i) &= \check{R}_{i-1} \cdots \check{R}_1 \check{R}_0^2 \check{R}_1 \cdots \check{R}_{i-1} = \check{R}_{M \otimes N \otimes V^{\otimes(i-1)}, V} \check{R}_{V, M \otimes N \otimes V^{\otimes(i-1)}} \otimes \text{id}_V^{\otimes(k-i)} \\ &= (C_{M \otimes N \otimes V^{\otimes(i-1)}} \otimes C_V) C_{M \otimes N \otimes V^{\otimes i}}^{-1} \otimes \text{id}_V^{\otimes(k-i)} \\ &= \sum_{\lambda, \mu, \nu} q^{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega, \omega + 2\rho \rangle} \pi_{\mu\nu}^\lambda \otimes \text{id}_V^{\otimes(k-i)}, \end{aligned} \tag{5.9}$$

where $\pi_{\mu\nu}^\lambda : M \otimes N \otimes \text{id}_V^{\otimes i} \rightarrow M \otimes N \otimes \text{id}_V^{\otimes i}$ is the projection onto the $L(\lambda)$ isotypic component of $(L(\mu) \otimes B_{i-1}^\mu) \otimes V$. Thus Z_i acts diagonally on the basis \hat{B}_k^λ and, by the definition of the labels of edges in the Bratteli diagram in (5.7), the eigenvalues of $Z_i v_S = q^{2e_i} v_S$ where e_i is the label on the edge $S^{(i)} \rightarrow S^{(i+1)}$ in the Bratteli diagram. □

5.3 Some tensor products for $\mathfrak{g} = \mathfrak{gl}_n$

The finite-dimensional irreducible polynomial representations $L(\lambda)$ of $U_q \mathfrak{gl}_n$ are indexed by elements of

$$P_{poly}^+ = \{\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n, \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

Use

$$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} = \sum_{i=1}^n (n-i)\varepsilon_i, \quad (5.10)$$

as in [Mac1, I (1.13)]. Identify each element $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$ in P_{poly}^+ with the corresponding partition having λ_i boxes in row i so that, for example,

$$\lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \end{array}.$$

The *content* of the box in row i and column j of a partition λ is

$$c(\text{box}) = j - i = (\text{diagonal number of box}), \quad (5.11)$$

where the diagonals are numbered by the elements of \mathbb{Z} from southwest to northeast, with the northwest corner box of a partition being in diagonal 0.

The representation $L(\varepsilon_1) = L(\square)$ is the standard n -dimensional representation of $U_q \mathfrak{gl}_n$. When $\nu = \varepsilon_1$, the decomposition in (5.5) is given by

$$L(\mu) \otimes L(\square) \cong \bigoplus_{\lambda \in \mu^+} L(\lambda), \quad (5.12)$$

where μ^+ is the set of partitions obtained by adding a box to μ . If $\lambda \in \mu^+$ and λ/μ is the box added to μ to obtain λ , then the action in (5.5) is given by

$$\begin{aligned} & \langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle \\ &= \langle \mu + \varepsilon_i, \mu + \varepsilon_i + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle = 2\mu_i + 1 + 2\rho_i - 1 - 2\rho_1 \\ &= 2\mu_i + 2(n-i) - 2(n-1) = 2\mu_i - 2i + 2 = 2c(\lambda/\mu) \end{aligned} \quad (5.13)$$

(see [Mac1, I (5.16) and (8.4)]). Since $\langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle = 2(n-1) + 1 = 2n-1$, it follows by induction on the number of boxes in a partition λ that

$$\langle \lambda, \lambda + 2\rho \rangle = (2n-1)|\lambda| + \sum_{\text{box} \in \lambda} 2c(\text{box}). \quad (5.14)$$

For $\mu, \nu \in P_{poly}^+$, the decomposition of the tensor product $L(\mu) \otimes L(\nu)$ can be calculated using the Littlewood-Richardson rule (see [Mac1, Ch. I (9.2)]). When μ and ν are rectangles the

decomposition is multiplicity free by the following theorem. In equation (5.15), \mathcal{A} consists of the boxes that are in the union of the rectangles (a^c) and (b^d) (placed with northwest corner at $(1, 1)$), and the dashed rectangular regions are the $\min(a, b) \times d$ rectangle \mathcal{B} with northwest corner box at $(\max(a, b) + 1, 1)$, and the $d \times \min(a, b)$ rectangle \mathcal{B}' with northwest corner at $(1, c + 1)$.

Proposition 5.3. (See [St, Lem. 3.3], [Ok, Thm 2.4]) *Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $c \geq d$. For $\mu \subseteq (\min(a, b)^d)$ let*

$$\begin{array}{ccc}
 \text{if } a \geq b: & & \text{if } a \leq b: \\
 \begin{array}{c}
 \begin{array}{c} a \quad b \\ \mu \\ \mathcal{B} \\ \mathcal{B}' \\ \mu^c \end{array} \\
 \begin{array}{c} c \\ \mathcal{A} \\ d \end{array} \\
 \dot{\mu} =
 \end{array} & & \begin{array}{c}
 \begin{array}{c} b \quad a \\ \mu \\ \mathcal{B} \\ \mathcal{B}' \\ \mu^c \end{array} \\
 \begin{array}{c} c \\ \mathcal{A} \\ d \end{array} \\
 \dot{\mu} =
 \end{array}
 \end{array} \tag{5.15}$$

so that μ^c is the 180° rotation of the complement of μ in a $\min(a, b) \times d$ rectangle. Denote the rectangular partition with c rows of length a by (a^c) . Then

$$L((a^c)) \otimes L((b^d)) \cong \bigoplus_{\mu \subseteq (\min(a, b)^d)} L(\dot{\mu}) \cong \bigoplus_{\nu \in \mathcal{P}^{(0)}} L(\nu), \tag{5.16}$$

where $\mathcal{P}^{(0)} = \{\dot{\mu} \mid \mu \subseteq ((\min(a, b)^d))\}$.

For an example of the decomposition in (5.16), see Figure 3, where the decomposition of $L(a^c) \otimes L(2^2)$ for $a, c \geq 2$ is indicated in level 0 of the Bratteli diagram (see the description following (5.23) for explanation of the Bratteli diagram).

The value in (5.5) for the product in (5.16) is given by using (5.14) to compute

$$\begin{aligned}
 & \langle \dot{\mu}, \dot{\mu} + 2\rho \rangle - \langle (a^c), (a^c) + 2\rho \rangle - \langle (b^d), (b^d) + 2\rho \rangle \\
 &= (2n - 1)(|\dot{\mu}| - |(a^c)| - |(b^d)|) + \left(\sum_{\text{box} \in \dot{\mu}} 2c(\text{box}) \right) - \sum_{\text{box} \in (a^c)} 2c(\text{box}) - \sum_{\text{box} \in (b^d)} 2c(\text{box}) \\
 &= 0 + \sum_{\text{box} \in \dot{\mu}} 2c(\text{box}) - ac(a - c) - bd(b - d).
 \end{aligned} \tag{5.17}$$

5.4 Irreducible H_k^{ext} -modules in $M \otimes N \otimes V^{\otimes k}$

In this subsection we provide, for $\mathfrak{g} = \mathfrak{gl}_n$, specific highest weight modules M , N , and V such that the $\mathcal{B}_k^{\text{ext}}$ -action factors through the extended two boundary Hecke algebra H_k^{ext} . In these cases the $\mathcal{B}_k^{\text{ext}}$ -modules B_k^λ in (5.8) are calibrated H_k^{ext} -modules. Theorem 5.5 identifies the B_k^λ for these cases explicitly in terms of the indexings of calibrated H_k^{ext} -modules given in Theorem 3.3 and Proposition 3.1.

Recall that, as defined in Section 2.2, the *extended two boundary Hecke algebra* H_k^{ext} is the quotient of the group algebra of the extended two boundary braid group $\mathbb{CB}_k^{\text{ext}}$ by the relations

$$(X_1 - a_1)(X_1 - a_2) = 0, \quad (Y_1 - b_1)(Y_1 - b_2) = 0, \quad \text{and} \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0, \quad (5.18)$$

$i = 1, \dots, k-1$, for fixed $a_1, a_2, b_1, b_2, t^{\frac{1}{2}} \in \mathbb{C}^\times$.

Theorem 5.4. *If $\mathfrak{g} = \mathfrak{gl}_n$, $M = L((a^c))$, $N = L((b^d))$, and $V = L(\square)$,*

$$a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d}, \quad \text{and} \quad t^{\frac{1}{2}} = q, \quad (5.19)$$

then the map Φ from Proposition 5.1 gives an action of H_k^{ext} on $M \otimes N \otimes V^{\otimes k}$ commuting with that of $U_q \mathfrak{gl}_n$.

Proof. The module $M \otimes V$ decomposes as

$$M \otimes V = L \left(\begin{array}{c} a \\ \square \\ c \end{array} \right) \oplus L \left(\begin{array}{c} a \\ c \\ \square \end{array} \right). \quad (5.20)$$

By (5.5) and (5.13), $\check{R}_{MV} \check{R}_{VM}$ acts on the first summand by the constant q^{2a} and on the second summand by the constant q^{-2c} . So

$$(\Phi(X_1) - q^{2a})(\Phi(X_1) - q^{-2c}) = 0; \quad \text{similarly} \quad (\Phi(Y_1) - q^{2b})(\Phi(Y_1) - q^{-2d}) = 0$$

by replacing (a^c) with (b^d) . The relation

$$(\Phi(T_i) - q)(\Phi(T_i) + q^{-1}) = 0$$

follows similarly by considering the tensor product $V \otimes V = L(\square) \otimes L(\square)$. □

From (2.17), (5.19), and (3.5),

$$\begin{aligned} a_1 &= q^{2a}, & a_2 &= q^{-2c}, & b_1 &= q^{2b}, & b_2 &= q^{-2d}, & t^{\frac{1}{2}} &= q, \\ t_k^{\frac{1}{2}} &= a_1^{\frac{1}{2}} (-a_2)^{-\frac{1}{2}} = -iq^{a+c} & \text{and} & & t_0^{\frac{1}{2}} &= b_1^{\frac{1}{2}} (-b_2)^{-\frac{1}{2}} = -iq^{b+d}, & & & & \\ -t^{r_1} &= -t_k^{\frac{1}{2}} t_0^{-\frac{1}{2}} = -q^{(a+c)-(b+d)}, & \text{and} & & -t^{r_2} &= t_k^{\frac{1}{2}} t_0^{\frac{1}{2}} = -q^{a+c+b+d}. \end{aligned} \quad (5.21)$$

Using these conversions, the genericity conditions in (4.1) become requirements that q is not a root of unity and

$$-q^{(a+c)-(b+d)}, -q^{a+c+b+d} \notin \{1, -1, q^{\pm 1}, -q^{\pm 1}, q^{\pm 2}, -q^{\pm 2}\} \quad \text{and} \quad -q^{(a+c)-(b+d)} \neq -q^{\pm(a+c+b+d)}.$$

In the context of Theorem 5.4, these genericity conditions are

$$q \text{ is not a root of unity, } a, b, c, d \in \mathbb{Z}_{>0} \quad \text{and} \quad (a+c) - (b+d) \notin \{0, \pm 1, \pm 2\}. \quad (5.22)$$

In the setting of Theorem 5.4, equation (5.8) provides H_k^{ext} -modules B_k^λ with

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^\lambda, \quad \text{as } (U_q \mathfrak{g}, \mathcal{H}_k^{\text{ext}})\text{-bimodules.} \quad (5.23)$$

Theorem 5.5 below will accomplish our primary goal for this paper by identifying the module B_k^λ explicitly as a calibrated H_k^{ext} -module $H_k^{(z,c,J)}$ as constructed in Theorem 3.3. The results of (5.12), (5.13), and Proposition 5.3 show that the Bratteli diagram of (5.7) has $\mathcal{P}^{(-1)} = \{(a^c)\}$, $\mathcal{P}^{(0)} = \{\hat{\mu} \mid \mu \subseteq ((\min(a,b))^d)\}$ as in Proposition 5.3 and, for $j \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{P}^{(j)} = \{ \text{partitions obtained by adding } j \text{ boxes to a partition in } \mathcal{P}^{(0)} \}.$$

By (5.17), if $\hat{\mu} \in \mathcal{P}^{(0)}$ then there is an edge

$$(a^c) \xrightarrow{e_0(\hat{\mu})} \hat{\mu} \text{ with label } e_0(\hat{\mu}) = -\frac{ac}{2}(a-c) - \frac{bd}{2}(b-d) + \sum_{\text{box} \in \hat{\mu}} c(\text{box}). \quad (5.24)$$

For $j \geq 0$, the edges $\mu \rightarrow \lambda$ from level j to level $j+1$ correspond to adding a single box to μ to get λ , and are labeled by $c(\lambda/\mu)$, the content of the box λ/μ :

$$\mu \xrightarrow{c(\lambda/\mu)} \lambda \quad \text{for edges from level } j \text{ to level } j+1. \quad (5.25)$$

The case when $M = L(a^c)$ and $N = L(2^2)$ with $a, c > 2$ is illustrated in Figure 3.

Let $\lambda \in \mathcal{P}^{(k)}$. Define

$$c_0 = -\frac{1}{2}(k(a-c+b-d) + ac(a-c) + bd(b-d)) + \sum_{\text{box} \in \lambda} c(\text{box}) \quad \text{and} \quad z = (-1)^k q^{2c_0}. \quad (5.26)$$

Using notation as in (5.15), let

$$\mu^c = \lambda \cap \mathcal{B}' \quad \text{and let } S_{\max}^{(0)} \text{ be the corresponding } \hat{\mu}. \quad (5.27)$$

Define the *shifted content* of a box by

$$\tilde{c}(\text{box}) = c(\text{box}) - \frac{1}{2}(a-c+b-d), \quad \text{and let } \mathbf{c} = (c_1, \dots, c_k) \text{ with } 0 \leq c_1 \leq c_2 \leq \dots \leq c_k \quad (5.28)$$

be the sequence of absolute values of the shifted contents of the boxes in $\lambda/S_{\max}^{(0)}$ arranged in increasing order. Index the boxes of $\lambda/S_{\max}^{(0)}$ with $1, 2, \dots, k$ so that

- (a) if $i < j$ then $|\tilde{c}(\text{box}_i)| \leq |\tilde{c}(\text{box}_j)|$,
- (b) if $i < j$ and $\tilde{c}(\text{box}_i) = \tilde{c}(\text{box}_j) < 0$ then box_i is SE of box_j ,
- (c) if $i < j$ and $\tilde{c}(\text{box}_i) = \tilde{c}(\text{box}_j) \geq 0$ then box_i is NW of box_j ,
- (d) if $i < j$ and $\tilde{c}(\text{box}_i) = -\tilde{c}(\text{box}_j)$, then $\tilde{c}(\text{box}_i) \leq 0 \leq \tilde{c}(\text{box}_j)$,

and define

$$J = \{ \varepsilon_i \mid \tilde{c}(\text{box}_i) \in \{-r_1, -r_2\} \} \\ \sqcup \left\{ \varepsilon_j - \varepsilon_i \mid \begin{array}{l} \tilde{c}(\text{box}_j) = \tilde{c}(\text{box}_i) + 1 > 0 \text{ and } \text{box}_j \text{ is NW of } \text{box}_i, \text{ or} \\ \tilde{c}(\text{box}_j) = \tilde{c}(\text{box}_i) - 1 < 0 \text{ and } \text{box}_j \text{ is SE of } \text{box}_i, \text{ or} \\ \tilde{c}(\text{box}_j) = -\tilde{c}(\text{box}_i) - 1 < 0 < \tilde{c}(\text{box}_i) \end{array} \right\} \\ \sqcup \left\{ \varepsilon_j + \varepsilon_i \mid \begin{array}{l} \tilde{c}(\text{box}_j) = -1 \text{ and } \tilde{c}(\text{box}_i) = 0 \text{ and } \text{box}_j \text{ is SE of } \text{box}_i, \text{ or} \\ \tilde{c}(\text{box}_j) = \frac{1}{2} \text{ and } \tilde{c}(\text{box}_i) = -\frac{1}{2} \text{ and } \text{box}_j \text{ is NW of } \text{box}_i, \text{ or} \\ \tilde{c}(\text{box}_j) = -\frac{1}{2} \text{ and } \tilde{c}(\text{box}_i) = -\frac{1}{2} \end{array} \right\}, \quad (5.29)$$

so that J is a subset of $P(\mathbf{c})$, where $P(\mathbf{c})$ is as defined in (3.7). See Examples 3 and 4 following the proof of Theorem 5.5.

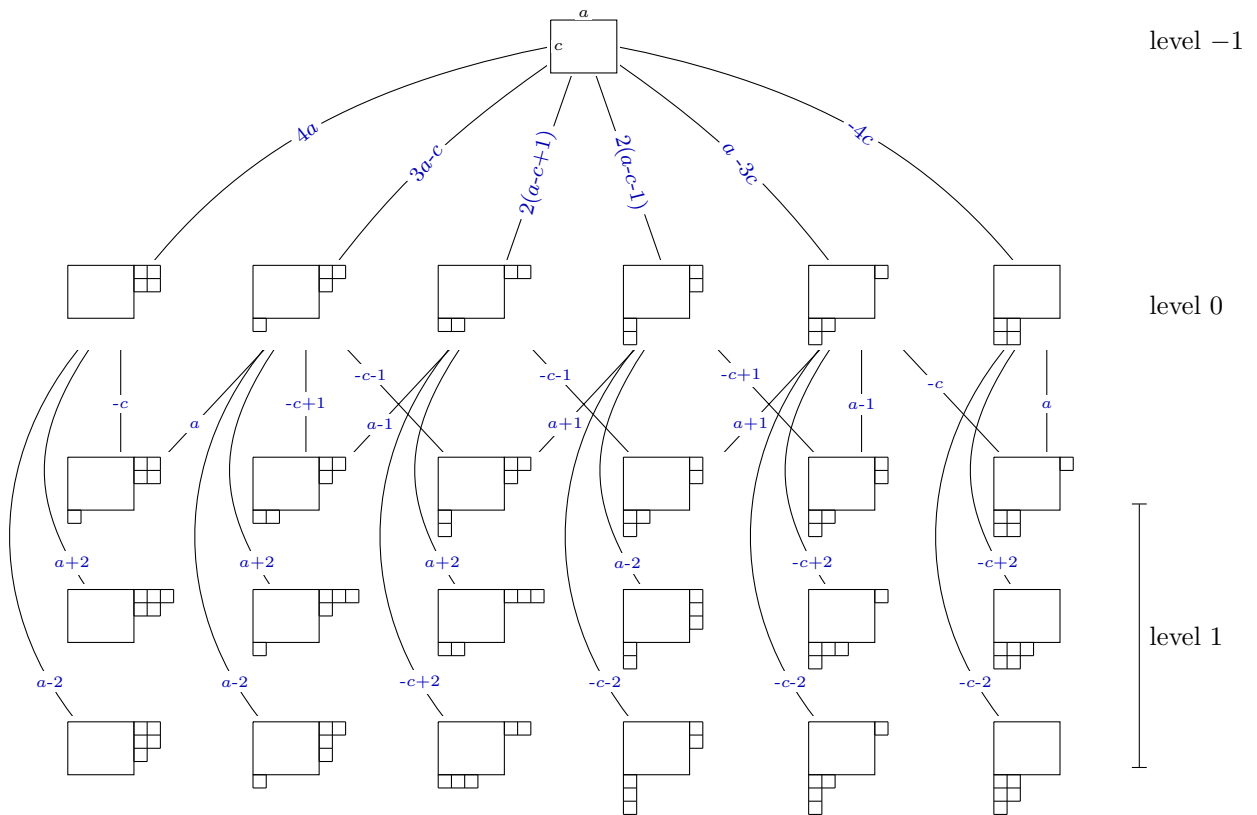


Figure 3: Levels -1 , 0 , and 1 of a Bratteli diagram encoding isotypic components of $M \otimes N \otimes V$ where $a, c > 2$ and $b = d = 2$. The edges from level -1 to level 0 are labeled by $e_0(T^{(0)})$ as in (5.17); the edges from level 0 to 1 are labeled by the content of the box added.

Theorem 5.5. Let $\mathfrak{g} = \mathfrak{gl}_n$ and let $M = L(a^c)$, $N = L(b^d)$ and $V = L(\square)$ so that H_k^{ext} acts on $M \otimes N \otimes V^{\otimes k}$ as in Theorem 5.4. Assume that the genericity conditions of (5.22) hold so that q is not a root of unity, $a, b, c, d \in \mathbb{Z}_{>0}$ and $(a+c) - (b+d) \notin \{0, \pm 1, \pm 2\}$. For $\lambda \in \mathcal{P}^{(k)}$, let B_k^λ be the H_k^{ext} -module of (5.23) and define z , \mathbf{c} and J as in (5.26), (5.28), and (5.29). Then

$$B_k^\lambda \cong H_k^{(z, \mathbf{c}, J)} \text{ as } H_k^{\text{ext}}\text{-modules.} \quad (5.30)$$

Proof. By Proposition 5.2, B_k^λ is a calibrated H_k^{ext} module. Therefore B_k^λ has a composition series with factors that are irreducible calibrated H_k^{ext} -modules. By Theorem 3.3, each factor is isomorphic to some $H_k^{(z, \mathbf{c}, J)}$ where (\mathbf{c}, J) is a skew local region, and (z, \mathbf{c}, J) is determined by the eigenvalues of the action of W_0, W_1, \dots, W_k . By Proposition 5.2, the simultaneous eigenbasis $\{v_S \mid S \in \mathcal{T}_k^\lambda\}$ B_k^λ is indexed by

$$\mathcal{T}_k^\lambda = \{\text{paths } S = ((a^c) \rightarrow S^{(0)} \rightarrow S^{(1)} \rightarrow \dots \rightarrow S^{(k)} = \lambda) \text{ in the Bratteli diagram}\}. \quad (5.31)$$

To determine which $H_k^{(z, \mathbf{c}, J)}$ appear as composition factors of B_k^λ it is necessary to compute the eigenvalues of the action of the W_i 's on the basis vectors v_S as follows.

By (5.24), (5.25), and the formulas in Proposition 5.2,

$$\Phi(P)v_S = q^{2e_0(S^{(0)})}v_S \quad \text{and} \quad \Phi(Z_i)v_S = q^{2c(S^{(i)}/S^{(i-1)})}v_S \quad \text{for } i = 1, \dots, k.$$

Using (2.18) and (5.19), $W_i = -(a_1 a_2 b_1 b_2)^{-\frac{1}{2}} Z_i$ with $a_1 = q^{2a}$, $a_2 = q^{-2c}$, $b_1 = q^{2b}$, and $b_2 = q^{-2d}$, and thus

$$\Phi(W_i)v_S = -(a_1 a_2 b_1 b_2)^{-\frac{1}{2}} \Phi(Z_i)v_S = -q^{-(a-c+b-d)} q^{2c(S^{(i)}/S^{(i-1)})} v_S = -q^{2\tilde{c}(S^{(i)}/S^{(i-1)})} v_S. \quad (5.32)$$

Then $\Phi(PW_1 \dots W_k)v_S = (-1)^k q^{2(e_0(S^{(0)})+c(S^{(1)}/S^{(0)})+\dots+c(S^{(k)}/S^{(k-1)})) - k(a-c+b-d)} v_S$ so that, with c_0 and z as in (5.26),

$$\Phi(W_0) = \Phi(PW_1 \dots W_k)v_S = (-1)^k q^{2c_0} v_S = z v_S. \quad (5.33)$$

Let $S = ((a^c) \rightarrow S^{(0)} \rightarrow S^{(1)} \rightarrow \dots \rightarrow S^{(k)} = \lambda)$ be a path to λ in the Bratteli diagram. In the context of the diagrams in (5.15), the partitions $S^{(0)}$ and $S_{\max}^{(0)}$ differ by moving some boxes from μ to μ^c (from the NW border of $\lambda/S_{\max}^{(0)}$ in \mathcal{B} to the NW border of $\lambda/S^{(0)}$ in \mathcal{B}'). Thus the sequence $\mathbf{c} = (c_1, \dots, c_k)$, where

$$c_1, \dots, c_k \text{ are the values } |\tilde{c}(S^{(1)}/S^{(0)})|, \dots, |\tilde{c}(S^{(k)}/S^{(k-1)})| \text{ arranged in increasing order,}$$

coincides with \mathbf{c} as defined in (5.28). Let $w_S \in \mathcal{W}_0$ be the minimal length element such that

$$w_S \mathbf{c} = w_S(c_1, \dots, c_k) = (c_{w_S^{-1}(1)}, \dots, c_{w_S^{-1}(k)}) = (\tilde{c}(S^{(1)}/S^{(0)}), \dots, \tilde{c}(S^{(k)}/S^{(k-1)})), \quad (5.34)$$

where $c_{-i} = -c_i$ for $i \in \{1, \dots, k\}$. The signed permutation w_S is the unique signed permutation such that

$$w_S \mathbf{c} = (\tilde{c}(S^{(1)}/S^{(0)}), \dots, \tilde{c}(S^{(k)}/S^{(k-1)})) \quad \text{and} \quad R(w_S) \cap Z(\mathbf{c}) = \emptyset,$$

where $Z(\mathbf{c})$ is as in (3.6). If the boxes of $\lambda/S^{(0)}$ are indexed according to the same conditions as just before (5.29), then w_S is the signed permutation given by

$$w_S(i) = \text{sgn}(\tilde{c}(\text{box}_i))(\text{entry in box}_i \text{ of } S),$$

where the path S is identified with the standard tableau of shape $\lambda/S^{(0)}$ that has $S^{(j)}/S^{(j-1)}$ filled with j .

The basis vector v_S appears in a composition factor isomorphic to $H_k^{(z, \mathbf{c}, J)}$ where

$$J = R(w_S) \cap P(\mathbf{c}), \quad \text{where } R(w_S) = R_1 \sqcup R_2 \sqcup R_3 \quad \text{and} \quad P(\mathbf{c}) = P_1 \sqcup P_2 \sqcup P_3,$$

as defined in (3.2) and (3.7), are given by

$$\begin{aligned} R_1 &= \{\varepsilon_i \mid i > 0 \text{ and } w_S(i) < 0\}, & P_1 &= \{\varepsilon_i \mid c_i \in \{r_1, r_2\}\}, \\ R_2 &= \{\varepsilon_j - \varepsilon_i \mid i < j \text{ and } w_S(i) > w_S(j)\}, & P_2 &= \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, c_j = c_i + 1\}, \\ R_3 &= \{\varepsilon_j + \varepsilon_i \mid i < j \text{ and } -w_S(i) > w_S(j)\}, & P_3 &= \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, c_j = -c_i + 1\}. \end{aligned}$$

To describe $J = (R_1 \cap P_1) \sqcup (R_2 \cap P_2) \sqcup (R_3 \cap P_3)$ in terms of the boxes in λ , first record that

$$R_1 \cap P_1 = \{\varepsilon_i \mid i > 0 \text{ and } w_S(i) < 0\} \cap \{\varepsilon_i \mid c_i \in \{r_1, r_2\}\} = \{\varepsilon_i \mid \tilde{c}(\text{box}_i) = \{-r_1, -r_2\}\}.$$

Next analyze

$$R_2 \cap P_2 = \{\varepsilon_j - \varepsilon_i \mid i < j \text{ and } w(i) > w(j)\} \cap \{\varepsilon_j - \varepsilon_i \mid 0 < i < j, c_j = c_i + 1\}.$$

Since $0 \leq c_i$ and $c_j = c_i + 1$, we have $c_j \geq 1$.

Case 1: $\tilde{c}(\text{box}_i) \geq 0$, so that $\tilde{c}(\text{box}_j) = \pm(\tilde{c}(\text{box}_i) + 1)$.

Case 1a: $\tilde{c}(\text{box}_j) = \tilde{c}(\text{box}_i) + 1$.

If box_j is NW of box_i then $w(j) < w(i)$ and $\varepsilon_j - \varepsilon_i \in J$.

If box_j is SE of box_i then $w(j) > w(i)$ and $\varepsilon_j - \varepsilon_i \notin J$.

Case 1b: $\tilde{c}(\text{box}_j) = -(\tilde{c}(\text{box}_i) + 1)$.

Then $w(j) < 0 < w(i)$ so that $w(j) < w(i)$ and $\varepsilon_j - \varepsilon_i \in J$.

Case 2: $\tilde{c}(\text{box}_i) < 0$, so that $\tilde{c}(\text{box}_j) = \pm(-\tilde{c}(\text{box}_i) + 1)$.

Case 2a: $\tilde{c}(\text{box}_j) = \tilde{c}(\text{box}_i) - 1 < \tilde{c}(\text{box}_i) < 0$.

If box_j is NW of box_i then $-w(j) < -w(i)$ so that $w(i) < w(j)$ and $\varepsilon_j - \varepsilon_i \notin J$.

If box_j is SE of box_i then $-w(j) > -w(i)$ so that $w(i) > w(j)$ and $\varepsilon_j - \varepsilon_i \in J$.

Case 2b: $\tilde{c}(\text{box}_j) = -\tilde{c}(\text{box}_i) + 1 > 0 > \tilde{c}(\text{box}_i)$.

Then $w(i) < 0$ and $0 < w(j)$ so that $\varepsilon_j - \varepsilon_i \notin J$.

Finally, analyze

$$R_3 \cap P_3 = \{\varepsilon_j + \varepsilon_i \mid i < j \text{ and } -w(i) > w(j)\} \cap \{\varepsilon_j + \varepsilon_i \mid 0 < i < j, c_j = -c_i + 1\}.$$

Since $0 \leq c_i$ and $c_j = -c_i + 1 \geq c_i$, we have $0 \leq c_i \leq 1/2$. Since the entries of \mathbf{c} are in \mathbb{Z} or in $\frac{1}{2} + \mathbb{Z}$, the possibilities for (c_i, c_j) are $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$, and the possibilities for $(\tilde{c}(\text{box}_i), \tilde{c}(\text{box}_j))$

are $(0, 1)$, $(0, -1)$, $(\frac{1}{2}, \pm\frac{1}{2})$, or $(-\frac{1}{2}, \pm\frac{1}{2})$.

Case 1: $\tilde{c}(\text{box}_j) = 1$ and $\tilde{c}(\text{box}_i) = 0$.

If box_j is NW of box_i then $0 < w(j) < w(i)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

If box_j is SE of box_i then $0 < w(i) < w(j)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

Case 2: $\tilde{c}(\text{box}_j) = -1$ and $\tilde{c}(\text{box}_i) = 0$.

If box_j is NW of box_i then $-w(j) < w(i)$ so that $-w(i) < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

If box_j is SE of box_i then $-w(j) > w(i)$ so that $-w(i) > w(j)$ and $\varepsilon_j + \varepsilon_i \in J$.

Case 3: $\tilde{c}(\text{box}_j) = \frac{1}{2}$ and $\tilde{c}(\text{box}_i) = \frac{1}{2}$.

Then $0 < w(i) < w(j)$ so that $-w(i) < 0 < w(j)$ and $\varepsilon_j + \varepsilon_i \notin J$.

Case 4: $\tilde{c}(\text{box}_j) = -\frac{1}{2}$ and $\tilde{c}(\text{box}_i) = \frac{1}{2}$.

This case cannot occur since, when indexing the boxes of $\lambda/S^{(0)}$,

the boxes of shifted content $-\frac{1}{2}$ are numbered before the boxes of shifted content $\frac{1}{2}$.

Case 5: $\tilde{c}(\text{box}_j) = \frac{1}{2}$ and $\tilde{c}(\text{box}_i) = -\frac{1}{2}$.

If box_j is NW of box_i then $w(i) < 0$ and $w(j) < -w(i)$ so that $\varepsilon_j + \varepsilon_i \in J$.

If box_j is SE of box_i then $w(i) < 0$ and $-w(i) < w(i)$ so that $\varepsilon_j + \varepsilon_i \notin J$.

Case 6: $\tilde{c}(\text{box}_j) = -\frac{1}{2}$ and $\tilde{c}(\text{box}_i) = -\frac{1}{2}$.

Then $0 < -w(j) < -w(i)$ and $w(j) < 0 < -w(i)$ so that $\varepsilon_j + \varepsilon_i \in J$.

This analysis shows that $J = R(w_S) \cap P(\mathbf{c}) = (R_1 \cap P_1) \sqcup (R_2 \cap P_2) \sqcup (R_3 \cap P_3)$ is as given in (5.29).

A consequence of the description of J in (5.29) is that $J = R(w_S) \cap P(\mathbf{c})$ is independent of the choice of $S \in \mathcal{T}_k^\lambda$. It follows that all composition factors of B_k^λ are isomorphic to $H_k^{(z, \mathbf{c}, J)}$.

Let $S, T \in \mathcal{T}_k^\lambda$ such that v_S and v_T have the same eigenvalues for W_0, \dots, W_k . By definition of \mathcal{T}_k^λ , $S^{(k)} = T^{(k)} = \lambda$. Since $W_k v_S = -q^{\tilde{c}(S^{(k)}/S^{(k-1)})} v_S = -q^{\tilde{c}(\lambda/S^{(k-1)})} v_S$ and $W_k v_T = -q^{\tilde{c}(T^{(k)}/T^{(k-1)})} v_T = -q^{\tilde{c}(\lambda/T^{(k-1)})} v_T$, we have $\tilde{c}(\lambda/T^{(k-1)}) = \tilde{c}(\lambda/S^{(k-1)})$ which implies that $T^{(k-1)} = S^{(k-1)}$. Using this and the fact that the eigenvalues of W_{k-1} on v_S and v_T are the same, implies similarly that $T^{(k-2)} = S^{(k-2)}$. Induction gives that

$$S^{(0)} = T^{(0)}, \quad \dots, \quad S^{(k)} = T^{(k)} \quad \text{so that} \quad S = T.$$

Thus $\dim((B_k^\lambda)_\gamma) \leq 1$ (in the notation of (3.1)) and $B_k^\lambda \cong H_k^{(z, \mathbf{c}, J)}$ as H_k^{ext} -modules. \square

In the course of the proof of Theorem 5.5 we have also established the following result, which deserves mention.

Corollary 5.6. *Keeping the notations of Theorem 5.5, let $\lambda \in \mathcal{P}^{(k)}$ and $S \in \mathcal{T}_k^\lambda$, and let w_S be the signed permutation defined in (5.34). Then*

$$\begin{array}{ccc} \mathcal{T}_k^\lambda & \longrightarrow & \mathcal{F}^{(\mathbf{c}, J)} \\ S & \longmapsto & w_S \end{array} \quad \text{is a bijection.}$$

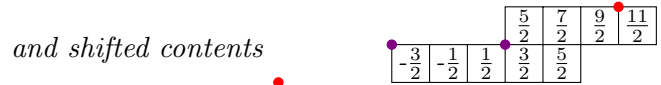
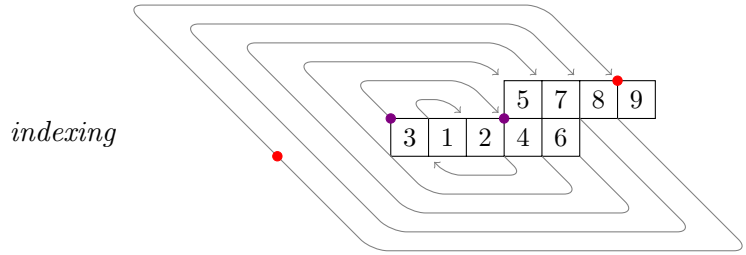
Example 3. *Let $M = L(a^c) = L(6)$ and $N = L(b^d) = L(3)$ so that*

$$a = 6, \quad c = 1, \quad b = 3, \quad d = 1, \quad r_1 = \frac{3}{2}, \quad \text{and} \quad r_2 = \frac{11}{2}.$$

The partition $\lambda = (10, 8)$ is in $\mathcal{P}^{(k)}$ with $k = 9$. Then we draw λ as the (marked) partition

$$\lambda = (10, 8) = \begin{array}{cccccccccc} \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} & \color{red}{\bullet} \\ \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} & \color{purple}{\bullet} \end{array}. \quad \text{Here,} \quad S_{\max}^{(0)} = (6, 3)$$

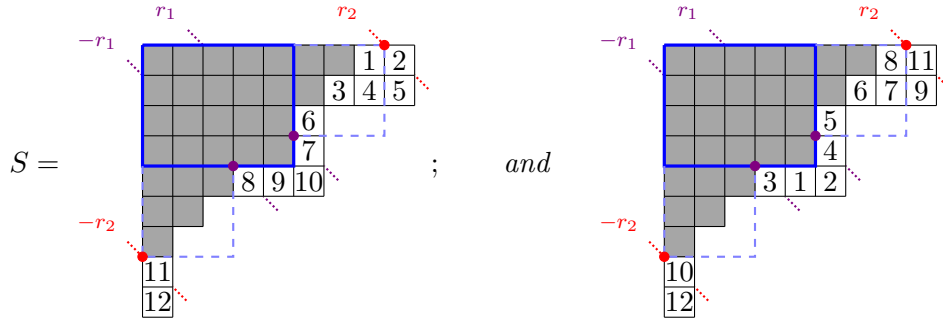
is indicated by the shaded boxes. The boxes of $\lambda/S_{\max}^{(0)}$ have



Example 4. Let $M = L(a^c) = L(5^4)$ and $N = L(b^d) = L(3^3)$ so that

$$a = 5, \quad c = 4, \quad b = 3, \quad d = 3, \quad r_1 = \frac{3}{2}, \quad \text{and} \quad r_2 = \frac{15}{2}.$$

The partition $\lambda = (9, 9, 6, 6, 6, 2, 1, 1, 1)$ is in $\mathcal{P}^{(k)}$ with $k = 12$. For this partition $S_{\max}^{(0)} = (7, 6, 5, 5, 3, 2, 1)$; and one tableau $S \in \mathcal{T}_k^\lambda$ with $S^{(0)} = S_{\max}^{(0)}$ (where the shaded portion of λ corresponds to $S^{(0)}$) is



indicates the indexing of the boxes in $\lambda/S_{\max}^{(0)}$. The contents of the boxes $S^{(i)}/S^{(i-1)}$ for $i = 1, \dots, k$ are 7, 8, 5, 6, 7, 3, 2, -1, 0, 1, -7, -8; and since $-\frac{1}{2}(a - c + b - d) = -\frac{1}{2}$, the shifted contents $\tilde{c}(S^{(i)}/S^{(i-1)})$ for $i = 1, \dots, k$ are

$$\frac{13}{2}, \frac{15}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{5}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{15}{2}, -\frac{17}{2},$$

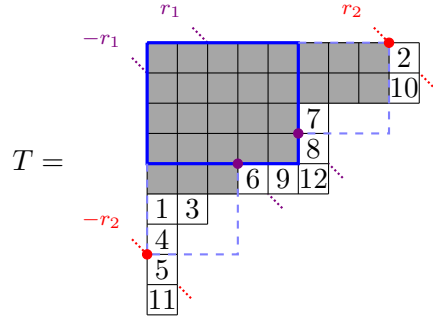
respectively. The sum of the contents of the boxes in $S_{\max}^{(0)}$ is 1, the sum of the contents of the boxes in λ is 23, $c_0 = -\frac{1}{2}(12(5 - 4 + 3 - 3) + 5 \cdot 4(5 - 4) + 3 \cdot 3(3 - 3)) + 24 = 8$,

$$z = q^{16}, \quad \text{and} \quad \mathbf{c} = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{13}{2}, \frac{15}{2}, \frac{15}{2}, \frac{17}{2}\right)$$

is the sequence of absolute values of the shifted contents, arranged in increasing order. Using (5.34),

$$\begin{aligned}
w_S &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ -9 & 10 & -8 & 7 & 6 & 3 & 4 & 1 & 5 & -11 & 2 & -12 \end{pmatrix}, \\
P(\mathbf{c}) &= \left\{ \begin{array}{l} \boxed{\varepsilon_3}, \varepsilon_4, \boxed{\varepsilon_{10}}, \varepsilon_{11}, \varepsilon_2 - \varepsilon_{-1}, \\ \varepsilon_3 - \varepsilon_1, \varepsilon_4 - \varepsilon_1, \boxed{\varepsilon_3 - \varepsilon_2}, \boxed{\varepsilon_4 - \varepsilon_2}, \varepsilon_5 - \varepsilon_3, \boxed{\varepsilon_5 - \varepsilon_4}, \\ \varepsilon_7 - \varepsilon_6, \boxed{\varepsilon_8 - \varepsilon_7}, \varepsilon_9 - \varepsilon_7, \\ \boxed{\varepsilon_{10} - \varepsilon_8}, \varepsilon_{11} - \varepsilon_8, \boxed{\varepsilon_{10} - \varepsilon_9}, \boxed{\varepsilon_{11} - \varepsilon_9}, \boxed{\varepsilon_{12} - \varepsilon_{10}}, \boxed{\varepsilon_{12} - \varepsilon_{11}}, \end{array} \right\}, \\
R(w_S) &= \left\{ \begin{array}{l} \varepsilon_1, \boxed{\varepsilon_3}, \boxed{\varepsilon_{10}}, \varepsilon_{12} \\ \varepsilon_{10} - \varepsilon_1, \varepsilon_{12} - \varepsilon_1, \boxed{\varepsilon_3 - \varepsilon_2}, \boxed{\varepsilon_4 - \varepsilon_2}, \varepsilon_5 - \varepsilon_2, \varepsilon_6 - \varepsilon_2, \varepsilon_7 - \varepsilon_2, \varepsilon_8 - \varepsilon_2, \varepsilon_9 - \varepsilon_2, \varepsilon_{10} - \varepsilon_2, \\ \varepsilon_{11} - \varepsilon_2, \varepsilon_{12} - \varepsilon_2, \varepsilon_{10} - \varepsilon_3, \varepsilon_{12} - \varepsilon_3, \boxed{\varepsilon_5 - \varepsilon_4}, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4, \varepsilon_8 - \varepsilon_4, \varepsilon_9 - \varepsilon_4, \varepsilon_{10} - \varepsilon_4, \\ \varepsilon_{11} - \varepsilon_4, \varepsilon_{12} - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5, \varepsilon_8 - \varepsilon_5, \varepsilon_9 - \varepsilon_5, \varepsilon_{10} - \varepsilon_5, \varepsilon_{11} - \varepsilon_5, \varepsilon_{12} - \varepsilon_5, \varepsilon_8 - \varepsilon_6, \\ \varepsilon_{10} - \varepsilon_6, \varepsilon_{11} - \varepsilon_6, \varepsilon_{12} - \varepsilon_6, \boxed{\varepsilon_8 - \varepsilon_7}, \varepsilon_{10} - \varepsilon_7, \varepsilon_{11} - \varepsilon_7, \varepsilon_{12} - \varepsilon_7, \boxed{\varepsilon_{10} - \varepsilon_8}, \varepsilon_{12} - \varepsilon_8, \\ \boxed{\varepsilon_{10} - \varepsilon_9}, \boxed{\varepsilon_{11} - \varepsilon_9}, \varepsilon_{12} - \varepsilon_9, \boxed{\varepsilon_{12} - \varepsilon_{10}}, \boxed{\varepsilon_{12} - \varepsilon_{11}}, \\ \varepsilon_3 + \varepsilon_1, \varepsilon_4 + \varepsilon_1, \varepsilon_5 + \varepsilon_1, \varepsilon_6 + \varepsilon_1, \varepsilon_7 + \varepsilon_1, \varepsilon_8 + \varepsilon_1, \varepsilon_9 + \varepsilon_1, \varepsilon_{10} + \varepsilon_1, \varepsilon_{11} + \varepsilon_1, \varepsilon_{12} + \varepsilon_1, \\ \varepsilon_{10} + \varepsilon_2, \varepsilon_{12} + \varepsilon_2, \varepsilon_4 + \varepsilon_3, \varepsilon_5 + \varepsilon_3, \varepsilon_6 + \varepsilon_3, \varepsilon_7 + \varepsilon_3, \varepsilon_8 + \varepsilon_3, \varepsilon_9 + \varepsilon_3, \varepsilon_{10} + \varepsilon_3, \varepsilon_{11} + \varepsilon_3, \\ \varepsilon_{12} + \varepsilon_3, \varepsilon_{10} + \varepsilon_4, \varepsilon_{12} + \varepsilon_4, \varepsilon_{10} + \varepsilon_5, \varepsilon_{12} + \varepsilon_5, \varepsilon_{10} + \varepsilon_6, \varepsilon_{12} + \varepsilon_6, \varepsilon_{10} + \varepsilon_7, \varepsilon_{12} + \varepsilon_7, \\ \varepsilon_{10} + \varepsilon_8, \varepsilon_{12} + \varepsilon_8, \varepsilon_{10} + \varepsilon_9, \varepsilon_{12} + \varepsilon_9, \varepsilon_{11} + \varepsilon_{10}, \varepsilon_{12} + \varepsilon_{10}, \varepsilon_{12} + \varepsilon_{11}, \end{array} \right\},
\end{aligned}$$

and $J = R(w_S) \cap P(\mathbf{c})$ consists of the outlined elements of $P(\mathbf{c})$ (which are the same as the outlined elements of $R(w_S)$). Another $T \in \mathcal{T}_k^\lambda$ is (again, with $T^{(0)}$ indicated by the shaded boxes)



Keeping the setting of Theorem 5.5, Proposition 3.1 associates a configuration of $2k$ boxes to (\mathbf{c}, J) . This configuration can be described in terms of the data of $\lambda \in \mathcal{P}^{(k)}$ as follows. With $S_{\max}^{(0)}$ as defined just before (5.28), let $\text{rot}(\lambda/S_{\max}^{(0)})$ be the 180° rotation of the skew shape $\lambda/S_{\max}^{(0)}$. Then

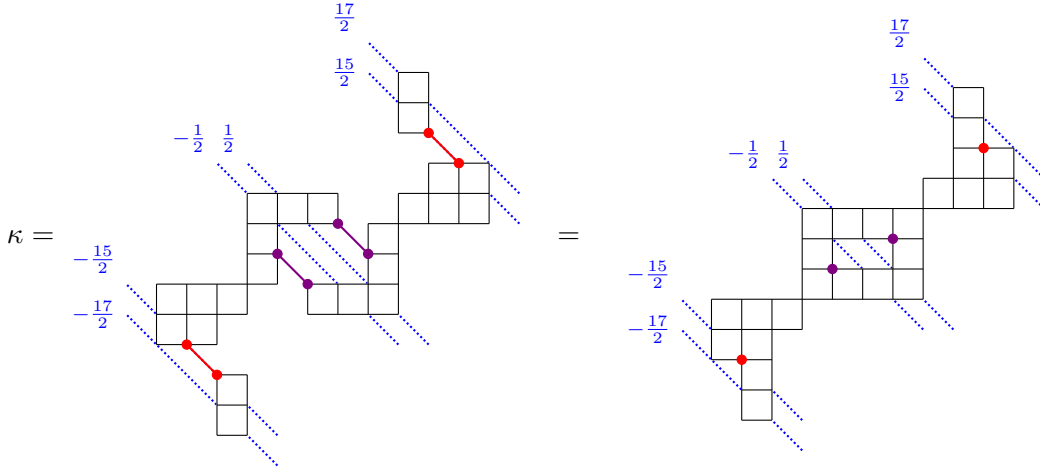
$$\text{the configuration of boxes } \kappa \text{ corresponding to } (\mathbf{c}, J) \text{ is } \kappa = \text{rot}(\lambda/S_{\max}^{(0)}) \cup \lambda/S_{\max}^{(0)}, \quad (5.35)$$

so that it is the (disjoint) union of two skew shapes $\lambda/S_{\max}^{(0)}$ and $\text{rot}(\lambda/S_{\max}^{(0)})$, placed with $\text{rot}(\lambda/S_{\max}^{(0)})$ northwest of $\lambda/S_{\max}^{(0)}$,

$\lambda/S_{\max}^{(0)}$ positioned so that the contents of its boxes are $(\tilde{c}(S^{(1)}/S^{(0)}), \dots, \tilde{c}(S^{(k)}/S^{(k-1)}))$,

$\text{rot}(\lambda/S_{\max}^{(0)})$ positioned so that the contents of its boxes are $(-\tilde{c}(S^{(k)}/S^{(k-1)}), \dots, -\tilde{c}(S^{(1)}/S^{(0)}))$,

and with markings placed at the NE corners of the rectangles \mathcal{B} and \mathcal{B}' corresponding to $\lambda/S^{(0)}$ (in the notation of (5.15)). The resulting doubled skew shape is symmetric under the 180° rotation which sends a box on diagonal c_i to a box on diagonal $-c_i$. In the case of Example 4 the corresponding configuration of boxes is

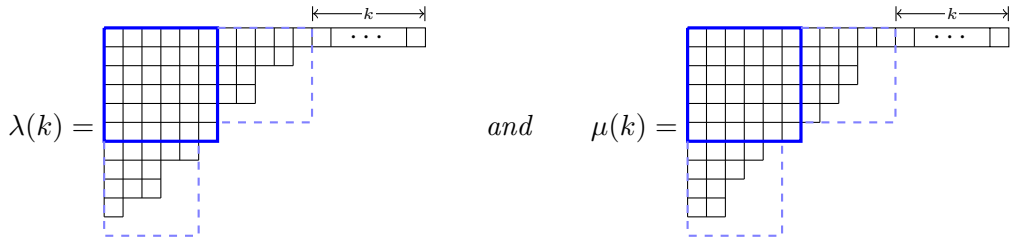


This configuration of boxes also appeared in Example 2.

For generically large a, b, c, d , there will be examples of $\lambda, \mu \in \mathcal{P}^{(k)}$ with $\lambda \neq \mu$ and $B_k^\lambda \cong B_k^\mu$ as H_k^{ext} -modules; see Example 5. This is because the eigenvalues of P on $M \otimes N$ are not sufficient to distinguish the components of $M \otimes N$ as a \mathfrak{gl}_n -module. It could be helpful to further extend H_k^{ext} and consider an algebra $Z(U_q \mathfrak{gl}_n) \otimes H_k$ acting on $M \otimes N \otimes V^{\otimes k}$.

Example 5. Let $a = c = 6$ and $b = d = 4$,

$$\lambda(k) = (11 + k, 10, 8, 8, 6, 6, 5, 3, 3, 1) \quad \text{and} \quad \mu(k) = (11 + k, 9, 9, 8, 7, 6, 4, 3, 2, 2), \quad \text{i.e.}$$



Then $\lambda(k) \neq \mu(k)$ but, as H_k^{ext} -modules,

$$B_k^{\lambda(k)} \cong B_k^{\mu(k)} \cong H_k^{(z, \mathbf{c}, \emptyset)}, \quad \text{where } \mathbf{c} = (11, 12, \dots, 11 + k - 1) \text{ and } z = q^{28+k(k+21)}.$$

Recall from (5.23) that

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^\lambda \quad \text{as } (U_q \mathfrak{g}, \mathcal{H}_k^{\text{ext}})\text{-bimodules.}$$

A consequence of Theorem 3.3(b) is the following construction of the irreducible H_k^{ext} -modules B_k^λ . Keeping the setting and notation of (5.31), for $\lambda \in \mathcal{P}^{(k)}$ and $S \in \mathcal{T}_k^\lambda$, let

$$s_j S \text{ be the path from } (a^c) \text{ to } \lambda \text{ that differs from } S \text{ only at } S^{(j)}. \quad (5.36)$$

The path $s_j S$ is unique if it exists: if $S = ((a^c) \rightarrow S^{(0)} \rightarrow S^{(1)} \rightarrow \dots \rightarrow S^{(k)})$ then $S^{(j+1)}$ is obtained by adding a box to $S^{(j)}$, and $(s_j S)^{(j)}$ is obtained by moving a box of $S^{(j)}$ to the position of the added box in $S^{(j+1)}$. In the case that $j = 0$, the paths $s_0 S$ and S satisfy $(s_0 S)^{(1)} = S^{(1)}$ and the partitions $(s_0 S)^{(0)}$ and $S^{(0)}$ in $\mathcal{P}^{(0)}$ differ by the placement of one box, with

$$\tilde{c}((s_0 S)^{(1)}/(s_0 S)^{(0)}) = -\tilde{c}(S^{(1)}/S^{(0)}), \quad (5.37)$$

where the shifted content of a box $\tilde{c}(\text{box})$ is as defined in (5.28).

Corollary 5.7. *Keep the conditions of Theorems 5.4 and 5.5. In particular, assume that the genericity conditions of (5.22) hold so that q is not a root of unity, $a, b, c, d \in \mathbb{Z}_{>0}$ and $(a+c) - (b+d) \notin \{0, \pm 1, \pm 2\}$. Let $\lambda \in \mathcal{P}^{(k)}$. Then B_k^λ has a basis $\{v_S \mid S \in \mathcal{T}_k^\lambda\}$ such that the H_k^{ext} -action is given by*

$$\begin{aligned} P v_S &= q^{2e_0(T)} v_S, & Z_i v_S &= q^{2c(S^{(i)}/S^{(i-1)})} v_S, \\ T_i v_S &= [T_i]_{S,S} v_S + \sqrt{-([T_i]_{S,S} - q)([T_i]_{S,S} + q^{-1})} v_{s_i S}, & \text{for } i = 1, \dots, k-1, \\ Y_1 v_S &= [Y_1]_{S,S} v_S + \sqrt{-([Y_1]_{S,S} - q^{-2d})([Y_1]_{S,S} - q^{2b})} v_{s_0 S}, \\ X_1 v_S &= [X_1]_{S,S} v_S + q^{-2c(S^{(1)}/S^{(0)})} q^{(a-c+b-d)} \sqrt{-([X_1]_{S,S} - q^{2a})([X_1]_{S,S} - q^{-2c})} v_{s_0 S}, \end{aligned}$$

where $v_{s_j S} = 0$ if $s_j S$ does not exist, and

$$\begin{aligned} [T_i]_{S,S} &= \frac{q - q^{-1}}{1 - q^{2(c(S^{(i)}/S^{(i-1)}) - c(S^{(i+1)}/S^{(i)}))}}, \\ [Y_1]_{S,S} &= \frac{(q^{2b} + q^{-2d}) - (q^{2a} + q^{-2c})q^{2(b-d)}q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)}q^{-4c(S^{(1)}/S^{(0)})}}, \\ [X_1]_{S,S} &= \frac{(q^{2a} + q^{-2c}) - (q^{2b} + q^{-2d})q^{2(a-c)}q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)}q^{-4c(S^{(1)}/S^{(0)})}}. \end{aligned}$$

Proof. The appropriate basis of B_k^λ is the one given in Proposition 5.2 and used also in the proof of Theorem 5.5. It is only necessary to convert from the notation v_w in Theorem 3.3 to the notation v_S using the bijection in Corollary 5.6. Recall from (5.21) that

$$a_1 = q^{2a}, \quad a_2 = q^{-2c}, \quad b_1 = q^{2b}, \quad b_2 = q^{-2d}, \\ t^{\frac{1}{2}} = q, \quad t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}}(-a_2)^{-\frac{1}{2}} = -iq^{a+c}, \quad \text{and} \quad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}}(-b_2)^{-\frac{1}{2}} = -iq^{b+d}.$$

From (3.12) and (5.32),

$$\gamma_{w^{-1}(i)} v_w = \Phi(W_i) v_S = -q^{-(a-c+b-d)} q^{2c(S^{(i)}/S^{(i-1)})} v_S.$$

From (2.18), (2.9), and (h), $Y_1 = b_1^{\frac{1}{2}}(-b_2)^{\frac{1}{2}} T_0 = iq^{b-d} T_0$ and $X_1 = (a_1 + a_2) - a_1 a_2 Y_1 Z_1^{-1} = q^{2a} + q^{-2c} - q^{2(a-c)} Y_1 Z_1^{-1}$. With these conversions, the formulas from (3.13) and (3.14) become

$$\begin{aligned} T_i v_S &= T_i v_w = [T_i]_{S,S} v_S + [T_i]_{s_i S, S} v_{s_i S}, & \text{for } i = 1, \dots, k-1, \\ Y_1 v_S &= iq^{b-d} T_0 v_w = [Y_1]_{S,S} v_S + [Y_1]_{s_0 S, S} v_{s_0 S}, & \text{and} \\ X_1 v_S &= \left(q^{2a} + q^{-2c} - q^{2(a-c)} Y_1 Z_1^{-1} \right) v_S = \left(q^{2a} + q^{-2c} - q^{2(a-c)} q^{-2c(S^{(1)}/S^{(0)})} Y_1 \right) v_S \\ &= [X_1]_{S,S} v_S - [X_1]_{s_0 S, S} v_{s_0 S}, \end{aligned}$$

with

$$\begin{aligned}
[T_i]_{S,S} &= [T_i]_{ww} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)} \gamma_{w^{-1}(i+1)}^{-1}} = \frac{q - q^{-1}}{1 - q^{2(c(S^{(i)}/S^{(i-i)}) - c(S^{(i+1)}/S^{(i)}))}}, \\
[Y_1]_{S,S} &= iq^{b-d}[T_0]_{ww} = iq^{b-d} \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) \gamma_{w^{-1}(1)}^{-1}}{1 - \gamma_{w^{-1}(1)}^{-2}} \\
&= iq^{b-d}(-i) \frac{(q^{(b+d)} + q^{-(b+d)}) - (q^{(a+c)} + q^{-(a+c)}) q^{a-c+b-d} q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)} q^{-4c(S^{(1)}/S^{(0)})}} \\
&= \frac{(q^{2b} + q^{-2d}) - (q^{2a} + q^{-2c}) q^{2(b-d)} q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)} q^{-4c(S^{(1)}/S^{(0)})}}, \text{ and} \\
[X_1]_{S,S} &= q^{2a} + q^{-2c} - q^{2(a-c)} q^{-2c(S^{(1)}/S^{(0)})} [Y_1]_{S,S} \\
&= q^{2a} + q^{-2c} - q^{2(a-c)} q^{-2c(S^{(1)}/S^{(0)})} \frac{(q^{2b} + q^{-2d}) - (q^{2a} + q^{-2c}) q^{2(b-d)} q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)} q^{-4c(S^{(1)}/S^{(0)})}} \\
&= \frac{(q^{2a} + q^{-2c}) - (q^{2b} + q^{-2d}) q^{2(a-c)} q^{-2c(S^{(1)}/S^{(0)})}}{1 - q^{2(a-c+b-d)} q^{-4c(S^{(1)}/S^{(0)})}}.
\end{aligned}$$

On the two-dimensional subspace $\text{span}_{\mathbb{C}}\{v_S, v_{s_0S}\}$ the action of T_0 in the basis $\{v_S, v_{s_0S}\}$ is a symmetric matrix $[T_0]$, and so the matrix of Y_1 in this basis is $[Y_1] = iq^{b-d}[T_0]$ is also symmetric. The action of Z_1 is by a diagonal matrix $[Z_1]$, so $[Z_1]^t = [Z_1]$. Therefore, using $X_1 = Z_1 Y_1^{-1}$ from (2.9) and $X_1 = (a_1 + a_2) - a_1 a_2 X_1^{-1}$ from (h), we have $([X_1]^{-1})^t = ([Y_1][Z_1]^{-1})^t = ([Z_1]^{-1})^t [Y_1]^t = [Z_1]^{-1} [Y_1]$ and so

$$[Z_1][X_1]^t[Z_1]^{-1} = [Z_1]((a_1 + a_2) - a_1 a_2 [Z_1]^{-1} [Y_1])[Z_1]^{-1} = [X_1].$$

Thus

$$[Z_1]_{S,S} [X_1]_{s_0S,S} [Z_1^{-1}]_{s_0L,s_0S} = [X_1]_{S,s_0S} \quad \text{and} \quad -[X_1]_{S,s_0S} [X_1]_{s_0S,S} = ([X_1]_{S,S} - a_1)([X_1]_{S,S} - a_2),$$

since $[X_1]$ is a 2×2 matrix with eigenvalues a_1 and a_2 (as in the proof of Theorem 3.3). Thus

$$\begin{aligned}
[X_1]_{s_0S,S} &= \sqrt{([X_1]_{s_0S,S})^2} = \sqrt{[X_1]_{S,s_0S} [Z_1]_{S,S}^{-1} [X_1]_{s_0S,S} [Z_1]_{s_0S,s_0S}} \\
&= \sqrt{[Z_1]_{S,S}^{-1} [Z_1]_{s_0S,s_0S}} \sqrt{-([X_1]_{S,S} - q^{2a})([X_1]_{S,S} - q^{-2c})}.
\end{aligned}$$

By (5.37), $c((s_0S)^{(1)}/(s_0S)^{(0)}) = -c(S^{(1)}/S^{(0)}) + (a - c + b - d)$, so that

$$\sqrt{[Z_1]_{S,S}^{-1} [Z_1]_{s_0S,s_0S}} = q^{-c(S^{(1)}/S^{(0)})} q^{c((s_0S)^{(1)}/(s_0S)^{(0)})} = q^{-2c(S^{(1)}/S^{(0)}) + (a-c+b-d)}.$$

Thus

$$[X_1]_{s_0S,S} = q^{-2c(S^{(1)}/S^{(0)})} q^{(a-c+b-d)} \sqrt{-([X_1]_{S,S} - q^{2a})([X_1]_{S,S} - q^{-2c})}.$$

□

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