

# Positive level, negative level and level zero

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*Dedicated to I.G. Macdonald and A.O. Morris*

## Abstract

This is a survey on the combinatorics and geometry of integrable representations of quantum affine Lie algebras with a particular focus on level 0. Pictures and examples are included to illustrate the affine Weyl group orbits, crystal graphs and Macdonald polynomials that provide detailed understanding of the structure of the extremal weight modules and their characters. The final section surveys the alcove walk method of working with the positive level, negative level and level zero affine flag varieties and describes the corresponding actions of the affine Hecke algebra.

**Key words**— affine flag varieties, integrable representations, quantum affine Lie algebras

## 0 Introduction

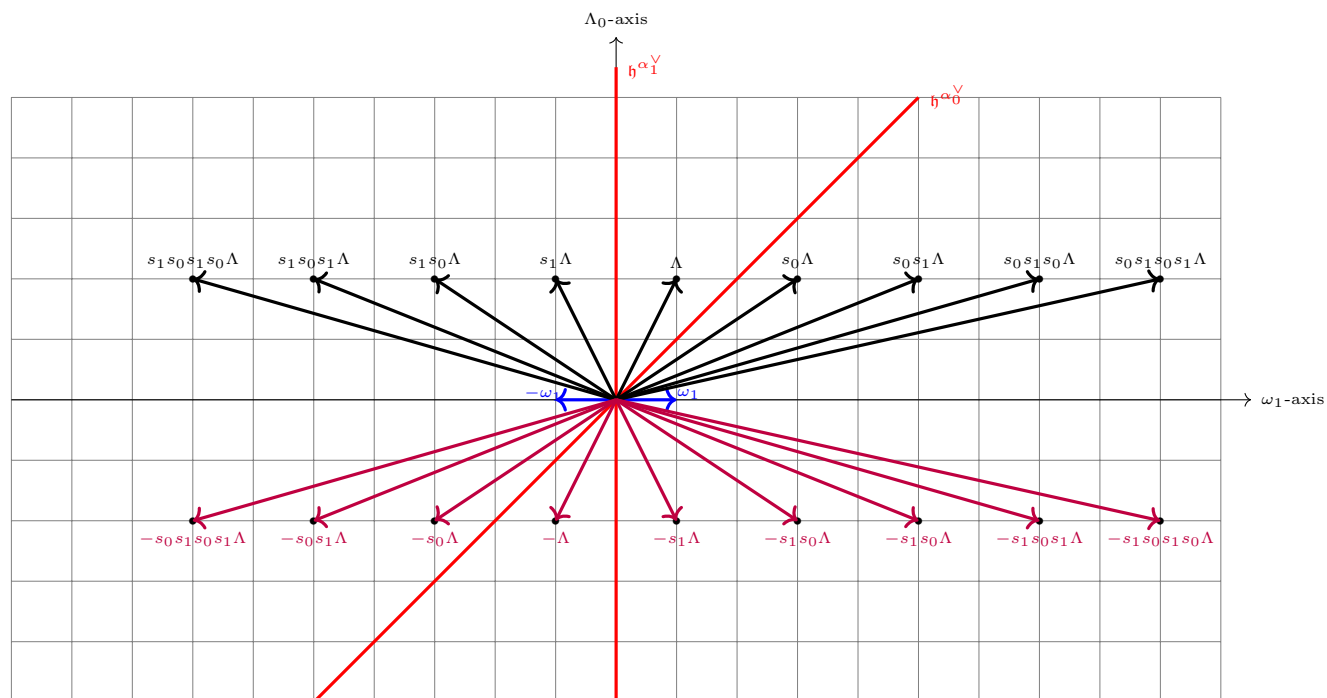
This paper is about positive level, negative level and level 0. It was motivated by the striking result of [KNS17], which establishes a Pieri-Chevalley formula for the K-theory of the semi-infinite (level 0) affine flag variety. This made us want to learn more about the level 0 integrable modules of quantum affine algebras. Our trek brought us face to face with a huge literature, including important contributions from Drinfeld, Kashiwara, Beck, Chari, Nakajima, Lenart-Schilling-Shimozono, Cherednik-Orr, Naito-Sagaki, Feigin-Makedonskyi, Kato, their coauthors and many others. It is a beautiful theory and we count ourselves lucky to have been drawn into it.

The main point is that the integrable modules for quantum affine algebras  $\mathbf{U}$  naturally partition themselves into families: positive level, negative level and level zero. Their structure is shadowed by the orbits of the affine Weyl group on the lattice of weights for the affine Lie algebra, which take the shape of a concave up paraboloid at positive level, a concave down paraboloid at negative level and a tube at level 0. These integrable modules have crystal bases which provide detailed control of their characters. At level 0 the characters are (up to a factor similar to a Weyl denominator) Macdonald polynomials specialised at  $t = 0$ . The next amazing feature is that there are Borel-Weil-Bott theorems for each case: positive level, negative level and level 0, where, respectively, the appropriate geometry is a positive level (thin) affine flag variety, a negative level (thick) affine flag variety, and a level zero (semi-infinite) affine flag variety.

This paper is a survey of the general picture of positive level, negative level, and level zero, in the context of the combinatorics of affine Weyl groups and crystals, of the representation theory of integrable modules for quantum affine algebras, and of the geometry of affine flag varieties. In recent years, the picture has become more and more rich and taken clearer focus. We hope that this paper will help to bring this story to a wider audience by providing pictures and some explicit small examples for  $\widehat{\mathfrak{sl}}_2$  and  $\widehat{\mathfrak{sl}}_3$ .

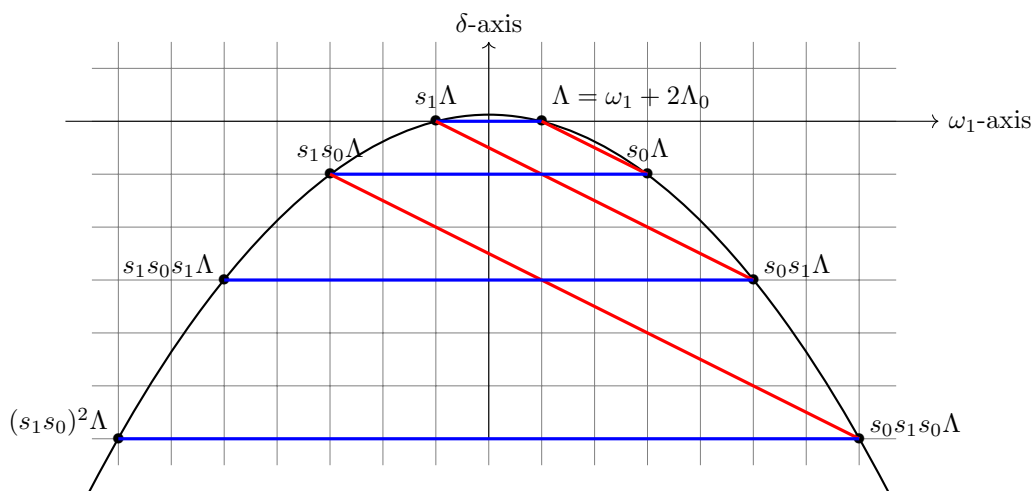
### 0.1 Orbits of the affine Weyl group $W$ action on $\mathfrak{h}^*$

For  $\widehat{\mathfrak{sl}}_2$  the vector space  $\mathfrak{h}^*$  is three dimensional with basis  $\{\delta, \omega_1, \Lambda_0\}$ , and the orbits of the action of the affine Weyl group  $W^{\text{ad}}$  on  $\mathfrak{h}^*$  on different levels look like



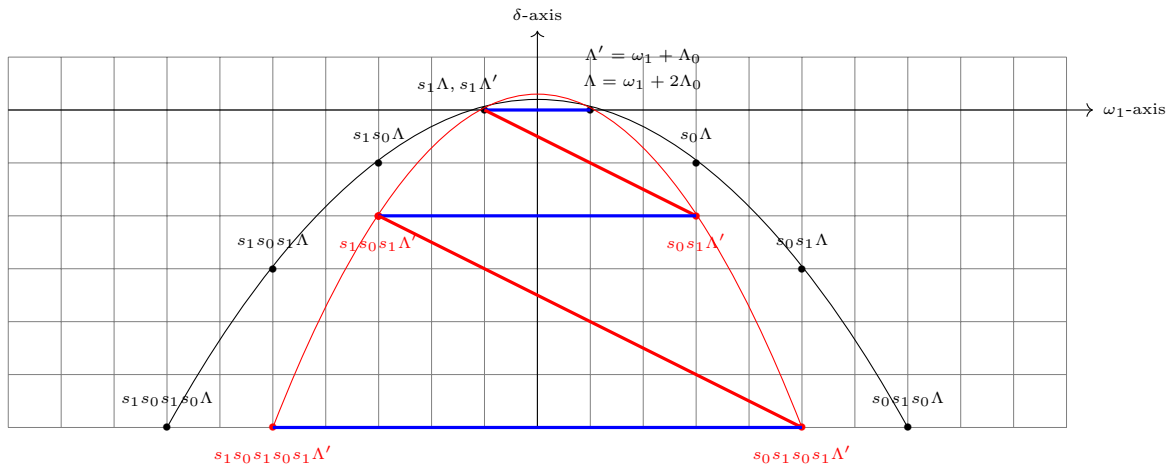
$W^{\text{ad}}(\omega_1 + 2\Lambda_0)$  (positive level),  $W^{\text{ad}}\omega_1$  (level 0), and  $W^{\text{ad}}(-\omega_1 - 2\Lambda_0)$  (negative level) mod  $\delta$

Although informative, the picture above is misleading as it is a two dimensional projection (mod  $\delta$ ) of what is actually going on. The  $W^{\text{ad}}$ -action fixes the level (the coefficient of  $\Lambda_0$ ) but it actually changes the  $\delta$  coordinate significantly. Let us look at the orbits in  $\delta$  and  $\omega_1$  coordinates (i.e. mod  $\Lambda_0$ ).

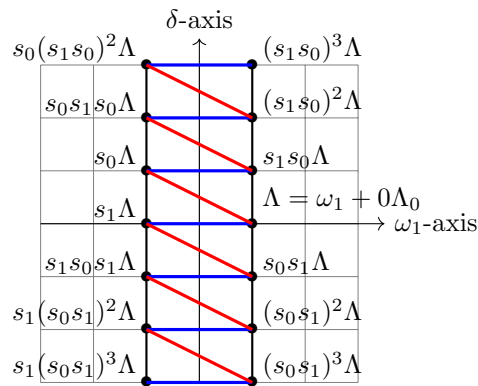


The positive level orbit  $W^{\text{ad}}(\omega_1 + 2\Lambda_0)$  mod  $\Lambda_0$

When the level (coefficient of  $\Lambda_0$ ) is large the parabola is wide, and it gets tighter as the level decreases.

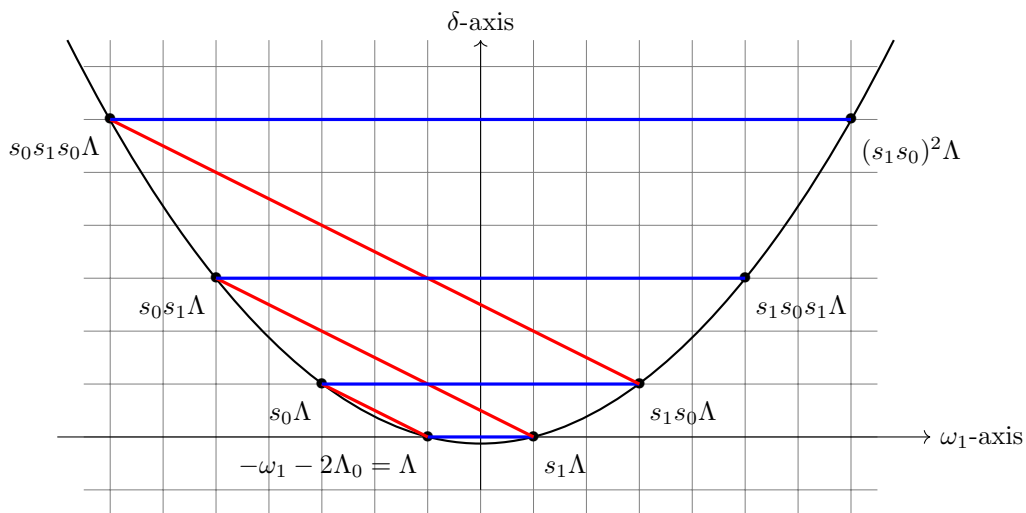


The parabolas bounding the orbits  $W^{\text{ad}}(\omega_1 + 2\Lambda_0)$  and  $W^{\text{ad}}(\omega_1 + \Lambda_0) \bmod \Lambda_0$   
 At level 0, the parabola pops and becomes two straight lines.



The level 0 orbit  $W^{\text{ad}}\omega_1 \bmod \Lambda_0$

At negative level the parabola forms again, but this time facing the opposite way, and getting wider as the level gets more and more negative.



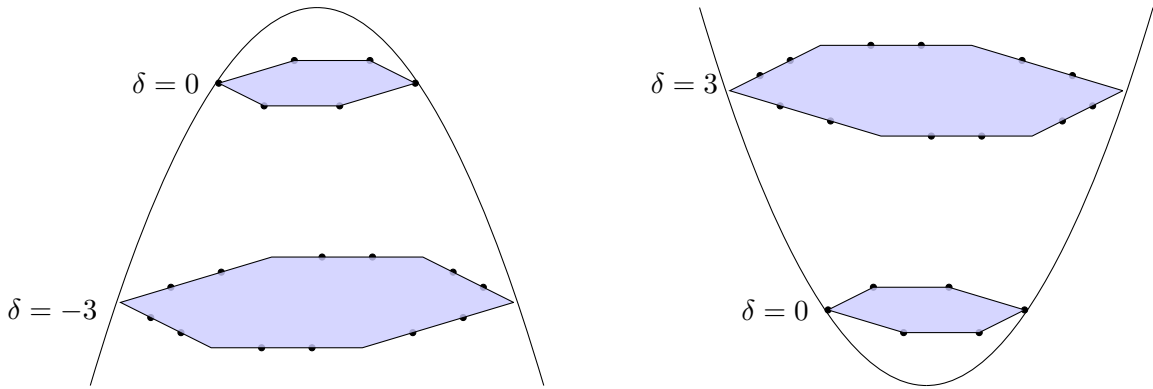
The orbit  $W^{\text{ad}}(-\omega_1 - 2\Lambda_0)$  (negative level)  $\bmod \Lambda_0$

The three different Bruhat orders on the affine Weyl group are visible on the  $W^{\text{ad}}$ -orbits:

$$\begin{aligned} v \preceq w & \quad \text{if } v(\omega_1 + \Lambda_0) \text{ is higher than } w(\omega_1 + \Lambda_0) \text{ in } W^{\text{ad}}(\omega_1 + \Lambda_0), \\ v \preceq^{\circ} w & \quad \text{if } v\omega_1 \text{ is higher than } w(\omega_1 + 0\Lambda_0) \text{ in } W^{\text{ad}}(\omega_1 + 0\Lambda_0). \\ v \leq w & \quad \text{if } v(-\omega_1 - \Lambda_0) \text{ is higher than } w(-\omega_1 - \Lambda_0) \text{ in } W^{\text{ad}}(-\omega_1 - \Lambda_0). \end{aligned}$$

The definitions of the Bruhat orders on  $W^{\text{ad}}$  and their relation to the closure order for Schubert cells in affine flag varieties is made precise in Section 1.3. The Hasse diagrams of the positive level, negative level, and level zero Bruhat orders for the Weyl group of  $\widehat{\mathfrak{sl}}_3$  are pictured in Plate C.

For the case of  $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$  the affine Weyl group orbits take a similar form, with the points sitting on a downward paraboloid at positive level, on an upward paraboloid at negative level and with the paraboloid popping and becoming a tube at level 0 (for an example tube see the picture of  $B(\omega_1 + \omega_2)$  for  $\widehat{\mathfrak{sl}}_3$  in Plate B).



Positive level orbit  $W^{\text{ad}}(\omega_1 + \omega_2 + 2\Lambda_0)$  for  $\widehat{\mathfrak{sl}}_3$     Negative level orbit  $W^{\text{ad}}(\omega_1 + \omega_2 - 2\Lambda_0)$  for  $\widehat{\mathfrak{sl}}_3$

## 0.2 Extremal weight modules $L(\Lambda)$ and their crystals $B(\Lambda)$

For the affine Lie algebra  $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$  the weights of integrable  $\mathfrak{g}$ -modules always lie in the set

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{C}\delta + \mathbb{Z}\omega_1 + \mathbb{Z}\Lambda_0.$$

A set of representatives for the orbits of the action of  $W^{\text{ad}}$  on  $\mathfrak{h}_{\mathbb{Z}}^*$  is

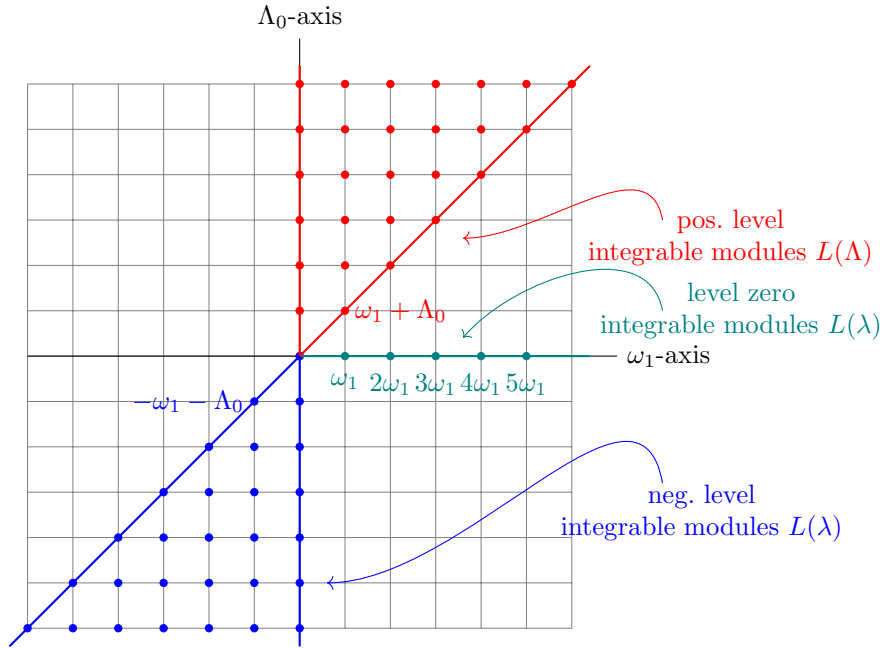
$$(\mathfrak{h}^*)_{\text{int}} = (\mathfrak{h}^*)_{\text{int}}^+ \cup (\mathfrak{h}^*)_{\text{int}}^0 \cup (\mathfrak{h}^*)_{\text{int}}^-, \quad \text{where} \quad \begin{aligned} (\mathfrak{h}^*)_{\text{int}}^0 &= \{a\delta + n\omega_1 \in \mathfrak{h}_{\mathbb{Z}}^* \mid 0 \leq n\}, \\ (\mathfrak{h}^*)_{\text{int}}^+ &= \{a\delta + m\omega_1 + n\Lambda_0 \in \mathfrak{h}_{\mathbb{Z}}^* \mid 0 \leq m \leq n\}, \\ (\mathfrak{h}^*)_{\text{int}}^- &= \{a\delta - m\omega_1 - n\Lambda_0 \in \mathfrak{h}_{\mathbb{Z}}^* \mid 0 \leq m \leq n\}. \end{aligned}$$

These sets are illustrated (mod  $\delta$ ) below.

For each of the  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}$ , there is a (universal) integrable *extremal weight* module  $L(\Lambda)$ , which is highest weight if  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}^+$ , is lowest weight (and not highest weight) if  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}^-$ , and which is neither highest or lowest weight when  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}^0$ . The module  $L(\Lambda)$  has a crystal  $B(\Lambda)$ .

At positive level and negative levels the crystals  $B(\Lambda)$  are connected, but the crystal  $B(\Lambda)$  is usually not connected in level 0. The connected components and their structure are known explicitly from a combination of results of Kashiwara, Beck-Chari-Pressley, Nakajima, Beck-Nakajima, Fourier-Littelmann, Ion and others. These results are collected in Theorem 2.4

and equation (2.12) expresses the characters of the  $L(\Lambda)$  in terms of Macdonald polynomials specialised at  $t = 0$ .



### 0.3 Affine flag varieties $G/I^+$ , $G/I^0$ and $G/I^-$

There are three kinds of affine flag varieties for the loop group  $G = G(\mathbb{C}((\epsilon)))$ : the positive level (thin) affine flag variety  $G/I^+$ , the negative level (thick) affine flag variety  $G/I^-$  and the level 0 (semi-infinite) affine flag variety  $G/I^0$ . A combination of results of Kumar, Mathieu, Kashiwara, Kashiwara-Tanisaki, Kashiwara-Shimozono, Varagnolo-Vasserot, Lusztig and Braverman-Finkelberg have made it clear that there is a Borel-Weil-Bott theorem for each of these:

$$\begin{aligned} H^0(G/I^+, \mathcal{L}_\Lambda) &\cong L(\Lambda), & \text{for positive level } \Lambda \in (\mathfrak{h}^*)_{\text{int}}^+, \\ H^0(G/I^0, \mathcal{L}_\lambda) &\cong L(\lambda), & \text{for level zero } \lambda \in (\mathfrak{h}^*)_{\text{int}}^0, \\ H^0(G/I^-, \mathcal{L}_{-\Lambda}) &\cong L(-\Lambda), & \text{for negative } -\Lambda \in (\mathfrak{h}^*)_{\text{int}}^-. \end{aligned}$$

These Borel-Weil-Bott theorems tightly connect the representation theory with the geometry. In all essential aspects the combinatorics of the positive level affine flag variety and the loop Grassmannian generalizes to the negative level and the level 0 affine flag varieties.

Section 5 extends the results of [PRS] and displays the alcove walk combinatorics for each of the three cases (positive level, negative level and level 0) in parallel. In addition it describes the method for deriving the natural affine Hecke algebra actions on the function spaces  $C(G/I^+)$ ,  $C(I^+ \backslash G/I^+)$ ,  $C(I^+ \backslash G/I^-)$  and  $C(I^- \backslash G/I^+)$ .

### 0.4 References, technicalities and acknowledgements

Section 1.1 introduces the affine Lie algebra and the homogeneous Heisenberg subalgebra following [Kac] and Section 1.2 gives explicit matrices describing the actions of the affine Weyl group  $W^{\text{ad}}$  on the affine Cartan  $\mathfrak{h}$  and its dual  $\mathfrak{h}^*$ . Section 1.3 defines the Bruhat orders on the

affine Weyl group and explains their relation to the corresponding affine flag varieties following [Kum] and [LuICM, §7 and 11]. Sections 1.4 and 1.5 introduce the affine braid groups and Macdonald polynomials following [RY11]. Section 1.6 treats the specializations of (nonsymmetric) Macdonald polynomials at  $t = 0$ ,  $t = \infty$ ,  $q = 0$  and  $q = \infty$  and reviews the result of Ion [Ion01] that relates Macdonald polynomials at  $t = 0$  to Demazure operators.

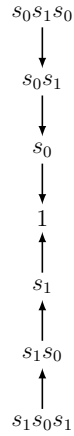
Section 2 follows [BN02], [B94], and [BCP98], introducing the quantum affine algebra  $\mathbf{U}$ , the conversion to its loop presentation, the PBW-type elements and the quantum homogeneous Heisenberg subalgebra. Section 2.2 defines integrable  $\mathbf{U}$ -modules and Section 2.3 introduces the extremal weight modules  $L(\Lambda)$  following [Kas94, (8.2.2)] and [Kas02, §3.1]. Section 2.4 reviews the Demazure character formulas for extremal weight modules. Section 2.5 discusses the loop presentation of the level 0 extremal weight modules and the fact that these coincide with the universal standard modules of [Nak99] and the global Weyl modules of [CP01]. Letting  $\mathbf{U}'$  be  $\mathbf{U}$  without the element  $D$ , Section 2.6 explains how to shrink the extremal weight module to a local Weyl module and how this provides a classification of finite dimensional simple  $\mathbf{U}'$ -modules by Drinfeld polynomials.

We have made a concerted effort to make a useful survey. In order to simplify the exposition we have brushed under the rug a number of technicalities which are wisely ignored when one learns the subject (particularly (a) the difference between simply laced cases and the general case requires proper attention to the diagonal matrix which symmetrizes the affine Cartan matrix [Kac, (2.1.1)] causing the constants  $d_i$  which pepper the quantum group literature and (b) the machinations necessary for allowing multiple parameters  $t_i^{\frac{1}{2}}$  in Macdonald polynomials). The reader who needs to sort out these features is advised to drink a strong double espresso to optimise clear thinking, consult the references (particularly [BN02] and [RY11]) and *not trust our exposition*. Perhaps in the future a more complete (probably book length) version of this paper will be completed which allows us to attend more carefully to these nuances and include more detailed proofs. Having made this point, we can say that a careful effort has been made to provide specific references to the literature at every step and we hope that this will be useful for the reader that wishes to go further.

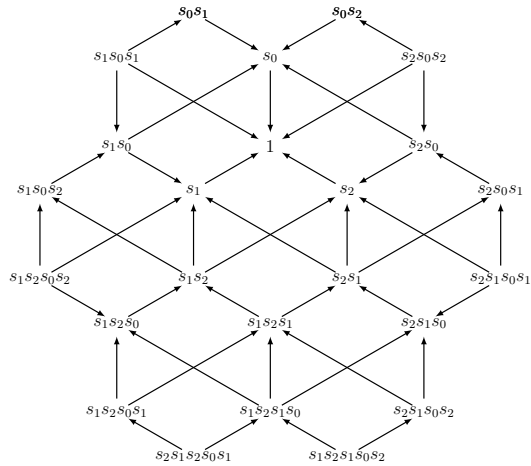
We thank Martha Yip for conversations, calculations and teaching us so much about Macdonald polynomials over the years. We thank all the institutions which have supported our work on this paper, particularly the University of Melbourne and the Australian Research Council (grants DP1201001942, DP130100674 and DE190101231). Arun Ram thanks IHP (Institut Henri Poincaré) and ICERM (Institute for Computational and Experimental Research in Mathematics) for support, hospitality and a stimulating working environments at the thematic programs “Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series” and “Combinatorics and Interactions”.

Finally, it is a pleasure to dedicate this paper to Ian Macdonald and Alun Morris who forcefully led the way to the kinds explorations of affine combinatorial representation theory that are happening these days.

PLATE A: Bruhat orders on the affine Weyl group



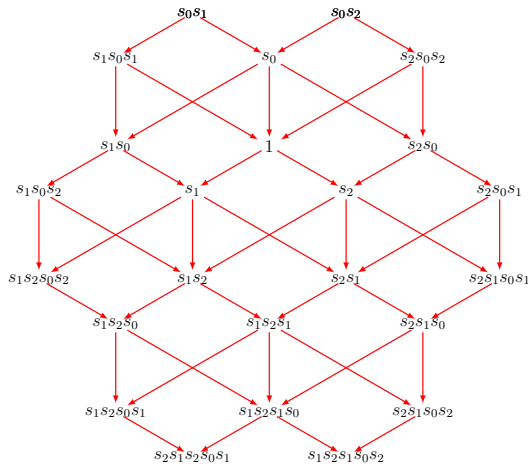
positive level Bruhat order for  $\widehat{\mathfrak{sl}}_2$   
1 is minimal



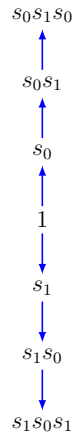
positive level Bruhat order for  $\widehat{\mathfrak{sl}}_3$   
1 is minimal



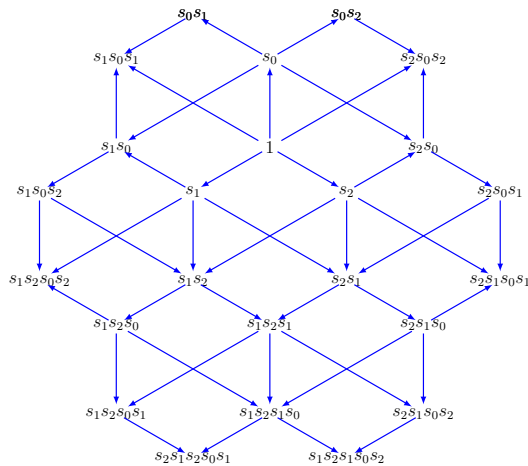
level zero Bruhat order for  $\widehat{\mathfrak{sl}}_2$   
translation invariant



level zero Bruhat order for  $\widehat{\mathfrak{sl}}_3$   
translation invariant

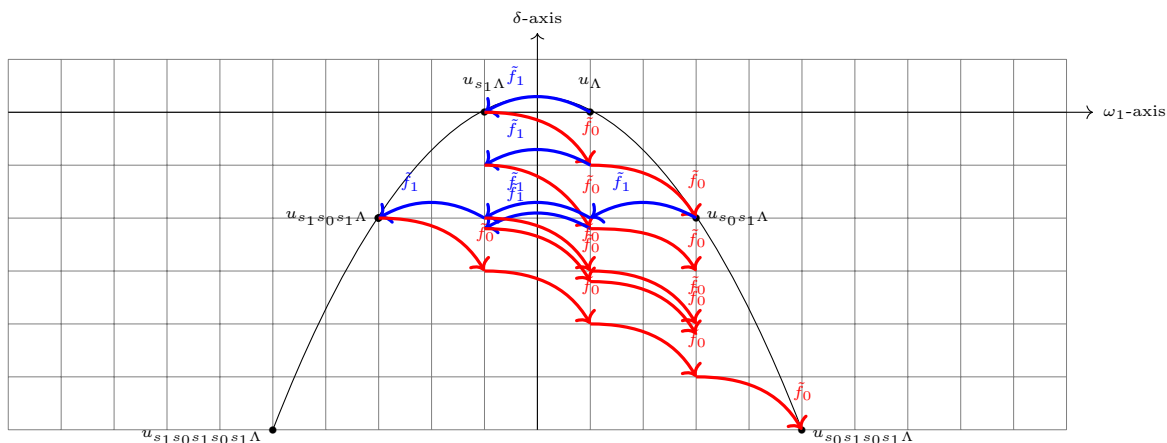


negative level Bruhat order for  $\widehat{\mathfrak{sl}}_2$   
1 is maximal

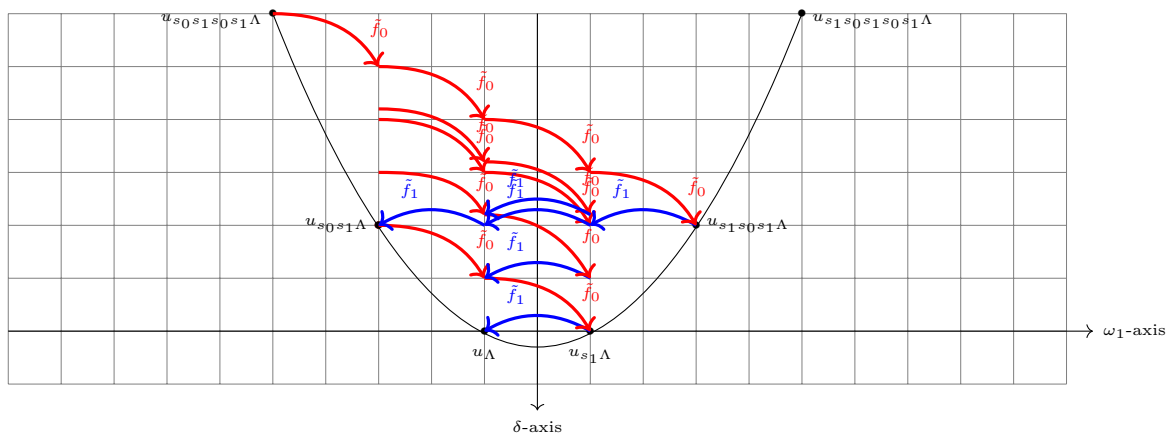


negative level Bruhat order for  $\widehat{\mathfrak{sl}}_3$   
1 is maximal

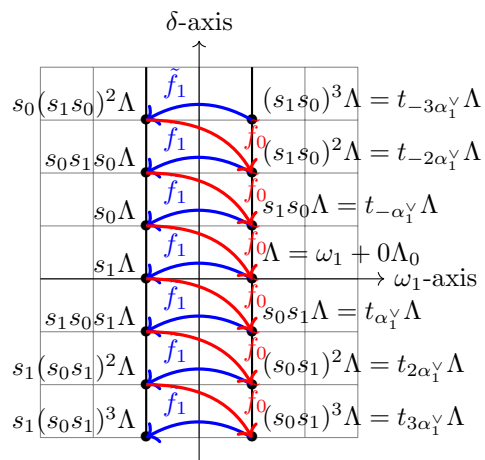
**PLATE B: Pictures of  $B(\omega_1 + \Lambda_0)$ ,  $B(\omega_1 + 0\Lambda)$  and  $B(-\omega_1 - \Lambda_0)$  for  $\widehat{\mathfrak{sl}}_2$**



Initial portion of the crystal graph of  $B(\omega_1 + \Lambda_0)$  for  $\widehat{\mathfrak{sl}}_2$



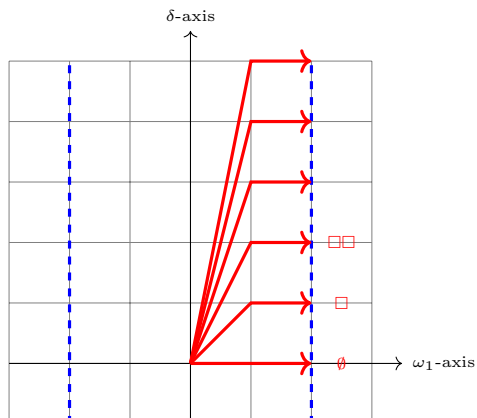
Final portion of the crystal graph of  $B(-\omega_1 - \Lambda_0)$  for  $\widehat{\mathfrak{sl}}_2$



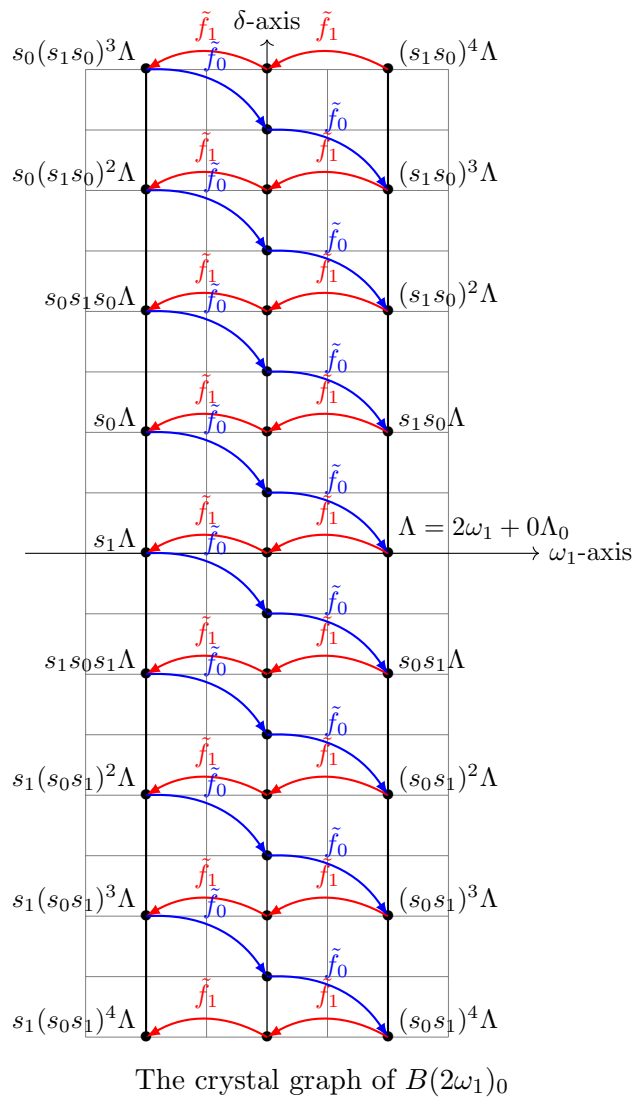
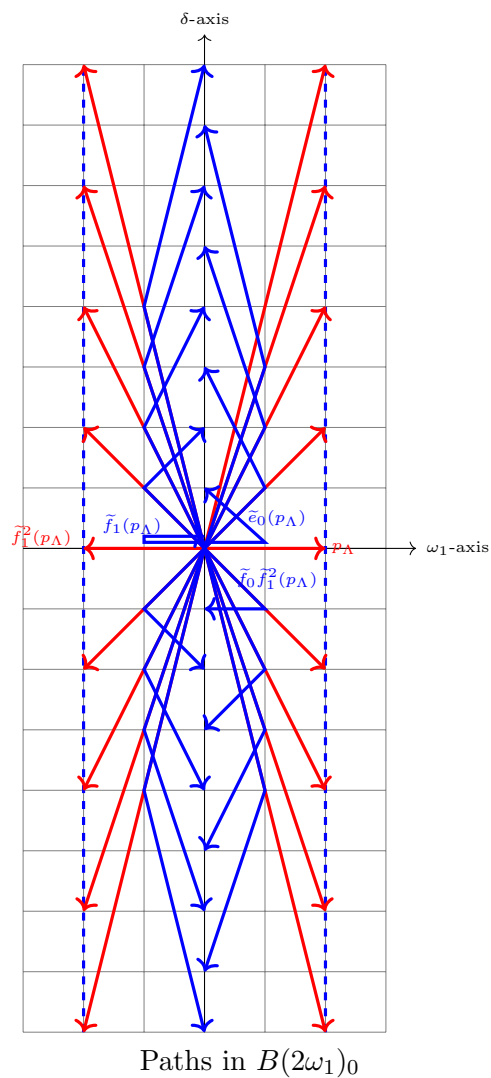
Middle portion of the crystal graph of  $B(\omega_1 + 0\Lambda_0)$  for  $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$



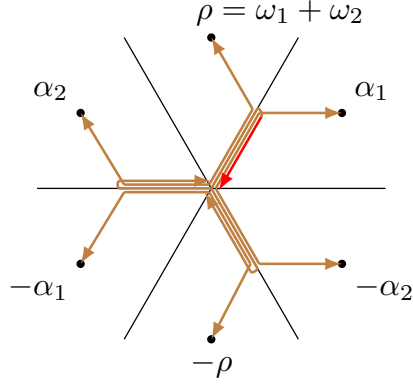
**PLATE C: Pictures for  $B(2\omega_1)$ .** Representative paths from the (first five) connected components of  $B(2\omega_1)$  are



and the paths in  $B(2\omega_1)_0 \subseteq B(\omega_1) \otimes B(\omega_1)$  are

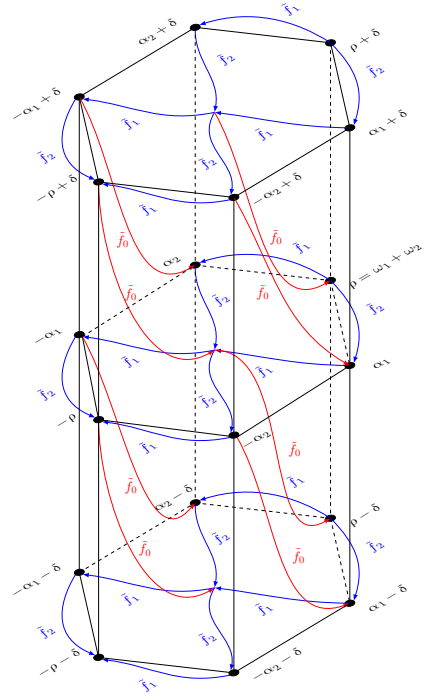


**PLATE D: Pictures and characters of  $B(\omega_1 + \omega_2)$  for  $\widehat{\mathfrak{sl}}_3$ .** The colour red indicates change in the  $\delta$ -coordinate.



$$B^{\text{fin}}(\omega_1 + \omega_2) \subseteq B(\omega_1) \otimes B(\omega_2)$$

$$\text{gchar}(B^{\text{fin}}(\omega_1 + \omega_2)) = X^\rho + X^{\alpha_1} + X^{\alpha_2} + X^{-\alpha_1} + X^{-\alpha_2} + X^{-\rho} + 2 + q^{-1}$$



$B(\omega_1 + \omega_2)$  crystal graph

At  $t = 0$  and  $t = \infty$  the normalized nonsymmetric Macdonald polynomials  $\tilde{E}_{s_1 s_2 s_1 \rho}(q, t)$  are

$$\tilde{E}_{s_1 s_2 s_1 \rho}(q, 0) = X^{s_1 s_2 s_1 \rho} + X^{s_1 s_2 \rho} + X^{s_2 s_1 \rho} + X^{s_2 \rho} + X^{s_1 \rho} + X^\rho + 2 + q,$$

$$\tilde{E}_{s_1 s_2 s_1 \rho}(q, \infty) = X^{s_1 s_2 s_1 \rho} + q^{-1}(X^{s_1 s_2 \rho} + X^{s_2 s_1 \rho}) + q^{-2}(X^{s_1 \rho} + X^{s_2 \rho} + X^\rho) + 2q^{-2} + q^{-1}.$$

Letting  $q = e^{-\delta}$  (as in [Kac, (12.1.9)]), the Demazure module  $L(\omega_1 + \omega_2)_{\leq s_1 s_2 s_1}$  has character

$$\text{char}(L(\omega_1 + \omega_2)_{\leq s_1 s_2 s_1}) = \frac{1}{(1 - q^{-1})^2} \tilde{E}_{s_1 s_2 s_1 \rho}(q^{-1}, 0) \quad \text{and}$$

$$\text{char}(L(\omega_1 + \omega_2)) = 0_q 0_q \tilde{E}_{s_1 s_2 s_1 \rho}(q^{-1}, 0) = 0_q 0_q \tilde{E}_{s_1 s_2 s_1 \rho}(q^{-1}, \infty),$$

where  $0_q = \dots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \dots$  as in Remark 2.2.

**Remark 0.1.** The expansion

$$\frac{1}{(1 - q^{-1})^2} = (1 + q^{-1} + q^{-2} + \dots)(1 + q^{-1} + q^{-2} + \dots) = 1 + 2q^{-1} + 3q^{-2} + 4q^{-3} + 5q^{-4} + \dots$$

show that the sizes of the weight spaces of  $L(\omega_1 + \omega_2)_{\leq s_1 s_2 s_1}$  are growing as  $\delta$  increases. Similarly, in the character formula of  $L(\omega_1 + \omega_2)$ , the factor  $0_q 0_q$  has coefficient of  $q^n$  equal to  $\text{Card}(\{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 + k_2 = n\}) = \infty$ . This shows that *every weight space of the extremal weight module  $L(\omega_1 + \omega_2)$  is infinite dimensional*.

# 1 Affine Weyl groups, braid groups and Macdonald polynomials

## 1.1 The affine Lie algebra $\mathfrak{g}$

Let  $\mathring{\mathfrak{g}}$  be a finite dimensional complex semisimple Lie algebra and fix a Cartan subalgebra  $\mathfrak{a} \subseteq \mathring{\mathfrak{g}}$  and a symmetric, ad-invariant, nondegenerate, bilinear form  $\langle \cdot, \cdot \rangle: \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \rightarrow \mathbb{C}$ . The *affine Kac-Moody algebra* is

$$\mathfrak{g} = \left( \bigoplus_{k \in \mathbb{Z}} \mathring{\mathfrak{g}} \epsilon^k \right) \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \text{with bracket given by } [K, x\epsilon^k] = 0, \quad [K, d] = 0,$$

$$[d, x\epsilon^k] = kx\epsilon^k, \quad \text{and} \quad [x\epsilon^k, y\epsilon^\ell] = [x, y]\epsilon^{k+\ell} + k\delta_{k,-\ell}\langle x, y \rangle K, \quad (1.1)$$

for  $x, y \in \mathring{\mathfrak{g}}$  and  $k, \ell \in \mathbb{Z}$  (see [Kac, (7.2.2)]). Let  $\theta$  be the highest root of  $\mathring{\mathfrak{g}}$  (the highest weight of the adjoint representation) and define

$$e_0 = f_\theta \epsilon, \quad f_0 = e_\theta \epsilon^{-1}, \quad \text{and} \quad h_0 = [e_0, f_0] = \alpha_0^\vee = -h_\theta + K.$$

The miracle is that  $\mathfrak{g}$  is a Kac-Moody Lie algebra with Chevalley generators

$$e_0, \dots, e_n, h_0, \dots, h_n, d, f_0, \dots, f_n, \quad \text{which satisfy Serre relations.} \quad (1.2)$$

Because of (1.2),  $\mathfrak{g}$  has a corresponding quantum enveloping algebra  $\mathbf{U} = U_q \mathfrak{g}$ .

The Cartan subalgebra of  $\mathfrak{g}$  is

$$\mathfrak{h} = \mathfrak{a} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \text{where } \mathfrak{a} \subseteq \mathring{\mathfrak{g}} \text{ is the Cartan subalgebra of } \mathring{\mathfrak{g}}.$$

Let  $\mathring{R}^+$  be the set of positive roots for  $\mathring{\mathfrak{g}}$ . For  $\alpha \in \mathring{R}^+$ ,  $k \in \mathbb{Z}$ ,  $\ell \in \mathbb{Z}_{\neq 0}$  and  $i \in \{1, \dots, n\}$ , let

$$x_{\alpha+k\delta} = e_\alpha \epsilon^k, \quad x_{-\alpha+k\delta} = f_\alpha \epsilon^k, \quad h_{i,\ell} = h_i \epsilon^\ell.$$

The *homogeneous Heisenberg subalgebra* (see [Kac, §8.4 and §14.8]) is

$$\mathbb{C}K \oplus \mathfrak{a}[\epsilon, \epsilon^{-1}] \quad \text{with} \quad [h_i \epsilon^k, h_j \epsilon^\ell] = k\delta_{k,-\ell} \frac{2}{\langle \alpha_i, \alpha_i \rangle} \langle \alpha_i, \alpha_j^\vee \rangle K, \quad (1.3)$$

and  $\mathfrak{a}[\epsilon]$  is a commutative Lie algebra with basis  $\{h_i \epsilon^k \mid i \in \{1, \dots, n\}, k \in \mathbb{Z}_{\geq 0}\}$ .

## 1.2 The affine Weyl group $W^{\text{ad}}$ and its action on $\mathfrak{h}^*$ and $\mathfrak{h}$

Let  $\delta, \omega_1, \dots, \omega_n, \Lambda_0$  be the basis in  $\mathfrak{h}^*$  which is the dual basis to the basis  $d, h_1, \dots, h_n, K$  of  $\mathfrak{h}$ . The affine Weyl group  $W^{\text{ad}}$  is the subgroup of  $GL(\mathfrak{h}^*)$  generated by the linear transformations  $s_0, s_1, \dots, s_n$  which, in the basis  $\delta, \omega_1, \dots, \omega_n, \Lambda_0$ , are

$$s_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -\alpha_i(h_1) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_2) & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_i(h_{i-1}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_{i+1}) & 1 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_n) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \text{for } i \in \{1, \dots, n\} \quad (1.4)$$

and, writing  $\theta = a_1\alpha_1 + \cdots + a_n\alpha_n$  and  $h_\theta = [e_\theta, f_\theta] = a_1^\vee h_1 + \cdots + a_n^\vee h_n$ ,

$$s_0 = \begin{pmatrix} 1 & a_1^\vee & a_2^\vee & \cdots & a_n^\vee & -1 \\ 0 & 1 - a_1 a_1^\vee & -a_1 a_2^\vee & \cdots & -a_1 a_n^\vee & a_1 \\ 0 & -a_2 a_1^\vee & 1 - a_2 a_2^\vee & \cdots & -a_2 a_n^\vee & a_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & -a_n a_1^\vee & -a_n a_2^\vee & \cdots & 1 - a_n a_n^\vee & a_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \quad (1.5)$$

Let  $\mathfrak{a}_{\mathbb{Z}}^{\text{ad}} = \mathbb{Z}\text{-span}\{h_1, \dots, h_n\}$ . The *finite Weyl group*  $W_{\text{fin}}$  generated by  $s_1, \dots, s_n$ . The translation presentation of the affine Weyl group is

$$W^{\text{ad}} = \mathfrak{a}_{\mathbb{Z}}^{\text{ad}} \rtimes W_{\text{fin}} = \{t_{\mu^\vee} u \mid \mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}, u \in W_{\text{fin}}\} \quad \text{with} \quad \begin{aligned} t_{\mu^\vee} t_{\nu^\vee} &= t_{\mu^\vee + \nu^\vee} \quad \text{and} \\ u t_{\mu^\vee} &= t_{u\mu^\vee} u, \end{aligned} \quad (1.6)$$

for  $\mu^\vee, \nu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}$  and  $u \in W_{\text{fin}}$ . Let  $\alpha_i^\vee$  be the image of  $h_i$  under the isomorphism  $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$  coming from the nondegenerate bilinear form on  $\mathfrak{a}$  which is the restriction of the nondegenerate bilinear form on  $\hat{\mathfrak{g}}$ . In matrix form with respect to the basis  $\delta, \omega_1, \dots, \omega_n, \Lambda_0$  of  $\mathfrak{h}^*$  the action of  $W$  on  $\mathfrak{h}^*$  is given by

$$t_{\mu^\vee} = \left( \begin{array}{c|ccc|c} 1 & k_1 & \cdots & k_n & -\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle \\ \vdots & & & & \mu_1^\vee \\ 0 & & 1 & & \vdots \\ \vdots & & & & \mu_n^\vee \\ \hline 0 & \cdots & 0 & \cdots & 1 \end{array} \right), \quad \text{for} \quad \begin{aligned} \mu^\vee &= k_1 h_1 + \cdots + k_n h_n \\ &= k_1 \alpha_1^\vee + \cdots + k_n \alpha_n^\vee \\ &= \frac{k_1}{d_1} \alpha_1 + \cdots + \frac{k_n}{d_n} \alpha_n \\ &= \mu_1^\vee \omega_1 + \cdots + \mu_n^\vee \omega_n, \end{aligned} \quad (1.7)$$

so that  $-\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle = -\frac{1}{2}(\mu_1^\vee k_1 + \cdots + \mu_n^\vee k_n)$ . (In (1.7)  $d_1, \dots, d_n$  are the minimal positive integers such that the product of the diagonal matrix  $\text{diag}(d_1, \dots, d_n)$  with the Cartan matrix is symmetric, see [Kac, (2.1.1)].)

The basis  $\{d, h_1, \dots, h_n, K\}$  of  $\mathfrak{h}$  is the dual basis to the basis  $\{\delta, \omega_1, \dots, \omega_n, \Lambda_0\}$  of  $\mathfrak{h}^*$ . Using the  $W^{\text{ad}}$ -action on  $\mathfrak{h}$  given by

$$s_i \mu^\vee = \mu^\vee - \alpha_i(\mu^\vee) h_i, \quad \text{for } i \in \{0, \dots, n\} \text{ and } \mu^\vee \in \mathfrak{h},$$

the matrices for the action of  $s_0, s_1, \dots, s_n$  on  $\mathfrak{h}$ , in the basis  $\{d, h_1, \dots, h_n, K\}$ , are the *transposes of the matrices in (1.4) and (1.5)*.

### 1.3 The positive level, negative level and level 0 Bruhat orders on $W^{\text{ad}}$

In the framework of Section 5, where  $G = \hat{G}(\mathbb{C}((\epsilon)))$  is the loop group, the closure orders for the Schubert cells in the positive level (thin) affine flag variety  $G/I^+$ , the negative level (thick) affine flag variety  $G/I^-$ , and the level 0 (semi-infinite) affine flag variety  $G/I^0$  give partial orders on the affine Weyl group  $W^{\text{ad}}$ :

$$\overline{I^+ w I^+} = \bigsqcup_{x \leq^+ w} I^+ x I^+, \quad \overline{I^+ w I^0} = \bigsqcup_{x \leq^0 w} I^+ x I^0, \quad \overline{I^+ w I^-} = \bigsqcup_{x \leq^- w} I^+ x I^-.$$

These orders can be described combinatorially as follows.

An element  $w \in W^{\text{ad}}$  is *dominant* if

$$w(\rho + \Lambda_0) \in \mathbb{R}_{\geq 0}\text{-span}\{\omega_1, \dots, \omega_n\} + \Lambda_0, \quad \text{where } \rho = \omega_1 + \dots + \omega_n.$$

In the identification of elements of  $W^{\text{ad}}$  with alcoves (fundamental regions for the action of  $W^{\text{ad}}$  on elements  $\mathfrak{a}^* + \Lambda_0 \bmod \delta$ , see [RY11], the dominant elements of  $W^{\text{ad}}$  are the alcoves in the dominant Weyl chamber.

Let  $x, w \in W^{\text{ad}}$  and let  $w = s_{i_1} \cdots s_{i_\ell}$  be a reduced word for  $w$  in the generators  $s_0, \dots, s_n$ . The *positive level Bruhat order* on  $W^{\text{ad}}$  is defined by

$$x \leq^+ w \quad \text{if } x \text{ has a reduced word which is a subword of } w = s_{i_1} \cdots s_{i_\ell}$$

The *negative level Bruhat order* on  $W^{\text{ad}}$  is defined by  $x \leq^- w$  if  $x \geq^+ w$ .

The *level 0 Bruhat order* on  $W^{\text{ad}}$  is determined by

- (a)  $\leq^0$  for dominant elements: If  $x, w$  are dominant then  $x \leq^0 w$  if and only if  $x \leq^+ w$ ,
- (b)  $\leq^0$  translation invariance: If  $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}$  and  $x, w \in W$  then  $x \leq^0 w$  if and only if  $xt_{\mu^\vee} \leq^0 wt_{\mu^\vee}$ .

The *positive level length* is  $\ell^+ : W^{\text{ad}} \rightarrow \mathbb{Z}_{\geq 0}$  given by  $\ell^+(w) = (\text{length of a reduced word for } w)$ .

The *negative level length* is  $\ell^- : W^{\text{ad}} \rightarrow \mathbb{Z}_{\leq 0}$  given by  $\ell^-(w) = -\ell^+(w)$ .

The *level 0 length* is  $\ell^0 : W^{\text{ad}} \rightarrow \mathbb{Z}$  given by

$$\ell^0(w) = \ell^+(w) \text{ if } w \text{ is dominant} \quad \text{and} \quad \ell^0(xt_{\mu^\vee}) - \ell^0(yt_{\mu^\vee}) = \ell^0(x) - \ell^0(y),$$

for  $x, y \in W^{\text{ad}}$  and  $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}$ . Using the formula for  $\ell^+$  given in [Mac96, (2.8)], gives a formula for  $\ell^0$ ,

$$\ell^0(ut_{\mu^\vee}) = \ell^+(u) + 2\langle \rho, \mu^\vee \rangle, \quad \text{for } u \in W_{\text{fin}}, \mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}. \quad (1.8)$$

The length functions  $\ell^+$ ,  $\ell^-$  and  $\ell^0$  return, respectively, the dimension, the codimension and the relative dimension of Schubert cells in the positive level, negative level and level 0 affine flag varieties.

#### 1.4 The affine braid groups $\mathcal{B}^{\text{sc}}$ and $\mathcal{B}^{\text{ad}}$

Let  $\omega_1^\vee, \dots, \omega_n^\vee$  be the basis of  $\mathfrak{a}$  which is dual to the basis  $\alpha_1, \dots, \alpha_n$  of  $\mathfrak{a}^*$ . Let

$$\mathfrak{a}_{\mathbb{Z}}^{\text{ad}} = \mathbb{Z}\text{-span}\{h_1, \dots, h_n\} \subseteq \mathfrak{a}_{\mathbb{Z}}^{\text{sc}} = \mathbb{Z}\text{-span}\{\omega_1^\vee, \dots, \omega_n^\vee\}.$$

The affine braid group  $\mathcal{B}^{\text{ad}}$  (resp.  $\mathcal{B}^{\text{sc}}$ ) is generated by  $T_1, \dots, T_n$  and  $Y^{\lambda^\vee}$ ,  $\lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}$  (resp.  $\lambda^\vee \in \mathfrak{a}^{\text{sc}}$ ), with relations

$$Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}, \quad \underbrace{T_i T_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i \cdots}_{m_{ij} \text{ factors}}, \quad \begin{aligned} T_i^{-1} Y^{\lambda^\vee} &= Y^{s_i \lambda^\vee} T_i^{-1}, & \text{if } \langle \lambda^\vee, \alpha_i \rangle = 0, \\ T_i^{-1} Y^{\lambda^\vee} T_i^{-1} &= Y^{s_i \lambda^\vee}, & \text{if } \langle \lambda^\vee, \alpha_i \rangle = 1, \end{aligned}$$

for  $i \in \{1, \dots, n\}$  and  $\lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}$  (resp.  $\mathfrak{a}_{\mathbb{Z}}^{\text{sc}}$ ) and  $m_{ij} = \alpha_i(h_j) \alpha_j(h_i)$  for  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ .

## 1.5 Macdonald polynomials

Let

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\delta, \omega_1, \dots, \omega_n, \Lambda_0\} \quad \text{and} \quad \mathfrak{a}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\omega_1, \dots, \omega_n\}.$$

The *double affine Hecke algebra*  $\tilde{H}$  is presented by generators  $T_0, \dots, T_n$  and  $X^\mu$ ,  $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ , with relations

$$X^\lambda X^\mu = X^{\lambda+\mu}, \quad \underbrace{T_i T_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i \cdots}_{m_{ij} \text{ factors}}, \quad T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \quad (1.9)$$

$$T_i X^\mu = X^{s_i \mu} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^\mu - X^{s_i \mu}}{1 - X^{-\alpha_i}}, \quad T_i^{-1} X^\mu = X^{s_i \mu} T_i^{-1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^\mu - X^{s_i \mu}}{1 - X^{-\alpha_i}}.$$

for  $i \in \{0, \dots, n\}$  and  $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ . For  $w \in W^{\text{ad}}$  put

$$Y^w = \begin{cases} Y^{ws_i} T_i^{-1}, & \text{if } w \triangleleft ws_i, \\ Y^{ws_i} T_i, & \text{if } w \triangleright ws_i, \end{cases} \quad \text{and let} \quad Y^{\lambda^\vee} = Y^{t\lambda^\vee} \text{ for } \lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}. \quad (1.10)$$

Putting  $q = X^\delta = Y^{-K}$  then, as an algebra over  $\mathbb{C}[q^{\pm 1}, t^{\pm \frac{1}{2}}]$ ,

$$\tilde{H} \text{ has basis } \{X^\mu T_u Y^{\lambda^\vee} \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^* + \mathbb{Z}\Lambda_0, u \in W_{\text{fin}}, \lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}\}, \quad \text{where } T_u = T_{i_1} \cdots T_{i_\ell},$$

for a reduced word  $u = s_{i_1} \cdots s_{i_\ell}$ . The *affine Hecke algebra* is the subalgebra  $H$  of  $\tilde{H}$  with basis  $\{T_u^\vee Y^{\lambda^\vee} \mid u \in W_{\text{fin}}, \lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}\}$ . The *polynomial representation* of  $\tilde{H}$  is

$$\mathbb{C}[X] = \text{Ind}_{\tilde{H}}^{\tilde{H}}(\mathbf{1}) \quad \text{with basis} \quad \{X^\mu \mathbf{1} \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^*\}, \quad (1.11)$$

and  $Y^K \mathbf{1} = q^{-1} \mathbf{1}$ ,  $Y^{-\alpha_i^\vee} \mathbf{1} = t \mathbf{1}$ , and  $T_i \mathbf{1} = t^{\frac{1}{2}} \mathbf{1}$  for  $i \in \{1, \dots, n\}$ .

Let

$$T_0^\vee = Y^{\alpha_0^\vee} X^{-\Lambda_0} T_0^{-1} X^{\Lambda_0} \quad \text{and} \quad T_i^\vee = T_i \text{ for } i \in \{1, \dots, n\}. \quad (1.12)$$

The automorphism of  $\tilde{H}$  given by conjugation by  $X^{-\Lambda_0}$  is the automorphism  $\tau: \tilde{H} \rightarrow \tilde{H}$  of [Ch96, (2.8)]. Extend  $\tilde{H}$  to allow rational functions in the  $Y^{\lambda^\vee}$ . For each  $i \in \{0, 1, \dots, n\}$ , the *intertwiner*  $\tau_i^\vee \in \tilde{H}$  is

$$\tau_i^\vee = T_i^\vee + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y^{-\alpha_i^\vee}} = (T_i^\vee)^{-1} + \frac{t^{-\frac{1}{2}}(1-t)Y^{-\alpha_i^\vee}}{1-Y^{-\alpha_i^\vee}} \quad \text{so that} \quad Y^{\lambda^\vee} \tau_i^\vee = \tau_i^\vee Y^{s_i \lambda^\vee}. \quad (1.13)$$

The *nonsymmetric Macdonald polynomial*  $E_\mu = E_\mu(q, t)$  is

$$E_\mu = E_\mu(q, t) = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1}, \quad \text{where } m_\mu = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word} \quad (1.14)$$

for the minimal length element in the coset  $t_\mu W_{\text{fin}}$ . The  $E_\mu$  form a basis of  $\mathbb{C}[X]$  consisting of eigenvectors for the  $Y^{\lambda^\vee}$  (the Cherednik-Dunkl operators).

Fix a reduced word  $m_\mu = s_{i_1} \cdots s_{i_\ell}$  as in (1.14). An alcove walk of type  $\vec{m}_\mu = (i_1, \dots, i_\ell)$  beginning at 1 is a sequence of steps, where a step of type  $j$  is (the signs - and + indicate that  $zs_j \triangleright z$ )

$$\begin{array}{cccc} \begin{array}{c} - \\ | \\ z \xrightarrow{\quad} zs_j \\ | \\ + \end{array} & \begin{array}{c} - \\ | \\ z \xleftarrow{\quad} zs_j \\ | \\ + \end{array} & \begin{array}{c} - \\ | \\ z \xrightarrow{\quad} zs_j \\ | \\ + \end{array} & \begin{array}{c} - \\ | \\ z \xleftarrow{\quad} zs_j \\ | \\ + \end{array} & \text{with } zs_j \triangleright z. \\ \text{positive } j\text{-crossing} & \text{negative } j\text{-crossing} & \text{positive } j\text{-fold} & \text{negative } j\text{-fold} & \end{array}$$

Let  $\mathcal{B}(1, \vec{m}_\mu)$  be the set of alcove walks of type  $\vec{m}_\mu = (i_1, \dots, i_\ell)$  beginning at 1. For a walk  $p \in \mathcal{B}(1, \vec{m}_\mu)$  let

$$\begin{aligned} f^+(p) &= \{k \in \{1, \dots, \ell\} \mid \text{the } k\text{th step of } p \text{ is a positive fold}\}, \\ f^-(p) &= \{k \in \{1, \dots, \ell\} \mid \text{the } k\text{th step of } p \text{ is a negative fold}\}, \\ f(p) &= f^+(p) \cup f^-(p) = \{k \in \{1, \dots, \ell\} \mid \text{the } k\text{th step of } p \text{ is a fold}\}. \end{aligned}$$

For  $p \in \mathcal{B}(1, \vec{m}_\mu)$  let  $\text{end}(p)$  be the endpoint of  $p$  (an element of  $W^{\text{ad}}$ ) and define the *weight*  $\text{wt}(p)$  and the *final direction*  $\varphi(p)$  of  $p$  by

$$X^{\text{end}(p)} = X^{\text{wt}(p)} T_{\varphi(p)}^\vee, \quad \text{with } \text{wt}(p) \in \mathfrak{a}_{\mathbb{Z}}^* \text{ and } \varphi(p) \in W_{\text{fin}}.$$

Using (1.13) and doing a left to right expansion of the terms of  $\tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1}$  produces the monomial expansion of  $E_\mu$  as sum over alcove walks as given in the following theorem.

**Theorem 1.1.** [RY11, Theorem 3.1 and Remark 3.3] *Let  $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$  and let  $m_\mu$  be the minimal length element in the coset  $t_\mu W_{\text{fin}}$ . Fix a reduced word  $\vec{m}_\mu = s_{i_1} \cdots s_{i_\ell}$ , let*

$$\beta_1^\vee = s_{i_\ell} \cdots s_{i_2} \alpha_{i_1}^\vee, \quad \beta_2^\vee = s_{i_\ell} \cdots s_{i_3} \alpha_{i_2}^\vee, \quad \dots, \quad \beta_\ell^\vee = \alpha_{i_\ell}^\vee,$$

and let  $\text{sh}(\beta_k^\vee)$  and  $\text{ht}(\beta_k^\vee)$  be defined by  $Y^{\beta_k^\vee} \mathbf{1} = q^{\text{sh}(\beta_k^\vee)} t^{\text{ht}(\beta_k^\vee)} \mathbf{1}$  for  $k \in \{1, \dots, \ell\}$ . Then

$$E_\mu(q, t) = \sum_{p \in \mathcal{B}(1, \vec{m}_\mu)} X^{\text{wt}(p)} t^{\frac{1}{2}(\ell(\varphi(p)))} \prod_{k \in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1 - q^{\text{sh}(\beta_k^\vee)} t^{\text{ht}(\beta_k^\vee)}} \prod_{k \in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t) q^{\text{sh}(\beta_k^\vee)} t^{\text{ht}(\beta_k^\vee)}}{1 - q^{\text{sh}(\beta_k^\vee)} t^{\text{ht}(\beta_k^\vee)}}.$$

## 1.6 Specializations of the normalized Macdonald polynomials $\tilde{E}_\mu(q, t)$

If  $m_\mu = t_\mu m$  with  $m \in W_{\text{fin}}$  then  $E_\mu(q, t)$  has top term  $t^{\frac{1}{2}\ell(m)} X^\mu$  (this term is the term corresponding to the unique alcove walk in  $\mathcal{B}(\vec{m}_\mu)$  with no folds). The *normalized nonsymmetric Macdonald polynomial* is

$$\tilde{E}_\mu(q, t) = t^{-\frac{1}{2}\ell(m)} E_\mu(q, t) \quad \text{so that } \tilde{E}_\mu(q, t) \text{ has top term } X^\mu.$$

A path  $p \in \mathcal{B}(1, \vec{m}_\mu)$  is *positively folded* if there are no negative folds, i.e.  $\#f^-(p) = 0$ .

A path  $p \in \mathcal{B}(1, \vec{m}_\mu)$  is *negatively folded* if there are no positive folds, i.e.  $\#f^+(p) = 0$ .

A path  $p \in \mathcal{B}(1, \vec{m}_\mu)$  is *positive semi-infinite* if  $\ell(\varphi(p)) - \ell(m) - \#f(p) + 2 \sum_{k \in f^-(p)} \text{ht}(\beta_k^\vee) = 0$ .

A path  $p \in \mathcal{B}(1, \vec{m}_\mu)$  is *negative semi-infinite* if  $\ell(m) - \ell(\varphi(p)) + \#f(p) + 2 \sum_{k \in f^+(p)} \text{ht}(\beta_k^\vee) = 0$ .

**Proposition 1.2.** *Let  $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ . The specializations  $q = 0$ ,  $t = 0$ ,  $q^{-1} = 0$  and  $t^{-1} = 0$  are well defined and given, respectively, by*

$$\begin{aligned} \tilde{E}_\mu(0, t) &= \sum_{\substack{p \in \mathcal{B}(1, \vec{m}_\mu) \\ p \text{ pos folded}}} t^{\frac{1}{2}(\ell(\varphi(p)) - \ell(m) - \#f(p))} (1 - t^{\#f(p)}) X^{\text{wt}(p)}, \\ \tilde{E}_\mu(q, 0) &= \sum_{\substack{p \in \mathcal{B}(1, \vec{m}_\mu) \\ p \text{ neg semi-inf}}} q^{\sum_{k \in f^-(p)} \text{sh}(\beta_k^\vee)} X^{\text{wt}(p)}, \\ \tilde{E}_\mu(\infty, t) &= \sum_{\substack{p \in \mathcal{B}(1, \vec{m}_\mu) \\ p \text{ neg folded}}} t^{-\frac{1}{2}(\ell(\varphi(p)) - \#f(p))} (1 - t^{-1})^{\#f(p)} X^{\text{wt}(p)}, \\ \tilde{E}_\mu(q, \infty) &= \sum_{\substack{p \in \mathcal{B}(1, \vec{m}_\mu) \\ p \text{ pos semi-inf}}} q^{-\sum_{k \in f^+(p)} \text{sh}(\beta_k^\vee)} X^{\text{wt}(p)}. \end{aligned}$$

For  $i \in \{0, 1, \dots, n\}$  and  $f \in \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*]$  define

$$\Delta_i f = \frac{f - s_i f}{1 - X^{-\alpha_i}} \quad \text{and} \quad D_i f = (1 + s_i) \frac{1}{1 - X^{-\alpha_i}} f = \frac{f - X^{-\alpha_i} s_i f}{1 - X^{-\alpha_i}}. \quad (1.15)$$

Equations (1.13), (1.12) and the last relation in (1.9) give that, as operators on the polynomial representation,

$$\begin{aligned} t^{\frac{1}{2}} \tau_i^\vee &= X^{-\Lambda_0} (D_i - t \Delta_i) X^{\Lambda_0} + \frac{(1-t) Y^{-\alpha_i^\vee}}{1 - Y^{-\alpha_i^\vee}} \quad \text{for } i \in \{1, \dots, n\}, \quad \text{and} \\ t^{\frac{1}{2}} \tau_0^\vee Y^{\alpha_0^\vee} &= Y^{-\alpha_0^\vee} t^{\frac{1}{2}} \tau_0^\vee = X^{-\Lambda_0} (D_0 - t \Delta_0) X^{\Lambda_0} + \frac{(1-t) Y^{-\alpha_0^\vee}}{1 - Y^{-\alpha_0^\vee}}. \end{aligned}$$

When applied in the formula of (1.14), these formulas for (normalized) intertwiners are specializable at  $t = 0$ , giving the following result (see the examples computed in Section 3.2).

**Theorem 1.3.** ([Ion01, §4.1]) *Let  $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ . There are unique  $\nu \in \mathfrak{a}_{\mathbb{Z}}^*$  and  $j \in \mathbb{Z}$  such that*

$$\Lambda_0 + \nu \in (\mathfrak{h}^*)_{\text{int}}^+ \quad \text{and} \quad -j\delta + \mu + \Lambda_0 \in W^{\text{ad}}(\Lambda_0 + \nu).$$

Let  $w \in W^{\text{ad}}$  be minimal length such that  $-j + \mu + \Lambda_0 = w(\Lambda_0 + \nu)$  and let  $w = s_{i_1} \cdots s_{i_\ell}$  be a reduced word. Letting  $D_0, \dots, D_n$  be the Demazure operators given in (1.15),

$$\tilde{E}_\mu(q, 0) = q^{-j} X^{-\Lambda_0} D_{i_1} \cdots D_{i_\ell} X^\nu X^{\Lambda_0} \mathbf{1}.$$

## 2 Quantum affine algebras $\mathbf{U}$ and integrable modules

### 2.1 The quantum affine algebra $\mathbf{U}$

The *quantum affine algebra*  $\mathbf{U}$  is the  $\mathbb{C}(q)$ -algebra generated by

$$E_0, \dots, E_n, F_0, \dots, F_n, K_0, \dots, K_n, C^{\pm \frac{1}{2}}, D^{\pm 1}$$

with Chevalley-Serre type relations corresponding to the affine Dynkin diagram. Following [Dr88, (11)], [Lu93, §39], [B94, end of §1], there is an action of the affine braid group  $\mathcal{B}^{\text{sc}}$  on  $\mathbf{U}$  by automorphisms.

Let  $\mathbf{U}^+$  be the subalgebra of  $\mathbf{U}$  generated by  $E_0, E_1, \dots, E_n$ . As explained in [BCP98, Lemma 1.1 (iv)] and [BN02, (3.1)], there is a doubly infinite ‘‘longest element’’ for the affine Weyl group with a favourite reduced expression  $w_\infty = \cdots s_{i_{-1}} s_{i_0} s_{i_1} \cdots$ . This reduced word is used, with the braid group action, to define *root vectors* in  $\mathbf{U}^+$  by

$$E_{\beta_0} = E_{i_0}, \quad \text{and} \quad E_{\beta_{-k}} = T_{i_0}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{-(k-1)}}^{-1} E_{i_{-k}} \quad \text{and} \quad E_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}} E_{i_k}, \quad (2.1)$$

for  $k \in \mathbb{Z}_{>0}$ . For  $i \in \{1, \dots, n\}$  and  $r, s \in \mathbb{Z}$  define the *loop generators* in  $\mathbf{U}^+$

$$\mathbf{x}_{i,r}^+ = E_{\alpha_i + r\delta} = Y^{-r\omega_i^\vee} E_i \quad \text{and} \quad \mathbf{x}_{i,s}^- = E_{-\alpha_i + s\delta} = Y^{s\omega_i^\vee} F_i, \quad (2.2)$$

where  $Y^{r\omega_i^\vee}$  and  $Y^{s\omega_i^\vee}$  are elements of the braid group  $\mathcal{B}^{\text{sc}}$  as defined in Section 1.4. For  $r \in \mathbb{Z}_{\geq 0}$  and  $s \in \mathbb{Z}_{>0}$  these are special cases of the root vectors in (2.1). For  $i \in \{1, \dots, n\}$  and  $r \in \mathbb{Z}_{>0}$  define  $\mathbf{q}_r^{(i)}$  by

$$\mathbf{q}_r^{(i)} = \mathbf{x}_{i,r}^- \mathbf{x}_{i,0}^+ - q^{-2} \mathbf{x}_{i,0}^+ \mathbf{x}_{i,r}^-, \quad \text{let} \quad \mathbf{q}^+(z) = 1 + (q - q^{-1}) \sum_{s \in \mathbb{Z}_{>0}} \mathbf{q}_s^{(i)} z^s, \quad (2.3)$$



and define  $\mathbf{p}_r^{(i)}$  and  $\mathbf{e}_r^{(i)}$  by

$$\mathbf{q}^+(z) = \exp\left(\sum_{r \in \mathbb{Z}_{>0}} (q - q^{-1}) \mathbf{p}_r^{(i)} z^r\right) \quad \text{and} \quad \exp\left(\sum_{r \in \mathbb{Z}_{>0}} \frac{\mathbf{p}_r^{(i)}}{[r]} z^r\right) = 1 + \sum_{k \in \mathbb{Z}_{>0}} \mathbf{e}_k^{(i)} z^k. \quad (2.4)$$

For a sequence of partitions  $\vec{\kappa} = (\kappa^{(1)}, \dots, \kappa^{(n)})$  define

$$\mathbf{s}_{\vec{\kappa}} = \mathbf{s}_{\kappa^{(1)}} \cdots \mathbf{s}_{\kappa^{(n)}}, \quad \text{where} \quad \mathbf{s}_{\kappa^{(i)}} = \det(\mathbf{e}_{(\kappa^{(i)})'_r, -r+s}^{(i)})_{1 \leq r, s \leq m_i}, \quad (2.5)$$

where  $(\kappa^{(i)})'_r$  is the length of the  $r$ th column of  $\kappa^{(i)}$  and  $m_i = \ell(\kappa^{(i)})$  (see [Mac, Ch. I (3.5)]). For a sequence  $\mathbf{c} = (\dots, c_{-3}, c_{-2}, c_{-1}, \vec{\kappa}, c_1, c_2, c_3, \dots)$  with  $c_i \in \mathbb{Z}_{\geq 0}$  and all but a finite number of  $c_i$  equal to 0. the corresponding *PBW-type element of  $\mathbf{U}^+$*  is

$$E_{\mathbf{c}} = (E_{\beta_0}^{(c_0)} E_{\beta_{-1}}^{(c_{-1})} E_{\beta_{-2}}^{(c_{-2})} \cdots) (\mathbf{s}_{\kappa^{(1)}} \cdots \mathbf{s}_{\kappa^{(n)}}) (\cdots E_{\beta_3}^{(c_3)} E_{\beta_2}^{(c_2)} E_{\beta_1}^{(c_1)}). \quad (2.6)$$

The *Cartan involution* is the  $\mathbb{C}$ -linear anti-automorphism  $\Omega: \mathbf{U} \rightarrow \mathbf{U}$  given by

$$\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_\alpha) = K_{-\alpha}, \quad \Omega(D) = D^{-1}, \quad \Omega(q) = q^{-1},$$

and  $\mathbf{U}^- = \Omega(\mathbf{U}^+)$ . Putting  $\mathbf{p}_{-r}^{(i)} = \Omega(\mathbf{p}_r^{(i)})$ , then (see [Dr88, Th. 2 (6)] and [B94, Th. 4.7 (2)])

$$[\mathbf{p}_k^{(i)}, \mathbf{p}_l^{(i)}] = \delta_{k, -l} \frac{1}{k} \left( \frac{q^{k\alpha_i(h_j)} - q^{-k\alpha_i(h_j)}}{q - q^{-1}} \right) \frac{C^k - C^{-k}}{q - q^{-1}}. \quad (2.7)$$

Define  $q_{-s}^{(i)} = \Omega(q_s^{(i)})$  and

$$\mathbf{q}^-(z^{-1}) = 1 + (q - q^{-1}) \sum_{s \in \mathbb{Z}_{>0}} \mathbf{q}_{-s}^{(i)} z^{-s} \quad (2.8)$$

The *Heisenberg subalgebra  $\mathbf{H}$*  is the subalgebra of  $\mathbf{U}$  generated by  $\{\mathbf{p}_k^{(i)} \mid i \in \{1, \dots, n\}, k \in \mathbb{Z}_{\neq 0}\}$ . In  $\mathbf{H} \cap \mathbf{U}^+$  the  $\mathbf{p}_r^{(i)}$  ( $r \in \mathbb{Z}_{>0}$ ) are the *power sums*, the  $\mathbf{q}_r^{(i)}$  ( $r \in \mathbb{Z}_{>0}$ ) are the *Hall-Littlewoods* and the  $\mathbf{e}_r^{(i)}$  ( $r \in \mathbb{Z}_{>0}$ ) are the *elementary symmetric functions* and the  $\mathbf{s}_{\vec{\kappa}}$  are the *Schur functions*.

## 2.2 Integrable $\mathbf{U}$ -modules

Let  $\mathfrak{h}_{\mathbb{Z}}^* = \{\Lambda \in \mathfrak{h}^* \mid \langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i \in \{0, 1, \dots, n\}\} = \mathbb{C}\delta + \mathbb{Z}\text{-span}\{\Lambda_0, \dots, \Lambda_n\}$ . A set of representatives for the  $W^{\text{ad}}$ -orbits on  $\mathfrak{h}_{\mathbb{Z}}^*$  is

$$\begin{aligned} (\mathfrak{h}^*)_{\text{int}}^+ &= \mathbb{C}\delta + \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \dots, \Lambda_n\}, \\ (\mathfrak{h}^*)_{\text{int}} &= (\mathfrak{h}^*)_{\text{int}}^+ \cup (\mathfrak{h}^*)_{\text{int}}^0 \cup (\mathfrak{h}^*)_{\text{int}}^-, \quad \text{where} \quad (\mathfrak{h}^*)_{\text{int}}^0 = \mathbb{C}\delta + 0\Lambda_0 + \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \dots, \omega_n\}, \\ (\mathfrak{h}^*)_{\text{int}}^- &= \mathbb{C}\delta + \mathbb{Z}_{\leq 0}\text{-span}\{\Lambda_0, \dots, \Lambda_n\}. \end{aligned} \quad (2.9)$$

For  $\widehat{\mathfrak{sl}}_2$  these sets are pictured (mod  $\delta$ ) in (0.2).

For  $i \in \{0, 1, \dots, n\}$  let  $\mathbf{U}_{(i)}$  be the subalgebra of  $\mathbf{U}$  generated by  $\{E_i, F_i, K_i\}$ . An *integrable  $\mathbf{U}$ -module* is a  $\mathbf{U}$ -module  $M$  such that if  $i \in \{0, \dots, n\}$  then

$$\text{Res}_{\mathbf{U}_{(i)}}^{\mathbf{U}}(M) \quad \text{is a direct sum of finite dimensional } \mathbf{U}_{(i)}\text{-modules,}$$

where  $\text{Res}_{\mathbf{U}(i)}^{\mathbf{U}}(M)$  denotes the restriction of the  $\mathbf{U}$ -module  $M$  to a  $\mathbf{U}(i)$ -module.

Let  $M$  be an integrable  $\mathbf{U}$ -module. Following [Lu93, §5], for each  $w \in W^{\text{ad}}$  there is a linear map

$$T_w: M \rightarrow M \quad \text{such that} \quad T_w(um) = T_w(u)T_w(m),$$

for  $u \in \mathbf{U}$  and  $m \in M$  (here  $T_w(u)$  refers to the braid group action on  $\mathbf{U}$ ). Thus, every integrable module  $M$  is a module for the semidirect product  $\mathcal{B}^{\text{ad}} \ltimes \mathbf{U}$  where  $\mathcal{B}^{\text{ad}}$  is the braid group of  $W^{\text{ad}}$ .

### 2.3 Extremal weight modules $L(\Lambda)$

Let  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}$ . Following [Kas94, (8.2.2)] and [Kas02, §3.1], the *extremal weight module*  $L(\Lambda)$  is the  $\mathbf{U}$ -module

$$\begin{aligned} &\text{generated by } \{u_{w\Lambda} \mid w \in W\} \quad \text{with relations} \quad K_i(u_{w\Lambda}) = q^{\langle w\Lambda, \alpha_i^\vee \rangle} u_{w\Lambda}, \\ &E_i u_{w\Lambda} = 0, \quad \text{and} \quad F_i^{\langle w\Lambda, \alpha_i^\vee \rangle} u_{w\Lambda} = u_{s_i w \Lambda}, \quad \text{if } \langle w\Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}, \\ &F_i u_{w\Lambda} = 0, \quad \text{and} \quad E_i^{-\langle w\Lambda, \alpha_i^\vee \rangle} u_{w\Lambda} = u_{s_i w \Lambda}, \quad \text{if } \langle w\Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\leq 0}, \end{aligned} \quad (2.10)$$

for  $i \in \{0, \dots, n\}$ . The module  $L(\Lambda)$  has a crystal  $B(\Lambda)$  ([Kas94, Prop. 8.2.2(ii)], [Kas02, §3.1]).

- If  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}^+$  then  $L(\Lambda)$  is the simple  $\mathbf{U}$ -module of highest weight  $\Lambda$  (see [Kac, (10.4.6)]).
- If  $\Lambda \notin (\mathfrak{h}^*)_{\text{int}}^+$  then  $L(\Lambda)$  is not a highest weight module.
- If  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}^-$  then  $L(\Lambda)$  is the simple  $\mathbf{U}$ -module of lowest weight  $\Lambda$ .

The finite dimensional simple modules  $L^{\text{fin}}(a(u))$  are integrable weight modules which are not extremal weight modules. The connection between the  $L^{\text{fin}}(a(u))$  and the  $L(\lambda)$  for  $\lambda \in (\mathfrak{h}^*)_{\text{int}}^0$  is given by Proposition 2.3 below.

The module  $L(\Lambda)$  is universal (see [Kas05, §2.6], [B02, §2.1], [Nak02, §2.5]). One way to formulate this universality is to let  $\mathbf{U}_0$  be the subalgebra generated by  $K_1, \dots, K_n, C, D$ , let  $\text{intInd}$  be an induction functor in the category of integrable  $\mathbf{U}$ -modules and write

$$L(\Lambda) = \text{intInd}_{\mathbf{U}_0 \rtimes \mathcal{B}^{\text{ad}}}^{\mathbf{U}}(S(\Lambda)), \quad \text{where} \quad S(\Lambda) = \text{span}\{u_{w\Lambda} \mid w \in W^{\text{ad}}\}$$

is the  $\mathbf{U}_0 \rtimes \mathcal{B}^{\text{ad}}$ -module with action given by  $T_i u_{w\Lambda} = (-q_i)^{\langle w\Lambda, \alpha_i^\vee \rangle} u_{s_i w \Lambda}$  and  $K_i u_{w\Lambda} = q^{\langle w\Lambda, \alpha_i^\vee \rangle} u_{w\Lambda}$ , for  $i \in \{0, 1, \dots, n\}$  and  $w \in W^{\text{ad}}$ .

### 2.4 Demazure submodules $L(\Lambda)_{\leq w}$

Let  $w \in W^{\text{ad}}$ . The *Demazure module*  $L(\Lambda)_{\leq w}$  is the  $\mathbf{U}^+$ -submodule of  $L(\Lambda)$  given by

$$L(\Lambda)_{\leq w} = \mathbf{U}^+ u_{w\Lambda} \quad \text{and} \quad \text{char}(L(\Lambda)_{\leq w}) = \sum_{p \in B(\Lambda)_{\leq w}} e^{\text{wt}(p)},$$

since  $L(\Lambda)_{\leq w}$  has a crystal  $B(\Lambda)_{\leq w}$ . The *BGG-Demazure operator* on  $\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{C}\text{-span}\{X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$  is given by

$$D_i = (1 + s_i) \frac{1}{1 - X^{-\alpha_i}}, \quad \text{for } i \in \{0, 1, \dots, n\}.$$

Let  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}$ ,  $w \in W$  and  $i \in \{0, 1, \dots, n\}$ .

$$\text{If } \Lambda \in (\mathfrak{h}^*)_{\text{int}}^+ \quad \text{then} \quad D_i \text{char}(L(\Lambda)_{\leq w}) = \begin{cases} \text{char}(L(\Lambda)_{\leq s_i w}), & \text{if } s_i w \succeq w, \\ \text{char}(L(\Lambda)_{\leq w}), & \text{if } s_i w \preceq w; \end{cases}$$

$$\begin{aligned} \text{if } \lambda \in (\mathfrak{h}^*)_{\text{int}}^0 \quad \text{then} \quad D_i \text{char}(L(\lambda)_{\leq w}) &= \begin{cases} \text{char}(L(\lambda)_{\leq s_i w}), & \text{if } s_i w \supseteq w, \\ \text{char}(L(\lambda)_{\leq w}), & \text{if } s_i w \leq w; \end{cases} \\ \text{if } \Lambda \in (\mathfrak{h}^*)_{\text{int}}^- \quad \text{then} \quad D_i \text{char}(L(\Lambda)_{\leq w}) &= \begin{cases} \text{char}(L(\Lambda)_{\leq s_i w}), & \text{if } s_i w \geq w, \\ \text{char}(L(\Lambda)_{\leq w}), & \text{if } s_i w \leq w; \end{cases} \end{aligned}$$

(see [Kum, Theorem 8.2.9], [Kas93], [Kas05, §2.8] and [Kt16, Theorems 4.7 and 4.11]).

## 2.5 An alternate presentation for level 0 extremal weight modules

For  $\lambda = m_1 \omega_1 + \cdots + m_n \omega_n \in (\mathfrak{h}^*)_{\text{int}}$  and let  $x_{1,1}, \dots, x_{m_1,1}, x_{1,2}, \dots, x_{m_2,2}, \dots, x_{1,n}, \dots, x_{m_n,n}$  be  $n$  sets of formal variables. Letting  $e_i^{(j)} = e_i(x_{1,j}, \dots, x_{m_j,j})$  denote the elementary symmetric function in the variables  $x_{1,j}, \dots, x_{m_j,j}$  define

$$\begin{aligned} RG_\lambda &= \mathbb{C}[x_{1,1}^{\pm 1}, \dots, x_{m_1,1}^{\pm 1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}^{\pm 1}, \dots, x_{m_n,n}^{\pm 1}]^{S_{m_n}} \\ &= \mathbb{C}[e_1^{(1)}, \dots, e_{m_1-1}^{(1)}, (e_{m_1}^{(1)})^{\pm 1}] \otimes \cdots \otimes \mathbb{C}[e_1^{(n)}, \dots, e_{m_n-1}^{(n)}, (e_{m_n}^{(n)})^{\pm 1}], \\ RG_\lambda^+ &= \mathbb{C}[x_{1,1}, \dots, x_{m_1,1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}, \dots, x_{m_n,n}]^{S_{m_n}}, \quad \text{and} \\ RG_\lambda^- &= \mathbb{C}[x_{1,1}^{-1}, \dots, x_{m_1,1}^{-1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}^{-1}, \dots, x_{m_n,n}^{-1}]^{S_{m_n}}. \end{aligned}$$

Let

$$\begin{aligned} e_+^{(i)}(u) &= (1 - x_{1,i}u)(1 - x_{2,i}u) \cdots (1 - x_{m_i,i}u) \quad \text{and} \\ e_-^{(i)}(u^{-1}) &= (1 - x_{1,i}^{-1}u^{-1})(1 - x_{2,i}^{-1}u^{-1}) \cdots (1 - x_{m_i,i}^{-1}u^{-1}). \end{aligned}$$

Let  $\mathbf{U}'$  be the subalgebra of  $\mathbf{U}$  without the generator  $D$ .

**Theorem 2.1.** (see [Nak02, §3.4]) *The extremal weight module  $L(\lambda)$  is the  $(\mathbf{U}' \otimes_{\mathbb{Z}} RG_\lambda)$ -module generated by a single vector  $m_\lambda$  with relations*

$$\begin{aligned} \mathbf{x}_{i,r}^+ m_\lambda &= 0, \quad K_i m_\lambda = q^{m_i} m_\lambda, \quad C m_\lambda = m_\lambda, \\ \mathbf{q}^+(u) m_\lambda &= K_i \frac{e_+^{(i)}(q^{-1}u)}{e_+^{(i)}(qu)} m_\lambda \quad \text{and} \quad \mathbf{q}^-(u^{-1}) m_\lambda = K_i \frac{e_-^{(i)}(qu^{-1})}{e_-^{(i)}(q^{-1}u^{-1})} m_\lambda, \end{aligned}$$

where  $\mathbf{q}^+(u)$  and  $\mathbf{q}^-(u^{-1})$  are as defined in (2.5) and (2.8).

In this form  $L(\lambda)$  has been termed the *universal standard module* [Nak02, §3.4] or the *global Weyl module* [CP01, §2]. See [Nak02, Theorem 2] and [Nak02, Remark 2.15] for discussion of how to see that the extremal weight module, the universal standard module and the global Weyl module coincide.

**Remark 2.2.** Let

$$0_q = \frac{1}{1-q} + \frac{q^{-1}}{1-q^{-1}} = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots,$$

(although  $\frac{q^{-1}}{1-q^{-1}} = \frac{1}{q-1} = \frac{-1}{1-q}$ , it is important to note that  $0_q$  is *not* equal to 0, it is a doubly infinite formal series in  $q$  and  $q^{-1}$ ). Since  $\deg(e_j^{(i)}) = j$ ,

$$\begin{aligned} \text{gchar}(RG_\lambda^+) &= \left( \prod_{i=1}^n \prod_{k=1}^{m_i} \frac{1}{1-q^k} \right) \quad \text{gchar}(RG_\lambda^-) = \left( \prod_{i=1}^n \prod_{k=1}^{m_i} \frac{1}{1-q^{-k}} \right) \quad \text{and} \\ \text{gchar}(RG_\lambda) &= \left( 0_{q^{m_1}} \prod_{k=1}^{m_1-1} \frac{1}{1-q^k} \right) \left( 0_{q^{m_2}} \prod_{k=1}^{m_2-1} \frac{1}{1-q^k} \right) \cdots \left( 0_{q^{m_n}} \prod_{k=1}^{m_n-1} \frac{1}{1-q^k} \right). \end{aligned}$$

## 2.6 Level 0 $L(\lambda)$ and finite dimensional simple $U$ -modules $L^{\text{fin}}(a(u))$

The loop presentation provides a triangular decomposition of  $\mathbf{U}$  (different from the usual triangular decomposition coming from the Kac-Moody presentation). The extremal weight module  $L(\lambda)$  is the standard (Verma type) module for the loop triangular decomposition (see [CP95, Theorem 2.3(b)], [Nak02, Lemma 2.14] and [CP94, outline of proof of Theorem 12.2.6]).

A *Drinfeld polynomial* is an  $n$ -tuple of polynomials  $a(u) = (a^{(1)}(u), \dots, a^{(n)}(u))$  with  $a^{(i)}(u) \in \mathbb{C}[u]$ , represented as

$$a(u) = a^{(1)}(u)\omega_1 + \dots + a^{(n)}(u)\omega_n, \quad \text{with} \quad a^{(i)}(u) = (u - a_{1,i}) \cdots (u - a_{m_i,i})$$

so that the coefficient of  $u^j$  in  $a^{(i)}(u)$  is  $e_{m_i-j}^{(i)}(a_{1,i}, \dots, a_{m_i,i})$ , the  $(m_i - j)$ th elementary symmetric function evaluated at the values  $a_{1,i}, \dots, a_{m_i,i}$ . The *local Weyl module* (a finite dimensional standard module) is defined by

$$M^{\text{fin}}(a(u)) = L(\lambda) \otimes_{RG_\lambda} m_{a(u)}, \quad \text{where} \quad e_k^{(i)}(x_{1,i}, x_{2,i}, \dots) m_{a(u)} = e_k^{(i)}(a_{1,i}, \dots, a_{m_i,i}) m_{a(u)}$$

specifies the  $RG_\lambda$ -action on  $m_{a(u)}$ . In other words, the module  $M(a(u))$  is  $L(\lambda)$  except that variables  $x_{j,i}$  from §2.5 have been specialised to the values  $a_{j,i}$ .

**Theorem 2.3.** (see [Dr88, Theorem 2] and [CP95, Theorem 3.3]) *The standard module  $M^{\text{fin}}(a(u))$  has a unique simple quotient  $L^{\text{fin}}(a(u))$  and*

$$\begin{array}{ccc} \{\text{Drinfeld polynomials}\} & \longrightarrow & \{\text{finite dimensional simple } \mathbf{U}'\text{-modules}\} \\ a(u) = a^{(1)}(u)\omega_1 + \dots + a^{(n)}(u)\omega_n & \longmapsto & L^{\text{fin}}(a(u)) \end{array}$$

is a bijection.

## 2.7 Path models for the crystals $B(\Lambda)$

The work of Littelmann [Li94], [Li95] provided a particularly convenient model for the crystals  $B(\Lambda)$  when  $\Lambda$  is positive or negative level. This model realizes the crystal as a set of paths  $p: \mathbb{R}_{[0,1]} \rightarrow \mathfrak{h}^*$  with combinatorially defined Kashiwara operators  $\tilde{e}_0, \dots, \tilde{e}_n, \tilde{f}_0, \dots, \tilde{f}_n$ . In the LS (Lakshmibai-Seshadri) model the generator of the crystal  $B(\Lambda)$  is the straight line path to  $\Lambda$ .

When  $\lambda \in (\mathfrak{h}^*)_{\text{int}}^0$  using

$$p_\lambda: \begin{array}{ccc} \mathbb{R}_{[0,1]} & \rightarrow & \mathfrak{h}^* \\ t & \rightarrow & t\lambda, \end{array} \quad \text{the straight line path from 0 to } \lambda,$$

as a generator for  $B(\lambda)$  may not be the optimal choice. Remarkably, Naito and Sagaki (see [INS16, Definition 3.1.4 and Theorem 3.2.1] and [NNS15, Theorem 4.6.1(b)]), have shown that  $B(\lambda)$  can be constructed with sequences of Weyl group elements and rational numbers as in [Li94, §1.2, 1.3 and 2.2] but with the positive level length  $\ell^+$  and Bruhat order  $\leq$  replaced by the level zero length  $\ell^0$  and Bruhat order  $\leq^0$ . However, when working with the Naito-Sagaki construction one must be very careful not to identify the Naito-Sagaki sequences with actual paths (piecewise linear maps from  $\mathbb{R}_{[0,1]}$  to  $\mathfrak{h}^*$ ) because the natural map from Naito-Sagaki sequences to paths is *not always injective* (an example is provided by [Kas02, Remark 5.10]).

If  $\Lambda \in (\mathfrak{h}^*)_{\text{int}}^+$  then  $-\Lambda \in (\mathfrak{h}^*)_{\text{int}}^-$  and

$$B(\Lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} p_\Lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } i_1, \dots, i_k \in \{0, 1, \dots, n\}\}$$

$$B(-\Lambda) = \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} p_{-\Lambda} \mid k \in \mathbb{Z}_{\geq 0} \text{ and } i_1, \dots, i_k \in \{0, 1, \dots, n\}\}.$$

are each a single connected component and their characters are determined by the Weyl-Kac character formula [Kac, Theorem 11.13.3].

## 2.8 Crystals for level 0 extremal weight modules $L(\lambda)$

For general  $\lambda \in (\mathfrak{h}^*)_{\text{int}}^0$  the crystal  $B(\lambda)$  is not connected (as a graph with edges determined by the Kashiwara operators  $\tilde{e}_0, \dots, \tilde{e}_n, \tilde{f}_0, \dots, \tilde{f}_n$ ). Let  $\lambda = m_1\omega_1 + \dots + m_n\omega_n$ , with  $m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}$ . By [BN02, Corollary 4.15], the map

$$\Phi_\lambda: \begin{array}{ccc} L(\lambda) & \longrightarrow & L(\omega_1)^{\otimes m_1} \otimes \dots \otimes L(\omega_n)^{\otimes m_n} \\ u_\lambda & \longmapsto & u_{\omega_1}^{\otimes m_1} \otimes \dots \otimes u_{\omega_n}^{\otimes m_n} \end{array} \quad \text{is injective} \quad (2.11)$$

and gives rise to an injection of crystals

$$B(\lambda) \hookrightarrow B(\omega_1)^{\otimes m_1} \otimes \dots \otimes B(\omega_n)^{\otimes m_n}$$

which takes the connected component of  $B(\lambda)$  containing  $b_\lambda$  to the connected component of  $B(\omega_1)^{\otimes m_1} \otimes \dots \otimes B(\omega_n)^{\otimes m_n}$  containing  $b_{\omega_1}^{\otimes m_1} \otimes \dots \otimes b_{\omega_n}^{\otimes m_n}$ . Kashiwara [Kas02, Theorem 5.15] fully described the structure of  $L(\omega_i)$  (see [BN02, Theorem 2.16]). Beck-Nakajima analyzed the PBW basis of (2.6) and use (2.11) to show that the connected components of  $B(\lambda)$  are labeled by  $n$ -tuples of partitions  $\kappa = (\kappa^{(1)}, \dots, \kappa^{(n)})$  such that  $\ell(\kappa^{(i)}) < m_i$ . Together with a result of Fourier-Littelmann which shows that the crystal of the level 0 module  $M^{\text{fin}}(a(u))$  is isomorphic to a level one Demazure crystal  $B(\nu + \Lambda_0)_{\leq w}$ , the full result is as detailed in Theorem 2.4 below.

A labeling set for a basis of  $RG_\lambda / \langle e_{m_i}^{(i)} = 1 \rangle$  is

$$S^\lambda = \{ \vec{\kappa} = (\kappa^{(1)}, \dots, \kappa^{(n)}) \mid \kappa^{(i)} \text{ is a partition with } \ell(\kappa^{(i)}) < m_i \text{ for } i \in \{1, \dots, n\} \}.$$

The connected component of  $b_\lambda$  in  $B(\lambda)$  is

$$B(\lambda)_0 = \left\{ \tilde{r}_{i_1} \dots \tilde{r}_{i_k} b_\lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } \tilde{r}_{i_1}, \dots, \tilde{r}_{i_k} \in \{ \tilde{e}_0, \dots, \tilde{e}_n, \tilde{f}_0, \dots, \tilde{f}_n \} \right\}.$$

Define  $B^{\text{fin}}(\lambda)$  to be the ‘‘crystal’’ which has a crystal graph which is the ‘‘quotient’’ of the crystal graph of  $L(\lambda)$  obtained by identifying the vertices  $b$  and  $b'$  if there is an element  $\mathbf{s} \in RG_\lambda$  such that  $\mathbf{s}G(b) = G(b')$ , where  $G(b)$  denotes the canonical basis element of  $L(\lambda)$  corresponding to  $b$ .

$B^{\text{fin}}(\lambda)$  is the crystal of the finite dimensional standard module  $M^{\text{fin}}(a(u))$ .

**Theorem 2.4.** (see [BN02, Theorem 4.16], [Nak02, §3.4]) and [FL05, Proposition 3]) *Let  $\lambda = m_1\omega_1 + \dots + m_n\omega_n \in (\mathfrak{h}^*)_{\text{int}}^0$ . As in Theorem 1.3, let  $\nu \in \mathfrak{a}^*$ ,  $j \in \mathbb{Z}_{\geq 0}$  and  $w \in W^{\text{ad}}$  such that  $w(\nu + \Lambda_0) = -j\delta + \lambda + \Lambda_0$  and  $w$  is minimal length. Then*

$$B(\lambda) \simeq B(\lambda)_0 \times S^\lambda \quad \text{and} \quad B(\lambda)_0 \simeq \mathbb{Z}^k \times B^{\text{fin}}(\lambda) \quad \text{and} \quad B^{\text{fin}}(\lambda) \simeq B(\nu + \Lambda_0)_{\leq w},$$

where  $k$  is the number of elements of  $m_1, \dots, m_n$  which are nonzero.

Additional useful references for Theorem 2.4 are [Nak02, Theorem 1] and [B02, Theorem 1]. The first two statements in Theorem 2.4 are reflections of the very important fact that  $L(\lambda)$  is free as an  $RG_\lambda$ -module. This fact that  $L(\lambda)$  is free as an  $RG_\lambda$ -module was deduced geometrically, via the quiver variety, in [Nak99, Theorem 7.3.5] (note that property  $(T_{G_{\mathbf{w}} \times \mathbb{C}^\times})$  there includes the freeness, see the definition of property  $(T_G)$  after [Nak99, (7.1.1)]). This freeness was understood more algebraically in the work of Fourier-Littelmann [FL05] and Chari-Ion [CI13, Cor. 2.10]. Further understanding of the  $RG_\lambda$ -action in terms of the geometry of the semi-infinite flag variety is in [BF13, §5.1].

The last statement of Theorem 2.4 is proved by considering the map

$$\begin{array}{ccc} B(\lambda) & \longrightarrow & B(\Lambda_0) \otimes B(\lambda) \\ \cup & & \cup \\ B^{\text{fin}}(\lambda) & \xrightarrow{\sim} & B(\nu + \Lambda_0)_{\leq w} \end{array} \quad \text{given by} \quad b \mapsto b_{\Lambda_0} \otimes b.$$

where  $b_{\Lambda_0}$  is the highest weight of the crystal  $B(\Lambda_0)$ . Combining the isomorphism  $B^{\text{fin}}(\lambda) \simeq B(\nu + \Lambda_0)_{\leq w}$  with Theorem 1.3 and the positive level formula in §2.4 gives

$$\text{char}(M(a(u))) = \text{char}(L(\nu + \Lambda_0)_{\leq w}) = q^{-j} X^{-\Lambda_0} \tilde{E}_\lambda(q, 0) \quad \text{and}$$

$$\text{char}(L(\lambda)_{\leq w_0}) = \text{gchar}(RG_\lambda^+) \tilde{E}_{w_0\lambda}(q, 0), \quad \text{char}(L(\lambda)) = \text{gchar}(RG_\lambda) \tilde{E}_{w_0\lambda}(q, 0). \quad (2.12)$$

### 3 Examples for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$

Let  $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$  with  $\mathfrak{h}^* = \mathbb{C}\omega_1 \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$  with affine Cartan matrix

$$\begin{pmatrix} \alpha_0(h_0) & \alpha_0(h_1) \\ \alpha_1(h_0) & \alpha_1(h_1) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{array}{ll} \theta = \alpha_1 = 2\omega_1, & \theta^\vee = \alpha_1^\vee = h_1, \\ \Lambda_1 = \omega_1 + \Lambda_0, & \alpha_0 = -\alpha_1 + \delta, \end{array}$$

Using that  $\langle \alpha_1, \alpha_1 \rangle = 2$ , if  $k \in \mathbb{Z}$  and  $\mu^\vee = k\alpha_1^\vee = k\alpha_1 = k2\omega_1 = 2k\omega_1$  so that  $\mu_1^\vee = 2k$  and  $-\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle = -\frac{1}{2}(k2k) = -k^2$ . Thus, following (1.7), (1.4) and (1.5), in the basis  $\{\delta, \omega_1, \Lambda_0\}$  of  $\mathfrak{h}^*$ ,

$$t_{k\alpha^\vee} = \begin{pmatrix} 1 & -k & -k^2 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_0 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

These matrices are used to compute the  $W$ -orbits pictured in Section 0.1.

#### 3.1 Macdonald polynomials

The Demazure operators are given by

$$D_1 f = \frac{f - X^{-2\omega_1}(s_1 f)}{1 - X^{-2\omega_1}} \quad \text{and} \quad D_0 f = \frac{f - X^{-\alpha_0}(s_0 f)}{1 - X^{-\alpha_0}} = \frac{f - q^{-1}X^{2\omega_1}(s_0 f)}{1 - q^{-1}X^{2\omega_1}}.$$

The normalized Macdonald polynomials  $\tilde{E}_{\omega_1}(q, t)$  and  $\tilde{E}_{-\omega_1}(q, t)$  are

$$\begin{array}{ccc} \begin{array}{c} | \quad | \quad | \quad | \\ \leftarrow \bullet \end{array} & & \begin{array}{c} | \quad | \quad | \quad | \\ \bullet \rightarrow \end{array} \\ \tilde{E}_{\omega_1}(q, t) = X^{\omega_1} & \text{and} & \tilde{E}_{-\omega_1}(q, t) = X^{-\omega_1} + \frac{(1-t)}{1-qt} X^{\omega_1} \\ & & = X^{-\omega_1} + \frac{(1-t^{-1})q^{-1}}{1-q^{-1}t^{-1}} X^{\omega_1} \end{array}$$

giving  $\tilde{E}_{\omega_1}(0, t) = \tilde{E}_{\omega_1}(\infty, t) = \tilde{E}_{\omega_1}(q, 0) = \tilde{E}_{\omega_1}(q, \infty) = X^{\omega_1}$  and

$$\begin{array}{ll} \tilde{E}_{-\omega_1}(0, t) = X^{-\omega_1} + (1-t)X^{\omega_1}, & \tilde{E}_{-\omega_1}(\infty, t) = X^{-\omega_1} \\ \tilde{E}_{-\omega_1}(q, 0) = X^{-\omega_1} + X^{\omega_1}, & \tilde{E}_{-\omega_1}(q, \infty) = X^{-\omega_1} + q^{-1}X^{\omega_1}. \end{array}$$

The normalized Macdonald polynomials  $\tilde{E}_{2\omega_1}(q, t)$  and  $\tilde{E}_{-2\omega_1}(q, t)$  are

$$\begin{array}{c}
 \begin{array}{|c|c|c|c|} \hline | & | & \bullet & | \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline | & | & \bullet & | \\ \hline \end{array} \\
 \\
 \tilde{E}_{2\omega_1}(q, t) = X^{2\omega_1} + \frac{(1-t)q}{1-qt} \quad \text{and} \\
 = X^{2\omega_1} + \frac{1-t^{-1}}{1-q^{-1}t^{-1}} \\
 \\
 \begin{array}{|c|c|c|c|} \hline | & | & \bullet & | \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline | & | & \bullet & | \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline | & | & \bullet & | \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline | & | & \bullet & | \\ \hline \end{array} \\
 \\
 \tilde{E}_{-2\omega_1}(q, t) = X^{-2\omega_1} + \frac{1-t}{1-qt} + \frac{1-t}{1-q^2t} X^{2\omega_1} + \frac{(1-t)(1-t)q}{(1-q^2t)(1-qt)} \\
 = X^{-2\omega_1} + \frac{(1-t^{-1})q^{-1}}{1-q^{-1}t^{-1}} + \frac{(1-t^{-1})q^{-2}}{1-q^{-2}t^{-1}} X^{2\omega_1} + \frac{(1-t^{-1})q^{-2}(1-t^{-1})}{(1-q^{-2}t^{-1})(1-q^{-1}t^{-1})}
 \end{array}$$

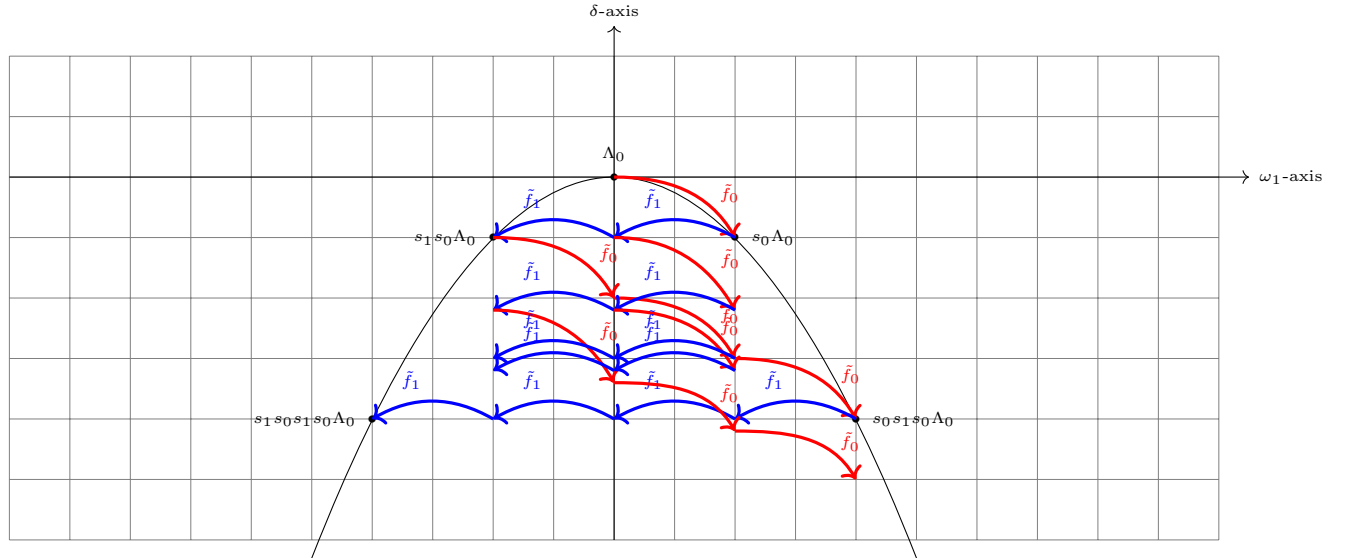
giving

$$\begin{array}{l}
 \tilde{E}_{2\omega_1}(0, t) = X^{2\omega_1}, \quad \tilde{E}_{2\omega_1}(\infty, t) = X^{2\omega_1} + (1-t^{-1}) \\
 \tilde{E}_{2\omega_1}(q, 0) = X^{2\omega_1} + q, \quad \tilde{E}_{2\omega_1}(q, \infty) = X^{2\omega_1} + 1
 \end{array}$$

and

$$\begin{array}{l}
 \tilde{E}_{-2\omega_1}(0, t) = X^{-2\omega_1} + (1-t)X^{2\omega_1} + (1-t), \quad \tilde{E}_{-2\omega_1}(\infty, t) = X^{-2\omega_1}, \\
 \tilde{E}_{-2\omega_1}(q, 0) = X^{-2\omega_1} + X^{2\omega_1} + 1 + q, \quad \tilde{E}_{-2\omega_1}(q, \infty) = X^{-2\omega_1} + q^{-2}X^{2\omega_1} + q^{-1} + q^{-2}.
 \end{array}$$

### 3.2 The crystal $B(\Lambda_0)$ and $B(\omega_1 + \Lambda_0)$



Initial portion of the crystal graph of  $B(\Lambda_0)$  for  $\widehat{\mathfrak{sl}}_2$

The characters of the first few Demazure modules in  $L(\Lambda_0)$  are

$$\begin{array}{l}
 \text{char}(L(\Lambda_0)_{\leq 1}) = X^{\Lambda_0} = X^{\Lambda_0} \tilde{E}_0(q^{-1}, 0), \\
 \text{char}(L(\Lambda_0)_{\leq s_0}) = D_0 X^{\Lambda_0} = X^{\Lambda_0} (1 + qX^{2\omega_1}) = qX^{\Lambda_0} (X^{2\omega_1} + q^{-1}) = qX^{\Lambda_0} \tilde{E}_{2\omega_1}(q^{-1}, 0), \\
 \text{char}(L(\Lambda_0)_{\leq s_1 s_0}) = D_1 D_0 X^{\Lambda_0} = X^{\Lambda_0} (1 + qX^{2\omega_1} + q + qX^{-2\omega_1}) \\
 = qX^{\Lambda_0} (X^{-2\omega_1} + X^{2\omega_1} + 1 + q^{-1}) = qX^{\Lambda_0} \tilde{E}_{-2\omega_1}(q^{-1}, 0).
 \end{array}$$

The crystal graph of  $B(\omega_1 + \Lambda_0)$  is pictured in Plate B and

$$\begin{aligned}\text{char}(L(\omega_1 + \Lambda_0)_{\leq 1}) &= X^{\Lambda_0} X^{\omega_1} = X^{\Lambda_0} \tilde{E}_{\omega_1}(q^{-1}, 0), \\ \text{char}(L(\omega_1 + \Lambda_0)_{\leq s_1}) &= D_1 X^{\omega_1 + \Lambda_0} = X^{\Lambda_0} (X^{\omega_1} + X^{-\omega_1}) = X^{\Lambda_0} \tilde{E}_{-\omega_1}(q^{-1}, 0), \\ \text{char}(L(\omega_1 + \Lambda_0)_{\leq s_0 s_1}) &= D_0 D_1 X^{\omega_1 + \Lambda_0} = X^{\Lambda_0} (X^{\omega_1} + X^{-\omega_1} + qX^{\omega_1} + q^2 X^{3\omega_1}) \\ &= q^2 X^{\Lambda_0} \tilde{E}_{3\omega_1}(q^{-1}, 0),\end{aligned}$$

### 3.3 The crystal $B(\omega_1)$

The crystal  $B(\omega_1) = \{p_{-\omega_1 + k\delta}, p_{\omega_1 + k\delta} \mid k \in \mathbb{Z}\}$  is a single connected component. The crystal graph of  $B(\omega_1)$  is pictured in Plate B. Following [Kac, (12.1.9)] and putting  $q = X^{-\delta}$  noting that  $\tilde{E}_{-\omega_1}(q^{-1}, 0) = X^{\omega_1} + X^{-\omega_1}$  and  $\tilde{E}_{-\omega_1}(q^{-1}, \infty) = X^{-\omega_1} + qX^{\omega_1}$ , then

$$\begin{aligned}\text{char}(L(\omega_1)_{\leq s_1}) &= \frac{1}{1 - q^{-1}} \tilde{E}_{-\omega_1}(q^{-1}, 0) \quad \text{and} \\ \text{char}(L(\omega_1)) &= 0_q \tilde{E}_{-\omega_1}(q^{-1}, 0) = 0_q \tilde{E}_{-\omega_1}(q^{-1}, \infty),\end{aligned}$$

where  $0_q = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots$  as in Remark 2.2.

### 3.4 The crystal $B(2\omega_1)$

On crystals, the injective  $\mathbf{U}$ -module homomorphism  $L(2\omega_1) \hookrightarrow L(\omega_1) \otimes L(\omega_1)$  given in (2.11) is the inclusion

$$\begin{aligned}B(\omega_1) \otimes B(\omega_1) &= \{p_{w_1\omega_1} \otimes p_{w_2\omega_1} \mid w_1, w_2 \in W^{\text{ad}}\} \\ \cup \\ B(2\omega_1) &= \{p_{w_1\omega_1} \otimes p_{w_2\omega_1} \mid w_1, w_2 \in W^{\text{ad}} \text{ with } w_1 \vartriangleright w_2\}\end{aligned}$$

The connected components of  $B(2\omega_1)$  are determined by

$$B(2\omega_1) = B(2\omega_1)_0 \times S^{2\omega_1}, \quad \text{where } S^{2\omega_1} = \{\text{partitions } \kappa \text{ with } \ell(\kappa) < 2\} = \mathbb{Z}_{\geq 0}.$$


For  $\kappa \in \mathbb{Z}_{\geq 0}$ , the connected component corresponding to  $\kappa$ , as a subset of  $B(\omega_1) \otimes B(\omega_1)$ , is

$$B(2\omega_1)_\kappa = \{p_{w_1\omega_1} \otimes p_{w_2\omega_1} \mid w_1 \vartriangleright w_2 \text{ and } \ell^0(w_1) - \ell^0(w_2) \in \{\kappa, \kappa + 1\}\}$$

Representatives of the components and the crystal graph of the connected component  $B(2\omega_1)_0$  are pictured in Plate C. Inspection of the crystal graphs gives

$$\begin{aligned}\text{char}(L(2\omega_1)_{\leq s_1}) &= \frac{1}{1 - q^{-2}} \frac{1}{1 - q^{-1}} \tilde{E}_{-2\omega_1}(q^{-1}, 0), \quad \text{and} \\ \text{char}(L(2\omega_1)) &= 0_q \frac{1}{1 - q} \tilde{E}_{-2\omega_1}(q^{-1}, 0) = 0_q \frac{1}{1 - q} \tilde{E}_{-2\omega_1}(q^{-1}, \infty).\end{aligned}$$

## 4 Examples for $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$

Let  $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$ . The affine Dynkin diagram is  and the affine Cartan matrix is

$$\begin{pmatrix} \alpha_0(h_0) & \alpha_0(h_1) & \alpha_0(h_2) \\ \alpha_1(h_0) & \alpha_1(h_1) & \alpha_1(h_2) \\ \alpha_2(h_0) & \alpha_2(h_1) & \alpha_2(h_2) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



and  $h_0 = -(h_1 + h_2) + K$ ,  $\alpha_0 = -(\alpha_1 + \alpha_2) + \delta$ ,  $\Lambda_1 = \omega_1 + \Lambda_0$ ,  $\Lambda_2 = \omega_2 + \Lambda_0$ .

In the basis  $\{\delta, \omega_1, \omega_2, \Lambda_0\}$  of  $\mathfrak{h}^*$  the action of the affine Weyl group  $W^{\text{ad}}$  is given by

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad s_0 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$t_{k_1 h_1 + k_2 h_2} = \begin{pmatrix} 1 & -k_1 & -k_2 & -k_1^2 - k_2^2 + k_1 k_2 \\ 0 & 1 & 0 & 2k_1 - k_2 \\ 0 & 0 & 1 & 2k_2 - k_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for } k_1, k_2 \in \mathbb{Z}.$$

These matrices are used to compute the orbits pictured at the end of Section 0.1.

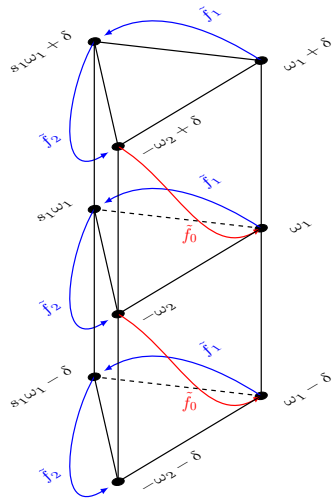
#### 4.1 The extremal weight modules $L(\omega_1)$ and $L(\omega_2)$

Letting  $\mathbb{C}^3 = \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$  be the standard representation of  $\hat{\mathfrak{g}} = \mathfrak{sl}_3$ , the extremal weight modules  $L(\omega_1)$  and  $L(\omega_2)$  for  $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$  are

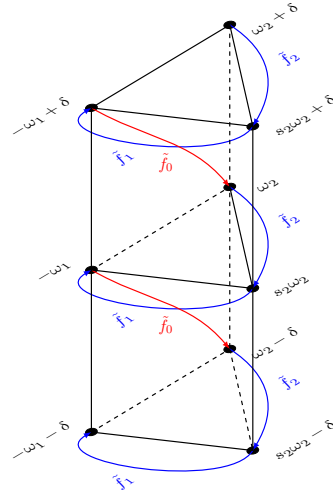
$$L(\omega_1) = \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}] \quad \text{and} \quad L(\omega_2) = (\Lambda^2 \mathbb{C}^3) \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}],$$

with  $u_{\omega_1} = v_1$  and  $u_{\omega_2} = v_1 \wedge v_2$ , respectively. The crystals  $B(\omega_1)$  and  $B(\omega_2)$  have realizations as sets of straight line paths:

$$B(\omega_1) = \{p_{\omega_1 + k\delta}, p_{s_1 \omega_1 + k\delta}, p_{-\omega_2 + k\delta} \mid k \in \mathbb{Z}\}, \quad B(\omega_2) = \{p_{\omega_2 + k\delta}, p_{s_2 \omega_2 + k\delta}, p_{-\omega_1 + k\delta} \mid k \in \mathbb{Z}\}.$$



$B(\omega_1)$  crystal graph



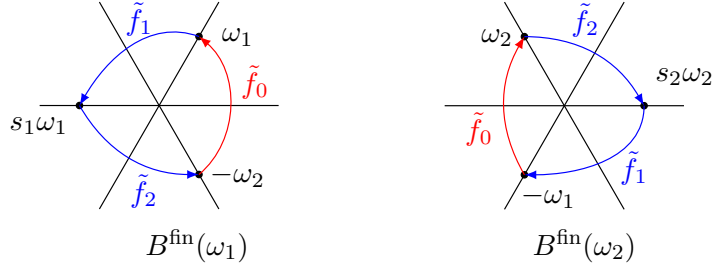
$B(\omega_2)$  crystal graph

Each of the crystals  $B(\omega_1)$  and  $B(\omega_2)$  has a single connected component, all weight spaces are one dimensional and

$$\begin{aligned} \text{char}(L(\omega_1)_{\leq s_2 s_1}) &= \frac{1}{1 - q^{-1}} (X^{\omega_1} + X^{s_1 \omega_1} + X^{-\omega_2}) = \frac{1}{1 - q^{-1}} \tilde{E}_{-\omega_2}(q^{-1}, 0), \quad \text{and} \\ \text{char}(L(\omega_1)) &= 0_q (X^{\omega_1} + X^{-\omega_2} + X^{s_1 \omega_1}) = 0_q \tilde{E}_{-\omega_2}(q^{-1}, 0) \\ &= 0_q (q^{-1} X^{\omega_1} + q^{-1} X^{\omega_2 - \omega_1} + X^{-\omega_2}) = 0_q \tilde{E}_{-\omega_2}(q^{-1}, \infty). \end{aligned}$$

where  $0_q = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots$  as in Remark 2.2.

For  $a \in \mathbb{C}$ , the crystals of  $M^{\text{fin}}((u-a)\omega_1) \cong \mathbb{C}^3$  and  $M^{\text{fin}}((u-a)\omega_2) \cong \Lambda^2(\mathbb{C}^3)$  have crystal graphs  $B^{\text{fin}}(\omega_1)$  and  $B^{\text{fin}}(\omega_2)$ .



## 4.2 The extremal weight module $L(\omega_1 + \omega_2)$

To construct the crystal  $B(\omega_1 + \omega_2)$  use

$$B(\omega_1) \otimes B(\omega_2) = \{p_{v\omega_1+k\delta} \otimes p_{w\omega_2+\ell\delta} \mid v \in \{1, s_1, s_2s_1\}, w \in \{1, s_2, s_1s_2\}, k \in \mathbb{Z}, \ell \in \mathbb{Z}\}$$

with the tensor product action for crystals given by (see, for example, [Ra06, Prop. 5.7])

$$\tilde{f}_i(p_1 \otimes p_2) = \begin{cases} \tilde{f}_i p_1 \otimes p_2, & \text{if } d_i^+(p_1) > d_i^-(p_2), \\ p_1 \otimes \tilde{f}_i p_2, & \text{if } d_i^+(p_1) \leq d_i^-(p_2), \end{cases} \quad \tilde{e}_i(p_1 \otimes p_2) = \begin{cases} \tilde{e}_i p_1 \otimes p_2, & \text{if } d_i^+(p_1) \geq d_i^-(p_2), \\ p_1 \otimes \tilde{e}_i p_2, & \text{if } d_i^+(p_1) < d_i^-(p_2). \end{cases}$$

where  $d_i^\pm(p)$  are determined by

$$\tilde{f}_i^{d_i^+(p)} p \neq 0 \text{ and } \tilde{f}_i^{d_i^+(p)+1} p = 0, \quad \text{and} \quad \tilde{e}_i^{d_i^-(p)} p \neq 0 \text{ and } \tilde{e}_i^{d_i^-(p)+1} p = 0.$$

The crystal  $B(\omega_1 + \omega_2)$  is realized as a subset of  $B(\omega_1) \otimes B(\omega_2)$  via the crystal embedding

$$\begin{array}{ccc} B(\omega_1 + \omega_2) & \hookrightarrow & B(\omega_1) \otimes B(\omega_2) \\ b_{\omega_1 + \omega_2} & \longmapsto & p_{\omega_1} \otimes p_{\omega_2} \end{array}$$

By Theorem 2.4,  $B(\omega_1 + \omega_2)$  is connected and is generated by  $b_{\omega_1 + \omega_2}$ . The crystal  $B(\omega_1 + \omega_2)$  is pictured and its character is computed in Plate D.

## 5 Alcove walks for affine flag varieties

### 5.1 The affine Kac-Moody group $G$

The most visible form of the loop group is  $G = \mathring{G}(\mathbb{C}((\epsilon)))$ , where  $\mathbb{C}((\epsilon))$  is the field of formal power series in a variable  $\epsilon$  and  $\mathring{G}$  is a reductive algebraic group. The favourite example is when

$$\text{when } \mathring{G} = GL_n \quad \text{and the loop group is } G = GL_n(\mathbb{C}((\epsilon))),$$

the group of  $n \times n$  invertible matrices with entries in  $\mathbb{C}((\epsilon))$ . A slightly more extended, and extremely powerful, point of view is to let  $G$  be the Kac-Moody group whose Lie algebra is the affine Lie algebra  $\mathfrak{g}$  of Section 1.1, so that  $G$  is a central extension of a semidirect product (where the semidirect product comes from the action of  $\mathbb{C}^\times$  on  $\mathring{G}(\mathbb{C}((\epsilon)))$  by “loop rotations”)

$$\{1\} \rightarrow \mathbb{C}^\times \rightarrow G \rightarrow \mathring{G}(\mathbb{C}((\epsilon))) \rtimes \mathbb{C}^\times \rightarrow \{1\} \quad \text{so that} \quad G = \exp(\mathfrak{g}). \quad (5.1)$$

In this section we wish to work with  $G$  via generators and relations. Fortunately, presentations of  $G$  are well established in the work of Steinberg [St67] and Tits [Ti87] and others (see [PRS, §3] for a survey). More specifically, up to an extra (commutative) torus  $T \cong (\mathbb{C}^\times)^k$  for some  $k$ , the group  $G$  is generated by subgroups isomorphic to  $SL_2(\mathbb{C})$ , the images of homomorphisms

$$\begin{aligned} \varphi_i: \quad SL_2(\mathbb{C}) &\longrightarrow G \\ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} &\longmapsto x_{\alpha_i}(c) \\ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} &\longmapsto x_{-\alpha_i}(c) \\ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} &\longmapsto h_{\alpha_i^\vee}(d) \\ \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix} &\longmapsto y_i(c) \end{aligned} \quad \begin{array}{l} \text{for each vertex } i \\ \text{of the affine Dynkin diagram.} \end{array} \quad (5.2)$$

For each  $\alpha \in \mathring{R}^+$  there is a homomorphism  $\varphi_\alpha: SL_2(\mathbb{C}((\epsilon))) \rightarrow G$  and we let

$$x_\alpha(f) = \varphi_\alpha \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{-\alpha}(f) = \varphi_\alpha \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \quad \text{for } f \in \mathbb{C}((\epsilon)). \quad (5.3)$$

For each  $z \in W^{\text{ad}}$  fix a reduced word  $z = s_{j_1} \cdots s_{j_r}$ , define

$$n_z = y_{j_1}(0) \cdots y_{j_r}(0) \quad \text{and define} \quad x_{\pm\alpha+r\delta}(c) = x_{\pm\alpha}(ce^r) \quad \text{for } \alpha \in \mathring{R}^+ \text{ and } r \in \mathbb{Z}. \quad (5.4)$$

In the following sections, for simplicity, we will work only with the loop group  $G = \mathring{G}(\mathbb{C}((\epsilon)))$  rather than the full affine Kac-Moody group as in (5.1). We shall also use a slightly more general setting where  $\mathbb{C}$  is replaced by an arbitrary field  $\mathbb{k}$  so that, in Proposition 5.4, we may let  $\mathbb{k} = \mathbb{F}_q$  be the finite field with  $q$  elements.

## 5.2 The affine flag varieties $G/I^+$ , $G/I^0$ and $G/I^-$

Let  $\mathbb{k}$  be a field. Define subgroups of  $\mathring{G}(\mathbb{k})$  by

$$\begin{array}{ll} \text{“unipotent upper triangular matrices”} & U^+(\mathbb{k}) = \langle x_\alpha(c) \mid \alpha \in \mathring{R}^+, c \in \mathbb{k} \rangle, \\ \text{“diagonal matrices”} & H(\mathbb{k}^\times) = \langle h_{\alpha_i^\vee}(c) \mid i \in \{1, \dots, n\}, c \in \mathbb{k}^\times \rangle, \\ \text{“unipotent lower triangular matrices”} & U^-(\mathbb{k}) = \langle x_{-\alpha}(c) \mid \alpha \in \mathring{R}^+, c \in \mathbb{k} \rangle. \end{array}$$

Let

$$\begin{array}{ll} \mathbb{F} = \mathbb{k}((\epsilon)), \text{ which has} & \mathfrak{o} = \mathbb{k}[[\epsilon]] \quad \text{and} \quad \mathfrak{o}^\times = \{p \in \mathbb{k}[[\epsilon]] \mid p(0) \neq 0\}, \\ \text{or let } \mathbb{F} = \mathbb{k}[\epsilon, \epsilon^{-1}], \text{ which has} & \mathfrak{o} = \mathbb{k}[\epsilon] \quad \text{and} \quad \mathfrak{o}^\times = \mathbb{k}^\times. \end{array}$$

Define subgroups of  $G = \mathring{G}(\mathbb{F})$  by

$$U^-(\mathbb{F}) = \langle x_{-\alpha+k\delta}(c) \mid c \in \mathbb{k}, \alpha \in \mathring{R}^+, k \in \mathbb{Z} \rangle \quad \text{and} \quad H(\mathfrak{o}) = \langle h_{\alpha_i^\vee}(c) \mid c \in \mathfrak{o}^\times \rangle.$$

Let  $g(0)$  denote  $g$  evaluated at  $\epsilon = 0$  and let  $g(\infty)$  denote  $g$  evaluated at  $\epsilon^{-1} = 0$ . Following, for example, [Pet97, Lect. 11 Theorem (B)] and [LuICM, §11], define subgroups of  $G(\mathbb{F})$  by

$$\begin{array}{ll} \text{(positive Iwahori)} & I^+ = \{g \in G \mid g(0) \text{ exists and } g(0) \in U^+(\mathbb{k})H(\mathbb{k}^\times)\}, \\ \text{(level 0 Iwahori)} & I^0 = U^-(\mathbb{F})H(\mathfrak{o}^\times), \\ \text{(negative Iwahori)} & I^- = \{g \in G \mid g(\infty) \text{ exists and } g(\infty) \in U^-(\mathbb{k})H(\mathbb{k}^\times)\}, \end{array}$$

Then

$G/I^+$  is the positive level (thin) affine flag variety,  
 $G/I^0$  is the level 0 (semi-infinite) affine flag variety,  
 $G/I^-$  is the negative level (thick) affine flag variety.

These are studied with the aid of the decompositions (relax notation and write  $zI^+$  for  $n_zI^+$ )

$$G = \bigsqcup_{x \in W^{\text{ad}}} I^+ x I^+, \quad G = \bigsqcup_{y \in W^{\text{ad}}} I^0 y I^+, \quad G = \bigsqcup_{z \in W^{\text{ad}}} I^- z I^+,$$

### 5.3 Labeling points of $I^+ w I^+$ , $I^+ w I^+ \cap I^0 v I^+$ and $I^+ w I^+ \cap I^- z I^+$

A blue labeled step of type  $i$  is

$$\begin{array}{c} \mathfrak{h}^{v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c \end{array} \quad \text{with } vs_i \geq v. \quad \text{Let } \Phi^+ \left( \begin{array}{c} \mathfrak{h}^{v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c \end{array} \right) = y_i(c),$$

where  $y_i(c)$  is as in (5.2). A blue labeled walk of type  $\vec{w} = (i_1, \dots, i_\ell)$  is a sequence  $(p_1, \dots, p_\ell)$  where  $p_k$  is a blue labeled step of type  $i_k$  and which begins at the alcove 1.

A red labeled step of type  $i$  is

$$\begin{array}{c} \mathfrak{h}^{v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c \end{array} \quad \text{or} \quad \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xleftarrow{vs_i} \\ 0 \end{array} \quad \text{or} \quad \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c^{-1} \end{array} \quad \text{where } vs_i \geq v. \\ c \in \mathbb{k} \qquad \qquad \qquad c \in \mathbb{k}^\times$$

With notations as in (5.4), let

$$\Phi^0 \left( \begin{array}{c} \mathfrak{h}^{v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c \end{array} \right) = x_{v\alpha_i}(c), \quad \Phi^0 \left( \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xleftarrow{vs_i} \\ 0 \end{array} \right) = x_{-v\alpha_i}(0), \quad \Phi^0 \left( \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c^{-1} \end{array} \right) = x_{-v\alpha_i}(c^{-1}).$$

A red labeled walk of type  $\vec{w} = (i_1, \dots, i_\ell)$  is a sequence  $(p_1, \dots, p_\ell)$  where  $p_k$  is a red labeled step of type  $i_k$  and which begins at the alcove 1.

A green labeled step of type  $i$  is

$$\begin{array}{c} \mathfrak{h}^{v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c \end{array} \quad \text{or} \quad \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xleftarrow{vs_i} \\ 0 \end{array} \quad \text{or} \quad \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c^{-1} \end{array} \quad \text{where } vs_i \geq v. \\ c \in \mathbb{k} \qquad \qquad \qquad c \in \mathbb{k}^\times$$

Let

$$\Phi^- \left( \begin{array}{c} \mathfrak{h}^{v\alpha_i} \\ vs_i \mid \xrightarrow{v} \\ c \end{array} \right) = x_{v\alpha_i}(c), \quad \Phi^- \left( \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xleftarrow{vs_i} \\ 0 \end{array} \right) = x_{-v\alpha_i}(0), \quad \Phi^- \left( \begin{array}{c} \mathfrak{h}^{-v\alpha_i} \\ v \mid \xrightarrow{vs_i} \\ c^{-1} \end{array} \right) = x_{-v\alpha_i}(c^{-1}).$$

A green labeled walk of type  $\vec{w} = (i_1, \dots, i_\ell)$  is a sequence  $(p_1, \dots, p_\ell)$  where  $p_k$  is a green labeled step of type  $i_k$  and which begins at the alcove 1.

**Theorem 5.1.** *Let  $v, w \in W^{\text{ad}}$  and fix a reduced expression  $w = s_{i_1} \dots s_{i_\ell}$  for  $w$ . The maps  $\Psi_{\vec{w}}^+$ ,  $\Psi_{\vec{w},v}^0$ ,  $\Psi_{\vec{w},v}^-$  are bijections.*

$$\begin{aligned} \Phi_{\vec{w}}^+ : \left\{ \begin{array}{l} \text{blue labeled paths of type} \\ \vec{w} = (i_1, \dots, i_\ell) \\ (p_1, \dots, p_\ell) \end{array} \right\} & \xrightarrow{\sim} (I^+ w I^+) / I^+ \\ & \mapsto \Phi^+(p_1) \dots \Phi^+(p_\ell) I^+ \\ \Phi_{\vec{w},v}^0 : \left\{ \begin{array}{l} \text{red labeled paths of type} \\ \vec{w} = (i_1, \dots, i_\ell) \text{ ending in } v \\ (p_1, \dots, p_\ell) \end{array} \right\} & \xrightarrow{\sim} (I^0 v I^+ \cap I^+ w I^+) / I^+ \\ & \mapsto \Phi^0(p_1) \dots \Phi^0(p_\ell) n_v I^+ \\ \Phi_{\vec{w},v}^- : \left\{ \begin{array}{l} \text{green labeled paths of type} \\ \vec{w} = (i_1, \dots, i_\ell) \text{ ending in } v \\ (p_1, \dots, p_\ell) \end{array} \right\} & \xrightarrow{\sim} (I^- v I^+ \cap I^+ w I^+) / I^+ \\ & \mapsto \Phi^-(p_1) \dots \Phi^-(p_\ell) n_v I^+ \end{aligned}$$

The proof of Theorem 5.1 is by induction on the length of  $w$  following [PRS, Theorem 4.1 and §7] where (a) and (b) are proved. The induction step can be formulated as the following proposition.

**Proposition 5.2.** *Let  $v, w \in W^{\text{ad}}$  and fix a reduced expression  $\vec{w} = s_{i_1} \dots s_{i_\ell}$  for  $w$ . Let  $j \in \{0, \dots, n\}$  and  $c \in \mathbb{C}$ . If  $ws_j \triangleleft w$  then assume that the reduced word for  $w$  is chosen with  $i_\ell = j$ . Let*

$$\tilde{c} \in \mathbb{C} \text{ and } \tilde{b}_1 \in I^+ \quad \text{be the unique elements such that } b_1 y_j(c) = y_j(\tilde{c}) \tilde{b}_1.$$

(a) *Let  $p = (p_1, \dots, p_\ell)$  be a blue labeled path of type  $(i_1, \dots, i_\ell)$  and let  $\Phi_{\vec{w}}^+(p) = y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell)$ . Then*

$$\begin{aligned} & (y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell) b_1) (y_j(c) b_2) \\ & = \begin{cases} y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell) y_j(\tilde{c}) \tilde{b}_1 b_2, & \text{if } w \triangleleft ws_j, \\ y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1}) y_{i_\ell}(c_\ell - \tilde{c}^{-1}) h_{\alpha_{i_\ell}}(\tilde{c}) x_{\alpha_{i_\ell}}(-\tilde{c}^{-1}) \tilde{b}_1 b_2, & \text{if } ws_j \triangleleft w \text{ and } \tilde{c} \neq 0, \\ y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1}) h_{\alpha_{i_\ell}}(-1) x_{\alpha_{i_\ell}}(c) \tilde{b}_1 b_2, & \text{if } ws_j \triangleleft w \text{ and } \tilde{c} = 0, \end{cases} \end{aligned}$$

(b) *Let  $p = (p_1, \dots, p_\ell)$  be a red labeled path of type  $(i_1, \dots, i_\ell)$  ending in  $v$  and let  $\Phi_{\vec{w},v}^0(p) = x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) n_v I^+$  where the notation is as in (5.4). Then*

$$\begin{aligned} & (x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) n_v b_1) (y_j(c) b_2) \\ & = \begin{cases} x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) x_{v\alpha_j}(\pm \tilde{c}) n_{vs_j} \tilde{b}_1 b_2, & \text{if } w \triangleleft ws_j, \\ x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) x_{-v\alpha_j}(\tilde{c}^{-1}) n_v x_{\alpha_j}(-\tilde{c}) h_{\alpha_j}(\tilde{c}) \tilde{b}_1 b_2, & \text{if } ws_j \triangleleft w \text{ and } \tilde{c} \neq 0, \\ x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) x_{-v\alpha_j}(0) n_{vs_j} \tilde{b}_1 b_2, & \text{if } ws_j \triangleleft w \text{ and } \tilde{c} = 0, \end{cases} \end{aligned}$$

(c) *Let  $p = (p_1, \dots, p_\ell)$  be a green labeled path of type  $(i_1, \dots, i_\ell)$  ending in  $v$  and let  $\Phi_{\vec{w},v}^-(p) = x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) n_v I^+$  where the notation is as in (5.4). Then*

$$\begin{aligned} & (x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) n_v b_1) (y_j(c) b_2) \\ & = \begin{cases} x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) x_{v\alpha_j}(\pm \tilde{c}) n_{vs_j} \tilde{b}_1 b_2, & \text{if } w \triangleleft ws_j, \\ x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) x_{-v\alpha_j}(\tilde{c}^{-1}) n_v x_{\alpha_j}(-\tilde{c}) h_{\alpha_j}(\tilde{c}) \tilde{b}_1 b_2, & \text{if } ws_j \triangleleft w \text{ and } \tilde{c} \neq 0, \\ x_{\gamma_1}(c_1) \dots x_{\gamma_\ell}(c_\ell) x_{-v\alpha_j}(0) n_{vs_j} \tilde{b}_1 b_2, & \text{if } ws_j \triangleleft w \text{ and } \tilde{c} = 0, \end{cases} \end{aligned}$$

## 5.4 Actions of the affine Hecke algebra

The *affine flag representation* is

$$C(G/I^+) = \mathbb{C}\text{-span} \left\{ y_{\vec{w}}(\vec{c})I^+ \mid w \in W, \vec{c} = (c_1, \dots, c_\ell) \in \mathbb{C}^{\ell(w)} \right\}.$$

Let

$$\begin{aligned} C(I^+ \backslash G/I^+) &= \mathbb{C}\text{-span}\{T_w \mid w \in W\}, & \text{where } T_w &= I^+ w I^+, \\ C(I^0 \backslash G/I^+) &= \mathbb{C}\text{-span}\{X^w \mid w \in W\}, & \text{where } X^w &= I^0 w I^+, \\ C(I^- \backslash G/I^+) &= \mathbb{C}\text{-span}\{L^w \mid w \in W\}, & \text{where } L^w &= I^- w I^+, \end{aligned}$$

The *affine Hecke algebra* is  $h_{I^+}(G/I^+) = \mathbb{C}\text{-span}\{T_w \mid w \in W\}$ ,

$$\text{where } T_w = I^+ n_w I^+ = \sum_{x \in I^+ n_w I^+} x \quad (\text{a formal sum}).$$

Proposition 5.3 computes the (right) action of  $h_{I^+}(G/I^+)$  on  $C(G/I^+)$ , and Proposition 5.4 computes the (right) action of  $h_{I^+}(G/I^+)$  on  $C(I^+ \backslash G/I^+)$  on  $C(I^0 \backslash G/I^+)$ , and on  $C(I^- \backslash G/I^+)$ . Proposition 5.3 follows from Proposition 5.2(a) by summing over  $c$  and Proposition 5.4 follows from Proposition 5.2 by summing over the appropriate double cosets. We use the convention that the normalization (Haar measure) is such that  $I^+ \cdot I^+ = I^+$ .

**Proposition 5.3.** *Let  $w \in W^{\text{ad}}$ , let  $\vec{w} = s_{i_1} \cdots s_{i_\ell}$  be a reduced word for  $w$  and let  $j \in \{0, \dots, n\}$ . Assume that if  $ws_j \triangleleft w$  then  $i_\ell = j$  and let*

$$y_{\vec{ws}_j}(\vec{c}) = y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1}) \quad \text{and} \quad y_{\vec{w}}(\vec{c}) = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) = y_{\vec{ws}_j}(\vec{c})y_j(c_\ell).$$

Then

$$y_{\vec{w}}(\vec{c})I^+ \cdot T_{s_j} = \begin{cases} y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1})I^+ + \sum_{\tilde{c} \in \mathbb{k}^\times} y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell - \tilde{c}^{-1})I^+, & \text{if } ws_j \triangleleft w, \\ \sum_{\tilde{c} \in \mathbb{k}} y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)y_j(\tilde{c})I^+, & \text{if } w \triangleleft ws_j. \end{cases}$$

**Proposition 5.4.** *Let  $\mathbb{k} = \mathbb{F}_q$  the finite field with  $q$  elements. Let  $w \in W^{\text{ad}}$  and  $j \in \{0, \dots, n\}$ . Then*

$$T_w T_{s_j} = \begin{cases} T_{ws_j}, & \text{if } w \triangleleft ws_j, \\ (q-1)T_w + qT_{ws_j}, & \text{if } ws_j \triangleleft w, \end{cases} \quad L^w T_{s_j} = \begin{cases} L^{ws_j}, & \text{if } w \triangleleft ws_j, \\ qL^{ws_j} + (q-1)L^w, & \text{if } ws_j \triangleleft w, \end{cases}$$

$$\text{and} \quad X^w T_{s_j} = \begin{cases} X^{ws_j}, & \text{if } w \triangleleft ws_j, \\ qX^{ws_j} + (q-1)X^w, & \text{if } ws_j \triangleleft w. \end{cases}$$

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