Monk rules for type GL_n Macdonald polynomials

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In memory of Georgia Benkart

Abstract

In this paper we give Monk rules for Macdonald polynomials which are analogous to the Monk rules for Schubert polynomials. These formulas are similar to the formulas given by Baratta [Ba08], but our method of derivation is to use Cherednik's interwiners. Deriving Monk rules by this technique addresses the relationship between the work of Baratta and the product formulas of Yip [Yi10]. Specializations of the Monk formula's at $q = 0$ and/or $t = 0$ provide Monk rules for Iwahori-spherical polynomials and for finite and affine key polynomials.

Key words — Macdonald polynomials, Schubert calculus

0 Introduction

In this paper, we use the term *electronic Macdonald polynomials* for what are commonly called 'non-symmetric' Macdonald polynomials in the literature (see [CR22] for motivation for this terminology). The (type GL_n) electronic Macdonald polynomials $\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ form a $\mathbb{C}$-basis for the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The goal of this paper is to give Monk type rules for the products

$$x_j E_\mu \qquad \text{and} \qquad (x_1 + \cdots + x_j)E_\mu \qquad \text{and} \qquad E_{\varepsilon_j} E_\mu,$$

$$x_j^{-1} E_\mu \qquad \text{and} \qquad (x_j^{-1} + \cdots + x_n^{-1})E_\mu \qquad \text{and} \qquad E_{-\varepsilon_j} E_\mu,$$

expanded in terms of electronic Macdonald polynomials (here $\varepsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the n-tuple with 1 in the jth entry and all other entries 0). We derive our formulas by viewing multiplication by x_j, multiplication by $(x_1 + \cdots + x_j)$, multiplication by E_{ε_j} etc. as operators on the ring $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm 1}]$. Expanding a product like $x_j E_\mu$ in terms of electronic Macdonald polynomials is equivalent to writing the operator of multiplication by x_j in terms of intertwiners and Cherednik-Dunkl operators. The expression of the operator x_j in terms of intertwiners and Cherednik-Dunkl operators can be viewed as a *universal formula* for multiplication by x_j in the basis of electronic Macdonald polynomials. These universal formulas are given in Theorem 2.1.

To obtain the explicit expansions of the products above it is then necessary to (carefully) "evaluate" the universal formula at μ. These explicit expansions of the products are given in Theorem 3.1.

Letting $e_1 = x_1 + \cdots + x_n$ and $e_{n-1} = x_1 \cdots x_n(x_1^{-1} + \cdots + x_n^{-1})$ be the first and $(n-1)$st elementary symmetric functions, Baratta [Ba08, Prop. 7, Prop. 8] gives formulas for

$$x_j E_\mu, \qquad e_1 E_\mu \qquad \text{and} \qquad e_{n-1} E_\mu$$

expanded in terms of electronic Macdonald polynomials. Baratta indicates that the formula for $x_j E_\mu$ also appears in Lascoux [La08]. The formulas of Baratta must be the same as ours, although unwinding

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and comparing the notations is not immediate (at least for us). In [Ba10] Baratta computes the products $e_r E_\mu$ where e_r denotes the rth elementary symmetric function. It might be possible to give alternate derivations of the products $e_r E_\mu$ for general r using the methods of this paper.

Baratta's approach is similar to that of Lascoux, using the interpolation Macdonald polynomials and results of Knop and Sahi [Kn96] and [Sa96]. Our approach uses the intertwiners and their relation with the Cherednik-Dunkl operators. Computing these formulas via intertwiners addresses the relationship between the formulas of Baratta and the methods of Yip [Yi10], who gives some related expansions, but in an alcove walk form.

The motivation for the term "Monk rules" comes from Schubert calculus. In Schubert calculus, Monk's rules for the Schubert polynomials $\mathfrak{S}_w$ are

$$x_j\mathfrak{S}_w = \Big(\sum_{\substack{1\le i<j\\ \ell(ws_{ij})=\ell(w)+1}} \mathfrak{S}_{ws_{ij}}\Big) - \Big(\sum_{\substack{j<i\le n\\ \ell(ws_{ji})=\ell(w)+1}} \mathfrak{S}_{ws_{ji}}\Big) \qquad \text{and} \qquad \mathfrak{S}_{s_{r,r+1}}\mathfrak{S}_w = \sum_{\substack{i\le r<j\\ \ell(ws_{ij})=\ell(w)+1}} \mathfrak{S}_{ws_{ij}}.$$

(here w is a permutation in the symmetric group S_n and s_{ij} denotes the transposition which switches i and j). These rules are proved in [Mac91, (4.15),(4.15′),(4.15″)]. A compendium of similar formulas for type GL_n Grothendieck polynomials is given in [LS04, §1,2, §1.3]. Though the analogies are tantalizing, we have not, in any generality, made a concrete connection between our 'Monk formulas' for Macdonald poylnomials and the formulas which appear in Schubert calculus. Part (c) of Corollary 4.4 provides a different formulation and proof of [AQ19, Theorem 3.3.6]. Other formulas related to Corollary 4.4 appear in Assaf [As21] and Gibson [Gib19] who, respectively, use the combinatorics of Kohnert diagrams and monomial crystals.

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1 Macdonald polynomials

Let $n \in \mathbb{Z}_{>0}$. The (Laurent) polynomial ring $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ has basis

$$\{x^\mu \mid \mu \in \mathbb{Z}^n\}, \qquad \text{where} \quad x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n} \quad \text{for } \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n.$$

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by permuting the variables $x_1, \ldots, x_n$. The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the positions of the entries. The two actions are related by $wx^\mu = x^{w\mu}$ for $w \in S_n$ and $\mu \in \mathbb{Z}^n$.

Let $q, t^{\frac12} \in \mathbb{C}^\times$. For $j \in \{1, \ldots, n\}$ let X_j be the operator on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ given by multiplication by x_j. For $i \in \{1, \ldots, n-1\}$ let $s_i \in S_n$ be the transposition which switches i and $i+1$. For $j \in \{1, \ldots, n\}$ let y_j be the operator on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ which replaces each occurrence of x_j with $q^{-1}x_j$. In formulas, if $f \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ then

$$\begin{aligned} (X_jf)(x_1, \ldots, x_n) &= x_j \cdot f(x_1, \ldots, x_n), \\ (s_if)(x_1, \ldots, x_n) &= f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+1}, \ldots, x_n), \\ (y_jf)(x_1, \ldots, x_n) &= f(x_1, \ldots, x_{j-1}, q^{-1}x_j, x_{j+1}, \ldots, x_n). \end{aligned} \tag{1.1}$$

Define operators $T_1, \ldots, T_{n-1}$, T_π and $T_\pi^\vee$ on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by

$$T_i = -t^{-\frac12} + t^{-\frac12}(1+s_i)\frac{1-tx_i^{-1}x_{i+1}}{1-x_i^{-1}x_{i+1}}, \qquad T_\pi = s_1s_2\cdots s_{n-1}y_n, \qquad T_\pi^\vee = X_1T_1\cdots T_{n-1}. \tag{1.2}$$

The *Cherednik-Dunkl operators* are

$$Y_1 = T_\pi T_{n-1}\cdots T_1, \quad Y_2 = T_1^{-1}Y_1T_1^{-1}, \quad Y_3 = T_2^{-1}Y_2T_2^{-1}, \quad \ldots, \quad Y_n = T_{n-1}^{-1}Y_{n-1}T_{n-1}^{-1}. \tag{1.3}$$

Given $\mu \in \mathbb{Z}^n$ let $v_\mu \in S_n$ be the minimal length permutation which rearranges μ into weakly increasing order. Explicitly, the permutation v_μ is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1,\ldots,r-1\} \mid \mu_{r'} \le \mu_r\} + \#\{r' \in \{r+1,\ldots,n\} \mid \mu_{r'} < \mu_r\}. \tag{1.4}$$

By definition, the electronic Macdonald polynomials E_μ are the simultaneous eigenvectors for the action of the Cherednik-Dunkl operators,

$$Y_i E_\mu = q^{-\mu_i} t^{-v_\mu(i)} t^{\frac{1}{2}(n+1)} E_\mu. \tag{1.5}$$

The "evaluate at μ" homomorphism $\mathrm{ev}^t_\mu \colon \mathbb{C}[Y_1^{\pm1},\ldots,Y_n^{\pm1}] \to \mathbb{C}$ is given by

$$\mathrm{ev}^t_\mu(Y_i) = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} \tag{1.6}$$

(so that ev^t_μ specializes Y_i to the value $q^{-\mu_i}t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)}$). Extend ev^t_μ to those elements of the field $\mathbb{C}(Y_1,\ldots,Y_n)$ for which the specialized denominator does not vanish. By (1.5), if $f(Y) \in \mathbb{C}(Y_1,\ldots,Y_n)$ and $\mathrm{ev}^t_\mu(f)$ is defined then

$$f(Y)E_\mu = \mathrm{ev}^t_\mu(f(Y))E_\mu. \tag{1.7}$$

The *interwiners* are

$$\tau^\vee_\pi = T^\vee_\pi, \quad \text{and} \quad \tau^\vee_i = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_i^{-1}Y_{i+1}} \quad \text{for } i \in \{1,\ldots,n-1\}. \tag{1.8}$$

Using the definition of $\tau^\vee_i$ and the relation $T_i - T_i^{-1} = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$,

$$\tau^\vee_i = T_i + f^+_{i+1,i} = T_i^{-1} + f^-_{i+1,i}, \tag{1.9}$$

where

$$\text{where} \quad f^+_{ij} = \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_iY_j^{-1}} \quad \text{and} \quad f^-_{ij} = \frac{t^{-\frac{1}{2}}(1-t)Y_iY_j^{-1}}{1-Y_iY_j^{-1}}, \tag{1.10}$$

for $i,j \in \{1,\ldots,n\}$ with $i \ne j$. The following key relations are proved (for example) in [GR21, Prop. 5.5],

$$Y_1\tau^\vee_\pi = q^{-1}\tau^\vee_\pi Y_n \quad \text{and} \quad Y_i\tau^\vee_\pi = \tau^\vee_\pi Y_{i-1} \quad \text{for } i \in \{2,\ldots,n\}, \text{ and} \tag{1.11}$$

$$Y_i\tau^\vee_i = \tau^\vee_i Y_{i+1}, \qquad Y_{i+1}\tau^\vee_i = \tau^\vee_i Y_i, \qquad \text{and} \qquad Y_k\tau^\vee_i = \tau^\vee_i Y_k, \tag{1.12}$$

for $i \in \{1,\ldots,n-1\}$ and $k \in \{1,\ldots,n\}$ with $k \notin \{i,i+1\}$.

The following Proposition gives an explicit expression for E_μ as a sequence of intertwiners acting on the polynomial 1.

Proposition 1.1. [GR21, Proposition 5.7 and Proposition 2.2(a)]

(a) *Let $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n_{\ge 0}$ and write $(r,c) \in \mu$ if $r \in \{1,\ldots,n\}$ and $c \in \{1,\ldots,\mu_r\}$. For $(r,c) \in \mu$ define*

$$u_\mu(r,c) = \#\{r' \in \{1,\ldots,r-1\} \mid \mu_{r'} < c \le \mu_r\} + \#\{r' \in \{r+1,\ldots,n\} \mid \mu_{r'} < c-1 < \mu_r\}.$$

Then

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})}\Big(\prod_{r=1}^{n}\prod_{c=1}^{\mu_r}(\tau^\vee_{u_\mu(r,c)}\cdots\tau^\vee_2\tau^\vee_1\tau^\vee_\pi)\Big)\cdot 1.$$

where the product is taken in order defined by $(r_1,c_1) < (r_2,c_2)$ if $c_1 < c_2$, and $(r_1,c) < (r_2,c)$ if $r_1 < r_2$.

(b) If $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ has a negative entry then

$$E_\mu = (x_1 \cdots x_n)^{-m} E_{(\mu_1+m,\ldots,\mu_n+m)}, \qquad \text{where } -m \text{ is the most negative entry of } \mu.$$

Proposition 1.2. *For $i \in \{1, \ldots, n\}$ let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ with 1 in the ith entry and all other entries 0. Then*

$$E_{\varepsilon_i} = x_i + \frac{(1-t)}{(1-qt^{n-i+1})}(x_{i-1} + \cdots + x_1) \quad \text{and}$$
$$E_{-\varepsilon_i} = x_i^{-1} + \Big(\frac{1-t}{1-qt^i}\Big)(x_{i+1}^{-1} + \cdots + x_n^{-1}).$$

Proof. Since $v_{-\varepsilon_{i+1}} = s_1 \cdots s_i$ then

$$Y_i^{-1} Y_{i+1} E_{-\varepsilon_{i+1}} = q^0 t^{v_{-\varepsilon_{i+1}}(i)} q^1 t^{-v_{-\varepsilon_{i+1}}(i+1)} E_{-\varepsilon_{i+1}} = qt^{i+1} t^{-1} E_{-\varepsilon_{i+1}} = qt^i E_{-\varepsilon_{i+1}}$$

The base case is $E_{-\varepsilon_n} = x_n^{-1}$ and the induction step is

$$\begin{aligned}
E_{-\varepsilon_i} &= t^{\frac{1}{2}} \tau_i^\vee E_{-\varepsilon_{i+1}} = \Big(t^{\frac{1}{2}} T_i + \frac{(1-t)}{1-Y_i^{-1}Y_{i+1}}\Big) E_{-\varepsilon_{i+1}} = \Big(t^{\frac{1}{2}} T_i + \frac{(1-t)}{1-qt^i}\Big) E_{-\varepsilon_{i+1}} \\
&= \Big(t^{\frac{1}{2}} T_i + \frac{(1-t)}{1-qt^i}\Big)(x_{i+1}^{-1} + \Big(\frac{1-t}{1-qt^{i+1}}\Big)(x_{i+2}^{-1} + \cdots + x_n^{-1}) \\
&= x_i^{-1} + \Big(\frac{1-t}{1-qt^i}\Big) x_{i+1}^{-1} + \Big(\frac{1-t}{1-qt^{i+1}}\Big)\Big(t + \frac{1-t}{1-qt^i}\Big)(x_{i+2}^{-1} + \cdots + x_n^{-1}) \\
&= x_i^{-1} + \Big(\frac{1-t}{1-qt^i}\Big)(x_{i+1}^{-1} + \cdots + x_n^{-1}).
\end{aligned}$$

The proof of the first statement is similar (see [GR21, Prop. 3.5] for details). □

Remark 1.3. The source of the statistics $v_\mu(r)$ and $u_\mu(r, c)$. The minimal length permutation which rearranges μ into weakly increasing order is $v_\mu = (v_\mu(1), \ldots, v_\mu(n))$. The affine Weyl group for type GL_n is the group of n-periodic permutations. If t_μ denotes the n-periodic permutation which is the translation in μ then $t_\mu = u_\mu v_\mu$ with $\ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu)$ and u_μ has a reduced word

$$u_\mu = \prod_{(r,c)\in\mu} (s_{u_\mu(r,c)} \cdots s_2 s_1 \pi),$$

where $s_i \in S_n$ is the transposition which switches i and $i + 1$ and π is the n-periodic permutation given by $\pi(i) = i + 1$. See [GR21, §2 and Prop. 2.2(a)]. □

2 Operator expansions

Let $j \in \{1, \ldots, n\}$ and let $C \subseteq \{1, \ldots, n\}$. Writing $C = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_m$ define

$$f_C(Y) = \frac{t^{-(m-1)/2}}{1 - qY_{a_1} Y_{a_m}^{-1}} \Big(\prod_{i=1}^{m-1} \frac{1-t}{1 - Y_{a_i} Y_{a_{i+1}}^{-1}}\Big). \tag{2.1}$$

Then define

$$F_{C,j}(Y) = \begin{cases} 0, & \text{if } j \notin C, \\ 1 - qY_{a_1} Y_{a_m}^{-1}, & \text{if } j = a_p \text{ and } p = 1, \\ Y_{a_1} Y_{a_p}^{-1} - Y_{a_1} Y_{a_{p-1}}^{-1}, & \text{if } j = a_p \text{ and } p \neq 1, \end{cases} \tag{2.2}$$

$$A_{C,j}(Y) = \begin{cases} 0, & \text{if } j < a_1, \\ Y_{a_1}Y_{a_p}^{-1} - qY_{a_1}Y_{a_m}^{-1}, & \text{if } a_p \le j < a_{p+1}, \\ (1-q)Y_{a_1}Y_{a_m}^{-1}, & \text{if } j > a_m, \end{cases} \tag{2.3}$$

$$\Phi_{C,j}(Y) = \begin{cases} 0, & \text{if } j \notin C, \\ 1 - qY_{a_1}Y_{a_m}^{-1}, & \text{if } j = a_p \text{ and } p = m, \\ Y_{a_1}Y_{a_p}^{-1} - Y_{a_1}Y_{a_{p-1}}^{-1}, & \text{if } j = a_p \text{ and } p \ne m, \end{cases} \tag{2.4}$$

$$\Psi_{C,j}(Y) = \begin{cases} 0, & \text{if } j \ge a_m, \\ Y_{a_p}Y_{a_m}^{-1} - qY_{a_1}Y_{a_m}^{-1}, & \text{if } a_{p-1} < j \le a_p, \\ (1-q)Y_{a_1}Y_{a_m}^{-1}, & \text{if } j \le a_1, \end{cases} \tag{2.5}$$

and

$$\begin{aligned} B_{C,j}(Y) &= F_{C,j}(Y) + \frac{1-t}{1-qt^{n-j+1}} A_{C,j}(Y) \quad \text{and} \\ \Omega_{C,j}(Y) &= \Phi_{C,j}(Y) + \Big(\frac{1-t}{1-qt^j}\Big)\Psi_{C,j+1}(Y). \end{aligned} \tag{2.6}$$

Write the complement of C in $\{1,\dots n\}$ as

$$C^c = \{b_1, \dots, b_{n-m}\} \quad \text{with} \quad b_1 < \cdots < b_r < j < b_{r+1} < \cdots < b_{n-m},$$

and define

$$\begin{aligned} \tau_{C,j} &= \tau^\vee_{b_r}\tau^\vee_{b_{r-1}} \cdots \tau^\vee_{b_1}\tau^\vee_\pi \tau^\vee_{b_{n-m}-1} \cdots \tau^\vee_{b_{r+1}-1} \quad \text{and} \\ \rho_{C,j} &= \tau^\vee_{b_{r+1}-1} \cdots \tau^\vee_{b_{n-m}-1} (\tau^\vee_\pi)^{-1} \tau^\vee_{b_1} \cdots \tau^\vee_{b_r}, \end{aligned} \tag{2.7}$$

where the $\tau_i^\vee$ are as in (1.8).

Example 2.1. Examples of $f_C(Y)$, $F_{C,j}(Y)$ and $\tau^\vee_{C,j}$. Let $n = 11$ and $C = \{2,5,7,9,10\}$. Then

$$f_C(Y) = \frac{1}{t^{-\frac{1}{2}}(1-t)} f^+_{2,10+K} f^+_{25} f^+_{57} f^+_{79} f^+_{9,10},$$

where f^+_{ij} is as in (1.10). Then

$$\begin{aligned} &F_{C,2}(Y) = 1 - qY_2Y_{10}^{-1}, \qquad F_{C,5}(Y) = Y_2Y_5^{-1} - 1, \qquad F_{C,7}(Y) = Y_2Y_7^{-1} - Y_2Y_5^{-1}, \\ &F_{C,9}(Y) = Y_2Y_9^{-1} - Y_2Y_7^{-1} \qquad F_{C,10}(Y) = Y_2Y_{10}^{-1} - Y_2Y_9^{-1}, \end{aligned}$$

and

$$\tau^\vee_{C,7} = \tau^\vee_6\tau^\vee_4\tau^\vee_3\tau^\vee_1\tau^\vee_\pi\tau^\vee_{11-1}\tau^\vee_{8-1} = \tau^\vee_6\tau^\vee_4\tau^\vee_3\tau^\vee_1\tau^\vee_\pi\tau^\vee_{10}\tau^\vee_7, \qquad \text{since} \quad C^c = \{1,3,4,6,8,11\}.$$

□

Theorem 2.1. ***(Monk rules: operator form)*** *Let $j \in \{1,\dots,n\}$. As in (1.1), let X_j denote the operator on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ given by multiplication by x_i and let E_{ε_j} and $E_{-\varepsilon_j}$ be the Macdonald polynomials of Proposition 1.2, identified with the operators on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ given by multiplication by E_{ε_j} and $E_{-\varepsilon_j}$, respectively. Use the notations of (2.1)-(2.7). Then, as operators on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$,*

(a) $X_j = \displaystyle\sum_{\substack{C\subseteq\{1,\dots,n\} \\ C\cap\{j\}\ne\emptyset}} \tau_{C,j}F_{C,j}(Y)f_C(Y),$

(b) $X_1 + \cdots + X_j = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{1,\ldots,j\}\neq\emptyset}} \tau_{C,j} A_{C,j}(Y) f_C(Y),$

(c) $E_{\varepsilon_j} = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{1,\ldots,j\}\neq\emptyset}} \tau_{C,j} B_{C,j}(Y) f_C(Y).$

(d) $X_j^{-1} = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ D\cap\{j\}\neq\emptyset}} \rho_{C,j} \Phi_{C,j}(Y) f_C(Y),$

(e) $X_j^{-1} + \cdots + X_n^{-1} = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{j,\ldots,n\}\neq\emptyset}} \rho_{C,j} \Psi_{C,j}(Y) f_C(Y)$

(f) $E_{-\varepsilon_j} = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ C\cap\{j,\ldots,n\}\neq\emptyset}} \rho_{C,j} \Omega_{C,j}(Y) f_C(Y).$

Proof. Since the proof of (a) is longer, let us first make remarks about the proofs of (b)-(f).
(b) This follows from (a) and the observation that $A_{C,j} = F_{C,1} + \cdots + F_{C,j}$.
(c) By the first identity in Proposition 1.2,

$$E_{\varepsilon_j} = x_j + \frac{1-t}{1-qt^{n-j+1}}(x_{j-1} + \cdots + x_1), \qquad \text{so that} \quad B_{C,j} = F_{C,j} + \frac{1-t}{1-qt^{n-j+1}} A_{C,j-1}.$$

(d) The proof is analogous to the proof of (a) by expanding

$$\begin{aligned} X_j^{-1} &= T_j \cdots T_{n-1} (\tau_\pi^\vee)^{-1} T_1^{-1} \cdots T_{j-1}^{-1} \\ &= (\tau_j^\vee - f_{j+1,j}^+) \cdots (\tau_{n-1}^\vee - f_{n,n-1}^+)(\tau_\pi^\vee)^{-1}(\tau_1^\vee - f_{2,1}^-) \cdots (\tau_{j-1}^\vee - f_{j,j-1}^-) \\ &= \sum_{\substack{L\subseteq\{j,\ldots,n-1\}\\ R\subseteq\{1,\ldots,j-1\}}} (-1)^{|L|+|R|} \varpi_L \tau_\pi^\vee \varpi_R, \end{aligned}$$

$$\varpi_L = \varpi_j \cdots \varpi_{n-1} \quad \text{with} \quad \varpi_i = \begin{cases} \tau_i^\vee, & \text{if } i \in \{j,\ldots,n-1\} \text{ and } i \notin L, \\ f_{i+1,i}^+, & \text{if } i \in \{j,\ldots,n-1\} \text{ and } i \in L, \end{cases} \qquad \text{and}$$

$$\varpi_R = \varpi_1 \cdots \varpi_{j-1} \quad \text{with} \quad \varpi_i = \begin{cases} \tau_i^\vee, & \text{if } i \in \{1,\ldots,j-1\} \text{ and } i \notin R, \\ f_{i+1,i}^-, & \text{if } i \in \{1,\ldots,j-1\} \text{ and } i \in R, \end{cases}$$

An example is provided in Example 2.3.
(e) This follows from (d) and the observation that $\Psi_{D,j} = \Phi_{D,j} + \cdots + \Phi_{D,n}$.
(f) By the second identity in Proposition 1.2

$$E_{-\varepsilon_j} = x_j^{-1} + \left(\frac{1-t}{1-qt^j}\right)(x_{j+1}^{-1} + \cdots + x_n^{-1}), \qquad \text{so that} \quad \Omega_{C,j} = \Phi_{D,j} + \left(\frac{1-t}{1-qt^j}\right)\Psi_{D,j+1}.$$

(a) As in (1.1), let X_j denote the operator on $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm 1}]$ given by mutliplication by x_j. The operator X_j can be written in terms of $\tau_\pi^\vee$ and $T_1, \ldots, T_{n-1}$ in the form

$$X_j = T_{j-1} \cdots T_1 \tau_\pi^\vee T_{n-1}^{-1} \cdots T_j^{-1}$$

(see, for example, [GR21, (5.3) and §5.6 and §5.7]). Then using (1.9) gives

$$\begin{aligned} X_j &= T_{j-1}\cdots T_1\tau_\pi^\vee T_{n-1}^{-1}\cdots T_j^{-1} \\ &= (\tau_{j-1}^\vee - f_{j,j-1}^+)\cdots(\tau_1^\vee - f_{2,1}^+)\tau_\pi^\vee(\tau_{n-1}^\vee - f_{n,n-1}^-)\cdots(\tau_j^\vee - f_{j+1,j}^-) \\ &= \sum_{\substack{L\subseteq\{j-1,\ldots,1\}\\ R\subseteq\{n-1,\ldots,j\}}} (-1)^{|L|+|R|}\omega_L\tau_\pi^\vee\omega_R, \end{aligned} \tag{2.8}$$

where

$$\omega_L = \omega_{j-1}\cdots\omega_1 \quad\text{with}\quad \omega_i = \begin{cases} \tau_i^\vee, & \text{if } i\in\{j-1,\ldots,1\} \text{ and } i\notin L,\\ f_{i+1,i}^+, & \text{if } i\in\{j-1,\ldots,1\} \text{ and } i\in L,\end{cases} \qquad \text{and}$$

$$\omega_R = \omega_{n-1}\cdots\omega_j \quad\text{with}\quad \omega_i = \begin{cases} \tau_i^\vee, & \text{if } i\in\{n-1,\ldots,j\} \text{ and } i\notin R,\\ f_{i+1,i}^-, & \text{if } i\in\{n-1,\ldots,j\} \text{ and } i\in R.\end{cases}$$

Write

$$\begin{matrix} L=\{\ell_1,\ldots,\ell_a\}, \\ R=\{r_1,\ldots,r_b\}, \end{matrix} \qquad \text{with} \qquad n>r_1>\ldots>r_b\ge j>\ell_1>\ldots>\ell_a>0,$$

and use the relations (1.11) and (1.12) to move all $\tau_i^\vee$ in $\omega_L\tau_\pi^\vee\omega_R$ to the left so that

$$(-1)^{|L|+|R|}\omega_L\tau_\pi^\vee\omega_R = \tau_{L^c,R^c}^\vee f_{L,R}, \tag{2.9}$$

where

$$\begin{aligned} L^c &= \{k_1,\ldots,k_{j-1-a}\} \text{ with } k_1>\cdots>k_{j-1-a} \text{ is the complement of } L \text{ in } \{j-1,\ldots,1\}, \\ R^c &= \{q_1,\ldots,q_{n-j-b}\} \text{ with } q_1>\cdots>q_{n-j-b} \text{ is the complement of } R \text{ in } \{n-1,\ldots,j\}, \end{aligned}$$

$$\tau_{L^r,R^c}^\vee = \tau_{k_1}^\vee\cdots\tau_{k_{j-1-a}}^\vee\tau_\pi^\vee\tau_{q_1}^\vee\cdots\tau_{q_{n-j-b}}^\vee,$$

and

$$f_{L,R} = (-1)^{|L|+|R|} f_{\ell_1,\ell_2}^+ f_{\ell_2,\ell_3}^+\cdots f_{\ell_{a-1},\ell_a}^+ f_{\ell_a,r_1+1+K}^+ f_{r_1+1,r_2+1}^- f_{r_2+1,r_3+1}^-\cdots f_{r_{b-1}+1,r_b+1}^- f_{r_b+1,j}^-,$$

where

$$f_{i,j+K}^+ = \frac{t^{-\frac12}(1-t)}{1-qY_iY_j^{-1}}$$

(the K in this expression is a formal notational symbol and has no other meaning in this context). An example of this process of using the relations (1.11) and (1.12) to move all the $\tau_i^\vee$ in $\omega_L\tau_\pi^\vee\omega_R$ to the left is given in Example 2.2.

Let $C=\{a_1,\ldots,a_m\}=\{\ell_a,\ldots,\ell_1,j,r_b+1,\ldots,r_1+1\}$ and let $C^c=\{b_1,\ldots,b_{n-m}\}$ be the complement of C in $\{1,\ldots,n\}$ so that

$$\begin{aligned} C&=\{a_1,\ldots,a_m\} \qquad \text{with} \qquad 1\le a_1<\cdots<a_m\le n \qquad \text{and} \qquad j\in C, \\ \text{and}\quad C^c&=\{b_1,\ldots,b_{n-m}\} \quad \text{with} \quad b_1<\cdots<b_r<j<b_{r+1}\cdots<b_{n-m}. \end{aligned}$$

Then

$$\tau_{L^c,R^c}^\vee = \tau_{b_r}^\vee\tau_{b_{r-1}}^\vee\cdots\tau_{b_1}^\vee\tau_\pi^\vee\tau_{b_{n-m}-1}^\vee\cdots\tau_{b_{r+1}-1}^\vee = \tau_{C,j}^\vee, \tag{2.10}$$

and letting p be such that $j=a_p$ and rearranging the factors in $f_{L,R}$ gives

$$\begin{aligned} f_{L,R} &= (-1)^{|L|+|R|} f_{\ell_a,r_1+1+K}^+ f_{\ell_{a-1},\ell_a}^+\cdots f_{\ell_2,\ell_3}^+ f_{\ell_1,\ell_2}^+ f_{r_b+1,j}^- f_{r_{b-1}+1,r_b+1}^-\cdots f_{r_2+1,r_3+1}^- f_{r_1+1,r_2+1}^- \\ &= (-1)^{|L|+|R|} f_{a_1,a_m+K}^+ f_{a_2a_1}^+\cdots f_{a_{p-2}a_{p-3}}^+ f_{a_{p-1}a_{p-2}}^+ f_{a_{p+1}a_p}^- f_{a_{p+2}a_{p+1}}^-\cdots f_{a_{m-1}a_{m-2}}^- f_{a_ma_{m-1}}^- \\ &= (-1)^{|L|+|R|} f_{a_1,a_m+K}^+ f_{a_2a_1}^+\cdots f_{a_{p-2}a_{p-3}}^+ f_{a_{p-1}a_{p-2}}^+ f_{a_{p+1}a_p}^- f_{a_{p+2}a_{p+1}}^-\cdots f_{a_{m-1}a_{m-2}}^- f_{a_ma_{m-1}}^-. \end{aligned}$$

Now use

$$f_{ki}^+ = \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_kY_i^{-1}} = -\frac{t^{-\frac{1}{2}}(1-t)Y_iY_k^{-1}}{1-Y_iY_k^{-1}} = -Y_iY_k^{-1}f_{ik}^+ \qquad \text{and}$$

$$f_{ki}^- = \frac{t^{-\frac{1}{2}}(1-t)Y_kY_i^{-1}}{1-Y_kY_i^{-1}} = -\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_iY_k^{-1}} = -f_{ik}^+.$$

If $p \neq 1$ then

$$\begin{aligned}
f_{L,R} &= (-1)^{|L|+|R|} f^+_{a_1,a_m+K} f^+_{a_2a_1} \cdots f^+_{a_{p-2}a_{p-3}} f^+_{a_{p-1}a_{p-2}} f^-_{a_{p+1}a_p} f^-_{a_{p+2}a_{p+1}} \cdots f^-_{a_{m-1}a_{m-2}} f^-_{a_ma_{m-1}} \\
&= (-1)^{|L|+|R|}(-1)^{|C|-2} f^+_{a_1,a_m+K}(Y_{a_1}Y_{a_2}^{-1}f^+_{a_1a_2}) \cdots (Y_{a_{p-3}}Y_{a_{p-2}}^{-1}f^+_{a_{p-3}a_{p-2}})(Y_{a_{p-2}}Y_{a_{p-1}}^{-1}f^+_{a_{p-2}a_{p-1}}) \\
&\qquad \cdot f^+_{a_pa_{p+1}} f^+_{a_{p+1}a_{p+2}} \cdots f^+_{a_{m-1}a_m} \\
&= (-1)\frac{f^+_{a_1,a_m+K}}{f^+_{a_{p-1}a_p}} Y_{a_1}Y_{a_{p-1}}^{-1} \Big(\prod_{i=1}^{m-1} f^+_{a_ia_{i+1}} \Big) \\
&= (-1)Y_{a_1}Y_{a_{p-1}}^{-1}(1-Y_{a_{p-1}}Y_{a_p}^{-1}) \frac{1}{t^{-\frac{1}{2}}(1-t)} f^+_{a_1,a_m+K} \Big(\prod_{i=1}^{m-1} f^+_{a_ia_{i+1}} \Big) \\
&= (Y_{a_1}Y_{a_p}^{-1} - Y_{a_1}Y_{a_{p-1}}^{-1}) \frac{t^{-\frac{m-1}{2}}(1-t)^{m-1}}{1-qY_{a_1}Y_{a_m}^{-1}} \Big(\prod_{i=1}^{m-1} \frac{1}{1-Y_{a_i}Y_{a_{i+1}}^{-1}} \Big) = F_{C,j}(Y) f_C(Y),
\end{aligned}$$

and if $p = 1$ then

$$\begin{aligned}
f_{L,R} &= (-1)^{|L|+|R|} f^-_{r_1+1,r_2+1} f^-_{r_2+1,r_3+1} \cdots f^-_{r_{b-1}+1,r_b+1} f^-_{r_b+1,j} \\
&= (-1)^{|L|+|R|} f^-_{a_m,a_{m-1}} f^-_{a_{m-1}a_{m-2}} \cdots f^-_{a_3a_2} f^-_{a_2a_1} \\
&= f^+_{a_{m-1}a_m} f^+_{a_{m-2}a_{m-1}} \cdots f^+_{a_2a_3} f^+_{a_1a_2} = \frac{1}{f^+_{a_1,a_m+K}} f^+_{a_1,a_m+K} \Big(\prod_{i=1}^{m-1} f^+_{a_ia_{i+1}} \Big) \\
&= (1-qY_{a_1}Y_{a_m}^{-1}) \frac{1}{t^{-\frac{1}{2}}(1-t)} f^+_{a_1,a_m+K} \Big(\prod_{i=1}^{m-1} f^+_{a_ia_{i+1}} \Big) \\
&= (1-qY_{a_1}Y_{a_m}^{-1}) \frac{t^{-\frac{m-1}{2}}(1-t)^{m-1}}{1-qY_{a_1}Y_{a_m}^{-1}} \Big(\prod_{i=1}^{m-1} \frac{1}{1-Y_{a_i}Y_{a_{i+1}}^{-1}} \Big) = F_{C,j}(Y) f_C(Y).
\end{aligned}$$

Inserting these expressions for $f_{L,R}$ and the expression for $\tau^\vee_{L^c,R^c}$ in (2.10) into (2.9) and (2.8) gives the formula in the statement. □

Remark 2.2. The $B_{C,j}$ defined in (2.6) are given by

$$B_{C,j} = \begin{cases}
0, & \text{if } j < a_1, \\
1-qY_{a_1}Y_{a_m}^{-1}, & \text{if } j = a_1, \\
\dfrac{1-t}{1-qt^{n-j+1}}(1-qY_{a_1}Y_{a_m}^{-1}), & \text{if } a_1 < j < a_2, \\
-q\dfrac{1-t}{1-qt^{n-j+1}}Y_{a_1}Y_{a_m}^{-1} + Y_{a_1}Y_{a_p}^{-1} - t\dfrac{1-qt^{n-j}}{1-qt^{n-j+1}}Y_{a_1}Y_{a_{p-1}}^{-1}, & \text{if } j = a_p \text{ and } p \neq 1, \\
-q\dfrac{1-t}{1-qt^{n-j+1}}Y_{a_1}Y_{a_m}^{-1} + \dfrac{1-t}{1-qt^{n-j+1}}Y_{a_1}Y_{a_{p-1}}^{-1}, & \text{if } a_{p-1} < j < a_p \text{ with } 2 < p, \\
\dfrac{(1-t)(1-q)}{1-qt^{n-j+1}}Y_{a_1}Y_{a_m}^{-1}, & \text{if } j > a_m.
\end{cases}$$

Similar expressions can be given for the $\Omega_{C,J}$. □

Example 2.2. A term in X_j for $n=11$ and $j=7$. This is an example of the computation for the proof of part (a) of Theorem 2.1. In the expansion of

$$\begin{aligned} X_7 &= T_6T_5T_4T_3T_2T_1\tau_\pi^\vee T_{10}^{-1}T_9^{-1}T_8^{-1}T_7^{-1} \\ &= (\tau_6^\vee - f_{76}^+)\cdots(\tau_1^\vee - f_{21}^+)\tau_\pi^\vee(\tau_{10}^\vee - f_{11,10}^-)\cdots(\tau_7^\vee - f_{87}^-), \end{aligned}$$

the term coming from choosing the $-f_{i,i-1}^\pm$ from the 2nd, 5th, 8th and 9th factors is

$$\begin{aligned} &\tau_6^\vee(-f_{6,5}^+)\tau_4^\vee\tau_3^\vee(-f_{3,2}^+)\tau_1^\vee\tau_\pi^\vee\tau_{10}^\vee(-f_{10,9}^-)(-f_{9,8}^-)\tau_7^\vee \\ &\qquad = (-1)^4\tau_6^\vee\tau_4^\vee\tau_3^\vee\tau_1^\vee f_{6,3}^+f_{3,1}^+\tau_\pi^\vee\tau_{10}^\vee\tau_7^\vee f_{10,9}^-f_{9,7}^- \\ &\qquad = (-1)^4\tau_6^\vee\tau_4^\vee\tau_3^\vee\tau_1^\vee\tau_\pi^\vee f_{5,2}^+f_{2,11+K}^+\tau_{10}^\vee\tau_7^\vee f_{10,9}^-f_{9,7}^- \\ &\qquad = (-1)^4\tau_6^\vee\tau_4^\vee\tau_3^\vee\tau_1^\vee\tau_\pi^\vee\tau_{10}^\vee\tau_7^\vee f_{5,2}^+f_{2,10+K}^+f_{10,9}^-f_{9,7}^- \\ &\qquad = (-1)^4\tau_{C,7}^\vee f_{5,2}^+f_{2,10+K}^+f_{10,9}^-f_{9,7}^-, \end{aligned}$$

where $C=\{2,5,7,9,10\}$ and $C^c=\{1,3,4,6,8,11\}$ so that

$$\tau_{C,7}^\vee = \tau_6^\vee\tau_4^\vee\tau_3^\vee\tau_1^\vee\tau_\pi^\vee\tau_{11-1}^\vee\tau_{8-1}^\vee = \tau_6^\vee\tau_4^\vee\tau_3^\vee\tau_1^\vee\tau_\pi^\vee\tau_{10}^\vee\tau_7^\vee.$$

Using

$$f_{52}^- = \frac{t^{-\frac{1}{2}}(1-t)Y_5Y_2^{-1}}{1-Y_5Y_2^{-1}} = -\frac{t^{-\frac{1}{2}}(1-t)}{1-Y_2Y_5^{-1}} = -f_{25}^+$$

and

$$f_{52}^+ = \frac{t^{-\frac{1}{2}}(1-t)}{1-Y_5Y_2^{-1}} = \frac{t^{-\frac{1}{2}}Y_2Y_5^{-1}(1-t)}{Y_2Y_5^{-1}-1} = -\frac{t^{-\frac{1}{2}}Y_2Y_5^{-1}(1-t)}{1-Y_2Y_5^{-1}} = -Y_2Y_5^{-1}f_{25}^+$$

gives

$$\begin{aligned} (-1)^4 f_{5,2}^+f_{2,10+K}^+f_{10,9}^-f_{9,7}^- &= (-1)Y_2Y_5^{-1}f_{25}^+f_{2,10+K}^+f_{9,10}^+f_{79}^+ = (-1)Y_2Y_5^{-1}\frac{1}{f_{57}^+}(f_{2,10+K}^+f_{25}^+f_{57}^+f_{79}^+f_{9,10}^+) \\ &= (-1)Y_2Y_5^{-1}\frac{t^{-\frac{1}{2}}(1-t)}{f_{57}^+}f_C = (Y_2Y_7^{-1} - Y_2Y_5^{-1})f_C(Y) = F_{C,7}(Y)f_C(Y). \end{aligned}$$

□

Example 2.3. An example of a term in X_j^{-1} for $n=11$ and $j=7$. This is an example of the computation for the proof of part (d) of Theorem 2.1. In the expansion of

$$\begin{aligned} X_7^{-1} &= T_7T_8T_9T_{10}(\tau_\pi^\vee)^{-1}T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1}T_5^{-1}T_6^{-1} \\ &= (\tau_7^\vee - f_{87}^+)\cdots(\tau_{10}^\vee - f_{11,10}^+)(\tau_\pi^\vee)^{-1}(\tau_1^\vee - f_{21}^-)\cdots(\tau_6^\vee - f_{76}^-) \end{aligned}$$

the term coming from choosing the $-f_{i,i-1}^\pm$ from the 2nd, 5th, 8th and 9th factors is

$$\begin{aligned} &\tau_7^\vee(-f_{9,8}^+)\tau_9^\vee\tau_{10}^\vee(\tau_\pi^\vee)^{-1}(-f_{2,1}^-)\tau_2^\vee\tau_3^\vee(-f_{5,4}^-)(-f_{6,5}^-)\tau_6^\vee \\ &\qquad = (-1)^4\tau_7^\vee\tau_9^\vee\tau_{10}^\vee f_{11,8}^+(\tau_\pi^\vee)^{-1}\tau_2^\vee\tau_3^\vee\tau_6^\vee f_{4,1}^-f_{5,4}^-f_{7,5}^- \\ &\qquad = (-1)^4\tau_7^\vee\tau_9^\vee\tau_{10}^\vee(\tau_\pi^\vee)^{-1}f_{1-K,9}^+\tau_2^\vee\tau_3^\vee\tau_6^\vee f_{4,1}^-f_{5,4}^-f_{7,5}^- \\ &\qquad = (-1)^4\tau_7^\vee\tau_9^\vee\tau_{10}^\vee(\tau_\pi^\vee)^{-1}\tau_2^\vee\tau_3^\vee\tau_6^\vee f_{1-K,9}^+f_{4,1}^-f_{5,4}^-f_{7,5}^- \\ &\qquad = (-1)^4\rho_{D,7}f_{1-K,9}^+f_{7,5}^-f_{5,4}^-f_{4,1}^- \end{aligned}$$

where $D = \{1,4,5,7,9\}$ and $D^c = \{2,3,6,8,10,11\}$ so that

$$\rho_{D,7} = \tau_7^\vee \tau_9^\vee \tau_{10}^\vee (\tau_\pi^\vee)^{-1} \tau_2^\vee \tau_3^\vee \tau_6^\vee = \tau_{8-1}^\vee \tau_{10-1}^\vee \tau_{11-1}^\vee (\tau_\pi^\vee)^{-1} \tau_2^\vee \tau_3^\vee \tau_6^\vee$$

Using $f_{ij}^- = -f_{ji}^+$ and $f_{ji}^+ = -Y_i Y_j^{-1} f_{ij}^+$ gives

$$\begin{aligned}(-1)^4 f^+_{1-K,9} f^-_{7,5} f^-_{5,4} f^-_{4,1} &= (-1) f^+_{1-K,9} f^+_{5,7} f^+_{4,5} f^+_{1,4} = (-1)\frac{1}{f^+_{7,9}} f^+_{1-K,9} f^+_{1,4} f^+_{4,5} f^+_{5,7} f^+_{7,9} \\ &= (Y_7 Y_9^{-1} - 1)\frac{1}{t^{-\frac{1}{2}}(1-t)} f^+_{1-K,9} f^+_{1,4} f^+_{4,5} f^+_{5,7} f^+_{7,9} = \Phi_{D,7}(Y) f_D(Y).\end{aligned}$$

□

3 Monk rules for Macdonald polynomials

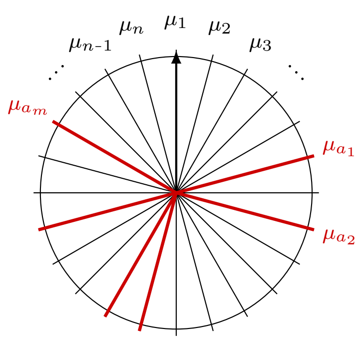
Let $k^\uparrow \colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the function which increments the kth coordinate by 1, and let $k^\downarrow \colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the function which decreases the kth coordinate by 1, so that if $\mu = (\mu_1, \ldots, \mu_n)$ then

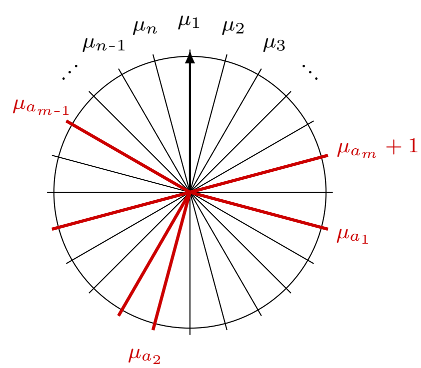
$$\begin{aligned} k^\uparrow \mu &= k^\uparrow(\mu_1,\ldots,\mu_n) = (\mu_1,\ldots,\mu_{k-1},\mu_k+1,\mu_{k+1},\ldots,\mu_n) \quad \text{and} \\ k^\downarrow \mu &= k^\downarrow(\mu_1,\ldots,\mu_n) = (\mu_1,\ldots,\mu_{k-1},\mu_k-1,\mu_{k+1},\ldots,\mu_n). \end{aligned}$$

Let $j \in \{1,\ldots,n\}$ and let $C \subseteq \{1,\ldots,n\}$. Write $C = \{a_1,\ldots,a_m\}$ with $a_1 < a_2 < \cdots < a_m$. For $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n$ define

$$\mathrm{rot}_C(\mu) = \gamma_C a_m^\uparrow \mu, \qquad \text{where, in cycle notation, } \gamma_C = (a_1,\ldots,a_m) \in S_n. \tag{3.1}$$

Thus, $\mathrm{rot}_C(\mu)$ is the same as μ except that in $\mathrm{rot}_C(\mu)$ the parts of μ indexed by the elements of C have been rotated and 1 has been added to μ_{a_m}.

$\mu =$ 

$\mathrm{rot}_C(\mu) =$  (3.2)

Let rrot_C be the inverse operation to rot_C so that $\mathrm{rrot}_C(\mathrm{rot_C}(\mu)) = \mu$ and $\mathrm{rrot}_C(\mu)$ is the same as μ except that in $\mathrm{rrot}_C(\mu)$ the parts of μ indexed by the elements of C have been rotated counterclockwise and 1 has been *subtracted* from μ_{a_1}.

For $k \in \{1,\ldots,n\}$ such that $k \notin C$ define

$$b(k) = \begin{cases} \mu_{a_m}+1, & \text{if } 1 \le k < a_1, \\ \mu_{a_i}, & \text{if } a_i < k < a_{i+1}, \\ \mu_{a_m}, & \text{if } a_m < k \le n, \end{cases} \quad \text{and} \quad c(k) = \begin{cases} v_{a_m^\uparrow \mu}(a_m), & \text{if } 1 \le k < a_1, \\ v_\mu(a_i), & \text{if } a_i < k < a_{i+1}, \\ v_\mu(a_m), & \text{if } a_m < k \le n, \end{cases} \tag{3.3}$$

and

$$d(k) = \begin{cases} \mu_{a_1} - 1, & \text{if } a_m < k \leq n, \\ \mu_{a_i}, & \text{if } a_{i-1} < k < a_i, \\ \mu_{a_1}, & \text{if } 1 \leq k < a_1, \end{cases} \quad \text{and} \quad e(k) = \begin{cases} v_{a_1^{\downarrow}\mu}(a_1), & \text{if } a_m < k \leq n, \\ v_\mu(a_i), & \text{if } a_{i-1} < k < a_i, \\ v_\mu(a_1), & \text{if } 1 \leq k < a_1, \end{cases}$$

Keeping $k \in \{1, \ldots, n\}$ such that $k \notin C$ define

$$\mathrm{wt}_\mu(C,k) = \begin{cases} 0, & \text{if } b(k) = \mu_k, \\ 1, & \text{if } b(k) > \mu_k, \\ t\dfrac{(1-q^{\mu_k - b(k)}t^{v_\mu(k)-c(k)+1})(1-q^{\mu_k-b(k)}t^{v_\mu(k)-c(k)-1})}{(1-q^{\mu_k-b(k)}t^{v_\mu(k)-c(k)})^2}, & \text{if } b(k) < \mu_k, \end{cases}$$

and

$$\mathrm{rwt}_\mu(C,k) = \begin{cases} 0, & \text{if } d(k) = \mu_k, \\ t^{-1}, & \text{if } d(k) > \mu_k, \\ \dfrac{(1-q^{\mu_k - d(k)}t^{v_\mu(k)-e(k)+1})(1-q^{\mu_k-d(k)}t^{v_\mu(k)-e(k)-1})}{(1-q^{\mu_k-d(k)}t^{v_\mu(k)-e(k)})^2}, & \text{if } d(k) < \mu_k. \end{cases}$$

For $k \in \{1, \ldots, n\}$ such that $k \in C$ define

$$\mathrm{wt}_\mu(C,k) = \mathrm{rwt}_\mu(C,k) = \begin{cases} \dfrac{1-t}{1-q^{\mu_{a_{i+1}}-\mu_{a_i}}t^{v_\mu(a_{i+1})-v_\mu(a_i)}}, & \text{if } k = a_i \text{ and } i \neq m, \\ \dfrac{1}{1-q^{\mu_{a_m}-\mu_{a_1}+1}t^{v_\mu(a_m)-v_\mu(a_1)}}, & \text{if } k = a_m. \end{cases} \tag{3.4}$$

Then define

$$\mathrm{wt}_\mu(C) = t^{-\#\{i \mid \mu_i > \mu_{a_m}\}} \prod_{i=1}^{n} \mathrm{wt}_\mu(C,k), \tag{3.5}$$

and

$$\mathrm{rwt}_\mu(C) = t^{\#\{i \mid \mu_i > \mu_{a_1}\}} \prod_{i=1}^{n} \mathrm{rwt}_\mu(C,k), \tag{3.6}$$

Theorem 3.1. ***(Monk rules for Macdonald polynomials)*** *Let $j \in \{1, \ldots, n\}$ and $\mu \in \mathbb{Z}^n_{\geq 0}$. Let E_μ denote the electronic Macdonald polynomial indexed by μ in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let*

$$F_\mu(C,j) = \begin{cases} 0, & \text{if } j \notin C. \\ 1 - q^{\mu_{a_m}-\mu_{a_1}+1}t^{v_\mu(a_m)-v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p = 1, \\ q^{\mu_{a_p}-\mu_{a_1}}t^{v_\mu(a_p)-v_\mu(a_1)} - q^{\mu_{a_{p-1}}-\mu_{a_1}}t^{v_\mu(a_{p-1})-v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p \neq 1, \end{cases}$$

$$A_\mu(C,j) = \begin{cases} 0, & \text{if } j < a_1, \\ q^{\mu_{a_p}-\mu_{a_1}}t^{v_\mu(a_p)-v_\mu(a_1)} - q^{\mu_{a_m}-\mu_{a_1}+1}t^{v_\mu(a_m)-v_\mu(a_1)}, & \text{if } a_p \leq j < a_{p+1}, \\ (1-q)q^{\mu_{a_m}-\mu_{a_1}}t^{v_\mu(a_m)-v_\mu(a_1)}, & \text{if } j > a_m, \end{cases}$$

and

$$B_\mu(C,j) = F_\mu(C,j) + \frac{1-t}{1-qt^{n-j+1}}A_\mu(C,j). \tag{3.7}$$

Let

$$\Phi_\mu(C,j) = \begin{cases} 0, & \text{if } j \notin C. \\ 1 - q^{\mu_{a_m}-\mu_{a_1}+1} t^{v_\mu(a_m)-v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p = m, \\ q^{\mu_{a_p}-\mu_{a_1}} t^{v_\mu(a_p)-v_\mu(a_1)} - q^{\mu_{a_{p-1}}-\mu_{a_1}} t^{v_\mu(a_{p-1})-v_\mu(a_1)}, & \text{if } j = a_p \text{ and } p \neq m, \end{cases}$$

$$\Psi_\mu(C,j) = \begin{cases} 0, & \text{if } j < a_1, \\ q^{\mu_{a_m}-\mu_{a_p}} t^{v_\mu(a_m)-v_\mu(a_p)} - q^{\mu_{a_m}-\mu_{a_1}+1} t^{v_\mu(a_m)-v_\mu(a_1)}, & \text{if } a_{p-1} < j \leq a_p, \\ (1-q) q^{\mu_{a_m}-\mu_{a_1}} t^{v_\mu(a_m)-v_\mu(a_1)}, & \text{if } j \leq a_1, \end{cases}$$

and

$$\Omega_\mu(C,j) = \Phi_\mu(C,j) + \frac{1-t}{1-qt^j}\Psi_\mu(C,j). \tag{3.8}$$

Let $\mathrm{rot}_\mu(C)$ *and* $\mathrm{wt}_\mu(C)$ *as in* (3.1) *and* (3.5), *and let* $\mathrm{rrot}_\mu(C)$ *and* $\mathrm{rwt}_\mu(C)$ *be as defined in* (3.1) *and* (3.6). *Then*

(a)

$$x_j E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\} \\ C\cap\{j\}\neq\emptyset}} F_\mu(C,j)\mathrm{wt}_\mu(C) E_{\mathrm{rot}_C(\mu)},$$

(b)

$$(x_1 + \cdots + x_j)E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\} \\ C\cap\{1,\ldots,j\}\neq\emptyset}} A_\mu(C,j)\mathrm{wt}_\mu(C) E_{\mathrm{rot}_C(\mu)},$$

(c)

$$E_{\varepsilon_j} E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\} \\ C\cap\{1,\ldots,j\}\neq\emptyset}} B_\mu(C,j)\mathrm{wt}_\mu(C) E_{\mathrm{rot}_C(\mu)},$$

(d)

$$x_j^{-1} E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\} \\ C\cap\{j\}\neq\emptyset}} \Phi_\mu(C,j)\mathrm{rwt}_\mu(C) E_{\mathrm{rrot}_C(\mu)},$$

(e)

$$(x_j^{-1} + \cdots + x_n^{-1})E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\} \\ C\cap\{j,\ldots,n\}\neq\emptyset}} \Psi_\mu(C,j)\mathrm{rwt}_\mu(C) E_{\mathrm{rrot}_C(\mu)},$$

(f)

$$E_{-\varepsilon_j} E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\} \\ C\cap\{j,\ldots,n\}\neq\emptyset}} \Omega_\mu(C,j)\mathrm{rwt}_\mu(C) E_{\mathrm{rrot}_C(\mu)},$$

Proof. From [GR21, (4.1) and (4.2)], if $\mu_i > \mu_{i+1}$ then

$$\begin{aligned} &t^{\frac12}\tau_i^\vee E_\mu = E_{s_i\mu} \quad \text{and} \\ &t^{\frac12}\tau_i^\vee E_{s_i\mu} = t\frac{(1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)+1})(1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)-1})}{(1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)})^2} E_\mu. \end{aligned} \tag{3.9}$$

From [GR21, (3.5)],

$$\tau_\pi^\vee E_\mu = t^{\frac12(n-1)-\#\{i\in\{1,\ldots,n-1\}\ |\ \mu_i\leq\mu_n\}} E_{\pi\mu}. \tag{3.10}$$

If the complement of C in $\{1,\ldots n\}$ is

$$C^c = \{b_1,\ldots,b_{n-m}\} \quad \text{with} \quad b_1 < \cdots < b_r < j < b_{r+1} < \cdots < b_{n-m}$$

then $\tau^\vee_{C,j} = \tau^\vee_{b_r}\tau^\vee_{b_{r-1}}\cdots\tau^\vee_{b_1}\tau^\vee_\pi\tau^\vee_{b_{n-1}-1}\cdots\tau^\vee_{b_{r+1}-1}$ and using (3.9) and (3.10) gives

$$\begin{aligned} t^{\frac12(n-m)}\tau^\vee_{C,j}E_\mu &= t^{\frac12}\tau^\vee_{b_r}t^{\frac12}\tau^\vee_{b_{r-1}}\cdots t^{\frac12}\tau^\vee_{b_1}\tau^\vee_\pi t^{\frac12}\tau^\vee_{b_{n-m}-1}\cdots t^{\frac12}\tau^\vee_{b_{r+1}-1}E_\mu \\ &= t^{\frac12(n-1)-\#\{\mu_i>\mu_{a_m}\}}\Big(\prod_{k\notin C}\mathrm{wt}_\mu(C,k)\Big)E_{\mathrm{rot}_C(\mu)}. \end{aligned} \tag{3.11}$$

An example of the step-by-step computation of $\tau^\vee_{C,j}E_\mu$ is given in Example 3.1.

Let $f_C(Y)$ and $F_{C,j}(Y)$ be as defined in (2.1) and (2.2), and let ev^t_μ be the evaluation map defined in (1.6). Since

$$\mathrm{ev}^t_\mu(Y_iY_j^{-1}) = q^{\mu_j-\mu_i}t^{v_\mu(j)-v_\mu(i)} \qquad \text{and} \qquad \mathrm{ev}^t_\mu\Big(\frac{1-t}{1-Y_iY_j^{-1}}\Big) = \frac{1-t}{1-q^{\mu_j-\mu_i}t^{v_\mu(j)-v_\mu(i)}}$$

then comparing (3.4) and (2.1) gives

$$\mathrm{ev}^t_\mu(f_C(Y)) = t^{-\frac12(m-1)}\prod_{k\in C}\mathrm{wt}_\mu(C,k) \qquad \text{and} \qquad \mathrm{ev}^t_\mu(F_{C,j}(Y)) = F_\mu(C,j). \tag{3.12}$$

Using (1.7) on the expression in Theorem 2.1(a) and inserting (3.12) and (3.11) gives

$$\begin{aligned} x_jE_\mu = X_jE_\mu &= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} \tau^\vee_{C,j}F_{C.j}(Y)f_C(Y)E_\mu \\ &= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} \tau^\vee_{C,j}\mathrm{ev}^t_\mu(F_{C,j}(Y))\mathrm{ev}^t_\mu(f_C(Y))E_\mu \\ &= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} F_\mu(C,j)t^{-\frac12(m-1)}\Big(\prod_{k\in C}\mathrm{wt}_\mu(C,k)\Big)\tau^\vee_{C,j}E_\mu \\ &= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} F_\mu(C,j)t^{-\frac12(m-1)}\Big(\prod_{k\in C}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12(n-1)-\#\{\mu_i<\mu_{a_m}\}}\Big(\prod_{k\notin C}\mathrm{wt}_\mu(C,k)\Big)t^{-\frac12(n-m)}E_{\mathrm{rot}_C(\mu)} \\ &= \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ j\in C}} F_\mu(C,j)\mathrm{wt}_\mu(C)E_{\mathrm{rot}_C(\mu)}. \end{aligned}$$

This completes the proof of (a). The proof of the remaining parts is similar, using parts (b)-(f) of Theorem 2.1. ☐

Example 3.1. An example of the computation of $\tau^\vee_{C,j}E_\mu$. Let $n = 11$ and $j = 7$ and

$$C = \{2,5,7,9,10\}, \qquad \text{so that} \quad \tau^\vee_{C,7} = \tau^\vee_6\tau^\vee_4\tau^\vee_3\tau^\vee_1\tau^\vee_\pi\tau^\vee_{10}\tau^\vee_7 = \tau^\vee_6\tau^\vee_4\tau^\vee_3\tau^\vee_1\tau^\vee_\pi\tau^\vee_{11-1}\tau^\vee_{8-1}$$

and $C^c = \{1,3,4,6,8,11\}$. Then, using (3.9) and (3.10),

$$\begin{aligned}
t^{\frac{6}{2}}\tau^\vee_{C,7}E_\mu &= t^{\frac12}\tau^\vee_6 t^{\frac12}\tau^\vee_4 t^{\frac12}\tau^\vee_3 t^{\frac12}\tau^\vee_1\tau^\vee_\pi t^{\frac12}\tau^\vee_{11-1}t^{\frac12}\tau^\vee_{8-1}E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10},\mu_{11})}\\
&= \mathrm{wt}_\mu(C,8)t^{\frac12}\tau^\vee_6 t^{\frac12}\tau^\vee_4 t^{\frac12}\tau^\vee_3 t^{\frac12}\tau^\vee_1\tau^\vee_\pi t^{\frac12}\tau^\vee_{11-1}E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{10},\mu_{11})}\\
&= \mathrm{wt}_\mu(C,8)\mathrm{wt}_\mu(C,11)t^{\frac12}\tau^\vee_6 t^{\frac12}\tau^\vee_4 t^{\frac12}\tau^\vee_3 t^{\frac12}\tau^\vee_1\tau^\vee_\pi E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11},\mu_{10})}\\
&= t^{\frac12(11-1)-\#\{\mu_i<\mu_{10}\}}\mathrm{wt}_\mu(C,8)\mathrm{wt}_\mu(C,11)t^{\frac12}\tau^\vee_6 t^{\frac12}\tau^\vee_4 t^{\frac12}\tau^\vee_3 t^{\frac12}\tau^\vee_1 E_{(\mu_{10}+1,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})}\\
&= t^{5-\#\{\mu_i<\mu_{10}\}}\Big(\prod_{k\in\{1,8,11\}}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12}\tau^\vee_6 t^{\frac12}\tau^\vee_4 t^{\frac12}\tau^\vee_3 E_{(\mu_1,\mu_{10}+1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})}\\
&= t^{5-\#\{\mu_i<\mu_{10}\}}\Big(\prod_{k\in\{1,3,8,11\}}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12}\tau^\vee_6 t^{\frac12}\tau^\vee_4 E_{(\mu_1,\mu_{10}+1,\mu_3,\mu_2,\mu_4,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})}\\
&= t^{5-\#\{\mu_i<\mu_{10}\}}\Big(\prod_{k\in\{1,3,4,8,11\}}\mathrm{wt}_\mu(C,k)\Big)t^{\frac12}\tau^\vee_6 E_{(\mu_1,\mu_{10}+1,\mu_3,\mu_4,\mu_2,\mu_5,\mu_6,\mu_8,\mu_7,\mu_9,\mu_{11})}\\
&= t^{5-\#\{\mu_i<\mu_{10}\}}\Big(\prod_{k\in\{1,3,4,6,8,11\}}\mathrm{wt}_\mu(C,k)\Big)E_{(\mu_1,\mu_{10}+1,\mu_3,\mu_4,\mu_2,\mu_6,\mu_5,\mu_8,\mu_7,\mu_9,\mu_{11})}\\
&= t^{5-\#\{\mu_i<\mu_{10}\}}\Big(\prod_{k\in C^c}\mathrm{wt}_\mu(C,k)\Big)E_{\mathrm{rot}_C(\mu)}.
\end{aligned}$$

The red entries correspond to the parts specified by C which are rotated to get $\mathrm{rot}_\mu(C)$ as in the picture in (3.2). ☐

Example 3.2. An example of the computation of $\rho_{D,j}E_\mu$. Let $n=11$ and $j=7$ and

$$D=\{1,4,5,7,9\},\qquad \text{so that}\quad \rho_{D,7}=\tau^\vee_7\tau^\vee_9\tau^\vee_{10}(\tau^\vee_\pi)^{-1}\tau^\vee_2\tau^\vee_3\tau^\vee_6$$

and $D^c=\{2,3,6,8,10,11\}$. Then, using (3.9) and (3.10),

$$\begin{aligned}
t^{\frac62}\rho_{D,7}E_\mu &= t^{\frac12}\tau^\vee_7 t^{\frac12}\tau^\vee_9 t^{\frac12}\tau^\vee_{10}(\tau^\vee_\pi)^{-1}t^{\frac12}\tau^\vee_2 t^{\frac12}\tau^\vee_3 t^{\frac12}\tau^\vee_6 E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10},\mu_{11})}\\
&= \mathrm{rwt}_\mu(D,6)t^{\frac12}\tau^\vee_7 t^{\frac12}\tau^\vee_9 t^{\frac12}\tau^\vee_{10}(\tau^\vee_\pi)^{-1}t^{\frac12}\tau^\vee_2 t^{\frac12}\tau^\vee_3 E_{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11})}\\
&= \mathrm{rwt}_\mu(D,3)\mathrm{rwt}_\mu(D,6)t^{\frac12}\tau^\vee_7 t^{\frac12}\tau^\vee_9 t^{\frac12}\tau^\vee_{10}(\tau^\vee_\pi)^{-1}t^{\frac12}\tau^\vee_2 E_{(\mu_1,\mu_2,\mu_4,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11})}\\
&= \Big(\prod_{k\in\{2,3,6\}}\mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau^\vee_7 t^{\frac12}\tau^\vee_9 t^{\frac12}\tau^\vee_{10}(\tau^\vee_\pi)^{-1}E_{(\mu_1,\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11})}\\
&= t^{-\frac12(11-1)+\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6\}}\mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau^\vee_7 t^{\frac12}\tau^\vee_9 t^{\frac12}\tau^\vee_{10}E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_{11},\mu_1-1)}\\
&= t^{-5+\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6,11\}}\mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau^\vee_7 t^{\frac12}\tau^\vee_9 E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_{10},\mu_1-1,\mu_{11})}\\
&= t^{-5+\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6,10,11\}}\mathrm{rwt}_\mu(D,k)\Big)t^{\frac12}\tau^\vee_7 E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_8,\mu_9,\mu_1-1,\mu_{10},\mu_{11})}\\
&= t^{-5+\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in\{2,3,6,8,10,11\}}\mathrm{rwt}_\mu(D,k)\Big)E_{(\mu_4,\mu_2,\mu_3,\mu_5,\mu_7,\mu_6,\mu_9,\mu_8,\mu_1-1,\mu_{10},\mu_{11})}\\
&= t^{-5+\#\{\mu_i<\mu_1\}}\Big(\prod_{k\in D^c}\mathrm{rwt}_\mu(D,k)\Big)E_{\mathrm{rrot}_D(\mu)}.
\end{aligned}$$

☐

4 Specializations of the Monk rule

Specializations of the electronic Macdonald polynomials at $q = 0$, $t = 0$, $q = \infty$ and $t = \infty$ are of interest. For example,

(a) $E_\mu(0,t)$ are the Iwahori-spherical functions of [Ion04], also called t-deformations of Demazure characters and Demazure atoms in [Al16];

(b) $E_\mu(q,0)$ are (level 1 or level 0) affine Demazure characters, or (affine) key polynomials, or non-symmetric q-Whittaker polynomials (see [Ion01], [MRY19] and [AG20]).

(c) $E_\mu(0,0)$ are Demazure characters or (finite) key polynomials. The finite key polynomials are special cases of the affine key polynomials.

By appropriately packaging the weights in the Monk formulas in Theorem 3.1 it is easy to specialize these formulas and obtain formulas at $t = 0$ and $q = 0$. Proposition 4.2 does this repackaging for the product $x_j E_\mu$ and the resulting formulas at $q = 0$ and $t = 0$ are given in Corollary 4.4. Similar formulas could be given for the other products in Theorem 3.1 and also for specializations at $t = \infty$ and $q = \infty$ (by packaging the coefficients in terms of q^{-1} and t^{-1} and then setting $q^{-1} = 0$ and/or $t^{-1} = 0$).

Let $\mu \in \mathbb{Z}^n$ and let $j \in \{1, \ldots, n\}$. Let $C \subseteq \{1, \ldots, n\}$ and let

$$C = \{a_1, a_2, \ldots, a_m\} \qquad \text{with} \quad 1 \le a_1 < a_2 < \cdots < a_m \le n.$$

For parsing the following definitions it is useful to note that

$$\begin{array}{lll} \mu_{a_{i-1}} > \mu_{a_i} & \text{if and only if} & v_\mu(a_{i-1}) > v_\mu(a_i), \qquad \text{and} \\ \mu_{a_1} < \mu_{a_m} + 1 & \text{if and only if} & v_\mu(a_1) < v_\mu(a_m). \end{array}$$

Assume $j \in C$ and let $p \in \{1, \ldots, m\}$ be given by $j = a_p$. Let

$$S' = \#\{i \in \{2, \ldots, m\} \mid \mu_{a_{i-1}} > \mu_{a_i}\} + \begin{cases} 1, & \text{if } v_\mu(a_1) > v_\mu(a_m), \\ 0, & \text{if } v_\mu(a_1) < v_\mu(a_m), \end{cases}$$

$$A' = \sum_{\substack{i \in \{2,\ldots,m\} \\ \mu_{a_{i-1}} > \mu_{a_i}}} (\mu_{a_{i-1}} - \mu_{a_i}) + \begin{cases} \mu_{a_1} - (\mu_{a_m} + 1), & \text{if } \mu_{a_1} \ge \mu_{a_m} + 1, \\ 0, & \text{if } \mu_{a_1} < \mu_{a_m} + 1, \end{cases}$$

$$\begin{aligned} B' = &-\{i \mid \mu_i > \mu_{a_m}\} + \{k \notin C \mid b(k) < \mu_k\} \\ &+ \sum_{\substack{i \in \{2,\ldots,m\} \\ v_\mu(a_{i-1}) > v_\mu(a_i)}} (v_\mu(a_{i-1}) - v_\mu(a_i)) + \begin{cases} v_\mu(a_1) - v_\mu(a_m), & \text{if } v_\mu(a_1) > v_\mu(a_m), \\ 0, & \text{if } v_\mu(a_1) < v_\mu(a_m). \end{cases} \end{aligned}$$

Then define

$$S_{j,\mu}(C) = S' + \begin{cases} 1, & \text{if } p \neq 1 \text{ and } \mu_{a_p} \geq \mu_{a_{p-1}}, \\ 0, & \text{if } p \neq 1 \text{ and } \mu_{a_p} < \mu_{a_{p-1}}, \\ 1, & \text{if } p = 1 \text{ and } \mu_{a_m} + 1 \leq \mu_{a_1}, \\ 0, & \text{if } p = 1 \text{ and } \mu_{a_m} + 1 > \mu_{a_1}, \end{cases} \tag{4.1}$$

$$A_{j,\mu}(C) = A' + \begin{cases} \mu_{a_{p-1}} - \mu_{a_1}, & \text{if } p \neq 1 \text{ and } \mu_{a_p} \geq \mu_{a_{p-1}}, \\ \mu_{a_p} - \mu_{a_1}, & \text{if } p \neq 1 \text{ and } \mu_{a_p} < \mu_{a_{p-1}}, \\ -(\mu_{a_1} - (\mu_{a_m} + 1)), & \text{if } p = 1 \text{ and } \mu_{a_m} + 1 \leq \mu_{a_1}, \\ 0, & \text{if } p = 1 \text{ and } \mu_{a_m} + 1 > \mu_{a_1}, \end{cases} \tag{4.2}$$

$$B_{j,\mu}(C) = B' + \begin{cases} v_\mu(a_{p-1}) - v_\mu(a_1), & \text{if } p \neq 1 \text{ and } v_\mu(a_{p-1}) < v_\mu(a_p), \\ v_\mu(a_p) - v_\mu(a_1), & \text{if } p \neq 1 \text{ and } v_\mu(a_{p-1}) > v_\mu(a_p), \\ -(v_\mu(a_1) - v_\mu(a_m)), & \text{if } p = 1 \text{ and } v_\mu(a_1) > v_\mu(a_p), \\ 0, & \text{if } p = 1 \text{ and } v_\mu(a_1) < v_\mu(a_p). \end{cases} \tag{4.3}$$

Remark 4.1. The statistics $S_{j,\mu}(C)$, $A_{j,\mu}(C)$ and $B_{j,\mu}(C)$ are interesting statistics on μ and on the permutation v_μ. What properties do these statistics have? How do they change when parts of μ are interchanged?

Proposition 4.2. *Let $\mu \in \mathbb{Z}^n$ and let $j \in \{1, \ldots, n\}$. Let $C \subseteq \{1, \ldots, n\}$ and let*

$$C = \{a_1, a_2, \ldots, a_m\} \qquad \text{with} \quad 1 \leq a_1 < a_2 < \cdots < a_m \leq n.$$

Assume $j \in C$ and let $p \in \{1, \ldots, m\}$ be given by $j = a_p$. Let

$$a_0 = a_m, \qquad \gamma_0 = \mu_{a_m} + 1, \qquad \text{and} \qquad \gamma_i = \mu_{a_i}, \quad \text{for } i \in \{1, \ldots, m\},$$

and define

$$W_{\mu,k\in C} = \left(\prod_{\substack{i \in \{1,\ldots,m\} \\ i \neq p}} \frac{1-t}{1 - q^{|\gamma_i - \gamma_{i-1}|} t^{|v_\mu(a_i) - v_\mu(a_{i-1})|}} \right)$$

For $k \notin C$ let $b(k)$ and $c(k)$ be as defined in (3.3) and let

$$W_{\mu,k\notin C} = \left(\prod_{\substack{k \notin C \\ \mu_k > b(k)}} \frac{(1 - q^{\mu_k - b(k)} t^{v_\mu(k) - c(k) + 1})(1 - q^{\mu_k - b(k)} t^{v_\mu(k) - c(k) - 1})}{(1 - q^{\mu_k - b(k)} t^{v_\mu(k) - c(k)})^2} \right).$$

Let $S_{j,\mu}(C)$, $A_{j,\mu}(C)$ and $B_{j,\mu}(C)$ be as defined in (4.1), (4.2) and (4.3). The coefficient of $E_{\mathrm{rot}_C(\mu)}$ in $x_j E_\mu$ is

$$(-1)^{S_{j,\mu}(C)} q^{A_{j,\mu}(C)} t^{B_{j,\mu}(C)} W_{\mu,k\in C} W_{\mu,k\notin C}.$$

Proof. By (3.5),

$$\prod_{k \notin C} \mathrm{wt}_\mu(C,k) = t^{\#\{k \notin C \mid b(k) < \mu_k\}} W_{\mu,k\notin C}.$$

Let $i \in \{2, \ldots, m\}$ and let $k = a_i$. If $v_\mu(a_i) < v_\mu(a_{i-1})$ then

$$\frac{1-t}{1 - q^{\mu_{a_i} - \mu_{a_{i-1}}} t^{v_\mu(a_i) - v_\mu(a_{i-1})}} = \frac{-q^{\mu_{a_{i-1}} - \mu_{a_i}} t^{v_\mu(a_{i-1}) - v_\mu(a_i)} (1-t)}{1 - q^{|\mu_{a_{i-1}} - \mu_{a_i}|} t^{|v_\mu(a_{i-1}) - v_\mu(a_i)|}}$$

and if $v_\mu(a_m) < v_\mu(a_1)$ then

$$\frac{1}{1-q^{\mu_{a_m}+1-\mu_{a_1}}t^{v_\mu(a_m)-v_\mu(a_1)}} = \frac{-q^{\mu_{a_1}-(\mu_{a_m}+1)}t^{v_\mu(a_1)-v_\mu(a_m)}}{1-q^{\mu_{a_1}-(\mu_{a_m}+1)}t^{v_\mu(a_1)-v_\mu(a_m)}} = \frac{-q^{\mu_{a_1}-(\mu_{a_m}+1)}t^{v_\mu(a_1)-v_\mu(a_m)}}{1-q^{|\mu_{a_m}+1-\mu_{a_1}|}t^{|v_\mu(a_m)-v_\mu(a_1)|}}.$$

These give that

$$\prod_{k\in C}\mathrm{wt}_\mu(C,k) = (-1)^{S'+\#\{i\ |\ \mu_i>\mu_{a_m}\}}q^{A'}t^{B''}W_{\mu,k\in C}\cdot\frac{1}{1-q^{|\gamma_p-\gamma_{p-1}|}t^{|v_\mu(a_p)-v_\mu(a_{p-1})|}}.$$

So

$$\begin{aligned}\mathrm{wt}_\mu(C) &= t^{-\#\{i\ |\ \mu_i>\mu_{a_m}\}}\Big(\prod_{k\in C}\mathrm{wt}_\mu(C,k)\Big)\Big(\prod_{k\notin C}\mathrm{wt}_\mu(C,k)\Big)\\ &= (-1)^{S'}q^{A'}t^{B'}W_{\mu,k\in C}W_{\mu,k\notin C}\cdot\frac{1}{1-q^{|\gamma_p-\gamma_{p-1}|}t^{|v_\mu(a_p)-v_\mu(a_{p-1})|}}.\end{aligned}$$

If $p\neq 1$ then

$$F_\mu(C,j) = \begin{cases}-q^{\mu_{a_{p-1}}-\mu_{a_1}}t^{v_\mu(a_{p-1})-v_\mu(a_1)}(1-q^{|\mu_{a_p}-\mu_{a_{p-1}}|}t^{|v_\mu(a_p)-v_\mu(a_{p-1})|}), & \text{if } \mu_{a_p}>\mu_{a_{p-1}},\\ q^{\mu_{a_p}-\mu_{a_1}}t^{v_\mu(a_p)-v_\mu(a_1)}(1-q^{|\mu_{a_p}-\mu_{a_{p-1}}|}t^{|v_\mu(a_p)-v_\mu(a_{p-1})|}), & \text{if } \mu_{a_p}<\mu_{a_{p-1}}.\end{cases}$$

If $p=1$ then

$$F_\mu(C,j) = \begin{cases}(1-q^{|\mu_{a_m}+1-\mu_{a_1}|}t^{|v_\mu(a_m)-v_\mu(a_1)|}), & \text{if } \mu_{a_m}+1>\mu_{a_1},\\ -q^{-(\mu_{a_1}-(\mu_{a_m}+1))}t^{-(v_\mu(a_1)-v_\mu(a_m))}(1-q^{|\mu_{a_m}+1-\mu_{a_1}|}t^{|v_\mu(a_m)-v_\mu(a_1)|}), & \text{if } \mu_{a_m}+1<\mu_{a_1}.\end{cases}$$

Thus

$$F_\mu(C,j)\mathrm{wt}_\mu(C) = (-1)^{S_{\mu,j}(C)}q^{A_{\mu_j}(C)}t^{B_{\mu,j}(C)}W_{\mu,k\in C}W_{\mu,k\notin C}$$

and the result now follows from Theorem 3.1(a). □

In order to specialize the coefficients in Proposition 4.2 at $q=0$ and $t=0$ it is important to know that the powers of q and t are nonnegative. This is established by the following Proposition.

Proposition 4.3. *Let $A_{j,\mu}(C)$ and $B_{j,\mu}(C)$ be as defined in* (4.2) *and* (4.3). *Then*

$$A_{j,\mu}(C)\geq 0 \qquad \textit{and} \qquad B_{j,\mu}(C)\geq 0.$$

Proof. To keep track of signs, write

$$A_{j,\mu}(C) = A' + \begin{cases}(\mu_{a_{p-1}}-\mu_{a_1}), & \text{if } p\neq 1 \text{ and } \mu_{a_1}\leq\mu_{a_{p-1}}\leq\mu_{a_p},\\ -(\mu_{a_1}-\mu_{a_{p-1}}), & \text{if } p\neq 1 \text{ and } \mu_{a_1}>\mu_{a_{p-1}}\leq\mu_{a_p},\\ (\mu_{a_p}-\mu_{a_1}), & \text{if } p\neq 1 \text{ and } \mu_{a_{p-1}}>\mu_{a_p}\geq a_1,\\ -(\mu_{a_1}-\mu_{a_p}), & \text{if } p\neq 1 \text{ and } \mu_{a_{p-1}}>\mu_{a_p}<\mu_{a_1},\\ -(\mu_{a_1}-(\mu_{a_m}+1)), & \text{if } p=1 \text{ and } \mu_{a_m}+1\leq\mu_{a_1},\\ 0, & \text{if } p=1 \text{ and } \mu_{a_m}+1>\mu_{a_1}.\end{cases}$$

Note that $A'\geq 0$ since it is a sum of positive integers. Let us consider the cases when the term added to A' is negative.

Case $\mu_{a_1} > \mu_{a_{p-1}} \le \mu_{a_p}$: Since the total of the descents of the sequence $(\mu_{a_1}, \mu_{a_2}, \ldots, \mu_{a_{p-1}})$ is at least as large as $(\mu_{a_1} - \mu_{a_p})$ then

$$\sum_{\substack{i\in\{2,\ldots,p-1\}\\ \mu_{a_{i-1}}>\mu_{a_i}}} (\mu_{a_{i-1}} - \mu_{a_i}) \ge (\mu_{a_1} - \mu_{a_{p-1}}), \quad \text{so that} \quad \sum_{\substack{i\in\{2,\ldots,p-1\}\\ \mu_{a_{i-1}}>\mu_{a_i}}} (\mu_{a_{i-1}} - \mu_{a_i}) - (\mu_{a_1} - \mu_{a_{p-1}}) \ge 0$$

and $A_{j,\mu}(C) \ge 0$.

Case $p \ne 1$ *and* $\mu_{a_{p-1}} > \mu_{a_p} < \mu_{a_1}$: Since the total of the descents of the sequence $(\mu_{a_1}, \mu_{a_2}, \ldots, \mu_{a_p})$ is at least as large as $(\mu_{a_1} - \mu_{a_p})$ then

$$\sum_{\substack{i\in\{2,\ldots,p\}\\ \mu_{a_{i-1}}>\mu_{a_i}}} (\mu_{a_{i-1}} - \mu_{a_i}) \ge (\mu_{a_1} - \mu_{a_p}) \qquad \text{so that} \qquad A_{j,\mu}(C) = A' - (\mu_{a_1} - \mu_{a_p}) \ge 0$$

and $A_{j,\mu}(C) \ge 0$.

Case $p = 1$ *and* $\mu_{a_m} + 1 > \mu_{a_1}$: In this case the last term in the definition of A' cancels with the added extra term $-(\mu_{a_1} - (\mu_{a_m} + 1))$ so that $A_{j,\mu}(C)$ is a sum of positive integers and is ≥ 0.

A similar argument shows shows that $B_{j,\mu}(C) \ge 0$. ☐

Now we are ready to specialize the result of Proposition 4.2 at $q = 0$ and $t = 0$.

Corollary 4.4. *Keep the same notations as in Propostion 4.2.*

(a) *If* $t = 0$ *then* $\quad x_j E_\mu = \displaystyle\sum_{\substack{C\subseteq\{1,\ldots,n\}\\ B_{j,\mu}(C)=0}} (-1)^{S_{j,\mu}(C)} q^{A_{j,\mu}(C)} E_{\mathrm{rot}_C(\mu)}.$

(b) *If* $q = 0$ *then*

$$x_j E_\mu = \sum_{\substack{C\subseteq\{1,\ldots,n\}\\ A_{j,\mu}(C)=0}} (-1)^{S_{j,\mu}(C)} t^{B_{j,\mu}(C)} (1-t)^{m-1} \Big(\prod_{\substack{\gamma_i=\gamma_{i-1}\\ i\ne p}} \frac{1}{1 - t^{v_\mu(a_i) - v_\mu(a_{i-1})}} \Big) E_{\mathrm{rot}_C(\mu)}.$$

(c) *If* $q = 0$ *and* $t = 0$ *then* $\quad x_j E_\mu = \displaystyle\sum_{\substack{C\subseteq\{1,\ldots,n\}\\ A_{j,\mu}(C)=0, B_{j,\mu}(C)=0}} (-1)^{S_{j,\mu}(C)} E_{\mathrm{rot}_C(\mu)}.$

References

[Al16] P. Alexandersson, *Non-symmetric Macdonald polynomials and Demazure-Lusztig operators*, Sém. Lothar. Combin. B76d (2019), MR4063906, arXiv:1602.05153. 4

[AQ19] S. Assaf and D. Quijada, *A Pieri rule for Demazure characters of the general linear group*, arXiv:1908.08502. 0

[AG20] S. Assaf and N. Gonzalez, *Affine Demazure crystals for specialized nonsymmetric Macdonald polynomials*, Algebr. Comb. **4** (2021) 777-793, MR4339352, arXiv:2002.04141. 4

[As21] S. Assaf, *An insertion algorithm for multiplying Demazure characters by Schur polynomials*, arXiv:2109.05651. 0

[Ba10] W. Baratta, *Further Pieri-type formulas for the nonsymmetric Macdonald polynomial*, J. Algebraic Combin. **36** (2012) 45–66, MR2927656, arXiv:1008.0892. 0

[Ba08] W. Baratta, *Pieri-type formulas for nonsymmetric Macdonald polynomials*, Int. Math. Res. Not. IMRN 2009 **15** 2829–2854, MR2525842, arXiv:0806.2695. (document), 0

[CR22] L. Colmenarejo and A. Ram, *c-functions and Macdonald polynomials*, arXiv:2212.03312. 0

[Gib19] J. Gibson, *A Demazure Character Formula for the Product Monomial Crystal*, Algebr. Comb. **4** (2021) 301-327, MR4244375, arXiv:1907.09681. 0

[GR21] W. Guo and A. Ram, *Comparing formulas for type GL_n Macdonald polynomials*, arXiv:2104.02942. 1, 1.1, 1, 1.3, 2, 3, 3

[Ion01] B. Ion, *Nonsymmetric Macdonald polynomials and Demazure characters*, Duke Math. J. **116** (2) (2003) 299-318, MR1953294, arXiv:math/0105061v2. 4

[Ion04] B. Ion, *Nonsymmetric Macdonald polynomials and matrix coefficients for unramified principal series*, Adv. Math. **201** (2006) 36-62, MR2204747, arXiv:math/0407044. 4

[Kn96] F. Knop, *Symmetric and nonsymmetric quantum Capelli polynomials*, Comment. Math. Helv., **72** (1997) 84–100, MR1456318, arXiv:q-alg/9603028. 0

[La08] A. Lascoux, *Schubert and Macdonald polynomials, a parallel*, (2008) http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.404.6012 0

[Len98] C. Lenart, *Combinatorial aspects of the K-theory of Grassmanians*, Ann. Comb. **4** (2000) 67–82, MR1763950.

[LS04] C. Lenart and F. Sottile, *A Pieri type formula for the K-theory of a flag manifold*, Trans. Amer. Math. Soc. **359** (2007) 2317-2342, MR2276622, arXiv:math.CO/0407412. 0

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford Mathematical Monographs, Oxford University Press, New York, 1995. ISBN: 0-19-853489-2, MR1354144.

[Mac91] I.G. Macdonald, *Notes on Schubert polynomials*, **6**, Publ. du Laboratoire Combinatoire et d'Informatique Mathématique, Université du Québec à Montréal, 1991. MR1161461. 0

[MRY19] F. McGlade, A. Ram and Y. Yang, *Positive level, negative level and level 0*, *Schubert calculus and its applications in combinatorics and representation theory*, Springer Proc. Math. Stat. **332**, Springer, Singapore (2020) 153-194, MR4167516, arXiv1907.11796. 4

[Pun16] A. Pun, *On decomposition of the product of Demazure atoms and Demazure characters*, Ph.D. Thesis, University of Pennsylvania. (2016), MR3553609, arXiv:1606.0229.

[Sa96] S. Sahi, *Interpolation, integrality and generalization of Macdonald's polynomials*, Int. Math. Res. Notices **10** (1996) 457-471, MR1399411. 0

[Yi10] M. Yip, *A Littlewood-Richardson rule for Macdonald polynomials*, Math. Z. **272** (2012) 1259-1290, MR2995168, arXiv:1010.0722. (document), 0