## CALIBRATED REPRESENTATIONS OF TWO BOUNDARY TEMPERLEY-LIEB ALGEBRAS

ZAJJ DAUGHERTY AND ARUN RAM

In memory of a friend and an inspiration, Vladimir Rittenberg 1934-2018

ABSTRACT. The two boundary Temperley-Lieb algebra  $TL_k$  arises in the transfer matrix formulation of lattice models in Statistical Mechanics, in particular in the introduction of integrable boundary terms to the six-vertex model. In this paper, we classify and study the calibrated representations—those for which all the Murphy elements (integrals) are simultaneously diagonalizable—which, in turn, corresponds to diagonalizing the transfer matrix in the associated model. Our approach is founded upon the realization of  $TL_k$  as a quotient of the type  $C_k$  affine Hecke algebra  $H_k$ . In previous work, we studied this Hecke algebra via its presentation by braid diagrams, tensor space operators, and related combinatorial constructions. That work is directly applied herein to give a combinatorial classification and construction of all irreducible calibrated  $TL_k$ -modules and explain how these modules also arise from a Schur-Weyl duality with the quantum group  $U_q \mathfrak{gl}_2$ .

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#### 1. INTRODUCTION

The paper [DR] studied the calibrated representations of affine Hecke algebras of type C with unequal parameters and developed their combinatorics and their role in Schur-Weyl duality. This paper applies that information to the study of two boundary Temperley-Lieb algebras. The two boundary Temperley-Lieb algebras appear in statistical mechanics for analysis of spin chains with generalized boundary conditions [GP, GNPR]. The spectrum of the Hamiltonian for these spin chains with boundaries can be determined via the representation theory of the two boundary Temperley-Lieb algebras. In fact, the need to understand the representation theory of the two boundary Temperley-Lieb algebra better was a primary motivation for our preceding papers [Dau, DR] on two boundary Hecke algebras.

DEPARTMENT OF MATHEMATICS, THE CITY COLLEGE OF NEW YORK, NAC 8/133, NEW YORK, NY 10031 DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE VIC 3010 AUSTRALIA *E-mail address:* zdaugherty@gc.cuny.edu, aram@unimelb.edu.au. *Date:* September 8, 2020.

#### ZAJJ DAUGHERTY AND ARUN RAM

In the first part of this paper, Section 2, we review the definition and structure of the two boundary Hecke algebra  $H_k$  (the affine Hecke algebra of type C with unequal parameters). Following this brief review we carefully analyze certain idempotents which, as we prove in Theorem 3.1, generate the ideal that one must quotient by to obtain the two boundary Temperley-Lieb algebra from the two boundary Hecke algebra. It is the expression of these idempotents in terms of the intertwiner presentation of  $H_k$  (see Proposition 2.3) that will eventually provide understanding of the weights that can appear in two boundary Temperley-Lieb modules (the possible eigenvalues of the "Murphy elements"  $W_i$ —see equation (2.9) and §2.4).

In Section 3 we define the two boundary Temperley-Lieb algebra (or symplectic blob algebra)  $TL_k$  following [GN, GMP07, GMP08, GMP12, Ree, KMP16, GMP17], and review the diagram algebra calculus for these algebras. Part of our contribution is to extend this calculus to make its connection to the diagrammatic calculus of the Hecke algebra  $H_k$  via braids. In Theorem 3.2 we use these diagrammatics to give a proof of a result of [GN] that provides an expansion of a certain central element of  $H_k$  inside  $TL_k$ . Using the Hecke algebra point of view, this result enables us to understand that the center of  $TL_k$  is a polynomial ring in one variable  $Z(TL_k) = \mathbb{C}[Z]$ , and that  $TL_k$  is of finite rank over this center. In retrospect, the algebra  $H_k$  has a similar structure and so perhaps this should not be surprising but, nonetheless, it is pleasant to see it come out in such a vivid and explicit form.

We have used a different normalization of the parameters of the two boundary Hecke and Temperley-Lieb algebra from those used in [GN, GMP12]. Our normalization will be helpful, for example, for future applications of these algebras to the theory of Macdonald polynomials and to the study of the exotic nilpotent cone. In both of these cases the affine Hecke algebra of type  $C_n$  plays an important role: the Koornwinder polynomials are the Macdonald polynomials for type  $(C_n^{\vee}, C_n)$  [M03], and the K-theory of the Steinberg variety of the exotic nilpotent cone provides a geometric construction of the representations of the two boundary Hecke and Temperley-Lieb algebras at unequal parameters (see [Kat]).

The calibrated representations are the irreducible representations of the two boundary Hecke algebra for which a large family of commuting operators (integrals, or Murphy elements) have a simple (joint) spectrum. This property makes these representations particularly attractive, and the detailed combinatorics of these representations has been worked out in [DR]. In Section 4 we use the detailed analysis of the idempotents done in Section 2 to determine exactly which calibrated irreducible representations of the two boundary Hecke algebra are representations of the two boundary Temperley-Lieb algebra (Theorem 4.3). In consequence, we obtain a full classification of the calibrated irreducible representations of the two boundary Temperley-Lieb algebras.

As explained in [DR], there is a Schur-Weyl type duality between the two boundary Hecke algebra and the quantum group  $U_q \mathfrak{gl}_n$ . The classical Schur-Weyl duality between  $U_q \mathfrak{gl}_n$  and the finite Hecke algebra of type A becomes a Schur-Weyl duality for the finite Temperley-Lieb algebra when n = 2. In Theorem 5.1 we show that at n = 2 the Schur-Weyl duality of [DR] gives a Schur-Weyl duality for the two boundary Temperley-Lieb algebra. This method (coming from R-matrices for the quantum group  $U_q \mathfrak{gl}_2$ ) provides many many irreducible calibrated representations of the two boundary Temperley-Lieb algebra  $TL_k$ . Using our results from Section 4, we determine exactly which irreducible calibrated representations of  $TL_k$  occur in the Schur-Weyl duality context.

The seeds of this work were sown in a conversation between Pavel Pyatov, Arun Ram and Vladimir Rittenberg at the Max Planck Institut in Bonn in 2006. Vladimir was the leader and provided the inspiration by introducing us to spin chains with boundaries. The seed has now grown from a concept into fully formed and fruitful mathematics. We thank all the institutions which have

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### 2. The two boundary Hecke algebra $H_k$

The two boundary Hecke algebra is often called the affine Hecke algebra of type  $(C, C^{\vee})$ . In this section we review the definitions of  $H_k^{\text{ext}}$  following our previous paper [DR]. In particular, we will need the basic diagrammatics and the "Bernstein" presentation with a Laurent polynomial ring  $\mathbb{C}[W_1^{\pm}, \ldots, W_k^{\pm}]$  and intertwiners  $\tau_1, \ldots, \tau_k$ . After this review we define the idempotent elements  $p_i^{(1^3)}, p_0^{(\emptyset, 1^2)}, p_{0^{\vee}}^{(\emptyset, 1^2)}, p_{0^{\vee}}^{(1^2, \emptyset)}$ , which we will need to quotient by in order to obtain the two boundary Temperley-Lieb algebra. We derive expressions of these elements in terms of the different choices of generators: the braid generators  $T_i$ , the cap/cup generators  $e_i$ , and the intertwiner generators  $\tau_i$  and  $W_j$ .

#### 2.1. Graph notation for braid relations. For generators g, h, encode relations graphically by

$$g \qquad h \\ \circ \qquad \circ \qquad \text{means } gh = hg,$$

$$g \qquad h \\ \circ \qquad \circ \qquad \text{means } ghg = hgh, \text{ and}$$

$$g \qquad h \\ \circ \qquad \circ \qquad \text{means } ghgh = hghg.$$

$$(2.1)$$

For example, the group of signed permutations,

$$\mathcal{W}_0 = \left\{ \begin{array}{l} \text{bijections } w \colon \{-k, \dots, -1, 1, \dots, k\} \to \{-k, \dots, -1, 1, \dots, k\} \\ \text{such that } w(-i) = -w(i) \text{ for } i = 1, \dots, k \end{array} \right\},$$
(2.2)

has a presentation by generators  $s_0, s_1, \ldots, s_{k-1}$ , with relations

2.2. The two boundary braid group. The two boundary braid group is the group  $\mathcal{B}_k$  generated by  $\overline{T}_0, \overline{T}_1, \ldots, \overline{T}_k$ , with relations

Pictorially, the generators of  $\mathcal{B}_k$  are identified with the braid diagrams

and the multiplication of braid diagrams is given by placing one diagram on top of another (multiplying generators left-to-right corresponds to stacking diagrams top-to-bottom).

In some applications (notably to the Schur-Weyl duality of  $[DR, \S5]$ ), it is useful to move the rightmost pole to the left by conjugating by the diagram

Define

and

$$X_{1} = T_{1}^{-1}T_{2}^{-1} \cdots T_{k-1}^{-1}\sigma \bar{T}_{k}\sigma^{-1}T_{k-1} \cdots T_{1} = \left\{ \bigcup_{i=1}^{k-1} \bigcup_{i$$

Define

for  $i = 2, \ldots, k$ . Let

The extended affine braid group is the group  $\mathcal{B}_k^{\text{ext}}$  generated by  $\mathcal{B}_k$  and P with the additional relations

$$PX_1P^{-1} = Z_1^{-1}X_1Z_1, \qquad PY_1P^{-1} = Z_1^{-1}Y_1Z_1, \tag{2.10}$$

$$PZ_1P^{-1} = Z_1$$
, and  $PT_iP^{-1} = T_i$  for  $i = 1, \dots, k-1$ . (2.11)

The element

$$Z_0 = PZ_1 \cdots Z_k \quad \text{is central in } \mathcal{B}_k^{\text{ext}} \tag{c0}$$

since the group  $\mathcal{B}_k^{\text{ext}}$  is a subgroup of the braid group on k+2 strands, and  $Z_0$  is the generator of the center of the braid group on k+2 strands (see [GM, Theorem 4.2]). So

if 
$$\mathcal{D} = \{Z_0^j \mid j \in \mathbb{Z}\}$$
 then  $\mathcal{B}_k^{\text{ext}} = \mathcal{D} \times \mathcal{B}_k$ , with  $\mathcal{D} \cong \mathbb{Z}$ . (2.12)

# 2.3. The extended affine Hecke algebra $H_k^{\text{ext}}$ of type $C_k$ . Fix $a_1, a_2, b_1, b_2, t^{\frac{1}{2}} \in \mathbb{C}^{\times}$ and let

$$t_k^{\frac{1}{2}} = a_1^{\frac{1}{2}} (-a_2)^{-\frac{1}{2}}, \qquad t_0^{\frac{1}{2}} = b_1^{\frac{1}{2}} (-b_2)^{-\frac{1}{2}}.$$
 (2.13)

The extended two boundary Hecke algebra  $H_k^{ext}$  with parameters  $t^{\frac{1}{2}}$ ,  $t_0^{\frac{1}{2}}$  and  $t_k^{\frac{1}{2}}$  is the quotient of  $\mathcal{B}_k^{ext}$  by the relations

$$(X_1 - a_1)(X_1 - a_2) = 0, \quad (Y_1 - b_1)(Y_1 - b_2) = 0, \text{ and } (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0,$$
 (H)

for i = 1, ..., k - 1. Let

$$T_0 = b_1^{-\frac{1}{2}} (-b_2)^{-\frac{1}{2}} Y_1, \qquad T_k = a_1^{-\frac{1}{2}} (-a_2)^{-\frac{1}{2}} T_{k-1} \cdots T_2 T_1 X_1 T_1^{-1} T_2^{-1} \cdots T_{k-1}^{-1}.$$
(2.14)

$$(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0, \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0, \quad (T_k - t_k^{\frac{1}{2}})(T_k + t_k^{-\frac{1}{2}}) = 0, \quad (2.15)$$

for  $i \in \{1, \dots, k-1\}$ .

Let  $a, a_0, a_k \in \mathbb{C}^{\times}$  and define

$$a_0 e_0 = T_0 - t_0^{\frac{1}{2}}, \qquad a e_i = T_i - t^{\frac{1}{2}}, \qquad a_k e_k = T_k - t_k^{\frac{1}{2}},$$
(2.16)

for  $i \in \{1, \ldots, k-1\}$ . The relations in (2.15) are equivalent to

$$T_0 e_0 = -t_0^{-\frac{1}{2}} e_0, \qquad T_i e_i = -t^{-\frac{1}{2}} e_i, \qquad T_k e_k = -t_k^{-\frac{1}{2}} e_k, \tag{2.17}$$

and to

$$e_0^2 = \frac{-(t_0^{\frac{1}{2}} + t_0^{-\frac{1}{2}})}{a_0} e_0, \qquad e_i^2 = \frac{-(t^{\frac{1}{2}} + t^{-\frac{1}{2}})}{a} e_i, \qquad e_k^2 = \frac{-(t_k^{\frac{1}{2}} + t_k^{-\frac{1}{2}})}{a_k} e_k, \tag{2.18}$$

for  $i \in \{1, \dots, k-1\}$ .

**Remark 2.1.** For  $i \in \{1, \ldots, k-2\}$ , using  $T_i = ae_i + t^{\frac{1}{2}}$  to expand  $T_i T_{i+1} T_i$  and  $T_{i+1} T_i T_{i+1}$  in terms of the  $e_i$  shows that in the presence of the relations (H),

 $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$  is equivalent to  $a^3 e_i e_{i+1} e_i - a e_i = a^3 e_{i+1} e_i e_{i+1} - a e_{i+1}$ .

Similarly,  $T_0T_1T_0T_1 = T_1T_0T_1T_0$  is equivalent to

$$a_0^2 a^2 e_0 e_1 e_0 e_1 - a_0 a (t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}) e_0 e_1 = a_0^2 a^2 e_1 e_0 e_1 e_0 - a_0 a (t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}) e_1 e_0.$$

In the case that  $a_0^2 a^2 = a_0 a (t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}})$  then

 $T_0T_1T_0T_1 = T_1T_0T_1T_0$  is equivalent to  $e_0e_1e_0e_1 - e_0e_1 = e_1e_0e_1e_0 - e_1e_0$ .

In the case that  $a^3 = a$  then

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$
 is equivalent to  $e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}$ .

This is the explanation for why the favorite choices of a,  $a_0$  and  $a_k$  satisfy

 $a = \pm 1,$   $a_0 a = t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}} = \llbracket t_0 t^{-1} \rrbracket$  and  $a_k a = t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}} = \llbracket t_k t^{-1} \rrbracket,$ 

where we use the notation

$$\llbracket t^s \rrbracket = (t^{\frac{s}{2}} + t^{-\frac{s}{2}}) = \left(\frac{t^s - t^{-s}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right) \left(\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{\frac{s}{2}} - t^{-\frac{s}{2}}}\right) = \frac{[2s]}{[s]}.$$
(2.19)

2.4. The Bernstein presentation of  $H_k^{\text{ext}}$ . The Murphy elements for  $H_k^{\text{ext}}$  are

$$W_1 = T_1^{-1}T_2^{-1}\cdots T_{k-1}^{-1}T_kT_{k-1}\cdots T_2T_1T_0$$
 and  $W_j = T_jW_{j-1}T_j$ ,

for  $j \in \{2, \ldots, k\}$ . Let

$$W_0 = PW_1 \cdots W_k.$$

**Theorem 2.2.** (See, for example, [DR, Theorem 2.2].) Fix  $t_0, t_k, t \in \mathbb{C}^{\times}$  and use notations for relations as defined in (2.1). The extended affine Hecke algebra  $H_k^{ext}$  defined in (H) is presented by generators,  $T_0, T_1, \ldots, T_{k-1}, W_0, W_1, \ldots, W_k$  and relations

$$W_0 \in Z(H_k^{\text{ext}}), \qquad \begin{array}{ccc} T_0 & T_1 & T_2 & T_{k-2} & T_{k-1} \\ \circ \underline{\qquad} = \underline{\qquad}$$

$$W_i W_j = W_j W_i, \quad for \ i, j = 0, 1, \dots, k;$$
 (B2)

$$T_0 W_j = W_j T_0, \quad for \ j \neq 1; \tag{B3}$$

$$T_i W_j = W_j T_i \text{ for } i = 1, \dots, k-1 \text{ and } j = 1, \dots, k \text{ with } j \neq i, i+1;$$
 (B4)

$$(T_0 - t_0^{\frac{1}{2}})(T_0 + t_0^{-\frac{1}{2}}) = 0, \quad and \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0 \text{ for } i = 1, \dots, k-1;$$
 (H)

for i = 1, ..., k - 1,

$$T_i W_i = W_{i+1} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_i - W_{i+1}}{1 - W_i W_{i+1}^{-1}}, \qquad T_i W_{i+1} = W_i T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{W_{i+1} - W_i}{1 - W_i W_{i+1}^{-1}}, \quad (C1)$$

and 
$$T_0 W_1 = W_1^{-1} T_0 + \left( (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}}) W_1^{-1} \right) \frac{W_1 - W_1^{-1}}{1 - W_1^{-2}}.$$
 (C2)

The two boundary Hecke algebra  $H_k$  with parameters  $t^{\frac{1}{2}}$ ,  $t^{\frac{1}{2}}_0$  and  $t^{\frac{1}{2}}_k$  is the subalgebra of  $H_k^{\text{ext}}$  generated by  $T_0, T_1, \ldots, T_k$ . Then

$$H_k^{\text{ext}} = H_k \otimes \mathbb{C}[W_0^{\pm 1}]$$
 as algebras, (2.20)

and, as proved for example in [DR, Theorem 2.3], the element

$$Z = W_1 + W_1^{-1} + W_2 + W_2^{-1} + \dots + W_k + W_k^{-1}$$
 is central in  $H_k^{\text{ext}}$ . (2.21)

#### 2.5. The elements $\tau_i$ . Define

$$\tau_0 = T_0 - \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})W_1^{-1}}{1 - W_1^{-2}}, \quad \text{and} \quad \tau_i = T_i - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - W_i W_{i+1}^{-1}}, \quad (2.22)$$

for  $i \in \{1, \ldots, k-1\}$ . Evoking the notation of [DR, §3], reviewed later in §4.1, let

$$\begin{aligned} f_{2\varepsilon_{i}} &= (1 - W_{i}^{-1})(1 + W_{i}^{-1}) = 1 - W_{i}^{-2}, \\ f_{\varepsilon_{i}-r_{2}} &= (1 - t_{0}^{\frac{1}{2}}t_{k}^{\frac{1}{2}}W_{i}^{-1}), \\ f_{-\varepsilon_{i}-r_{2}} &= (1 - t_{0}^{\frac{1}{2}}t_{k}^{\frac{1}{2}}W_{i}), \\ f_{-\varepsilon_{i}-\varepsilon_{j}} &= 1 - W_{i}W_{j}^{-1}, \end{aligned} \qquad \begin{aligned} f_{\varepsilon_{i}-\varepsilon_{j}+1} &= (1 + t_{0}^{\frac{1}{2}}t_{k}^{-\frac{1}{2}}W_{i}^{-1}), \\ f_{\varepsilon_{i}-\varepsilon_{j}} &= 1 - tW_{i}W_{j}^{-1}, \end{aligned}$$

$$(2.23)$$

for  $i, j \in \{1, \ldots, k\}$ . Then

$$a_0 e_0 = \tau_0 - t_0^{-\frac{1}{2}} \frac{f_{\varepsilon_1 - r_1} f_{\varepsilon_1 - r_2}}{f_{2\varepsilon_1}} \quad \text{and} \quad a_i e_i = \tau_i - t^{-\frac{1}{2}} \frac{f_{\varepsilon_i - \varepsilon_{i+1} + 1}}{f_{\varepsilon_i - \varepsilon_{i+1}}}, \tag{2.24}$$

and, as proved in [DR, Proposition 2.4],  $\overset{\tau_0}{\circ} \overset{\tau_1}{=} \overset{\tau_2}{\circ} \overset{\tau_{k-2}}{=} \overset{\tau_{k-1}}{\circ}$  and

$$\begin{aligned} \tau_0^2 &= W_1^{-2} t_0^{-1} \frac{f_{\varepsilon_1 - r_1} f_{-\varepsilon_1 - r_1} f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_2}}{f_{2\varepsilon_1}^2}, \qquad W_1 \tau_0 = \tau_0 W_1^{-1}, \qquad W_r \tau_0 = \tau_0 W_r, \\ \tau_i^2 &= t^{-1} \frac{f_{\varepsilon_i - \varepsilon_{i+1} + 1} f_{\varepsilon_{i+1} - \varepsilon_i + 1}}{f_{\varepsilon_i - \varepsilon_{i+1}} f_{\varepsilon_{i+1} - \varepsilon_i}}, \qquad W_i \tau_i = \tau_i W_{i+1}, \quad W_{i+1} \tau_i = \tau_i W_i, \quad W_j \tau_i = \tau_i W_j, \end{aligned}$$

for  $r, j \in \{1, \ldots, k\}$  with  $r \neq 1$  and  $j \neq i, i + 1$ .

2.6. The elements  $p_i^{(1^3)}$ ,  $p_0^{(\emptyset,1^2)}$  and  $p_0^{(1^2,\emptyset)}$ . Fix  $i \in \{1, \ldots, k-2\}$ . Let

- $HS_3$  be the subalgebra of  $H_k^{\text{ext}}$  generated by  $T_i$  and  $T_{i+1}$ , and let
- $HB_2$  be the subalgebra of  $H_k^{\text{ext}}$  generated by  $T_0$  and  $T_1$ .

The idempotent  $p_i^{(1^3)}$  in  $HS_3$  and the idempotents  $p_0^{(\emptyset,1^2)}$  and  $p_0^{(1^2,\emptyset)}$  in  $HB_2$  are uniquely determined by the equations

$$(p_i^{(1^3)})^2 = p_i^{(1^3)}, \qquad (p_0^{(\emptyset, 1^2)})^2 = p_0^{(\emptyset, 1^2)} \qquad (p_0^{(1^2, \emptyset)})^2 = p_0^{(1^2, \emptyset)}, \qquad (2.25)$$

and

$$T_{i}p_{i}^{(1^{3})} = -t^{-\frac{1}{2}}p_{i}^{(1^{3})}, \qquad T_{i+1}p_{i}^{(1^{3})} = -t^{-\frac{1}{2}}p_{i}^{(1^{3})}, T_{0}p_{0}^{(\emptyset,1^{2})} = -t_{0}^{-\frac{1}{2}}p_{0}^{(\emptyset,1^{2})} \qquad T_{1}p_{0}^{(\emptyset,1^{2})} = -t^{-\frac{1}{2}}p_{0}^{(\emptyset,1^{2})}, T_{0}p_{0}^{(1^{2},\emptyset)} = t_{0}^{\frac{1}{2}}p_{0}^{(1^{2},\emptyset)}, \qquad T_{1}p_{0}^{(1^{2},\emptyset)} = -t^{-\frac{1}{2}}p_{0}^{(1^{2},\emptyset)}.$$

$$(2.26)$$

The conditions in (2.26) are equivalent to

$$ae_{i}p_{i}^{(1^{3})} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{i}^{(1^{3})}, \qquad ae_{i+1}p_{i}^{(1^{3})} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{i}^{(1^{3})}, a_{0}e_{0}p_{0}^{(\emptyset,1^{2})} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{0}^{(\emptyset,1^{2})}, \qquad ae_{1}p_{0}^{(\emptyset,1^{2})} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{0}^{(\emptyset,1^{2})}, a_{0}e_{0}p_{0}^{(1^{2},\emptyset)} = 0, \qquad ae_{1}p_{0}^{(1^{2},\emptyset)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{0}^{(1^{2},\emptyset)}.$$
(2.27)

**Proposition 2.3.** Let  $p_i^{(1^3)}$ ,  $p_0^{(\emptyset,1^2)}$  and  $p_0^{(1^2,\emptyset)}$  be as defined in (2.25) and (2.26) and let

$$N = t^{-\frac{1}{2}}(1+t)(1+t+t^2)$$
 and  $N_0 = N'_0 = t_0^{-1}t^{-1}(1+t_0)(1+t)(1+t_0t).$ 

Then the expansions of these idempotents in terms of the three favored generating sets is given by

$$\begin{split} Np_{i}^{(1^{3})} &= T_{i}T_{i+1}T_{i} - t^{\frac{1}{2}}T_{i}T_{i+1} - t^{\frac{1}{2}}T_{i+1}T_{i} + tT_{i} + tT_{i+1} - t^{\frac{3}{2}} \\ &= a^{3}e_{i}e_{i+1}e_{i} - ae_{i} = a^{3}e_{i+1}e_{i}e_{i+1} - ae_{i+1} \\ &= \tau_{i}\tau_{i+1}\tau_{i} - t^{-\frac{1}{2}}\tau_{i+1}\tau_{i}\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}} - t^{-\frac{1}{2}}\tau_{i}\tau_{i+1}\frac{f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i}}} \\ &+ t^{-1}\tau_{i}\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}f_{\varepsilon_{i+2}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}f_{\varepsilon_{i+2}-\varepsilon_{i}}} + t^{-1}\tau_{i+1}\frac{f_{\varepsilon_{i+2}-\varepsilon_{i}+1}f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+2}-\varepsilon_{i}}f_{\varepsilon_{i+1}-\varepsilon_{i}}} \\ &- t^{-\frac{3}{2}}\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}f_{\varepsilon_{i+2}-\varepsilon_{i}+1}f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}f_{\varepsilon_{i+2}-\varepsilon_{i}}f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}, \end{split}$$

$$\begin{split} N_{0}p_{0}^{(\emptyset,1^{2})} &= T_{0}T_{1}T_{0}T_{1} - t_{0}^{\frac{1}{2}}T_{1}T_{0}T_{1} - t^{\frac{1}{2}}T_{0}T_{1}T_{0} + t_{0}^{\frac{1}{2}}t^{\frac{1}{2}}T_{0}T_{1} + t_{0}^{\frac{1}{2}}t^{\frac{1}{2}}T_{1}T_{0} - t_{0}t^{\frac{1}{2}}T_{1} - t_{0}^{\frac{1}{2}}t^{\frac{1}{2}}t_{0} + t_{0}t \\ &= a_{0}^{2}a^{2}e_{0}e_{1}e_{0}e_{1} - a_{0}a(t_{0}^{-\frac{1}{2}}t^{\frac{1}{2}} + t_{0}^{\frac{1}{2}}t^{-\frac{1}{2}})e_{0}e_{1} = a_{0}^{2}a^{2}e_{1}e_{0}e_{1}e_{0} - a_{0}a(t_{0}^{-\frac{1}{2}}t^{\frac{1}{2}} + t_{0}^{\frac{1}{2}}t^{-\frac{1}{2}})e_{1}e_{0} \\ &= \tau_{0}\tau_{1}\tau_{0}\tau_{1} - t_{0}^{\frac{1}{2}}\tau_{1}\tau_{0}\tau_{1}\frac{f_{\varepsilon_{1}-r_{2}}f_{\varepsilon_{1}-r_{1}}}{f_{2\varepsilon_{1}}} - t^{-\frac{1}{2}}\tau_{0}\tau_{1}\tau_{0}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{\varepsilon_{2}-\varepsilon_{1}}} \\ &+ t_{0}^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_{0}\tau_{1}\frac{f_{-\varepsilon_{2}-\varepsilon_{1}+1}}{f_{-\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{1}-r_{2}}f_{\varepsilon_{1}-r_{1}}}{f_{2\varepsilon_{1}}} - t_{0}^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_{1}\tau_{0}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{\varepsilon_{2}-\varepsilon_{1}}} \\ &- t_{0}t^{-\frac{1}{2}}\tau_{1}\frac{f_{\varepsilon_{2}-r_{2}}f_{\varepsilon_{2}-r_{1}}}{f_{2\varepsilon_{2}}}\frac{f_{-\varepsilon_{2}-\varepsilon_{1}+1}}{f_{-\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{1}}}f_{\varepsilon_{2}-\varepsilon_{1}}} - t_{0}^{\frac{1}{2}}t^{-\frac{1}{2}}\tau_{0}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{\varepsilon_{2}-\varepsilon_{1}}}} \\ &+ t_{0}t^{-\frac{1}{2}}\frac{f_{\varepsilon_{1}-r_{2}}f_{\varepsilon_{2}-r_{1}}}{f_{2\varepsilon_{2}}}\frac{f_{-\varepsilon_{2}-\varepsilon_{1}+1}}{f_{-\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}f_{\varepsilon_{2}-\varepsilon_{1}}} \\ &+ t_{0}t^{-\frac{1}{2}}\frac{f_{\varepsilon_{1}-r_{1}}f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{-\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}}}f_{\varepsilon_{2}-\varepsilon_{1}}}{f_{2\varepsilon_{2}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}f_{\varepsilon_{2}-\varepsilon_{1}}} \\ &+ t_{0}t^{-\frac{1}{2}}\frac{f_{\varepsilon_{1}-r_{2}}f_{\varepsilon_{1}-r_{1}}}{f_{2\varepsilon_{1}}}\frac{f_{-\varepsilon_{2}-\varepsilon_{1}+1}}{f_{-\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}}}f_{\varepsilon_{2}-\varepsilon_{1}}}{f_{2\varepsilon_{2}}}f_{\varepsilon_{2}-\varepsilon_{1}}}} \\ &+ t_{0}t^{-\frac{1}{2}}\frac{f_{\varepsilon_{1}-r_{1}}}{f_{2\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{-\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}}}f_{\varepsilon_{2}-\varepsilon_{1}}}} \\ &+ t_{0}t^{-\frac{1}{2}}\frac{f_{\varepsilon_{1}-r_{1}}}{f_{2\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2\varepsilon_{2}-\varepsilon_{1}}}\frac{f_{\varepsilon_{2}-\varepsilon_{1}+1}}{f_{2$$

and

$$\begin{split} N_0' p_0^{(1^2,\emptyset)} &= T_0 T_1 T_0 T_1 + t_0^{-\frac{1}{2}} T_1 T_0 T_1 - t^{\frac{1}{2}} T_0 T_1 T_0 - t_0^{-\frac{1}{2}} t^{\frac{1}{2}} T_0 T_1 - t_0^{-\frac{1}{2}} t^{\frac{1}{2}} T_1 T_0 - t_0 t^{\frac{1}{2}} T_1 + t_0^{-\frac{1}{2}} t T_0 + t_0 t \\ &= (a_0^2 a^2 e_0 e_1 e_0 e_1 - a_0 a (t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}) e_0 e_1) - (a_0 a^2 e_1 e_0 e_1 - a (t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}) e_1) \\ &= \tau_0 \tau_1 \tau_0 \tau_1 - t_0^{\frac{1}{2}} \tau_1 \tau_0 \tau_1 W_1^{-2} \frac{f_{-\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} - t^{-\frac{1}{2}} \tau_0 \tau_1 \tau_0 \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+ t_0^{\frac{1}{2}} t^{-\frac{1}{2}} \tau_0 \tau_1 W_1^{-2} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{-\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}} \tau_1 \tau_0 W_2^{-2} \frac{f_{-\varepsilon_2 - r_2} f_{-\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &- t_0 t^{-\frac{1}{2}} \tau_1 W_1^{-2} W_2^{-2} \frac{f_{-\varepsilon_2 - r_2} f_{-\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_1}} \\ &- t_0^{\frac{1}{2}} t^{-1} \tau_0 W_2^{-2} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2} - \varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2} - \varepsilon_1}} \frac{f_$$

*Proof.* The expressions in terms of  $T_i$  are proved by using the relations  $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$  and  $T_0^2 = (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})T_0 + 1$  to show that the equations in (2.26) are satisfied. In view of the conditions (2.25), using the equations (2.26) to compute the product of the expansion in terms of the  $T_i$  with each element  $p_i^{(1^3)}$ ,  $p_0^{(\emptyset,1^2)}$  and  $p_0^{(1^2,\emptyset)}$  respectively, determines the normalizing constants

$$N = -t^{-\frac{3}{2}} - t^{-\frac{1}{2}} - t^{-\frac{1}{2}} - t^{\frac{1}{2}} - t^{\frac{1}{2}} - t^{\frac{3}{2}} = t^{-\frac{1}{2}}(1+t)(1+t+t^2), \text{ and}$$
$$N_0 = N_0' = t_0^{-1}t^{-1} + t^{-1} + t_0^{-1} + 1 + t_0 + t + t_0t = t_0^{-1}t^{-1}(1+t_0)(1+t)(1+t_0t).$$

Checking the conditions (2.27) verifies that the expressions in terms of the  $e_i$  for the elements  $Np_i^{(1^3)}$ ,  $N_0p_0^{(\emptyset,1^2)}$  and  $N'_0p_0^{(1^2,\emptyset)}$  are correct. Similarly, using the expressions for  $a_0e_0$  and  $ae_i$  in terms of  $\tau_i$  given in (2.24) to check these same conditions verifies that the expressions for the elements  $Np_i^{(1^3)}$ ,  $N_0p_0^{(\emptyset,1^2)}$  and  $N'_0p_0^{(1^2,\emptyset)}$  in terms of the  $\tau_i$  are correct.

2.7. Setting up the relation  $a_k a^2 e_{k-1} e_k e_{k-1} - a(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}) e_{k-1} = 0$ . As in [DR, Remark 2.3], let  $w_A$  be the longest element of  $WA_k = \langle s_1, \ldots, s_{k-1} \rangle$ . Let

and note that  $T_{w_A}^{-1}T_{k-1}T_{w_A} = T_1$ . Then

$$(T_{0^{\vee}} - t_k^{\frac{1}{2}})(T_{0^{\vee}} + t_k^{-\frac{1}{2}}) = 0$$
 and  $T_{0^{\vee}}T_1 T_{0^{\vee}} T_1 = T_1 T_{0^{\vee}} T_1 T_{0^{\vee}}.$ 

Let  $HB_2^{\vee}$  be the subalgebra of  $H_k^{\text{ext}}$  generated by  $T_{0^{\vee}}$  and  $T_1$  and define idempotents  $p_{0^{\vee}}^{(\emptyset,1^2)}$  and  $p_{0^{\vee}}^{(1^2,\emptyset)}$  in  $HB_2^{\vee}$  by the equations

$$(p_{0^{\vee}}^{(\emptyset,1^2)})^2 = p_{0^{\vee}}^{(\emptyset,1^2)}, \qquad (p_{0^{\vee}}^{(1^2,\emptyset)})^2 = p_{0^{\vee}}^{(1^2,\emptyset)}; \tag{2.28}$$

and

$$T_{0^{\vee}} p_{0^{\vee}}^{(\emptyset,1^2)} = -t_k^{-\frac{1}{2}} p_{0^{\vee}}^{(\emptyset,1^2)}, \qquad T_1 p_{0^{\vee}}^{(\emptyset,1^2)} = -t^{-\frac{1}{2}} p_{0^{\vee}}^{(\emptyset,1^2)}, T_{0^{\vee}} p_{0^{\vee}}^{(1^2,\emptyset)} = t_k^{\frac{1}{2}} p_{0^{\vee}}^{(1^2,\emptyset)}, \qquad \text{and} \qquad T_1 p_{0^{\vee}}^{(1^2,\emptyset)} = -t^{-\frac{1}{2}} p_{0^{\vee}}^{(1^2,\emptyset)}.$$

$$(2.29)$$

Let  $a_k \in \mathbb{C}^{\times}$  and define

$$a_k e_{0^{\vee}} = T_{0^{\vee}} - t_k^{\frac{1}{2}}, \quad \text{so that} \quad e_{0^{\vee}} = T_{w_A} e_k T_{w_A}^{-1} \quad \text{and} \quad e_1 = T_{w_A} e_{k-1} T_{w_A}^{-1}.$$
 (2.30)

The conditions in (2.29) are equivalent to

$$a_{k}e_{0^{\vee}}p_{0^{\vee}}^{(\emptyset,1^{2})} = -(t_{k}^{\frac{1}{2}} + t_{k}^{-\frac{1}{2}})p_{0^{\vee}}^{(\emptyset,1^{2})}, \qquad ae_{1}p_{0^{\vee}}^{(\emptyset,1^{2})} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{0^{\vee}}^{(\emptyset,1^{2})}, \\ a_{k}e_{0^{\vee}}p_{0^{\vee}}^{(1^{2},\emptyset)} = 0, \qquad \text{and} \qquad ae_{1}p_{0^{\vee}}^{(1^{2},\emptyset)} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})p_{0^{\vee}}^{(\emptyset,1^{2})},$$

$$(2.31)$$

Using  $a_k e_{0^{\vee}} = W_1 T_0^{-1} - t_k^{\frac{1}{2}} = W_1 (T_0 - (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})) - t_k^{\frac{1}{2}} = W_1 (\tau_0 + t_0^{\frac{1}{2}} - c_{\alpha_0} - (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})) - t_k^{\frac{1}{2}}$ , a short computation gives

$$a_k e_{0^{\vee}} = \tau_0 W_1^{-1} - t_0^{-\frac{1}{2}} W_1^{-1} \frac{f_{\varepsilon_1 - r_2} f_{-\varepsilon_1 - r_1}}{f_{2\varepsilon_1}}.$$

and

$$\begin{split} N_k p_{0^\vee}^{(1^2,\emptyset)} &= (a_k^2 a^2 e_{0^\vee} e_1 e_{0^\vee} e_1 - a_k a (t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}) e_{0^\vee} e_1) - (a_k a^2 e_1 e_{0^\vee} e_1 - a (t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}) e_1) \\ &= \tau_0 \tau_1 \tau_0 \tau_1 (W_1 W_2)^{-1} - t_0^{-\frac{1}{2}} \tau_1 \tau_0 \tau_1 (W_1 W_2)^{-1} \frac{f_{-\varepsilon_1 - r_2} f_{\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} + t^{-\frac{1}{2}} \tau_0 \tau_1 \tau_0 (W_1 W_2)^{-1} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &- t_0^{-\frac{1}{2}} t^{-\frac{1}{2}} \tau_0 \tau_1 (W_1 W_2)^{-1} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{-\varepsilon_1 - r_2} f_{\varepsilon_1 - r_1}}{f_{2\varepsilon_2}} \\ &- t_0^{-\frac{1}{2}} t^{-\frac{1}{2}} \tau_1 \tau_0 (W_1 W_2)^{-1} \frac{f_{-\varepsilon_2 - r_2} f_{\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+ t_0^{-1} t^{-\frac{1}{2}} \tau_1 (W_1 W_2)^{-1} \frac{f_{-\varepsilon_2 - r_2} f_{\varepsilon_2 - r_1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_1}} \\ &- t_0^{-\frac{1}{2}} t^{-1} \tau_0 (W_1^{-1} W_2)^{-1} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{-\varepsilon_2 - \varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1} f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \\ &+ t_0^{-1} t^{-1} (W_1 W_2)^{-1} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+ t_0^{-1} t^{-1} (W_1 W_2)^{-1} \frac{f_{-\varepsilon_1 - r_2} f_{\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{-\varepsilon_2 - \varepsilon_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1}}{f_{\varepsilon_2 - \varepsilon_1}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+ t_0^{-1} t^{-1} (W_1 W_2)^{-1} \frac{f_{-\varepsilon_1 - r_2} f_{\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+ t_0^{-1} t^{-1} (W_1 W_2)^{-1} \frac{f_{-\varepsilon_1 - r_2} f_{\varepsilon_1 - r_1}}{f_{2\varepsilon_1}} \frac{f_{-\varepsilon_2 - \varepsilon_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+ t_0^{-1} t^{-1} (W_1 W_2)^{-1} \frac{f_{\varepsilon_1 - \varepsilon_1 - \varepsilon_2} f_{\varepsilon_1 - \varepsilon_1}}{f_{2\varepsilon_1}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{2\varepsilon_2}} \frac{f_{\varepsilon_2 - \varepsilon_1 + 1}}{f_{\varepsilon_2 - \varepsilon_1}} \\ &+$$

in analogy with (and with the same proof as) Proposition 2.3.

## 3. The two boundary Temperley-Lieb algebra $TL_k$

In this section we define the two boundary Temperley-Lieb algebra  $TL_k$  (also called the symplectic blob algebra, see [GMP07, GMP08, GMP12, Ree, KMP16, GMP17]) and review its diagrammatic calculus. We extend the diagrammatic calculus to make clear the relationship to the two boundary Hecke algebra and to set the stage for the proof of Theorem 3.2. Although Theorem 3.2 takes the form of a computation, it is a computation that has amazing consequences as it determines the relationship between the center of  $H_k^{\text{ext}}$  and the center of  $TL_k$ . The center of  $H_k^{\text{ext}}$  is a ring of symmetric functions (see [DR, Theorem 2.3]) and the center of  $TL_k$  turns out to be a polynomial ring  $\mathbb{C}[Z]$  in a single variable Z. We shall see that, in the same way that  $H_k^{\text{ext}}$  is finite rank over its center, the algebra  $TL_k$  is finite rank over  $\mathbb{C}[Z]$ . However, whereas the former has the easily classified rank of  $(2^k k!)^2$  over its center, the rank of  $TL_k$  is as yet unclassified combinatorially. For example, dim $(TL_k(b)) = 5$ , 19, 84, 335, and 1428, for k = 1, 2, 3, 4, and 5, respectively.

3.1. The extended two boundary Temperley-Lieb algebra  $TL_k^{\text{ext}}$ . Let  $H_k^{\text{ext}}$  be the extended two boundary Hecke algebra as defined in (2.15). The extended two boundary Temperley-Lieb algebra  $TL_k^{\text{ext}}$  is the quotient of  $H_k^{\text{ext}}$  by the relations

$$p_{0^{\vee}}^{(\emptyset,1^2)} = p_{0^{\vee}}^{(1^2,\emptyset)}, \qquad p_0^{(\emptyset,1^2)} = p_0^{(1^2,\emptyset)} \text{ and } p_i^{(1^3)} = 0 \text{ for } i \in \{1,\ldots,k-2\}.$$

**Theorem 3.1.** The algebra  $TL_k^{\text{ext}}$  is the quotient of  $H_k^{\text{ext}}$  by the relations

$$a_k a^2 e_{k-1} e_k e_{k-1} - a(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}) e_{k-1} = 0, \qquad a_0 a^2 e_1 e_0 e_1 - a(t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}) e_1 = 0,$$

and

$$a^{3}e_{i}e_{i+1}e_{i} - ae_{i} = a^{3}e_{i+1}e_{i}e_{i+1} - ae_{i+1} = 0$$
 for  $i \in \{1, \dots, k-2\}$ .

*Proof.* Let  $F_i = a^3 e_i e_{i+1} e_i - a e_i = a^3 e_{i+1} e_i e_{i+1} - a e_{i+1}$  for  $i \in \{1, \dots, k-2\}$ ,

 $F_k = a_k a^2 e_{k-1} e_k e_{k-1} - a(t_k^{-\frac{1}{2}} t^{\frac{1}{2}} + t_k^{\frac{1}{2}} t^{-\frac{1}{2}}) e_{k-1}, \quad \text{and} \quad F_0 = a_0 a^2 e_1 e_0 e_1 - a(t_0^{-\frac{1}{2}} t^{\frac{1}{2}} + t_0^{\frac{1}{2}} t^{-\frac{1}{2}}) e_1.$ By Proposition 2.3

By Proposition 2.3,

$$N_0 p_0^{(1^2,\emptyset)} = e_0 F_0, \qquad N_0 p_0^{(\emptyset,1^2)} = (e_0 - 1) F_0, \qquad F_0 = N_0 (p_0^{(1^2,\emptyset)} - p_0^{(\emptyset,1^2)}), \quad \text{and} \quad N p_i^{(1^3)} = F_i;$$

and, by (2.30),

$$T_{w_A}F_kT_{w_A}^{-1} = N_0^{\vee}(p_{0^{\vee}}^{(1^2,\emptyset)} - p_{0^{\vee}}^{(\emptyset,1^2)}), \qquad T_{w_A}^{-1}p_{0^{\vee}}^{(1^2,\emptyset)}T_{w_A} = e_kF_k, \text{ and } T_{w_A}^{-1}p_{0^{\vee}}^{(\emptyset,1^2)}T_{w_A} = (e_k - 1)F_k.$$
  
Thus, provided N, N<sub>0</sub> and N<sub>k</sub> are invertible, the ideal  $H_k^{\text{ext}}F_kH_k^{\text{ext}}$  is the same as the ideal generated

Thus, provided N,  $N_0$  and  $N_k$  are invertible, the ideal  $H_k^{\text{ext}} F_k H_k^{\text{ext}}$  is the same as the ideal generated by  $(p_{0^{\vee}}^{(1^2,\emptyset)} \text{ and } p_{0^{\vee}}^{(\emptyset,1^2)};$  the ideal  $H_k^{\text{ext}} F_0 H_k^{\text{ext}}$  is the same as the ideal generated by  $p_0^{(1^2,\emptyset)}$  and  $p_0^{(\emptyset,1^2)};$ and  $H_k^{\text{ext}} p_i^{(1^3)} H_k^{\text{ext}} = H_k^{\text{ext}} F_i H_k^{\text{ext}}.$ 

3.2. The two boundary Temperley-Lieb algebra  $TL_k$ . The two boundary Temperley-Lieb algebra  $TL_k$  is the subalgebra of  $TL_k^{\text{ext}}$  generated by  $a_0e_0, ae_1, \ldots, ae_{k-1}, a_ke_k$  (as defined in (2.16)). As in (2.12) and (2.20), where  $\mathcal{B}_k^{\text{ext}} = \mathcal{B}_k \times \mathcal{D}$  and  $H_k^{\text{ext}} = H_k \otimes \mathbb{C}[W_0^{\pm 1}]$ , the extended two boundary Temperley-Lieb algebra is

$$TL_k^{\text{ext}} = TL_k \otimes \mathbb{C}[W_0^{\pm 1}],$$
 as algebras, where  $W_0 = PW_1 \cdots W_k.$ 

3.3. Diagrammatic calculus for  $TL_k$ . Pictorially, identify

for  $i \in \{1, \ldots, k-1\}$ . Recall the notation

$$\llbracket x \rrbracket = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

from (2.19). With  $i \in \{1, ..., k-1\}$ , the relations (2.16), (2.17) and (2.18) are

In the quotient by  $(ae_i)(ae_{i+1})(ae_i) = ae_i$ , we have

$$ae_{i}T_{i+1}T_{i} = aT_{i+1}T_{i}e_{i+1} = t^{\frac{1}{2}}a^{2}e_{i}e_{i+1}, \qquad ae_{i}T_{i+1}^{-1}T_{i}^{-1} = aT_{i+1}^{-1}T_{i}^{-1}e_{i+1} = t^{-\frac{1}{2}}a^{2}e_{i}e_{i+1}, \qquad (3.1)$$

$$ae_{i+1}T_{i}T_{i+1} = aT_{i}^{-1}T_{i+1}^{-1}e_{i} = t^{-\frac{1}{2}}a^{2}e_{i+1}e_{i}, \qquad (3.1)$$

$$ae_{i+1}T_{i}^{-1}T_{i+1}^{-1} = aT_{i}^{-1}T_{i+1}^{-1}e_{i} = t^{-\frac{1}{2}}a^{2}e_{i+1}e_{i}, \qquad (3.1)$$

which are proved by using  $T_i^{\pm 1} = ae_i + t^{\pm \frac{1}{2}}$  to expand both sides in terms of  $e_i$ . When  $a_0(ae_1)e_0(ae_1) - \llbracket t_0t^{-1} \rrbracket (ae_1) = 0$  and  $a_k(ae_{k-1})e_k(ae_{k-1}) - \llbracket t_kt^{-1} \rrbracket ae_{k-1} = 0$ , then

$$(ae_1)T_0T_1 = t^{\frac{1}{2}}(ae_1)T_0^{-1}, \qquad T_1T_0(ae_1) = t^{\frac{1}{2}}T_0^{-1}(ae_1), (ae_{k-1})T_kT_{k-1} = t^{\frac{1}{2}}(ae_{k-1})T_k^{-1}, \quad T_{k-1}T_k(ae_{k-1}) = t^{\frac{1}{2}}T_k^{-1}(ae_{k-1}),$$
(3.2)

$$(ae_{1})T_{0}(ae_{1}) = -t^{\frac{1}{2}}(t_{0}^{\frac{1}{2}} - t_{0}^{-\frac{1}{2}})(ae_{1}), \qquad (ae_{k-1})T_{k}(ae_{k-1}) = -t^{\frac{1}{2}}(t_{k}^{\frac{1}{2}} - t_{k}^{-\frac{1}{2}})(ae_{k-1}), \quad (3.4)$$

$$(\int_{\mathbb{C}}^{\mathbb{C}} = -t^{\frac{1}{2}}(t_{0}^{\frac{1}{2}} - t_{0}^{-\frac{1}{2}}) \int_{\mathbb{C}}^{\mathbb{C}} \qquad (\int_{\mathbb{C}}^{\mathbb{C}} = -t^{\frac{1}{2}}(t_{k}^{\frac{1}{2}} - t_{k}^{-\frac{1}{2}})(ae_{k-1}), \quad (3.4)$$

and

$$e_{0}T_{1}^{-1}T_{0}^{-1}T_{1}^{-1}e_{0} = -t^{-\frac{1}{2}} \llbracket t_{0} \rrbracket e_{0}(ae_{1})e_{0} - t^{-1}t_{0}^{\frac{1}{2}}e_{0}^{2}$$

$$\overbrace{\parallel \downarrow }^{\parallel \downarrow} = -t^{-\frac{1}{2}} \llbracket t_{0} \rrbracket \overleftarrow{\smile}^{-1} - t^{-1}t_{0}^{\frac{1}{2}} \overleftarrow{\bigcirc}^{-1}$$

3.4.  $TL_k$  as a diagram algebra. Using the pictorial notation, the algebra  $TL_k$  has a basis (see [GMP12, Theorem 3.4]) of non-crossing diagrams with k dots in the top row, k dots in the bottom row, edges connecting pairs of dots, an even number of left boundary to right boundary edges, and

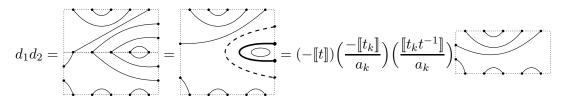
 $(-1)^{\#\{\text{left boundary edges}\}} = 1$  and  $(-1)^{\#\{\text{right boundary edges}\}} = 1$ .

For example,



are both basis elements of  $TL_k$ . Multiplication of basis elements can be computed pictorially by vertical concatenation, with self-connected loops and strands with both ends on the left or on the right replaced by constant coefficients according to the following local rules:

For example with  $d_1$  and  $d_2$  as above,



(where the dashed strand is removed with a coefficient of  $\frac{[t_k t^{-1}]}{a_k}$ , and the thick strand is removed with a coefficient of  $\frac{-[t_k]}{a_k}$ ).

3.5. The through-strand filtration of  $TL_k$ . A through-strand is an edge that connects a top vertex to a bottom vertex. Define the ideals

 $TL_k^{(\leq j)} = \mathbb{C}\text{-span}\{\text{diagrams with } \leq j \text{ through-strands}\}.$ 

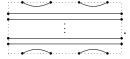
Then the algebra  $\mathcal{T\!L}_k$  is filtered by ideals as

$$TL_k = TL_k^{(\leq k)} \supseteq TL_k^{(\leq k-1)} \supseteq \cdots \supseteq TL_k^{(\leq 1)} \supseteq TL_k^{(\leq 0)} \supseteq 0.$$
(3.5)

If

$$TL_{k}^{(j)} = \frac{TL_{k}^{(\leq j)}}{TL_{k}^{(\leq j-1)}}, \quad \text{then} \quad \dim(TL_{k}^{(j)}) < \infty, \text{ for } j \ge 1, \quad \text{and} \quad \dim(TL_{k}^{(\leq 0)}) = \infty,$$

as there can be an arbitrarily large number of edges which connect the left and right sides in diagrams with no through strands:



# 3.6. The elements $I_1$ and $I_2$ . As in [GN, §3.2], define

$$I_1 = \begin{cases} (ae_1)(ae_3)\cdots(ae_{k-1}), & \text{if } k \text{ is even,} \\ (ae_1)(ae_3)\cdots(ae_{k-2})e_k, & \text{if } k \text{ is odd,} \end{cases} = \begin{cases} & \ddots & & \text{if } k \text{ is even,} \\ & \ddots & & & \text{if } k \text{ is odd,} \end{cases}$$
(3.6)

and

Up to a constant multiple the elements  $I_1$  and  $I_2$  are idempotents and

Proposition 3.2 gives another striking formula for the elements  $I_1I_2I_1$  and  $I_2I_1I_2$ .

3.7. The element  $ZI_1$  in  $TL_k$ . Conceptually, the diagram

if it represented a true element of the algebra  $H_k$ . Though the diagram F does not naturally represent an element of  $H_k$ , the diagrams

do appear in the algebra  $TL_k$  and play an important role in the proof of the following theorem. See also [GN, Thm. 4.1], using Remark 3.4 below as a guide.

**Theorem 3.2.** Let  $Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}$  which, as noted in (2.21) is a central element of  $H_k$ . As elements of  $TL_k$ ,

*if* k *is even, then*  $D^{\text{even}} = a_0 a_k I_1 I_2 I_1 + [t_0 t_k t^{-1}] I_1$  and  $ZI_1 = [k] D^{\text{even}}$ , and *if* k *is odd, then*  $D^{\text{odd}} = t^{-\frac{1}{2}} \left(\frac{-[t_0]}{a_0}\right) \left(a_0 a_k I_2 I_1 I_2 - [t_0 t_k^{-1}] I_2\right)$  and  $t^{-\frac{1}{2}} \left(\frac{-[t_0]}{a_0}\right) ZI_2 = [k] D^{\text{odd}}$ .

*Proof.* Case: k even. Let

$$L^{\text{even}} = I_1((ae_2)(ae_4)\cdots(ae_{k-2})e_k)I_1 = \bigcup_{i=1}^{|C|} (ae_2)(ae_4)\cdots(ae_{k-2})I_1 = \bigcup_{i=1}^{|C|} ($$

Using  $T_0^{-1} = a_0 e_0 + t_0^{-\frac{1}{2}}$  for the left pole and  $T_k = a_k e_k + t_k^{\frac{1}{2}}$  for the right pole,  $D^{\text{even}} = a_0 a_k I_1 I_2 I_1 + a_0 t_k^{\frac{1}{2}} M^{\text{even}} + a_k t_0^{-\frac{1}{2}} L^{\text{even}} + t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} P^{\text{even}}$   $= a_0 a_k I_1 I_2 I_1 + (t_k^{\frac{1}{2}} [t_0 t^{-1}]] + t_0^{-\frac{1}{2}} [t_k t^{-1}]] - t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}} [t_l]) I_1$  $= a_0 a_k I_1 I_2 I_1 + [t_0 t_k t^{-1}]] I_1,$ 

which completes the proof of the first statement.

Using  $(ae_1)T_1^{-1} = (-t^{\frac{1}{2}})(ae_1)$  and  $(ae_1)T_1T_0(ae_1) = t^{\frac{1}{2}}(ae_1)T_0^{-1}(ae_1)$  gives

and using  $T_{k-1}(ae_{k-1}) = (-t^{-\frac{1}{2}})(ae_{k-1})$  and  $T_{k-1}^{-1}T_k^{-1}(ae_{k-1}) = t^{-\frac{1}{2}}T_k(ae_{k-1})$  gives

Pictorially,

Working left to right removing loops,

$$I_{1}W_{1+2i}I_{1} = \left(t^{\frac{1}{2}}t^{\frac{1}{2}}(-\llbracket t\rrbracket)\right)^{i}\left(t^{-\frac{1}{2}}t^{\frac{1}{2}}(-\llbracket t\rrbracket)\right)^{\frac{k}{2}-1-i}R^{\text{even}} = \left(-\llbracket t\rrbracket)^{\frac{k}{2}-1}t^{i+\frac{1}{2}}(-t^{\frac{1}{2}})D^{\text{even}},$$
$$I_{1}W_{1+2i}^{-1}I_{1} = \left(t^{-\frac{1}{2}}t^{-\frac{1}{2}}(-\llbracket t\rrbracket)\right)^{i}\left(t^{-\frac{1}{2}}t^{\frac{1}{2}}(-\llbracket t\rrbracket)\right)^{\frac{k}{2}-1-i}S = \left(-\llbracket t\rrbracket)^{\frac{k}{2}-1}t^{-(i+\frac{1}{2})}(-t^{-\frac{1}{2}})D^{\text{even}},$$

for  $i \in \{0, \ldots, \frac{k}{2} - 1\}$ . Since  $I_1 W_{1+2i} I_1$  and  $I_1 W_{2+2i} I_1$  only differ by two twists (similarly  $I_1 W_{1+2i}^{-1} I_1$ and  $I_1 W_{2+2i}^{-1} I_1$  only differ by two twists) the relations  $T_i^{\pm 1}(ae_i) = (ae_i)T_i^{\pm 1} = (-t^{\pm \frac{1}{2}})(ae_i)$  give

$$I_1 W_{2+2i} I_1 = (-t^{-\frac{1}{2}})(-t^{-\frac{1}{2}})t^{-1} I_1 W_{1+2i} I_1 = (-\llbracket t \rrbracket)^{\frac{k}{2}-1} t^{i+\frac{1}{2}} (-t^{-\frac{1}{2}}) D^{\text{even}} \text{ and}$$
$$I_1 W_{2+2i}^{-1} I_1 = (-t^{\frac{1}{2}})(-t^{-\frac{1}{2}}) I_1 W_{1+2i}^{-1} I_1 = (-\llbracket t \rrbracket)^{\frac{k}{2}-1} t^{-(i+\frac{1}{2})} (-t^{\frac{1}{2}}) D^{\text{even}}, \text{ for } i \in \{0, \dots, \frac{k}{2}-1\}.$$

Thus

$$\begin{split} (-\llbracket t \rrbracket)^{\frac{k}{2}} ZI_1 &= ZI_1^2 = I_1 ZI_1 = \sum_{i=0}^{\frac{k}{2}-1} I_1 (W_{1+2i} + W_{2+2i} + W_{1+2i}^{-1} + W_{2+2i}^{-1}) I_1 \\ &= -(-\llbracket t \rrbracket)^{\frac{k}{2}-1} D^{\text{even}} \sum_{i=0}^{\frac{k}{2}-1} \left( t^{i+\frac{1}{2}} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) + t^{-(i+\frac{1}{2})} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \right) \\ &= (-\llbracket t \rrbracket)^{\frac{k}{2}} D^{\text{even}} \left( \frac{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \right) = (-\llbracket t \rrbracket)^{\frac{k}{2}} [k] D^{\text{even}}. \end{split}$$

Case: k odd. Let

$$L^{\text{odd}} = I_2((ae_3)(ae_5)\cdots(ae_{k-2})e_k)I_2 = \bigcup_{i=1}^{n} \bigcup_$$

Using 
$$e_0 T_1^{-1} T_0^{-1} T_1^{-1} e_0 = -t^{-\frac{1}{2}} \llbracket t_0 \rrbracket e_0(ae_1) e_0 - t^{-1} t_0^{\frac{1}{2}} e_0^2$$
 and  $T_k = a_k e_k + t_k^{\frac{1}{2}}$  gives  
 $D^{\text{odd}} = -t^{-\frac{1}{2}} \llbracket t_0 \rrbracket a_k I_2 I_1 I_2 - t^{-1} t_0^{\frac{1}{2}} a_k L^{\text{odd}} - t^{-\frac{1}{2}} \llbracket t_0 \rrbracket t_k^{\frac{1}{2}} M^{\text{odd}} - t^{-1} t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} P^{\text{odd}}$   
 $= t^{-\frac{1}{2}} \left( \frac{-\llbracket t_0 \rrbracket}{a_0} \right) \left( a_0 a_k I_2 I_1 I_2 + \left( -t^{-\frac{1}{2}} t_0^{\frac{1}{2}} \llbracket t_k t^{-1} \rrbracket - t_k^{\frac{1}{2}} \llbracket t_0 \rrbracket + t^{-\frac{1}{2}} t_0^{\frac{1}{2}} \llbracket t_k \rrbracket \rrbracket \right) I_2 \right)$   
 $= t^{-\frac{1}{2}} \left( \frac{-\llbracket t_0 \rrbracket}{a_0} \right) \left( a_0 a_k I_2 I_1 I_2 - \llbracket t_0 t_k^{-1} \rrbracket I_2 \right),$ 

which completes the proof of the first statement.

Using 
$$(ae_2)T_2^{-1} = -t^{\frac{1}{2}}(ae_2)$$
 and  $T_2T_1T_0T_1(ae_2) = t^{\frac{3}{2}}T_1^{-1}T_0^{-1}T_1^{-1}(ae_2),$   
 $R^{\text{odd}} = I_2(T_2^{-1}(ae_3)(ae_5)\cdots(ae_{k-2})T_kT_2T_1T_0T_1)I_2 = \underbrace{\left( \bigcup_{i=1}^{k} \bigcup_{i=1$ 

Using 
$$T_{k-1}(ae_{k-1}) = -t^{-\frac{1}{2}}(ae_{k-1})$$
 and  $T_{k-1}^{-1}T_{k}^{-1}(ae_{k-1}) = t^{-\frac{1}{2}}T_{k}(ae_{k-1})$  gives  

$$S^{\text{odd}} = I_{2}(T_{1}^{-1}T_{0}^{-1}T_{1}^{-1}(ae_{3})(ae_{5})\cdots(ae_{k-2})T_{k-1}^{-1}T_{k}^{-1}T_{k-1})I_{2} = \overbrace{\bigcup}^{0} \underbrace{\bigcup}_{0} \underbrace{U} \underbrace{\bigcup}_{$$

Pictorially,

$$I_2 W_{1+2i}^{-1} I_2 = \underbrace{\left[ \begin{array}{c} | \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} & \mathbf{y} \\ | \mathbf{y} & \mathbf{y} \\ |$$

Working left to right removing loops,

$$I_2 W_{2+2i} I_2 = \left( t^{\frac{1}{2}} t^{\frac{1}{2}} (-\llbracket t \rrbracket) \right)^i \left( t^{-\frac{1}{2}} t^{\frac{1}{2}} (-\llbracket t \rrbracket) \right)^{\frac{k-1}{2}-1-i} R^{\text{odd}} = \left( -\llbracket t \rrbracket)^{\frac{k-3}{2}} t^{i+1} (-t) D^{\text{odd}},$$
  
$$I_2 W_{2+2i}^{-1} I_2 = \left( t^{-\frac{1}{2}} t^{-\frac{1}{2}} (-\llbracket t \rrbracket) \right)^i \left( t^{-\frac{1}{2}} t^{\frac{1}{2}} (-\llbracket t \rrbracket) \right)^{\frac{k-1}{2}-1-i} S^{\text{odd}} = \left( -\llbracket t \rrbracket)^{\frac{k-3}{2}} t^{-(i+1)} (-1) D^{\text{odd}},$$

for  $i \in \{0, \ldots, \frac{k-1}{2} - 1\}$ . Since  $I_2 W_{2+2i} I_2$  and  $I_2 W_{3+2i} I_2$  only differ by two twists (similarly  $I_2 W_{2+2i}^{-1} I_2$  and  $I_2 W_{3+2i}^{-1} I_2$  only differ by two twists) the relations  $T_i^{\pm 1} e_i = e_i T_i^{\pm 1} = (-t^{\pm \frac{1}{2}}) e_i$  give

$$I_2 W_{3+2i} I_2 = (-t^{-\frac{1}{2}})(-t^{-\frac{1}{2}}) I_2 W_{2+2i} I_2 = (-\llbracket t \rrbracket)^{\frac{k-3}{2}} t^{i+1} (-1) D^{\text{odd}}, \text{ and} I_2 W_{3+2i}^{-1} I_2 = (-t^{\frac{1}{2}})(-t^{\frac{1}{2}}) I_2 W_{2+2i}^{-1} I_2 = (-\llbracket t \rrbracket)^{\frac{k-3}{2}} t^{-(i+1)} (-t) D^{\text{odd}}, \text{ for } i \in \{0, \dots, \frac{k-1}{2} - 1\}.$$

Next,

Using 
$$-t_0^{-\frac{1}{2}}T_k - t_0^{\frac{1}{2}}T_k^{-1} = -t_0^{-\frac{1}{2}}(a_k e_k + t_k^{\frac{1}{2}}) - t_0^{\frac{1}{2}}(a_k e_k + t_k^{-\frac{1}{2}}) = -\llbracket t_0 \rrbracket a_k e_k - \llbracket t_0 t_k^{-1} \rrbracket,$$
  
 $I_2(W_1 + W_1^{-1})I_2 = (-\llbracket t \rrbracket)^{\frac{k-1}{2}} \left( -\llbracket t_0 \rrbracket a_k I_2 I_1 I_2 - \llbracket t_0 t_k^{-1} \rrbracket M^{\text{odd}} \right)$   
 $= (-\llbracket t \rrbracket)^{\frac{k-1}{2}} \left( -\llbracket t_0 \rrbracket a_k I_2 I_1 I_2 - \llbracket t_0 t_k^{-1} \rrbracket \left( \frac{-\llbracket t_0 \rrbracket}{a_0} \right) I_2 \right)$   
 $= (-\llbracket t \rrbracket)^{\frac{k-1}{2}} \left( \frac{-\llbracket t_0 \rrbracket}{a_0} \right) \left( a_0 a_k I_2 I_1 I_2 - \llbracket t_0 t_k^{-1} \rrbracket I_2 \right) = -(t+1)(-\llbracket t \rrbracket)^{\frac{k-3}{2}} D^{\text{odd}}.$ 

Thus

$$\begin{split} \left(\frac{-\llbracket t_0 \rrbracket}{a_0}\right) (-\llbracket t \rrbracket)^{\frac{k-1}{2}} ZI_2 &= ZI_2^2 = I_2 ZI_2 \\ &= I_2 (W_1 + W_1^{-1}) I_2 + \sum_{i=0}^{\frac{k-1}{2}-1} I_2 (W_{2+2i} + W_{3+2i} + W_{2+2i}^{-1} + W_{3+2i}^{-1}) I_2 \\ &= -(t+1) (-\llbracket t \rrbracket)^{\frac{k-3}{2}} t^{\frac{1}{2}} D^{\text{odd}} + (-\llbracket t \rrbracket)^{\frac{k-3}{2}} \left(\sum_{i=0}^{\frac{k-3}{2}} (t^{i+1} - t^{-(i+1)})(-t-1) D^{\text{odd}}\right) \\ &= -(-\llbracket t \rrbracket)^{\frac{k-3}{2}} (t+1) D^{\text{odd}} \left(1 + \sum_{i=0}^{\frac{k-3}{2}} (t^{i+1} - t^{-(i+1)})\right) = (-\llbracket t \rrbracket)^{\frac{k-1}{2}} t^{\frac{1}{2}} D^{\text{odd}} [k]. \end{split}$$

**Corollary 3.3.** Let  $Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}$  and let  $I_1$  and  $I_2$  be as defined in (3.6) and (3.7). If k is even, then

$$a_0 a_k I_1 I_2 I_1 = \left(\frac{1}{[k]}Z - \llbracket t_0 t_k t^{-1} \rrbracket\right) I_1 \quad and \quad a_0 a_k I_2 I_1 I_2 = \left(\frac{1}{[k]}Z - \llbracket t_0 t_k t^{-1} \rrbracket\right) I_2.$$

If k is odd, then

$$a_0 a_k I_1 I_2 I_1 = \left(\frac{1}{[k]}Z + [t_0 t_k^{-1}]\right) I_1 \quad and \quad a_0 a_k I_2 I_1 I_2 = \left(\frac{1}{[k]}Z + [t_0 t_k^{-1}]\right) I_2.$$

*Proof.* As observed in the proof of Theorem 3.2, the products  $I_1ZI_1$  and  $I_2Z_2$  reduce to computation of the diagram with a single string going around all the poles ( $D^{\text{even}}$  or  $D^{\text{odd}}$ ). These diagrammatics give that there are constants  $C, C_1, C_2$  and  $D, D_1, D_2$  such that

$$I_1^2 = CI_1, \quad I_1I_2I_1 = (C_1Z + C_2)I_1, \quad I_2^2 = DI_2, \quad I_1I_2I_1 = (D_1Z + D_2)I_2.$$

Then, computing  $(I_1I_2I_1)^2$  in two different ways, we have

$$I_1I_2I_1I_1I_2I_1 = CI_1I_2I_1I_2I_1 = C(D_1Z + D_2)I_1I_2I_1, \text{ and} I_1I_2I_1I_1I_2I_1 = (C_1Z + C_2)I_1I_1I_2I_1 = C(C_1Z + C_2)I_1I_2I_1,$$

which indicates that  $C_1Z + C_2 = D_1Z + D_2$ .

Theorem 3.2 gives that, if k is even, then

$$a_0 a_k I_1 I_2 I_1 = D^{\text{even}} - \llbracket t_0 t_k t^{-1} \rrbracket I_1 = \frac{1}{[k]} Z I_1 - \llbracket t_0 t_k t^{-1} \rrbracket I_1,$$

and if k is odd, then

$$a_0 a_k I_2 I_1 I_2 = t^{\frac{1}{2}} \left( \frac{a_0}{-\llbracket t_0 \rrbracket} \right) D^{\text{odd}} + \llbracket t_0 t_k^{-1} \rrbracket I_2 = \frac{1}{[k]} Z I_2 + \llbracket t_0 t_k^{-1} \rrbracket I_2.$$

**Remark 3.4. Comparison to de Gier-Nichols.** Let us explain how to relate the constants in Corollary 3.3 and Proposition 4.4 to the values which appear in [GN]. Let

$$\begin{array}{ll} t_0^{\frac{1}{2}} = -iq^{\omega_1}, & t^{\frac{1}{2}} = q^{-1}, & t_k^{\frac{1}{2}} = -iq^{\omega_2}, \\ T_0 = -ig_0, & T_i = -g_i, & T_k = -ig_k, \\ e_0 = e_0, & e_i = e_i, & e_k = e_k. \end{array}$$

Then

$$(g_0 - q^{\omega_1})(g_0 - q^{-\omega_1}) = 0, \quad (g_i + q^{-1})(g_i - q) = 0, \quad (g_k - q^{\omega_1})(g_- q^{-\omega_1}) = 0,$$

as in [GN, Definitions 2.4, 2.6, and 2.8], and

$$g_0 = q^{\omega_1} - (q^{1+\omega_1} - q^{-(1+\omega_1)})e_0, \quad g_i = e_i - q^{-1}, \quad g_k = q^{\omega_2} - (q^{1+\omega_2} - q^{-(1+\omega_2)})e_k,$$

as in [GN, (5)]. Following [GN, Definitions 2.8 and (9)],

$$\begin{aligned} \mathcal{I}_{0}^{(C)} &= g_{1}^{-1} \cdots g_{k-1}^{-1} g_{k} g_{k-1} \cdots g_{2} g_{1} g_{0} = (-1)^{k-1} (-i) (-i) (-1)^{k-1} T_{1}^{-1} \cdots T_{k-1}^{-1} T_{k} T_{k-1} \cdots T_{1} T_{0} = -W_{1}, \\ \mathcal{I}_{i}^{(C)} &= g_{i} \mathcal{I}_{i-1}^{(C)} g_{i} = (-1)^{2} T_{i} (-W_{i}) T_{i} = -W_{i+1} \quad \text{for } i \in \{1, \dots, k-1\}, \text{ and} \\ \mathcal{I}_{k} &= \sum_{i=0}^{k-1} (\mathcal{I}_{i}^{(C)} + (\mathcal{I}_{i}^{(C)})^{-1}) = -(W_{1} + w_{1}^{-1} + \dots + W_{k} + W_{k}^{-1}) = -Z. \end{aligned}$$

Use the notation  $[x] = \frac{t^{\frac{x}{2}} - t^{-\frac{x}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{q^x - q^{-x}}{q - q^{-1}}$  and let  $a_0, a$  and  $a_k$  take the favorite values from Remark 2.1 so that

$$a = -1$$
,  $a_0 = -[t_0 t^{-1}]$ , and  $a_k = -[t_k t^{-1}]$ , and set  $\theta = c + \frac{k-1}{2}$  and  $z = [t^{\theta}][k]$ .

as in Proposition 4.4. Following [GN, Theorem 4.1] and remembering that  $Z_k = -Z$ , let

$$\Theta = \theta + \frac{1}{\log q} i\pi \quad \text{so that} \quad \begin{array}{l} -[k] \llbracket t^{\theta} \rrbracket &= -[k] (t^{\frac{\theta}{2}} + t^{-\frac{\theta}{2}}) = [k] (-q^{-\theta} - q^{\theta}) \\ &= [k] (q^{-(\theta + \frac{1}{\log q}i\pi)} + q^{\theta + \frac{1}{\log q}i\pi}) = [k] (q^{-\Theta} + q^{\Theta}) = [k] \frac{[2\Theta]}{[\Theta]} . \end{array}$$

Note that

$$\begin{split} a_0 a_k &= [\![t_0 t^{-1}]\!] [\![t_k t^{-1}]\!] = (t_0^{\frac{1}{2}} t^{-\frac{1}{2}} + t_0^{-\frac{1}{2}} t^{\frac{1}{2}})(t_k^{\frac{1}{2}} t^{-\frac{1}{2}} + t_k^{-\frac{1}{2}} t^{\frac{1}{2}}) \\ &= (-iq^{-\omega_1 - 1} + iq^{\omega_1 + 1})(-iq^{-\omega_2 - 1} + iq^{\omega_2 + 1}) = -[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2. \end{split}$$

Then the constant b that appears in [GN, Definition 3.6 and Theorem 4.1] to make  $I_1I_2I_1 = bI_1$ and  $I_2I_1I_2 = bI_2$  as operators on a simple  $TL_k$ -module is computed from Corollary 3.3 as follows:

$$\begin{split} b &= \frac{\frac{1}{[k]} z - \llbracket t_0 t_k t^{-1} \rrbracket}{a_0 a_k} = \frac{\frac{1}{[k]} [k] \llbracket t^{\theta} \rrbracket - \llbracket t_0 t_k t^{-1} \rrbracket}{\llbracket t_0 t^{-1} \rrbracket \llbracket t_k t^{-1} \rrbracket} = \frac{\llbracket t^{\theta} \rrbracket - \llbracket t_0 t_k t^{-1} \rrbracket}{\llbracket t_0 t^{-1} \rrbracket \llbracket t_k t^{-1} \rrbracket} \\ &= -\frac{(q^{\Theta} + q^{-\Theta}) + (-iq^{\omega_1})(-iq^{\omega_2})q + (iq^{-\omega_1})(iq^{-\omega_2})q^{-1}}{-[\omega_1 + 1] [\omega_2 + 1] (q - q^{-1})^2} \\ &= \frac{q^{\Theta} + q^{-\Theta} - q^{\omega_1 + \omega_2 + 1} - q^{-(\omega_1 + \omega_2 + 1)}}{[\omega_1 + 1] [\omega_2 + 1] (q - q^{-1})^2} \\ &= \frac{((q^{\omega_1 + \omega_2 + 1 + \Theta})^{\frac{1}{2}} - (q^{\omega_1 + \omega_2 + 1 + \Theta})^{-\frac{1}{2}})((q^{\omega_1 + \omega_2 + 1 - \Theta})^{\frac{1}{2}} - (q^{\omega_1 + \omega_2 + 1 - \Theta})^{-\frac{1}{2}})}{[\omega_1 + 1] [\omega_2 + 1] (q - q^{-1})^2} \\ &= \frac{[\frac{1}{2} (\omega_1 + \omega_2 + 1 + \Theta)][\frac{1}{2} (\omega_1 + \omega_2 + 1 - \Theta)]}{[\omega_1 + 1] [\omega_2 + 1]} \quad \text{when $k$ is even, and} \end{split}$$

$$\begin{split} b &= \frac{\frac{1}{[k]}z + [t_0t_k^{-1}]]}{a_0a_k} = \frac{\frac{1}{[k]}[k][t^{\theta}] + [t_0t_k^{-1}]]}{[t_0t^{-1}]][t_kt^{-1}]]} = \frac{[t^{\theta}]] + [t_0t_k^{-1}]]}{[t_0t^{-1}]][t_kt^{-1}]]} \\ &= \frac{-(q^{\Theta} + q^{-\Theta}) + (-iq^{\omega_1})(iq^{-\omega_2}) + (iq^{-\omega_1})(-iq^{\omega_2})}{-[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= \frac{-q^{\Theta} - q^{-\Theta} + q^{\omega_1 - \omega_2} + q^{-(\omega_1 - \omega_2)}}{-[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= -\frac{((q^{\omega_1 - \omega_2 - \Theta})^{\frac{1}{2}} - (q^{\omega_1 - \omega_2 - \Theta})^{-\frac{1}{2}})((q^{\omega_1 - \omega_2 + \Theta})^{\frac{1}{2}} - (q^{\omega_1 - \omega_2 + \Theta})^{-\frac{1}{2}})}{[\omega_1 + 1][\omega_2 + 1](q - q^{-1})^2} \\ &= -\frac{[\frac{1}{2}(\omega_1 - \omega_2 - \Theta)][\frac{1}{2}(\omega_1 - \omega_2 + \Theta)]}{[\omega_1 + 1][\omega_2 + 1]} \quad \text{when } k \text{ is odd.} \end{split}$$

# 4. Calibrated representations of $H_k^{\text{ext}}$ and $TL_k^{\text{ext}}$

In this section we classify and construct all irreducible calibrated representations of the extended two boundary Temperley-Lieb algebras  $TL_k^{\text{ext}}$ . This is done by using the classification of irreducible calibrated  $H_k^{\text{ext}}$ -modules from [DR], which we begin by reviewing in Sections 4.1 and 4.2. Using the formulas for the elements  $p_i^{(1^3)}$ ,  $p_0^{(\emptyset,1^2)}$ ,  $p_0^{(\emptyset,1^2)}$ ,  $p_{0^{\vee}}^{(\emptyset,1^2)}$ , and  $p_{0^{\vee}}^{(1^2,\emptyset)}$  that one quotients  $H_k^{\text{ext}}$  by to obtain  $TL_k^{\text{ext}}$ , we determine exactly which irreducible calibrated representations of  $H_k^{\text{ext}}$  factor through the quotient, thus providing a full classification of irreducible calibrated representations of  $TL_k^{\text{ext}}$ .

4.1. Calibrated representations of  $H_k^{\text{ext}}$ . A calibrated  $H_k^{\text{ext}}$ -module is an  $H_k^{\text{ext}}$ -module M such that  $W_0, W_1, \ldots, W_k$  are simultaneously diagonalizable as operators on M. Let  $r_1, r_2 \in \mathbb{C}$  such that

$$-t^{r_1} = -t_k^{\frac{1}{2}} t_0^{-\frac{1}{2}} \quad \text{and} \quad -t^{r_2} = t_k^{\frac{1}{2}} t_0^{\frac{1}{2}}.$$
(4.1)

For  $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{C}^k$  let  $c_{-i} = -c_i$  and define

$$Z(\mathbf{c}) = \{\varepsilon_i \mid c_i = 0\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = 0\},$$
  
$$\sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = 0\},$$
  
$$P(\mathbf{c}) = \{\varepsilon_i \mid c_i \in \{\pm r_1, \pm r_2\}\} \sqcup \{\varepsilon_j - \varepsilon_i \mid 0 < i < j \text{ and } c_j - c_i = \pm 1\}$$

$$(4.2)$$

$$\sqcup \{\varepsilon_j + \varepsilon_i \mid 0 < i < j \text{ and } c_j + c_i = \pm 1\},\tag{4.3}$$

where  $\{\varepsilon_1, \ldots, \varepsilon_n\}$  is an orthonormal basis for the weights corresponding to  $\mathfrak{gl}_n$  (see [DR, §3]). A *local region* is a pair  $(\mathbf{c}, J)$  with  $\mathbf{c} \in \mathbb{C}^k$  and  $J \subseteq P(\mathbf{c})$ . The set of *standard tableaux* of shape  $(\mathbf{c}, J)$  is

$$\mathcal{F}^{(\mathbf{c},J)} = \{ w \in \mathcal{W}_0 \mid R(w) \cap Z(\mathbf{c}) = \emptyset, \ R(w) \cap P(\mathbf{c}) = J \}$$
(4.4)

(see the following section for a visualization of this set as fillings of box arrangements). A skew local region is a local region  $(\mathbf{c}, J), \mathbf{c} = (c_1, \ldots, c_k)$ , such that

if  $w \in \mathcal{F}^{(\mathbf{c},J)}$  then  $w\mathbf{c} = ((w\mathbf{c})_1, \dots, (w\mathbf{c})_n)$  satisfies

$$(w\mathbf{c})_{1} \neq 0, \quad (w\mathbf{c})_{2} \neq 0, \quad (w\mathbf{c})_{1} \neq -(w\mathbf{c})_{2},$$
  
 $(w\mathbf{c})_{i} \neq (w\mathbf{c})_{i+1} \text{ for } i = 1, \dots, k-1, \text{ and } (w\mathbf{c})_{i} \neq (w\mathbf{c})_{i+2} \text{ for } i = 1, \dots, k-2.$  (4.5)

The following theorem constructs and classifies the calibrated irreducible representations of  $H_k^{\text{ext}}$ .

**Theorem 4.1.** [DR, Theorem 3.3] Assume  $t^{\frac{1}{2}}$ ,  $t^{\frac{1}{2}}_{0}$ , and  $t^{\frac{1}{2}}_{k}$  are invertible,  $t^{\frac{1}{2}}$  is not a root of unity, and

$$t_0^{\frac{1}{2}} t_k^{\frac{1}{2}}, -t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}} \notin \{1, -1, t^{\pm \frac{1}{2}}, -t^{\pm \frac{1}{2}}, t^{\pm 1}, -t^{\pm 1}\} \quad and \quad t_0^{\frac{1}{2}} t_k^{\frac{1}{2}} \neq (-t_0^{-\frac{1}{2}} t_k^{\frac{1}{2}})^{\pm 1}$$
  
Let  $r_1, r_2$  be as in (4.1).

(a) Let  $(\mathbf{c}, J)$  be a skew local region and let  $z \in \mathbb{C}^{\times}$ . Define

$$H_k^{(z,\mathbf{c},J)} = \operatorname{span}_{\mathbb{C}} \{ v_w \mid w \in \mathcal{F}^{(\mathbf{c},J)} \},$$
(4.6)

so that the symbols  $v_w$  are a labeled basis of the vector space  $H_k^{(z,\mathbf{c},J)}$ . Let

$$\gamma_i = -t^{c_i} \text{ for } i = 1, 2, \dots, k, \text{ and } \gamma_0 = z \gamma_{w^{-1}(1)}^{-1} \cdots \gamma_{w^{-1}(k)}^{-1}.$$

Then the following formulas make  $H_k^{(z,\mathbf{c},J)}$  into an irreducible  $H_k^{ext}$ -module:

$$PW_1 \cdots W_k v_w = zv_w, \qquad Pv_w = \gamma_0 v_w, \qquad W_i v_w = \gamma_{w^{-1}(i)} v_w, \tag{4.7}$$

$$T_{i}v_{w} = [T_{i}]_{ww}v_{w} + \sqrt{-([T_{i}]_{ww} - t^{\frac{1}{2}})([T_{i}]_{ww} + t^{-\frac{1}{2}})} v_{s_{i}w}, \quad for \ i = 1, \dots, k-1,$$
(4.8)

$$T_0 v_w = [T_0]_{ww} v_w + \sqrt{-([T_0]_{ww} - t_0^{\frac{1}{2}})([T_0]_{ww} + t_0^{-\frac{1}{2}})} v_{s_0 w},$$
(4.9)

where  $v_{s_iw} = 0$  if  $s_iw \notin \mathcal{F}^{(\mathbf{c},J)}$ , and

$$[T_i]_{ww} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - \gamma_{w^{-1}(i)}\gamma_{w^{-1}(i+1)}^{-1}} \quad and \quad [T_0]_{ww} = \frac{(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}) + (t_k^{\frac{1}{2}} - t_k^{-\frac{1}{2}})\gamma_{w^{-1}(1)}^{-1}}{1 - \gamma_{w^{-1}(1)}^{-2}}.$$
 (4.10)

(b) The map

$$\mathbb{C}^{\times} \times \{skew \ local \ regions \ (\mathbf{c}, J)\} \quad \longleftrightarrow \quad \{irreducible \ calibrated \ H_k^{ext}\text{-}modules\}$$

$$(z, \mathbf{c}, J) \qquad \longmapsto \qquad H_k^{(z, \mathbf{c}, J)}$$

is a bijection.

### 4.2. Configurations of boxes. Let $(\mathbf{c}, J)$ be a local region with $\mathbf{c} = (c_1, \ldots, c_k)$ ,

$$\mathbf{c} \in \mathbb{Z}^k$$
 or  $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$ , and  $0 \le c_1 \le \dots \le c_k$ . (4.11)

Start with an infinite arrangement of NW to SE diagonals, numbered consecutively from  $\mathbb{Z}$  or  $\mathbb{Z} + \frac{1}{2}$ , increasing southwest to northeast (see Example 4.2). The *configuration*  $\kappa$  of boxes corresponding to the local region ( $\mathbf{c}$ , J) has 2k boxes (labeled box<sub>-k</sub>,..., box<sub>-1</sub>, box<sub>1</sub>,..., box<sub>k</sub>) with the following conditions.

- ( $\kappa$ 1) Location: box<sub>i</sub> is on diagonal  $c_i$ , where  $c_{-i} = -c_i$  for  $i \in \{-k, \ldots, -1\}$ .
- ( $\kappa 2$ ) Same diagonals: box<sub>i</sub> is NW of box<sub>i</sub> if i < j and box<sub>i</sub> and box<sub>i</sub> are on the same diagonal.
- $(\kappa 3)$  Adjacent diagonals:

If  $\varepsilon_j - \varepsilon_i \in J$ , then box<sub>i</sub> is NW (strictly north and weakly west) of box<sub>i</sub>:

If  $\varepsilon_j - \varepsilon_i \in P(\mathbf{c}) - J$ , then box<sub>j</sub> is SE (weakly south and strictly east) of box<sub>i</sub>:  $[i \mid j]$ 

( $\kappa 4$ ) Markings: There is a marking on each of the diagonals  $r_1$ ,  $-r_1$ ,  $r_2$  and  $-r_2$ . If  $\varepsilon_i \in J$ , box<sub>i</sub> is NW of the marking on diagonal  $c_i$ :

If  $\varepsilon_i \in P(\mathbf{c}) - J$ , then box<sub>i</sub> is SE of the marking in diagonal  $c_i$ :

A standard filling of the boxes of  $\kappa$  is a bijective function  $S: \kappa \to \{-k, \ldots, -1, 1, \ldots, k\}$  such that

- (S1) Symmetry:  $S(box_{-i}) = -S(box_i)$ .
- (S2) Same diagonals:

If 0 < i < j and  $box_i$  and  $box_j$  are on the same diagonal then  $S(box_i) < S(box_j)$ .

(S3) Adjacent diagonals:

If 0 < i < j, box<sub>i</sub> and box<sub>j</sub> are on adjacent diagonals, and box<sub>j</sub> is NW of box<sub>i</sub>, then  $S(\text{box}_j) < S(\text{box}_i)$ .

If 0 < i < j, box<sub>i</sub> and box<sub>j</sub> are on adjacent diagonals, and box<sub>j</sub> is SE of box<sub>i</sub>, then  $S(\text{box}_j) > S(\text{box}_i)$ .

(S4) Markings:

If  $box_i$  is on a marked diagonal and is SE of the marking, then  $S(box_i) > 0$ .

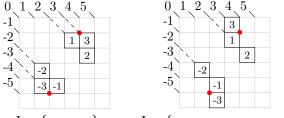
If  $box_i$  is on a marked diagonal and is NW of the marking, then  $S(box_i) < 0$ .

The *identity filling* of a configuration  $\kappa$  is the filling F of the boxes of  $\kappa$  given by  $F(\text{box}_i) = i$ , for  $i = -k, \ldots, -1, 1, \ldots, k$ . The identity filling of  $\kappa$  is usually not a standard filling of  $\kappa$  (see Example 4.2).

**Example.** Let k = 4,  $r_1 = 1$ , and  $r_2 = 3$ . Consider  $\mathbf{c} = (2, 2, 3)$ . Then

$$Z(\mathbf{c}) = \{\varepsilon_1 - \varepsilon_2\} \qquad and \qquad P(\mathbf{c}) = \{\varepsilon_3, \ \varepsilon_3 - \varepsilon_1, \ \varepsilon_3 - \varepsilon_2\}.$$

The box configurations corresponding to  $J = \{\varepsilon_3 - \varepsilon_2\}$  and  $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$  (filled with their identity fillings) are



 $J = \{\varepsilon_3 - \varepsilon_2\} \qquad J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$ 

For both configurations, the identity filling is not a standard filling. Examples of standard fillings of the configuration corresponding to  $J = \{\varepsilon_2 - \varepsilon_3\}$  include



**Proposition 4.2.** [DR, Proposition 3.1] Let  $\kappa$  be a configuration of boxes corresponding to a local region  $(\mathbf{c}, J)$  with  $\mathbf{c} \in \mathbb{Z}^k$  or  $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$ . For  $w \in \mathcal{W}_0$  let  $S_w$  be the filling of the boxes of  $\kappa$  given by

 $S_w(\text{box}_i) = w(i), \text{ for } i = -k, \dots, -1, 1, \dots, k.$ 

The map

$$\begin{array}{ccc} \mathcal{F}^{(\mathbf{c},J)} & \longrightarrow & \{standard \ fillings \ S \ of \ the \ boxes \ of \ \kappa\} \\ w & \longmapsto & S_w \end{array} \quad is \ a \ bijection.$$

4.3. Calibrated representations of  $TL_k^{\text{ext}}$ . The following theorem determines which calibrated irreducible representations of  $H_k^{\text{ext}}$  are  $TL_k^{\text{ext}}$ -modules. In Theorem 4.3 the answer is stated in terms of the configuration of boxes  $\kappa$ . By  $(\kappa 1)-(\kappa 4)$  of Section 4.2 the local region  $(\mathbf{c}, J)$  is determined by  $\kappa$ . See Theorem 5.1 for the explicit conversion from  $\kappa$  to  $(\mathbf{c}, J)$  for the irreducible calibrated  $TL_k$ -modules.

**Theorem 4.3.** Assume that if  $r_1, r_2 \in \mathbb{Z}$  or  $r_1, r_2 \in \mathbb{Z} + \frac{1}{2}$ , then  $r_2 > r_1 + 1$ . Let  $\kappa$  be the configuration of boxes corresponding to a skew local region  $(\mathbf{c}, J)$  with  $\mathbf{c} \in \mathbb{Z}^k$  or  $\mathbf{c} \in (\mathbb{Z} + \frac{1}{2})^k$ . The irreducible calibrated  $H_k^{\text{ext}}$ -module  $H_k^{(z,\mathbf{c},J)}$  is a  $TL_k^{\text{ext}}$ -module if and only if  $\kappa$  is a 180° rotationally symmetric skew shape with two rows of k boxes each (with or without markings),

Proof. Let  $P = \{p_0^{(\emptyset,1^2)}, p_0^{(1^2,\emptyset)}, p_{0^{\vee}}^{(\emptyset,1^2)}, p_1^{(1^3)}, p_2^{(1^3)}, \dots, p_{k-2}^{(1^3)}\}$  so that  $TL_k$  is the quotient of  $H_k$  by the ideal generated by the set P. For  $w \in \mathcal{F}^{(\mathbf{c},J)}$  let  $S_w$  be the standard tableau of shape  $\kappa$  corresponding to w as given in Proposition 4.2. For  $j \in \{-k, \dots, -1, 1, \dots, k\}$ ,

 $(w\mathbf{c})_j$  is the diagonal number of  $S_w(j)$ ,

where  $S_w(j)$  is the box containing j in  $S_w$ .

Step 1: Rewriting of the conditions for  $pv_w = 0$ . By the construction of  $H_k^{(z,\mathbf{c},J)}$  in Theorem 4.1, the module  $H_k^{(z,\mathbf{c},J)}$  has basis  $\{v_w \mid w \in \mathcal{F}^{(\mathbf{c},J)}\}$  and, if  $w \in \mathcal{F}^{(\mathbf{c},J)}$  then

$$\tau_i v_w = 0 \quad \text{if and only if} \quad (w\mathbf{c})_{i+1} = (w\mathbf{c})_i \pm 1,$$
  
$$f_{\varepsilon_i - r_2} v_w = 0 \quad \text{if and only if} \quad (w\mathbf{c})_i = r_2, \quad \text{and}$$
  
$$f_{\varepsilon_i - \varepsilon_j + 1} v_w = 0 \quad \text{if and only if} \quad (w\mathbf{c})_i = (w\mathbf{c})_j - 1.$$

Let  $i \in \{1, \ldots, k-2\}$ . Using the expansion of  $p_i^{(1^3)}$  in terms of the  $\tau_i$  from Proposition 2.3,

$$p_i^{(1^3)}v_w = \tau_i\tau_{i+1}\tau_iv_w - t^{-\frac{1}{2}}\tau_{i+1}\tau_i\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}}v_w - t^{-\frac{1}{2}}\tau_i\tau_{i+1}\frac{f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+1}-\varepsilon_i}}v_w + t^{-1}\tau_i\frac{f_{\varepsilon_{i+1}-\varepsilon_{i+1}}}{f_{\varepsilon_{i+1}-\varepsilon_i}}v_w + t^{-1}\tau_{i+1}\frac{f_{\varepsilon_{i+2}-\varepsilon_{i}+1}f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+2}-\varepsilon_i}f_{\varepsilon_{i+1}-\varepsilon_i}}v_w + t^{-1}\tau_{i+1}\frac{f_{\varepsilon_{i+2}-\varepsilon_{i}+1}f_{\varepsilon_{i+1}-\varepsilon_{i}+1}}{f_{\varepsilon_{i+2}-\varepsilon_i}f_{\varepsilon_{i+1}-\varepsilon_i}}v_w$$

we consider the condition  $p_i^{(1^3)}v_w = 0$  term-by-term. First,  $\tau_i\tau_{i+1}\tau_i v_w = 0$  exactly when  $(wc)_{i+1} = (wc)_i \pm 1$  or  $(s_iwc)_{i+2} = (s_iwc)_{i+1} \pm 1$  or  $(s_{i+1}s_iw)_{i+1} = (s_{i+1}s_iw)_i = \pm 1$ , i.e. when

$$(w\mathbf{c})_{i+1} = (w\mathbf{c})_i \pm 1$$
 or  $(w\mathbf{c})_{i+2} = (w\mathbf{c})_i \pm 1$  or  $(w\mathbf{c})_{i+2} = (w\mathbf{c})_{i+1} \pm 1$ 

Next,  $-t^{-\frac{3}{2}} \frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}f_{\varepsilon_{i+2}-\varepsilon_i+1}f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}f_{\varepsilon_{i+2}-\varepsilon_i}f_{\varepsilon_{i+1}-\varepsilon_i+1}} v_w = 0$  exactly when

$$(w\mathbf{c})_{i+1} = (w\mathbf{c})_i + 1$$
 or  $(w\mathbf{c})_{i+2} = (w\mathbf{c})_i + 1$  or  $(w\mathbf{c})_{i+1} = (w\mathbf{c})_{i+1} + 1$ .

Thus  $-t^{-\frac{3}{2}} \frac{f_{\varepsilon_{i+1}-\varepsilon_{i+2}+1}f_{\varepsilon_{i+2}-\varepsilon_i+1}f_{\varepsilon_{i+1}-\varepsilon_i+1}}{f_{\varepsilon_{i+1}-\varepsilon_{i+2}}f_{\varepsilon_{i+2}-\varepsilon_i}f_{\varepsilon_{i+1}-\varepsilon_i+1}} v_w = 0$  already implies  $\tau_i \tau_{i+1} \tau_i v_w = 0$ , and similarly for the other terms in the expansion of  $p_i^{(1^3)} v_w = 0$ . Thus  $p_i^{(1^3)} v_w = 0$  if and only if

$$(w\mathbf{c})_i = (w\mathbf{c})_{i+1} - 1 \text{ or } (w\mathbf{c})_i = (w\mathbf{c})_{i+2} - 1 \text{ or } (w\mathbf{c})_{i+1} = (w\mathbf{c})_{i+2} - 1.$$
 (4.13)

Similarly,  $p_0^{(\emptyset,1^2)}v_w = 0$  if and only if

 $(w\mathbf{c})_1 \in \{r_1, r_2\} \text{ or } (w\mathbf{c})_2 \in \{r_1, r_2\} \text{ or } (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \text{ or } (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1;$  (4.14)

 $p_0^{(1^2,\emptyset)}v_w = 0$  if and only if

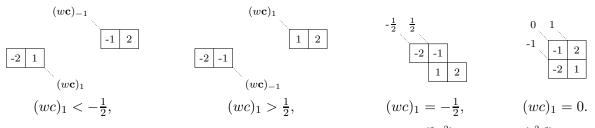
 $(w\mathbf{c})_1 \in \{-r_1, -r_2\}$  or  $(w\mathbf{c})_2 \in \{-r_1, -r_2\}$  or  $(w\mathbf{c})_2 = (w\mathbf{c})_1 + 1$  or  $(w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1;$  (4.15)  $p_{0\vee}^{(\emptyset,1^2)}v_w = 0$  if and only if

$$(w\mathbf{c})_1 \in \{-r_1, r_2\} \text{ or } (w\mathbf{c})_2 \in \{-r_1, r_2\} \text{ or } (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \text{ or } (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1;$$
 (4.16)

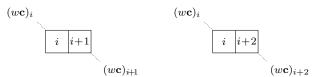
and  $p_{0^{\vee}}^{(1^2,\emptyset)}v_w = 0$  if and only if

$$(w\mathbf{c})_1 \in \{r_1, -r_2\} \text{ or } (w\mathbf{c})_2 \in \{r_1, -r_2\} \text{ or } (w\mathbf{c})_2 = (w\mathbf{c})_1 + 1 \text{ or } (w\mathbf{c})_2 = (w\mathbf{c})_{-1} + 1.$$
 (4.17)

Step 2: If  $\kappa$  is as in (4.12) and  $w \in \mathcal{F}^{(\mathbf{c},J)}$  and  $p \in P$  then  $pv_w = 0$ . Assume  $\kappa$  has the form given in (4.12) and let  $w \in \mathcal{F}^{(\mathbf{c},J)}$ . Since  $\kappa$  has only two rows the positions of (-2, -1, 1, 2) in  $S_w$  take one of the following forms:



In each of these cases, the conditions in (4.14)–(4.17) give that  $p_0^{(\emptyset,1^2)}v_w = 0$ ,  $p_0^{(1^2,\emptyset)}v_w = 0$ ,  $p_{0^{\vee}}^{(\emptyset,1^2)}v_w = 0$  and  $p_{0^{\vee}}^{(1^2,\emptyset)}v_w = 0$ . Next, let  $i \in \{1,\ldots,k-2\}$ . Since  $\kappa$  has only two rows, then either i or i+1 are in the same row

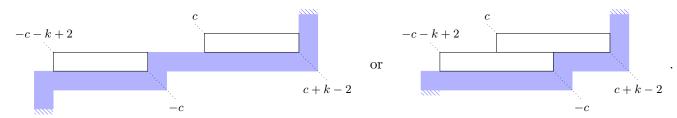


or *i* and i + 2 are in the same row. Thus, by (4.13),  $p_i v_w = 0$ . This completes the proof that if  $\kappa$  is of the form (4.12) then  $H_k^{(z,\mathbf{c},J)}$  is a  $TL_k^{\text{ext}}$ -module.

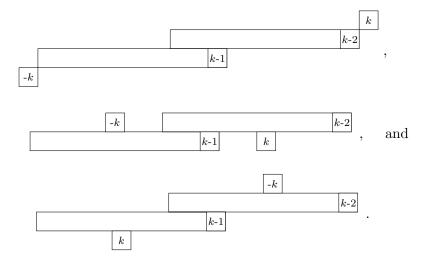
Step 3: If  $\kappa$  is not as in (4.12) then there exists  $w \in \mathcal{F}^{(\mathbf{c},J)}$  and  $p \in P$  such that  $pv_w \neq 0$ . Let 2k be the number of boxes in  $\kappa$ . The proof is by induction on k.

First, if k = 2, then the condition (4.13) does not apply. If  $\mathbf{c} = (r_1, r_2)$  then there are 8 possibilities for  $w\mathbf{c}$ :  $(r_1, r_2)$ ,  $(-r_1, r_2)$ ,  $(r_1, -r_2)$ ,  $(-r_1, -r_2)$ ,  $(r_2, r_1)$ ,  $(-r_2, r_1)$ ,  $(r_2, -r_1)$  and  $(-r_2, -r_1)$ . None of these satisfy all of the conditions (4.14)–(4.17). If  $\mathbf{c} = (c_1, c_1 + 1)$ , then  $s_1\mathbf{c} = (c_1 + 1, c_1)$  does not satisfy (4.14) and  $s_0s_1s_0s_1\mathbf{c} = (-c, -c - 1)$  does not satisfy (4.17). Thus that only the shaded local regions in Figure 1 can have  $pv_w = 0$  for all  $p \in P$  and all  $w \in \mathcal{F}^{(\mathbf{c},J)}$ . For all of these,  $\kappa$  is as in (4.12).

Next, assume k > 2 and proceed inductively. If  $H_k^{(z,\mathbf{c},J)}$  is a calibrated  $TL_k^{\text{ext}}$ -module then  $\operatorname{Res}_{TL_{k-1}^{\text{ext}}}^{TL_k^{\text{ext}}}(H_k^{(z,\mathbf{c},J)})$  is calibrated  $TL_{k-1}^{\text{ext}}$ -module. This means that if  $S_w$  is a standard tableau of shape  $\kappa$  and  $S'_w$  is  $S_w$  except with the boxes  $S_w(k)$  and  $S_w(-k)$  removed and  $\kappa'$  is the shape of  $S'_w$ , then  $\kappa'$  must be as in (4.12) and have only two rows. The box  $S_w(k)$  is in a SE corner of  $\kappa$  and the box  $S_w(-k)$  is in a NW corner of  $\kappa$ .



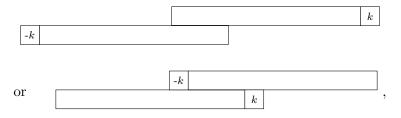
Given that  $\kappa'$  has only two rows and  $\kappa$  is obtained from  $\kappa'$  by adding boxes that could contain k and -k in a standard tableau, the following are possibilities that we discard for  $\kappa$ :



Namely, in each case there is a standard tableaux that has k - 2, k - 1 and k in positions that do not satisfy the conditions in (4.13). Thus, in these cases, there exists an  $S_w$  of shape  $\kappa$  for which  $p_{k-2}^{(1^3)}v_w \neq 0$ . Further, in the case

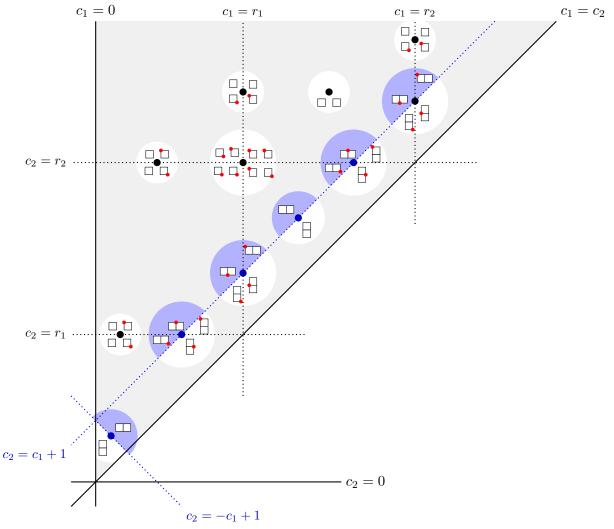


the shape  $\kappa$  does not satisfy the  $(w\mathbf{c})_{k-2} \neq (w\mathbf{c})_k$  from (4.5) and the module  $H_k^{(z,\mathbf{c},J)}$  is not calibrated. In summary, unless  $\kappa$  is of the form given in (4.12)



then either  $H_k^{(z,\mathbf{c},J)}$  is not calibrated or there exists an  $S_w$  of shape  $\kappa$  for which  $p_{k-2}^{(1^3)}v_w \neq 0$ .  $\Box$ 

FIGURE 1. Calibrated representations of  $H_2$  have regular central character. For each ( $\mathbf{c}, J$ ) the corresponding configuration of boxes  $\kappa$  is displayed in the local region of chambers corresponding to the elements of  $\mathcal{F}^{(\mathbf{c},J)}$ ; only the boxes on positive diagonals are shown, since they determine  $\kappa$  when  $\mathbf{c}$  is regular. The local regions marked in blue are those that factor through the Temperley-Lieb quotient.



The following proposition determines the action of the central element Z on each of the irreducible calibrated  $TL_k^{\text{ext}}$ -modules.

**Proposition 4.4.** Let  $Z = W_1 + W_1^{-1} + \cdots + W_k + W_k^{-1}$  be the central element of  $TL_k^{\text{ext}}$  studied in Theorem 3.2. Assume that  $\mathbf{c} = (c, c+1, \ldots, c+k-1)$  and  $H_k^{(z, \mathbf{c}, J)}$  is an irreducible calibrated  $TL_k^{\text{ext}}$  as in Theorem 4.3. If  $v \in H_k^{(z, \mathbf{c}, J)}$ , then

$$Zv = \llbracket t^{\theta} \rrbracket [k]v, \qquad where \quad \theta = c + \frac{k-1}{2}, \quad \llbracket t^{\theta} \rrbracket = t^{\frac{\theta}{2}} + t^{-\frac{\theta}{2}} \quad and \quad [k] = \frac{t^{\frac{\kappa}{2}} - t^{-\frac{\kappa}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}$$

Proof. Let  $v \in H_k^{(z,\mathbf{c},J)}$  be such that  $W_i v = q^{c+i-1}$  for  $i \in \{1,\ldots,k\}$ . Then  $Zv_w = zv_w$  where  $z = t^{-(c+k-1)} + \cdots + t^{-(c+1)} + t^{-c} + t^c + t^{c+1} + \cdots + t^{c+k-1}$  $= (t^{c+\frac{k-1}{2}} + t^{-(c+\frac{k-1}{2})})(t^{-\frac{k-1}{2}} + \cdots + t^{\frac{k-1}{2}}) = (t^{\frac{\theta}{2}} + t^{-\frac{\theta}{2}})\frac{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \llbracket t^{\theta} \rrbracket [k].$ 

Since Z is a central element of  $H_k^{\text{ext}}$  and  $H_k^{(z,\mathbf{c},J)}$  is a simple  $H_k^{\text{ext}}$ -module, Schur's lemma implies that if  $v \in H_k^{(z,\mathbf{c},J)}$  then Zv = zv.

# 5. Schur-Weyl duality between $TL_k^{\text{ext}}$ and $U_q\mathfrak{gl}_2$

In this section we show that the Schur-Weyl duality studied in [DR] provides calibrated irreducible representations of the two boundary Temperley-Lieb algebra. We classify these representations using the combinatorial classification of irreducible calibrated  $TL_k^{\text{ext}}$  modules obtained in Theorem 4.3. We follow the combinatorics of [Dau, §4] and [DR, §5]. Note that similar constructions hold for replacing  $\mathfrak{gl}_2$  with  $\mathfrak{sl}_2$ —see, for example, [Dau, §4]

The irreducible finite dimensional representations  $L(\lambda)$  of  $U_q(\mathfrak{gl}_2)$  are indexed by  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$  with  $\lambda_1 \geq \lambda_2$ . By the Clebsch-Gordan formula or the Littlewood-Richardson rule (see [Mac, (5.16)])

$$L(a,0) \otimes L(b,0) = L(a+b,0) \oplus L(a+b-1,1) \oplus \cdots \oplus L(a+1,b-1) \oplus L(a,b),$$

and

$$L(\lambda_1, \lambda_2) \otimes L(1, 0) = \begin{cases} L(\lambda_1 + 1, \lambda_2) \oplus L(\lambda_1, \lambda_2 + 1), & \text{if } \lambda_1 > \lambda_2, \\ L(\lambda_1 + 1, \lambda_2), & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Let  $a, b \in \mathbb{Z}_{\geq 0}$  with  $a \geq b$  and fix the simple  $U_q \mathfrak{gl}_2$ -modules

$$M = L(a, 0),$$
  $N = L(b, 0)$   $V = L(1, 0).$  (5.1)

We identify  $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$  with a left-justified arrangement of boxes with  $\lambda_i$  boxes in the *i*th row. As in [DR, (5.28)] the *shifted content* of a box in row *i* and column *j* as

$$\tilde{c}(box) = j - i - \frac{1}{2}(a + b - 2)$$
(5.2)

i.e. the shifted content is its diagonal number, where the box in the upper left corner has shifted content  $-\frac{1}{2}(a+b-2)$ .

For  $j \in \mathbb{Z}_{\geq -1}$  let  $\mathcal{P}^{(j)}$  be an index set for the irreducible  $U_q \mathfrak{gl}_2$ -modules that appear in  $M \otimes N \otimes V^{\otimes j}$ . In particular,

$$\mathcal{P}^{(-1)} = (a,0), \quad \mathcal{P}^{(0)} = \{(a+b-j,j) \mid j=0,1,\dots,b\} \text{ and } \\ \mathcal{P}^{(j)} = \{(a+b+j-\ell,\ell) \mid 0 \le \ell \le \frac{1}{2}(j+a+b)\}, \text{ for } j \ge 1.$$

Following [DR, §5.4], the associated Bratteli diagram has

vertices on level j labeled by the partitions in  $\mathcal{P}^{(j)}$ , an edge  $(a, 0) \longrightarrow \mu$  for each  $\mu \in \mathcal{P}^{(0)}$ , and for each  $j \ge 0, \mu \in \mathcal{P}^{(j)}$  and  $\lambda \in \mathcal{P}^{(j+1)}$ , there is

an edge  $\mu \to \lambda$  if  $\lambda$  is obtained from  $\mu$  by adding a box.

The case when a = 6 and b = 3 is illustrated in Figure 2.

Assume  $q \in \mathbb{C}^{\times}$  and a > b + 2 so that the generality condition  $(a + 1) - (b + 1) \notin \{0, \pm 1, \pm 2\}$  of [DR, Theorem 5.5] is satisfied. Define

$$r_1 = \frac{1}{2}(a-b)$$
 and  $r_2 = \frac{1}{2}(a+b+2),$  (5.3)

and let  $H_k^{\text{ext}}$  be the extended two boundary Hecke algebra with parameters  $t_0^{\frac{1}{2}}$ ,  $t_k^{\frac{1}{2}}$ , and  $t^{\frac{1}{2}}$  given by

$$t^{\frac{1}{2}} = q, \quad t_0 = -t^{r_2 - r_1} = -q^{(b+1)}, \quad \text{and} \quad t_k = -t^{r_2 + r_1} = -q^{2(a+1)},$$
 (5.4)

so that  $-t_k^{\frac{1}{2}}t_0^{-\frac{1}{2}} = -t^{r_1}$  and  $t_k^{\frac{1}{2}}t_0^{\frac{1}{2}} = -t^{r_2}$  as in [DR, (3.5), (5.21)]. By [DR, Theorem 5.4 and (5.21)] there are

commuting actions of  $U_q \mathfrak{gl}_2$  and  $H_k^{\text{ext}}$  on  $M \otimes N \otimes V^{\otimes k}$ ,

where the  $H_k^{\text{ext}}$  action is given via R-matrices for the quantum group  $U_q \mathfrak{gl}_2$ .

**Theorem 5.1.** Let  $a, b \in \mathbb{Z}_{\geq 0}$  with a > b+2. Let  $q \in \mathbb{C}^{\times}$  not a root of unity and let  $H_k^{\text{ext}}$  be the two boundary Hecke algebra with parameters  $t_0^{\frac{1}{2}}$ ,  $t_k^{\frac{1}{2}}$  and  $t^{\frac{1}{2}}$  as in (5.4). Let  $U_q \mathfrak{gl}_2$  be the Drinfeld-Jimbo quantum group corresponding to  $\mathfrak{gl}_2$  and let M, N and V be the simple  $U_q \mathfrak{gl}_2$ -modules given in (5.1). Then the  $H_k^{\text{ext}}$  action factors through  $TL_k^{\text{ext}}$  and, as  $(U_q \mathfrak{gl}_2, TL_k^{\text{ext}})$ -bimodules,

$$M \otimes N \otimes V^{\otimes k} \cong \bigoplus_{\lambda \in \mathcal{P}^{(k)}} L(\lambda) \otimes B_k^{\lambda} \qquad with \quad B_k^{(a+b+k-\ell,\ell)} \cong H^{(z,\mathbf{c},J)},$$

where  $z = (-1)^k q^{(a+b-\ell)(a+b-\ell-1)+\ell(\ell-3)-a(a-1)-b(b-1)-k(a+b-2)}$  and  $(\mathbf{c}, J)$  is the local region corresponding to the configuration  $\kappa$  of 2k boxes

$$\ell + 1 - r_2 - k$$

$$(5.5)$$

that has k boxes in each row, the shifted content of the leftmost box in the first row is  $r_2 - \ell$ , the shifted content of the leftmost box in the second row is  $\ell + 1 - r_2 - k$ . Between the rows there are blue markers in diagonals with shifted content  $\pm r_1$  and there are red markers in diagonals with shifted content  $\pm r_2$ , as pictured. Explicitly,  $\mathbf{c} = (c_1, c_2, \dots, c_k)$  is the sequence of

absolute values of 
$$c, c+1, \cdots, c+k-1, where c = \frac{1}{2}(a+b) - \ell + 1$$

arranged in increasing order; and J is the union of

$$J_1 = \begin{cases} \emptyset, & \text{if } a \ge b \ge \ell, \\ \{\varepsilon_{\ell-b}\}, & \text{if } a \ge \ell > b, \\ \{\varepsilon_{a-b}\}, & \text{if } \ell > a > b, \end{cases}$$

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and

$$J_{2} = \begin{cases} \emptyset, & \text{if } \frac{1}{2}(a+b+2) > \ell, \\ \{\varepsilon_{2} - \varepsilon_{1}, \varepsilon_{4} - \varepsilon_{3}, \dots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}\}, & \text{if } \ell \ge \frac{1}{2}(a+b+2) \text{ and } a+b \text{ even}, \\ \{\varepsilon_{3} - \varepsilon_{2}, \varepsilon_{5} - \varepsilon_{4}, \dots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}\}, & \text{if } \ell \ge \frac{1}{2}(a+b+2) \text{ and } a+b \text{ odd}. \end{cases}$$

*Proof.* Fix  $\lambda = (a + b + k - \ell, \ell) \in \mathcal{P}^{(k)}$ . The sum of the contents of the boxes in  $\lambda$  is

$$\sum_{box\in\lambda} c(box) = (0+1+\ldots+(a+b+k-\ell-1)) + (-1+0+\cdots+\ell-2)$$
$$= \frac{1}{2}(a+b+k-\ell-1)(a+b+k-\ell) + \frac{1}{2}\ell(\ell-3).$$

By [DR, Theorem 5.5 and (5.35)],  $\mathcal{B}_k^\lambda \cong H_k^{(z,\mathbf{c},J)}$  where

$$z = (-1)^k q^{2c_0}$$
, where  $c_0 = -\frac{1}{2}(k(a+b-2) + a(a-1) + b(b-1)) + \sum_{box \in \lambda} c(box)$ ,

and **c** and J and the corresponding configuration  $\kappa$  of 2k boxes are determined as follows.

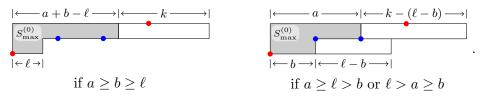
Place markers at the NW corner of the boxes at positions (1, a + b + 1), (2, a + 1), (2, b + 1), and (3, 1) so that these markers are in the diagonals with shifted contents  $\pm r_1$  and  $\pm r_2$ .

$$\lambda = (a + b + k - \ell, \ell) =$$

Following [DR, (5.27)], let

$$S_{\max}^{(0)} = \begin{cases} (a+b-\ell,\ell), & \text{if } a \ge b \ge \ell, \\ (a,b), & \text{if } a \ge \ell \ge b \end{cases}$$

(since  $a \ge b$  we are in the left case of [DR, (5.15)] with c = d = 1 so that  $\mu^c = \min(\ell, b)$  and  $S_{\max}^{(0)} = \mathring{\mu} = (a + b - \mu^c, \mu^c)$ ):



By [DR, (5.35)], the corresponding configuration of boxes is  $\kappa = \operatorname{rot}(\lambda/S_{\max}^{(0)}) \cup \lambda/S_{\max}^{(0)}$ , as pictured above in (5.5).

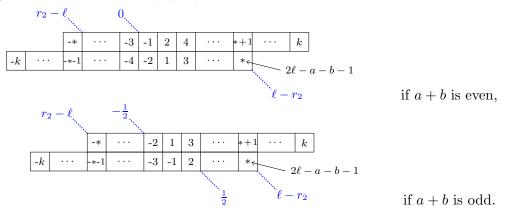
To determine  $(\mathbf{c}, J)$ , use the conditions  $(\kappa 1)-(\kappa 4)$  of Section 4.2 which specify the relation between  $\kappa$  and  $(\mathbf{c}, J)$ . First index the boxes of  $\kappa$  with  $-k, \ldots, -1, 1, \ldots, k$  by diagonals, left to right, and NW to SE along diagonals. The sequence  $\mathbf{c} = (c_1, \ldots, c_k)$  with  $0 \leq c_1 \leq c_2 \leq \cdots \leq c_k$  is the sequence of the absolute values of the shifted contents of boxes in the first row of  $\kappa$ . Next, the set J is determined as follows.

1. By ( $\kappa 4$ ), the set J contains  $\varepsilon_i$  if i > 0 and box<sub>i</sub> is NW of the marker in the diagonal with shifted content  $r_1$  or  $r_2$  in  $\kappa$ . This occurs on diagonal  $r_1$  whenever  $\ell > b$  (marked in blue),

$$\varepsilon_{\ell-b} \in J \text{ if } a \ge \ell > b \quad \text{and} \quad \varepsilon_{a-b} \in J \text{ if } \ell > a \ge b;$$

and J contains no roots of the form  $\varepsilon_j$  when  $a \ge b \ge \ell$ .

2. By ( $\kappa$ 3), the set J contains  $\varepsilon_j - \varepsilon_i$  if j > i > 0 and box<sub>i</sub> and box<sub>j</sub> are in the same column of  $\kappa$  (so that box<sub>i</sub> and box<sub>j</sub> are in adjacent diagonals and box<sub>j</sub> is NW of box<sub>i</sub>). This occurs exactly when  $0 \ge r_2 - \ell = \frac{1}{2}(a+b+2) - \ell$ . If  $\ell \ge \frac{1}{2}(a+b+2)$  and a+b is even then the boxes indexed  $1, 3, \ldots, 1 + 2(\ell - \frac{1}{2}(a+b+2)) = 2\ell - (a+b+1)$  are in the second row directly below boxes of index  $2, 4, \ldots, 2\ell - a - b$ . If  $\ell \ge \frac{1}{2}(a+b+2)$  and a+b is odd then boxes  $2, 4, \ldots, 2(\ell - \frac{1}{2}(a+b+1))$ , directly below boxes of index  $3, 5, \ldots, 2\ell - a - b$ :



So J contains

$$\varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \dots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}$$
 if  $\ell \ge \frac{1}{2}(a+b+2)$  and  $a+b$  is even, or  $\varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \dots, \varepsilon_{2\ell-a-b} - \varepsilon_{2\ell-a-b-1}$  if  $\ell \ge \frac{1}{2}(a+b+2)$  and  $a+b$  is odd.

3. Also by ( $\kappa$ 3), the set J contains  $\varepsilon_j + \varepsilon_i$  if j > i > 0, and box<sub>j</sub> is directly above box<sub>-i</sub>, which does not occur.

In this way **c** and J are determined from  $\kappa$ . Since all of these  $H_k^{(z,\mathbf{c},J)}$  satisfy the conditions of Theorem 4.3, it follows that the  $H_k^{\text{ext}}$ -action on  $M \otimes N \otimes V^{\otimes k}$  factors through  $TL_k^{\text{ext}}$ .

**Remark 5.2.** The dimension of  $B_k^{(a+b+k-\ell,\ell)}$  is the number of paths in the Bratteli diagram from a shape on level 0 to the shape  $\lambda = (a+b+k-\ell,\ell)$  on level k. Summing over the shapes on level 0 for which there is a path to  $\lambda$  gives

$$\dim(B^{(a+b+k-\ell,\ell)}) = \sum_{c=\max(0,\ell-k)}^{\min(b,\ell)} f^{\lambda/(a+b-c,c)}$$

where  $f^{\lambda/\mu}$  is the number of standard tableaux of skew shape  $\lambda/\mu$ . If  $\ell \leq a + b - c$  then the second row of  $\lambda/(a + b - \ell, \ell)$  does not overlap the first row and thus

$$f^{\lambda/(a+b-c,c)} = \binom{k}{\ell-c}$$
 if  $\ell \le a+b-c$ 

Since  $c \leq \min(b, \ell)$ , the case  $\ell > a + b - c$  can occur only when  $\ell > a \geq b$ , in which case

$$(a+b+k-\ell,\ell)/(a+b-c,c) = \underbrace{\underbrace{\leftarrow a+b-c \longrightarrow \leftarrow k-\ell+c \longrightarrow}_{\ell \leftarrow (a+b-c)}}_{\ell \leftarrow (a+b-c)}$$

so that

$$f^{(a+b+k-\ell,\ell)/(a+b-c,c)} = \sum_{j=\ell-(a+b-c)}^{k+\ell-c} f^{(k-j,j)} = \sum_{j=\ell-(a+b-c)}^{\min(k-(\ell-c),\ell-c)} \binom{k}{j} - \binom{k}{j-1} = \binom{k}{\ell-c} - \binom{k}{\ell-(a+b-c)-1}$$

Namely, the first equality comes from the Pieri formula and the expansion of a skew Schur function by Littlewood-Richardson coefficients (see [Mac, (5.16)] for the Pieri formula and [Mac, (5.2) and (5.3)] for Littlewood-Richardson coefficients) and the second equality comes from the number of standard tableaux of a two row shape as given, for example, in [GHJ, Theorem 2.8.5 and Lemma 2.8.4].

The following examples reference the node label styles in Figure 2.

**Example.** Let a = 7 and b = 3. The markers are in the diagonals with shifted contents  $\pm r_1$  and  $\pm r_2$ , where  $r_1 = 2$  and  $r_2 = 6$ . An example where  $\ell > a \ge b$ : Let  $\ell = 11$  and k = 14, then

11 
$$\lambda = (13, 11) =$$
 with  $S_{\max}^{(0)} = (7, 3)$ 

The boxes of  $\lambda/S_{\max}^{(0)}$  have

Then **c** is the rearrangement of the absolute values of (-2, -1, 0, 1, 2, 3, 3, 4, 4, 5, 5, 6, 7, 8) into increasing order and  $J = \{\varepsilon_4, \varepsilon_2 - \varepsilon_1, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{11}\}$ . The configuration of boxes  $\kappa$  corresponding to  $(\mathbf{c}, J)$  has indexing of boxes

-11	-9	-7	-5	-3	-1	2	4	6	8	10	12	13	14
-14-13-12	-10	-8	-6	-4	-2	1	3	5	7	9	11		

**Example.** Let a = 6 and b = 3 to take advantage of the setting and notation of Figure 2. The markers are in the diagonals with shifted contents  $\pm r_1$  and  $\pm r_2$ , where  $r_1 = \frac{3}{2}$  and  $r_2 = \frac{11}{2}$ . (1) An example where  $\ell > a \ge b$ : Let  $\ell = 8$  and k = 9, then

(8) 
$$\lambda = (10,8) =$$
 with  $S_{\max}^{(0)} = (6,3).$ 

The boxes of  $\lambda/S_{\max}^{(0)}$  have

shifted contents: 
$$\begin{bmatrix} \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

Then **c** is the rearrangement of the absolute values of  $\left(-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}\right)$  into increasing order and  $J = \{\varepsilon_3, \varepsilon_3 - \varepsilon_2, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6\}$ . The configuration of boxes  $\kappa$  corresponding to  $(\mathbf{c}, J)$  has indexing of boxes

(2) An example with  $a \ge \ell > b$ : Let k = 3 and  $\ell = 5$ , so that  $a + b + k - \ell = 7$ .

$$\langle 5 \rangle$$
  $\lambda = (7,5) =$  with  $S_{\max}^{(0)} = (6,3).$ 

The boxes of  $\lambda/S_{\max}^{(0)}$  have

shifted contents: 
$$\begin{bmatrix} \frac{5}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Then **c** is the rearrangement of the absolute values of  $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2})$  in increasing order and  $J = \{\varepsilon_2\}$ . The configuration of boxes  $\kappa$  corresponding to  $(\mathbf{c}, J)$  is

$$\begin{array}{c|c} 1 & 2 & 3 \\ \hline -3 & -2 & -1 \end{array} \quad with \quad P(\mathbf{c}) = \{\varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\} \end{array}$$

(3) An example with  $a \ge b \ge \ell$ : Let k = 3 and  $\ell = 2$ , so that  $a + b + k - \ell = 10$ . Then

2 
$$\lambda = (10,2) =$$
 with  $S_{\max}^{(0)} = (7,2).$ 

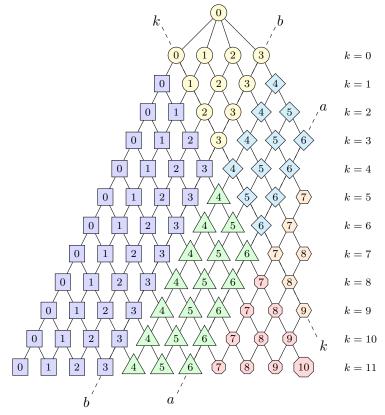
The boxes of  $\lambda/S_{\max}^{(0)}$  have

shifted	contents:	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	
ontojeca	0011110111101	4	4	- 2	•

Then **c** is the rearrangement of the absolute values of  $(\frac{7}{2}, \frac{9}{2}, \frac{11}{2})$  in increasing order and  $J = \emptyset$ . The configuration of boxes  $\kappa$  corresponding to  $(\mathbf{c}, J)$  is

$$\boxed{-3-2-1} \bullet \qquad \boxed{1 \ 2 \ 3} \quad with \quad P(\mathbf{c}) = \{\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.$$

FIGURE 2. The Temperley-Lieb Bratteli diagram for a = 6 and b = 3, levels 0–9. Partitions  $\lambda = (a + b + k - \ell, \ell)$  are labeled by  $\ell$ . The dimension formulas are consequences of Remark 5.2.



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