Invent. math. 106, 461–488 (1991)

Inventiones mathematicae

A Frobenius formula for the characters of the Hecke algebras

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Oblatum 10-XII-1990

Summary. This paper uses the theory of quantum groups and the quantum Yang-Baxter equation as a guide in order to produce a method of computing the irreducible characters of the Hecke algebra. This approach is motivated by an observation of M. Jimbo giving a representation of the Hecke algebra on tensor space which generates the full centralizer of a tensor power of the "standard" representation of the quantum group $U_q(\mathfrak{sl}(n))$. By rewriting the solutions of the quantum Yang-Baxter equation for $U_q(\mathfrak{sl}(n))$ in a different form one can avoid the quantum group completely and produce a "Frobenius" formula for the characters of the Hecke algebra by elementary methods. Using this formula we derive a combinatorial rule for computing the irreducible characters of the Hecke algebra. This combinatorial rule is a q-extension of the Murnaghan-Nakayama for computing the irreducible characters of the symmetric group. Along the way one finds connections, apparently unexplored, between the irreducible characters of the Hecke algebra and Hall-Littlewood symmetric functions and Kronecker products of symmetric groups.

0 Introduction

In 1900, in a remarkable paper [Fr], Frobenius gave a formula and a method of computing the characters of the symmetric group. This method was later used to give a completely combinatorial rule for the computation of the characters of the symmetric group, often referred to as the Murnaghan-Nakayama rule, see [Mac, Chapter I. Example 9]. In his study of the representations of the general linear group $Gl(n)$, I. Schur [Sc1, Sc2] showed that the Frobenius method can be obtained by way of a reciprocity between the $Gl(n)$ and the symmetric group, now known as Schur-Weyl duality. Specifically, there are actions of each group on tensor space under which each group action generates the full centralizer algebra of the other.

* Work partially supported by an NSF grant at the University of California, San Diego

In 1986, Jimbo [Ji] recognized that there exists a similar duality between the quantum group $U_q(\mathfrak{gl}(n))$ and the Hecke algebras of type A. Guided by this observation, we develop a "Frobenius" formula and a combinatorial rule for computing the characters of the Hecke algebras. The presentation given in this paper allows one to avoid the quantum group completely. The derivation of the combinatorial rule shows that there are connections, apparently unexplored, between characters of Hecke algebras, Hall-Littlewood symmetric functions, and Kronecker products of symmetric group representations. The combinatorial rule which is derived in this paper is essentially a q-analogue of the Murnaghan-Nakayama rule, in the sense that when one specializes $q = 1$ it trivially reduces to the classical rule for the symmetric group.

The paper is organized as follows. In the first section we develop the needed notation and give the necessary basic facts about the symmetric group and the Hecke algebra. In Sect. 2 we give a brief description of the duality between the quantum group and the Hecke algebra and the motivation behind our approach to the characters of the Hecke algebra. In Sect. 3 we show that the irreducible characters of the Hecke algebra are determined by traces arising from a certain action of the Hecke algebra on tensor space. In Sect. 4 we show that these traces are, up to a scalar multiple, Hall-Littlewood symmetric functions and give a Frobenius type formula for the characters of the Hecke algebra. In Sect. 5 we use the Frobenius formula to derive formulas for the irreducible characters of the Hecke algebra. In Sect. 6 we develop a connection between the characters of the Hecke algebra and Kronecker products of symmetric group representations and derive a combinatorial rule for computing characters of the Hecke algebras. Section 7 gives a brief summary of work of King and Wybourne and of Vershik and Kerov on the characters of the Hecke algebra. In the final section we give explicit formulas for some special cases and tables of characters.

Acknowledgements. I would like to thank H. Wenzl for teaching me about quantum groups and for explaining to me how the calculation I wanted to do could actually be done. I would like to thank A. Garsia for showing me how to do computations via λ-ring notation and for all the support (both grant support and otherwise) and encouragement he has given me. I don't think I would have pushed this all the way through without it. I would also like to thank M. Haiman and S.T. Whitehead for enlightening discussions.

1 The symmetric group, S_f, and the Hecke algebra, H_f

Notation

We shall adopt the notations in [Mac] for partitions and symmetric functions. In particular, if $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geqq \lambda_2 \geqq \ldots$ is a partition, then $l(\lambda)$ denotes the length (number of parts) of λ, $|\lambda|$, the weight (sum of the parts) of λ. If $|\lambda| = f$ we write $\lambda \vdash f$. Often we shall use the notation $(1^{m_1} 2^{m_2} \ldots)$ for a partition, so that m_i denotes the number of parts equal to i in the partition. m_λ shall denote the monomial symmetric function, h_λ the homogeneous symmetric function, s_λ the Schur function, and p_λ the power symmetric function associated to the partition λ. If F is a symmetric function then $F|_{s_\lambda}$ shall denote the coefficient of s_λ in an expansion of F in terms of Schur functions.

Let f be a positive integer. A composition of f, $c \models f$, is a sequence of positive integers $c = (c_1, c_2, \ldots, c_k)$, such that $\sum_i c_i = f$. As in the case of partitions the c_i are called the parts of the composition c. The partition $\lambda(c)$ given by arranging the parts of c in decreasing order is called the shape of c.

The symmetric group, S_f

The symmetric group S_f can be defined as the group generated by generators $1, s_1, s_2, \ldots, s_{f-1}$ and relations

(B1) $$s_i s_j = s_j s_i, \quad \text{if } |i-j| > 1 ,$$

(B2) $$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} ,$$

(S) $$s_i^2 = 1 .$$

The elements s_i are called simple transpositions. The length, $\ell(\sigma)$, of an element σ in S_f is the minimum number of simple transpositions necessary to express σ. Any product of $\ell(\sigma)$ transpositions equal to σ is called a reduced decomposition of σ. By viewing s_i as the operation that switches i and $i+1$ each element of S_f can be viewed as a permutation of $\{1, 2, \ldots, f\}$. We write $\sigma(i)$ for the image of i under σ.

There is an embedding of $S_m \times S_n$ into S_{m+n}, $(\sigma, \pi) \mapsto \sigma \times \pi$, given by making S_m act on $\{1, 2, \ldots, m\}$ and S_n act on $\{m+1, m+2, \ldots, m+n\}$. The r-cycle is the element
$$\gamma_r = s_{r-1} s_{r-2} \ldots s_2 s_1$$
of S_r. The 1-cycle is the identity $1 \in S_1$. For each $c = (c_1, c_2, \ldots)$, composition of f, define $\gamma_c \in S_f$ by
$$\gamma_c = \gamma_{c_1} \times \gamma_{c_2} \times \ldots .$$
Given any permutation $\sigma \in S_f$ there exists some permutation π such that $\pi\sigma\pi^{-1} = \gamma_\lambda$ for some $\lambda \vdash f$. The partition λ is the cycle type of the permutation σ. Any two permutations with the same cycle type are said to be in the same conjugacy class. The number of permutations in the conjugacy class C_λ is determined by the partition λ and is equal to $f!/\lambda?$ where $\lambda?$ is given by
$$\lambda? = \prod_{i \geqq 1} i^{m_i} \cdot m_i! , \tag{1.1}$$
where m_i is the number of parts of λ equal to i.

Let $\mathbb{C}S_f$ denote the group algebra of the symmetric group. A character of S_f is a linear functional $\chi_{S_f} \colon \mathbb{C}S_f \to \mathbb{C}$ such that $\chi_{S_f}(ab) = \chi_{S_f}(ba)$ for all $a, b \in \mathbb{C}S_f$. Two permutations $\sigma, \pi \in S_f$ are in the same conjugacy class if and only if $\chi_{S_f}(\sigma) = \chi_{S_f}(\pi)$ for all characters χ_{S_f} of S_f.

The irreducible representations of S_f are indexed by partitions of f. The dimension of the irreducible representation indexed by the partition λ will be denoted d_λ, and the irreducible character determined by this representation by $\chi^\lambda_{S_f}$. Corresponding to each $\lambda \vdash f$ there is a unique minimal central idempotent $z_\lambda \in \mathbb{C}S_f$ given by
$$z_\lambda = \frac{d_\lambda}{f!} \sum_{\sigma \in S_f} \chi^\lambda_{S_f}(\sigma)\sigma . \tag{1.2}$$

To each minimal idempotent $p \in \mathbb{C}S_f$ there is one and only one $\lambda \vdash f$ such that $pz_\lambda = p$. A partition of unity is a set of minimal idempotents $\{p_i^\lambda\}$, $\lambda \vdash f$, $1 \leqq i \leqq d_\lambda (p_i^\lambda z_\lambda = p_i^\lambda)$ such that $p_i^\lambda p_j^\mu = 0$ unless $\lambda = \mu$ and $i = j$ and $1 = \sum_\lambda \sum_i p_i^\lambda$.

The Hecke algebra H_f

Let $\mathbb{C}(q)$ be the field of rational functions in the variable q. The Hecke algebra H_f is the $\mathbb{C}(q)$ algebra given by generators $1, g_1, g_2, \ldots, g_{f-1}$ and relations

(B1) $$g_i g_j = g_j g_i, \quad \text{if } |i-j| > 1 ,$$

(B2) $$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} ,$$

(H) $$g_i^2 = (q-1) g_i + q .$$

For each $\sigma \in S_f$, the symmetric group, let $T_\sigma = g_{i_1} g_{i_2} \ldots g_{i_k}$ where $\sigma = s_{i_1} s_{i_2} \ldots s_{i_k}$ is a reduced decomposition of σ. T_σ is well defined since only the relations (B1) and (B2) are necessary to prove that two reduced decompositions of w are equal [Bou]. The T_σ form a basis of H_f. If we specialize $q = 1$, then the relation (H) is the same as relation (S) above.

Remark. Although here we have chosen to work over the field $\mathbb{C}(q)$ of rational functions everything goes through in exactly the same fashion if we work over a field k of characteristic 0 and let $q \in k$ such that $1 + q + q^2 + \ldots + q^r \neq 0$ for any $r = 1, 2, \ldots, f$, see [Bou].

A character of H_f is a $\mathbb{C}(q)$ linear functional $\chi\colon H_f \to \mathbb{C}(q)$ such that for all $h_1, h_2 \in H_f$,

$$\chi(h_1 h_2) = \chi(h_2 h_1) .$$

H_f is a semisimple $\mathbb{C}(q)$ algebra [Bou, H, Wz1]. The irreducible representations of H_f are indexed by partitions of f. For each $\lambda \vdash f$ the dimension of the irreducible representation indexed by λ is d_λ as in the case of S_f, and determines an irreducible character χ^λ of H_f.

2 Remarks on the quantum group $U_q(\mathfrak{sl}(n))$ and the solutions of the quantum Yang-Baxter equation

The purpose of this section is to describe the algebraic motivation behind the approach to the characters of the Hecke algebra given in this paper.

In [Ji] M. Jimbo describes a reciprocity between, in his notation, the quantum group $\hat{U}(\mathfrak{gl}(N+1))$ and the Hecke algebra. In view of the fact that the "modern" approach to the theory (and notation) of quantum groups and the quantum Yang-Baxter equation is slightly different from that used in Jimbo's original paper, I shall endeavor to give a short summary explaining how this reciprocity comes about. The excellent (and dense) papers [Dr] and [Ji2] contain further information on quantum groups and the quantum Yang-Baxter equation.

Define $\mathscr{A} = U_q(\mathfrak{sl}(n))$ to be the associative algebra over $\mathbb{C}(q)$ defined by generators k_i, k_i^{-1}, X_i^+, X_i^-, $1 \leqq i \leqq n$, and the relations:

$$\begin{aligned} &k_i k_i^{-1} = k_i^{-1} k_i = 1 , \\ &k_i k_j = k_j k_i , \\ &k_i X_j^{\pm} k_i^{-1} = q^{\pm a_{ij}/2} X_j^{\pm} , \\ &[X_i^+, X_j^-] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}} , \\ &X_i^{\pm} X_j^{\pm} = X_j^{\pm} X_i^{\pm} , \qquad |i-j| \geqq 2 , \\ &(X_i^{\pm})^2 X_j^{\pm} - (q + q^{-1}) X_i^{\pm} X_j^{\pm} X_i^{\pm} + X_j^{\pm} (X_i^{\pm})^2 = 0, \qquad j = i \pm 1 , \end{aligned} \tag{2.1}$$

where $a_{ii} = 2$, $a_{ij} = -1$ if $j = i \pm 1$, and $a_{ij} = 0$ if $|i - j| \geqq 2$.

$\mathscr{A}$ becomes a Hopf algebra with coproduct $\Delta\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$, antipode $S\colon \mathscr{A} \to \mathscr{A}$ and counit $\varepsilon\colon \mathscr{A} \to \mathbb{C}(q)$ given by

$$\begin{aligned} &\Delta(k_i) = k_i \otimes k_i, && \Delta(X_i^{\pm}) = k_i \otimes X_i^{\pm} + X_i^{\pm} \otimes k_i^{-1} , \\ &S(k_i) = k_i^{-1}, && S(X_i^{\pm}) = -q^{\mp 1} X_i^{\pm} , \\ &\varepsilon(k_i) = 1, && \varepsilon(X_i^{\pm}) = 0 . \end{aligned} \tag{2.2}$$

For any invertible element $\mathscr{R} \in \mathscr{A} \otimes \mathscr{A}$ given by $\mathscr{R} = \sum a_i \otimes b_i$ define $\mathscr{R}_{12}, \mathscr{R}_{13}, \mathscr{R}_{23} \in \mathscr{A}^{\otimes 3}$ to be the elements

$$\mathscr{R}_{12} = \sum a_i \otimes b_i \otimes 1, \quad \mathscr{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \quad \mathscr{R}_{23} = \sum 1 \otimes a_i \otimes b_i ,$$

$\mathscr{R}$ satisfies the quantum Yang-Baxter equation (QYBE) if

$$\mathscr{R}_{12} \mathscr{R}_{13} \mathscr{R}_{23} = \mathscr{R}_{23} \mathscr{R}_{13} \mathscr{R}_{12} . \tag{2.3}$$

Let $T\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ be given by

$$T(a \otimes b) = b \otimes a, \quad \text{for all } a, b \in \mathscr{A} . \tag{2.4}$$

$\mathscr{R}$ is a universal R-matrix if $\mathscr{R}$ satisfies the relations,

$$\begin{aligned} T\Delta(a) &= \mathscr{R} \Delta(a) \mathscr{R}^{-1}, \quad \text{for all } a \in \mathscr{A} , \\ (\Delta \otimes \mathrm{id})(\mathscr{R}) &= \mathscr{R}_{13} \mathscr{R}_{23} , \\ (\mathrm{id} \otimes \Delta)(\mathscr{R}) &= \mathscr{R}_{13} \mathscr{R}_{12} . \end{aligned} \tag{2.5}$$

If $\mathscr{R}$ is a universal R-matrix then $\mathscr{R}$ satisfies the QYBE.

Let $V = \mathbb{C}(q)^n$ and let E_{ij} denote the $n \times n$ matrix that is 1 in the ith row and jth column and 0 everywhere else. The fundamental representation (ρ, V), $\rho\colon \mathscr{A} \to \mathrm{End}_{\mathbb{C}(q)}(V)$, of $U_q(\mathfrak{sl}(n))$ is given by setting

$$\begin{aligned} \rho(X_i^+) &= E_{i,i+1} , \\ \rho(X_i^-) &= E_{i+1,i} , \\ \rho(k_i) &= q E_{ii} + q^{-1} E_{i+1,i+1} + \sum_{\substack{1 \leqq j \leqq n \\ j \neq i, i+1}} E_{jj} . \end{aligned} \tag{2.6}$$

Since $\mathscr{A}$ is a Hopf algebra the tensor representation $(\rho^{\otimes f}, V^{\otimes f})$ is well defined for every $f \geqq 1$.

Let $\mathscr{R}$ be the universal R-matrix and let $\check{R}_\rho = \rho(T\mathscr{R})$, where T is as in (2.4). The form of the matrix $\check{R}_\rho$ is well known [Ji2] and is given by

$$\check{R}_\rho = \sum_i qE_{ii} \otimes E_{ii} + \sum_{i<j} (q-1)E_{ii} \otimes E_{jj} + q^{1/2}(E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij}) .$$

For each $1 \leqq i \leqq f-1$, define $\check{R}_i \in \mathrm{End}(V^{\otimes f})$ as

$$\check{R}_i = 1 \otimes \ldots \otimes 1 \otimes \check{R}_\rho \otimes 1 \otimes \ldots \otimes 1 ,$$

$\check{R}$ appearing as the matrix in the ith and $i+1$st factor. Writing the QYBE (2.3) in terms of the matrices $\check{R}_i$ gives that

$$\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1} ,$$

which implies that the $\check{R}_i$ satisfy the braid relation (B2) in the definition of the Hecke algebra. Jimbo [Ji] noticed that defining a map $\pi: H_f \to \mathrm{End}_{\mathbb{C}(q)}(V)$ given by $\pi(g_i) = \check{R}_i$ gives a representation of the Hecke algebra H_f.

The first equality in (2.5) can be used to show that matrices $\check{R}_i$ all commute with the elements of $\rho^{\otimes f}(\mathscr{A})$. Jimbo [Ji] observed that the algebras $\pi(H_f)$ and $\rho^{\otimes f}(\mathscr{A})$ are mutual commutants of one another. This given, the double centralizer theory gives that

(2.7) Theorem. *As an $H_f \otimes U_q(\mathfrak{sl}(n))$ representation*

$$V^{\otimes f} \cong \bigoplus_{\substack{\lambda \vdash f \\ \ell(\lambda) \leqq n}} H_\lambda \otimes V_\lambda ,$$

where H_λ is a irreducible H_f representation and V_λ is an irreducible $U_q(\mathfrak{sl}(n))$ representation.

Remark. Actually Theorem (2.7) follows from the double centralizer theory only for some parametrization of the irreducible representations of the Hecke algebra. To show that the correspondence between the irreducibles is as given above with the conventional indexing of irreducible representations one may use a Zariski argument to reduce to the classical Schur-Weyl duality between the symmetric group and the general linear group. An alternate approach is to use the results of § 3 to give a proof of Theorem (2.7).

Let I denote the identity matrix and note that the spectral projection

$$\frac{\rho(k_i) - q^{-1}I}{q - q^{-1}} \cdot \frac{\rho(k_i) - I}{q - 1} = E_{ii}$$

of $\rho(k_i)$ is an element of $\rho(\mathscr{A})$. Since $E_{ii} \in \rho(\mathscr{A})$ for every i one has that the matrix $d = \sum_i x_i E_{ii}$ is in $\rho(\mathscr{A})$ for all $x_1, \ldots, x_n \in \mathbb{C}(q)$. Since the value of d depends only on the elements $\rho(k_i)$ and the k_i are grouplike elements in the Hopf algebra $\mathscr{A}$ (i.e. $\Delta(k_i) = k_i \otimes k_i$) we have that the matrix $D = d^{\otimes f}$ is an element of $\rho^{\otimes f}(\mathscr{A})$. I must give thanks to H. Wenzl for alerting me to this fact.

The results in this paper are obtained by computing explicitly, for certain special elements $T_{\gamma_\mu} \in H_f$, the trace of the action of the element $T_{\gamma_\mu} \otimes D \in H_f \otimes U_q(\mathfrak{sl}(n))$ on each side of the isomorphism in Theorem (2.7).

3 The action of H_f on $V^{\otimes f}$

For the remainder of this paper fix positive integers f and n with $n > f$.

An n-composition of f is a sequence of nonnegative integers $c = (c_1, c_2, \ldots, c_n)$, $c_i \geqq 0$, such that $c_1 + c_2 + \ldots + c_n = f$. Note that an n-composition of f is not a true composition of f since we allow $c_i = 0$. Let $x_1, x_2, \ldots, x_n$ be n independent (commuting) variables. For any n-composition, $c = (c_1, c_2, \ldots, c_n)$, of f, define $x^c = x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$.

Let $v_1, v_2, \ldots, v_n$ be an alphabet of n letters (noncommuting variables). The content $c(w)$ of a word $w = v_{i_1} v_{i_2} \ldots v_{i_f}$ of length f is the n-composition of f, $c(w) = (c_1, c_2, \ldots, c_n)$, such that c_i is the number of letters equal to v_i in the word w. The length of a word w will be denoted $|w|$. Let $V^{\otimes f}$ denote the $\mathbb{C}(q^{1/2})$ span of the words of length f from the alphabet $v_1, v_2, \ldots, v_n$.

For each n-composition c of f define a projection operator E_c on $V^{\otimes f}$ by

$$E_c w = \begin{cases} w & \text{if } c(w) = c \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that for any n-composition c of f, $E_c^2 = E_c$, and that if c and c' are two n-compositions of f such that $c \neq c'$, then $E_c E_{c'} = 0$. The identity operator on $V^{\otimes f}$ is

$$I = \sum_c E_c \ ,$$

where the sum is over all n-compositions of f. Let D be the operator

$$D = \sum_c x^c E_c \ , \tag{3.1}$$

where, as before, the sum is over all n-compositions of f. For each word $w = v_{i_1} v_{i_2} \ldots v_{i_f}$ of length f we have $Dw = x_{i_1} x_{i_2} \ldots x_{i_f} w = x^{c(w)} w$.

The symmetric group S_f acts on the words $w = v_{i_1} v_{i_2} \ldots v_{i_f}$, $1 \leqq i_j \leqq n$, of length f by permuting the letters v_{i_j} of w. We shall write the elements of S_f as operators which act on the right so that

$$v_{i_1} v_{i_2} \ldots v_{i_f} s_j = v_{i_1} \ldots v_{i_{j-1}} v_{i_{j+1}} v_{i_j} v_{i_{j+2}} \ldots v_{i_f} \ ,$$

gives the action of the simple transposition s_j.

For each generator g_j of H_f and each word $w = v_{i_1} v_{i_2} \ldots v_{i_f}$ of length f, define the action of g_j on w by

$$w g_j = \begin{cases} qw & \text{if } i_j = i_{j+1} \ , \\ (q-1)w + q^{1/2} w s_j & \text{if } i_j < i_{j+1} \ , \\ q^{1/2} w s_j & \text{if } i_j > i_{j+1} \ . \end{cases} \tag{3.2}$$

(3.3) Proposition. *The action defined above extends to a well defined action of H_f on $V^{\otimes f}$.*

Proof. Checking that the above action satisfies the defining relations for H_f is a straight forward, albeit slightly lengthy, calculation which we leave to the reader. □

The action above was obtained by rewriting the solutions of the quantum Yang-Baxter equation. M. Jimbo [Ji] observed that this gives a representation of the Hecke algebra.

Note that the action of g_j, and therefore of H_f, commutes with the action of D and of E_c for each n-composition c of f, since all words in the expansion of wg_j have the same content as w.

Since the words of length f form a basis of $V^{\otimes f}$, the trace of a linear operator L on $V^{\otimes f}$ can be given by

$$\operatorname{tr}(L) = \sum_{w} L(w)|_w \,, \tag{3.4}$$

where the sum is over all words w of length f and $a|_w$, $a \in V^{\otimes f}$ denotes the coefficient of w in the expansion of a as a linear combination of words.

The setup here is analogous to that in [Wz2, Sect. 5]. In particular the following lemma is a special case of Lemma 5.2 in [Wz2].

(3.5) Lemma. *For any idempotent* $p \in H_f$, $\operatorname{tr}(Dp) = \sum_w Dwp|_w$ *is independent of* q.

Proof. Let c be an n-composition of f. Since the actions of H_f and E_c commute, $(E_c p)^2 = E_c p E_c p = E_c^2 p^2 = E_c p$. The trace of any idempotent operator is just the rank of the operator, so for every n-composition c, $\operatorname{tr}(E_c p) \in \mathbb{Z}$. So $\operatorname{tr}(E_c p)$ is independent of q. Since $\operatorname{tr}(Dp) = \sum_c x^c \operatorname{tr}(E_c p)$, $\operatorname{tr}(Dp)$ is also independent of q. $\square$

Lemma 3.5 shows that, for an idempotent $p \in H_f$, one can compute $\operatorname{tr}(Dp)$ by specializing $q = 1$. When $q = 1$, the relations in the definition of H_f reduce to the relations defining S_f and the action of H_f on words reduces to the S_f action on words.

(3.6) Lemma. *Recall that* $\gamma_r = s_{r-1} s_{r-2} \dots s_1 \in S_r$. *Then, viewing* $D\gamma_r$ *as an operator on* $V^{\otimes f}$,

$$\operatorname{tr}(D\gamma_r) = p_r(x) = \sum_{i=1}^{n} x_i^r \,,$$

where $p_r(x)$ *is the power symmetric function in the variables* $x_1, \dots, x_n$.

Proof. It is sufficient to note that

$$\begin{aligned} Dv_{i_1} \dots v_{i_r} \gamma_r|_{v_{i_1} \dots v_{i_r}} &= Dv_{i_r} v_{i_1} \dots v_{i_{r-1}}|_{v_{i_1} \dots v_{i_r}} \\ &= \begin{cases} x_{i_1}^r, & \text{if } i_1 = i_2 = \dots = i_r \,, \\ 0 & \text{otherwise.} \end{cases} \quad \square \end{aligned}$$

(3.7) Lemma. *Let* p_λ *be a minimal idempotent of* $\mathbb{C}S_f$ *such that* $p_\lambda z_\lambda = p_\lambda$, *where* z_λ *is the minimal central idempotent of* $\mathbb{C}S_f$ *indexed by* λ. *Then*

$$\operatorname{tr}(Dp_\lambda) = s_\lambda(x) \,,$$

where $s_\lambda(x)$ *is the Schur function in the variables* $x_1, x_2, \dots, x_n$.

Proof. Let d_λ be the dimension of the irreducible representation of $\mathbb{C}S_f$ corresponding to λ. Using (1.2) we have that

$$\begin{aligned}\operatorname{tr}(Dp_\lambda) &= \frac{1}{d_\lambda}\operatorname{tr}(Dz_\lambda)\\ &= \frac{1}{d_\lambda}\frac{d_\lambda}{f!}\sum_{\sigma\in S_f}\chi^\lambda_{S_f}(\sigma)\operatorname{tr}(D\sigma)\,.\end{aligned}$$

Since both $\chi^\lambda_{S_f}(\sigma)$ and $\operatorname{tr}(D\sigma)$ are constant on conjugacy classes C_μ in S_f,

$$\begin{aligned}\operatorname{tr}(Dp_\lambda) &= \frac{1}{f!}\sum_{\mu\vdash f}\chi^\lambda(\mu)\operatorname{tr}(D\gamma_\mu)|C_\mu|\\ &= \sum_{\mu\vdash f}\chi^\lambda(\mu)\frac{p_\mu(x)}{\mu?}\,,\end{aligned}$$

where $\mu?$ is given by (1.1). Thus, by [Mac, (7.6)], $\operatorname{tr}(Dp_\lambda)$ is the Schur function $s_\lambda(x)$. □

(3.8) Theorem. *For any $h\in H_f$,*

$$\operatorname{tr}(Dh) = \sum_\lambda \chi^\lambda(h)s_\lambda(x)\,,$$

where, for each $\lambda\vdash f$, χ^λ is the corresponding irreducible character of H_f and $s_\lambda(x)$ is the corresponding Schur function in the variables $x_1,\dots,x_n$.

Proof. Let $\{p_i^\lambda\}$, $\lambda\vdash f$, $1\leqq i\leqq d_\lambda$ be a partition of unity in H_f with the property that when we specialize $q=1$ each p_i^λ is well defined and that, at $q=1$, $\{p_i^\lambda|_{q=1}\}$ is a partition of unity of $\mathbb{C}S_f$. Partitions of unity for H_f that satisfy this property are known, see the remark below. For each λ and each $1\leqq i\leqq d_\lambda$ let h_{ii}^λ be the constant in $\mathbb{C}(q)$ such that $p_i^\lambda h p_i^\lambda = h_{ii}^\lambda p_i^\lambda$. Note that the h_{ii}^λ are the diagonal elements of the matrix of h in the irreducible representation corresponding to λ determined by this partition of unity. Thus, for each λ, $\sum_i h_{ii}^\lambda = \chi^\lambda(h)$.

Since the action of D and p_i^λ commute in all cases, the trace property gives that $\operatorname{tr}(Dp_i^\lambda hp_j^\mu) = \operatorname{tr}(Dp_j^\mu p_i^\lambda h) = 0$ unless $\lambda=\mu$ and $i=j$.

$$\begin{aligned}\operatorname{tr}(Dh) &= \sum_{\substack{\lambda,\mu\vdash f\\ 1\leqq i,j\leqq d_\lambda}}\operatorname{tr}(Dp_i^\lambda hp_j^\mu)\\ &= \sum_{\substack{\lambda\vdash f\\ 1\leqq i\leqq d_\lambda}}\operatorname{tr}(Dp_i^\lambda hp_i^\lambda)\\ &= \sum_{\substack{\lambda\vdash f\\ 1\leqq i\leqq d_\lambda}}h_{ii}^\lambda\operatorname{tr}(Dp_i^\lambda)\,.\end{aligned}$$

By virtue of Lemma 3.5, $\operatorname{tr}(Dp_i^\lambda) = s_\lambda(x)$ for all $1\leqq i\leqq d_\lambda$. Since $\sum_i h_{ii}^\lambda = \chi^\lambda(h)$ we have that

$$\operatorname{tr}(Dh) = \sum_{\lambda\vdash f}\chi^\lambda(h)s_\lambda(x)\,. \;\square$$

Remark. Some possible choices for the partition of unity in the proof of the above theorem can be given explicitly. [Wz1] constructs a partition of unity in H_f with the property that at $q = 1$ it reduces to the partition of unity given by the Young orthogonal minimal idempotents of S_f. [Gy] constructs a partition of unity in H_f with the property that at $q = 1$ it is the partition of unity in $\mathbb{C}S_f$ given by Young symmetrizers. Either of these is sufficient for our purposes. The only advantage of the [Gy] approach is that it is immediate from the definition of the minimal idempotents that they are the same as the Young symmetrizers at $q = 1$.

4 The Frobenius formula for the characters of H_f

The crucial step in this development is to evaluate the left hand side of the expression in Theorem 3.8 by another means.

(4.1) Theorem. *For $1 \leqq r \leqq f$ the trace of the operator DT_{γ_r} on $V^{\otimes r}$ is given by*

$$\operatorname{tr}(DT_{\gamma_r}) = \sum_I q^{e(I)}(q-1)^{l(I)} x_{i_1} x_{i_2} \ldots x_{i_r} ,$$

where the sum is over all sequences $I = (i_1, i_2, \ldots, i_r)$ such that $1 \leqq i_1 \leqq \ldots \leqq i_r$, $e(I)$ denotes the number of $i_j = i_{j+1}$ in I and $l(I)$ denotes the number of $i_j < i_{j+1}$ in I.

Proof. The proof is by induction on r.

Let $w = v_{i_1} v_{i_2} \ldots v_{i_r}$, $w' = v_{i_1} v_{i_2} \ldots v_{i_{r-1}}$ and $w'' = v_{i_1} v_{i_2} \ldots v_{i_{r-2}}$.

Case 1 $i_{r-1} > i_r$:

$$Dwg_{r-1} \ldots g_1|_w = q^{1/2} D(w'' v_{i_r} v_{i_{r-1}}) g_{r-2} \ldots g_1|_{v_{i_1} \ldots v_{i_r}} .$$

Since $g_{r-2} \ldots g_1$ acts only on the letters in $w'' v_{i_r}$ and $v_{i_{r-1}} \neq v_{i_r}$, $DwT_{\gamma_r}|_w = 0$.

Case 2 $i_{r-1} < i_r$:

$$\begin{aligned} DwT_{\gamma_r}|_w &= (q-1) DwT_{\gamma_{r-1}}|_w + q^{1/2} w'' v_{i_r} v_{i_{r-1}} T_{\gamma_{r-1}}|_w \\ &= (q-1) x_{i_r} (Dw' T_{\gamma_{r-1}})|_{w'} + 0 . \end{aligned}$$

Case 3 $i_{r-1} = i_r$:

$$DwT_{\gamma_r}|_w = qDwT_{\gamma_{r-1}}|_w = qx_{i_r}(Dw' T_{\gamma_{r-1}})|_{w'} .$$

So, by induction,

$$\begin{aligned} \operatorname{tr}(DT_{\gamma_r}) &= \sum_{\substack{w \\ |w| = r}} DwT_{\gamma_r}|_w \\ &= \sum_{1 \leqq i_1, i_2, \ldots, i_r \leqq n} Dv_{i_1} v_{i_2} \ldots v_{i_r} T_{\gamma_r}|_{v_{i_1} v_{i_2} \ldots v_{i_r}} \\ &= \sum_{i_1 \leqq i_2 \leqq \ldots \leqq i_r} q^{\#\text{ of } i_j = i_{j+1}} (q-1)^{\#\text{ of } i_j < i_{j+1}} x_{i_1} x_{i_2} \ldots x_{i_r} . \ \square \end{aligned}$$

(4.2) Proposition. *For any composition $c \models f$,*

$$\operatorname{tr}(DT_{\gamma_c}) = \prod_i \operatorname{tr}(DT_{\gamma_{c_i}}) .$$

Proof. It is sufficient to prove that for permutations $\rho \in S_{r_1}$ and $\sigma \in S_{r_2}$, $r_1 + r_2 = f$,

$$\operatorname{tr}(DT_{\rho\times\sigma}) = \operatorname{tr}(DT_\rho)\operatorname{tr}(DT_\sigma) .$$

By direct computation, since T_ρ acts only on the first r_1 letters of a word w and T_σ acts only on the remaining letters of w,

$$\begin{aligned}\sum_w DwT_{\rho\times\sigma}|_w &= \sum_{\substack{w_1w_2=w\\ |w_1|=r_1,\,|w_2|=r_2}} Dw_1w_2T_{\rho\times\sigma}|_{w_1w_2}\\ &= \Bigg(\sum_{\substack{w_1\\ |w_1|=r_1}} Dw_1T_\rho|_{w_1}\Bigg)\Bigg(\sum_{\substack{w_2\\ |w_2|=r_2}} Dw_2T_\sigma|_{w_2}\Bigg) . \quad \square\end{aligned}$$

Remark. The above proposition is just a proof of the fact that the trace of the action of $H_{\lambda_1}\otimes H_{\lambda_2}\otimes\ldots$ on the tensor product representation $V^{\otimes\lambda_1}\otimes V^{\otimes\lambda_2}\otimes\ldots$ is the product of the traces of the actions of the H_{λ_i} on $V^{\otimes\lambda_i}$.

λ-ring notation and Hall-Littlewood symmetric functions

In order to simplify the derivation of formulas for the irreducible characters of the Hecke algebras we shall use λ-ring notation for symmetric functions. The following is a short exposition of λ-ring notation. The identities (4.3–4.10) are all well known and can be found in [Mac].

An alphabet is a sum of commuting variables so that, for example, $X = x_1 + x_2 + \ldots + x_n$ is the alphabet of commuting variables $x_1, \ldots, x_n$. In this notation, if $X = x_1 + \ldots + x_n$ and $Y = y_1 + \ldots + y_n$ then XY represents the alphabet of variables $\{x_iy_j\}_{1\leqq i,j\leqq n}$.

For each $r \geqq 1$ the power symmetric function p_r is given by

$$\begin{aligned} p_r(0) &= 0 ,\\ p_r(x) &= x^r ,\\ p_r(X+Y) &= p_r(X) + p_r(Y) , \qquad (4.3)\\ p_r(XY) &= p_r(X)p_r(Y) ,\end{aligned}$$

where x is any single variable and X and Y are any two alphabets. For each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ define

$$p_\mu(X) = p_{\mu_1}(X)p_{\mu_2}(X)\ldots p_{\mu_k}(X) .$$

For an alphabet X, we define

$$\Omega(X) = \exp\Bigg(\sum_{r\geqq 1}\frac{p_r(X)}{r}\Bigg) .$$

For each $r \geqq 0$ define the homogeneous symmetric function $h_r(X)$ to be the coefficient of t^r in $\Omega(Xt)$, i.e.

$$\Omega(Xt) = \sum_{r \geqq 0} h_r(X) t^r . \tag{4.4}$$

For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ define

$$h_\mu(X) = h_{\mu_1}(X) h_{\mu_2}(X) \ldots h_{\mu_k}(X) .$$

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the Schur function $s_\lambda(X)$ in the alphabet X can be given by

$$s_\lambda(X) = \det(h_{\lambda_i - i + j}(X))_{1 \leqq i, j \leqq n} ,$$

where $n \geqq \ell(\lambda)$, and $h_r(X) = 0$ for any $r < 0$.

Let X, Y be two alphabets. The *addition formula* for Schur functions is

$$s_\lambda(X + Y) = \sum_{\mu \subseteq \lambda} s_\mu(X) s_{\lambda/\mu}(Y) . \tag{4.5}$$

The *duality formula* for Schur functions is

$$s_\lambda(-X) = (-1)^{|\lambda|} s_{\lambda'}(X) \tag{4.6}$$

where λ' denotes the conjugate partition to λ.

For a brief moment let S_λ, S_μ, S_ν denote the irreducible representations of the symmetric group S_f corresponding to the partitions $\lambda, \mu, \nu \vdash f$ respectively. Set $c_{\lambda\mu\nu}$ to be the multiplicity of the representation S_λ in the Kronecker product of symmetric group representations $S_\mu \otimes S_\nu$. The numbers $c_{\lambda\mu\nu}$ are symmetric with respect to all three partitions λ, μ, ν. This given, the *product formula* for Schur functions is

$$s_\lambda(XY) = \sum_{\mu, \nu} c_{\lambda\mu\nu} s_\mu(X) s_\nu(Y) . \tag{4.7}$$

Let $m_\lambda(X)$ be the monomial symmetric function in the alphabet X. For any alphabet X,

$$h_\mu(X) = \sum_\lambda s_\lambda(X) K_{\lambda\mu}, \quad \text{and}$$

$$s_\lambda(X) = \sum_\mu K_{\lambda\mu} m_\mu(X) , \tag{4.8}$$

where $K_{\lambda\mu}$ is the Kostka number, the number of column strict fillings of the shape λ with weight μ.

For alphabets $X = x_1 + \ldots + x_n$, $Y = y_1 + \ldots + y_n$ and a variable t,

$$\Omega(XY(1-t)) = \prod_{1 \leqq i,j \leqq n} \frac{1 - t x_i y_j}{1 - x_i y_j} . \tag{4.9}$$

The following expansions of $\Omega(XY(1-t))$ follow from the expansions in [Mac, Chapter I, §4].

$$\begin{aligned}\Omega(XY(1-t)) &= \sum_\lambda s_\lambda(X)s_\lambda(Y(1-t)) \\ &= \sum_\lambda h_\lambda(X(1-t))m_\lambda(Y) \qquad (4.10)\\ &= \sum_\lambda \frac{p_\lambda(1-t)}{\lambda?}\, p_\lambda(X)p_\lambda(Y)\,,\end{aligned}$$

where $\lambda?$ is given by (1.1).

Chapter III of [Mac] is devoted to the study of Hall-Littlewood symmetric functions. In particular the Hall-Littlewood symmetric function $q_r(x_1, \ldots, x_n; t)$ is defined by

$$q_0(x; t) = 1\,,$$

$$q_r(x; t) = (1-t)\sum_{i=1}^{n} x_i^r \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \quad (r \geqq 1)\,.$$

For a partition μ,

$$q_\mu(x; t) = \prod_{i=1}^{\ell(\mu)} q_{\mu_i}(x; t)\,.$$

The generating function for the $q_r(x; t)$ is ([Mac, Chapter III (2.10)])

$$\sum_{r=0}^{\infty} q_r(x; t)y^r = \prod_i \frac{1 - tx_iy}{1 - x_iy}\,. \tag{4.11}$$

Expressions for $\mathrm{tr}(DT_{\gamma_r})$

(4.12) Lemma. *Let t be a variable and $\lambda \vdash f$. Then, in λ-ring notation,*

$$s_\lambda(1-t) = \begin{cases} (1-t)(-t)^{f-m}, & \text{if } \lambda = (1^{f-m}m) \text{ for some } m \geqq 1\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

Proof. By the addition formula

$$s_\lambda(1-t) = \sum_{\mu \subseteq \lambda} s_\mu(1)s_{\lambda/\mu}(-t)\,.$$

$s_\mu(1) = 0$ unless $\mu = (k)$ for some k. Using duality,

$$s_{\lambda/(k)}(-t) = (-1)^{f-k}s_{(\lambda/(k))'}(t) = \begin{cases} (-t)^{f-k}, & \text{if } \lambda/(k) \text{ is a vertical strip,} \\ 0, & \text{if } \lambda/(k) \text{ is not a vertical strip}\,. \end{cases}$$

There are only two cases when $\lambda/(k)$ is a vertical strip: either $\lambda = (1^{f-k}k)$ or $\lambda = (1^{f-k-1}(k+1))$. So if $s_\lambda(1-t) \neq 0$, $\lambda = (1^{f-m}m)$ for some m.

$$\begin{aligned} s_{1^{f-m}m}(1-t) &= \sum_{(k) \subseteq (1^{f-m}m)} s_{(k)}(1) s_{(1^{f-m}m)/(k)}(-t) \\ &= s_{(1^{f-m}m)/(m)}(-t) + s_{(1^{f-m}m)/(m-1)}(-t) \\ &= (-t)^{f-m} + (-t)^{f-m+1} = (1-t)(-t)^{f-m} . \ \square \end{aligned}$$

(4.13) Theorem. *Let $X = x_1 + x_2 + \ldots + x_n$. Then, in λ-ring notation, for $r \geqq 1$,*

(a) $\operatorname{tr}(DT_{\gamma_r}) = \sum_{\mu \vdash r} q^{r-\ell(\mu)}(q-1)^{\ell(\mu)-1} m_\mu(X)$,

(b) $\operatorname{tr}(DT_{\gamma_r}) = \dfrac{q^r}{q-1} q_r(X; q^{-1})$,

(c) $\operatorname{tr}(DT_{\gamma_r}) = \dfrac{q^r}{q-1} h_r(X(1-q^{-1})$,

(d) $\operatorname{tr}(DT_{\gamma_r}) = \dfrac{1}{q-1} \sum_{\mu \vdash r} \dfrac{\prod_i (q^{\mu_i}-1)}{\mu?} p_\mu(X)$,

(e) $\operatorname{tr}(DT_{\gamma_r}) = \sum_{k=1}^{r} (-1)^{r-k} q^{k-1} s_{(1^{r-k}k)}(X)$,

(f) *For* $\mu \vdash f$, $\operatorname{tr}(DT_{\gamma_\mu}) = \dfrac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} h_\mu(X(1-q^{-1}))$.

Proof. (a) follows immediately from Theorem 4.1. Let $\tilde{q}_r(X; q) = \operatorname{tr}(DT_{\gamma_r})$ and rewrite $\tilde{q}_r(x; q)$ in the form

$$\tilde{q}_r(x; q) = \frac{q^r}{q-1} \sum_{\mu \vdash r} \left(\frac{q-1}{q}\right)^{\ell(\mu)} m_\mu(X) .$$

If $\tilde{q}_0(x; q) = 1/(q-1)$ then the generating function for the $\tilde{q}_r(x, q)$ is

$$\begin{aligned} \sum_{r \geqq 0} \tilde{q}_r(x; q) y^r &= \frac{1}{q-1} \sum_{r \geqq 0} q^r y^r \sum_{\mu \vdash r} \left(\frac{q-1}{q}\right)^{\ell(\mu)} m_\mu(X) \\ &= \frac{1}{q-1} \prod_i \left(\frac{\frac{q-1}{q}}{1-qyx_i} - \frac{q-1}{q} + 1 \right) \\ &= \frac{1}{q-1} \prod_i \left(\frac{1 - 1/q + (1/q)(1-qyx_i)}{1-yx_i} \right) \\ &= \frac{1}{q-1} \prod_i \left(\frac{1-(1/q)qyx_i}{1-qyx_i} \right) , \end{aligned}$$

giving (b).

$$\prod_i \left(\frac{1-(1/q)qyx_i}{1-qyx_i} \right) = \Omega(Xqy(1-q^{-1})) ,$$

and the expansions in (4.10) give

$$\begin{aligned}\sum_{r\geqq 0} \tilde{q}_r(x;q)y^r &= \frac{1}{q-1}\,\Omega(Xqy(1-q^{-1}))\\ &= \frac{1}{q-1}\sum_\mu s_\mu(X)s_\mu(qy(1-q^{-1}))\\ &= \frac{1}{q-1}\sum_{r\geqq 0} h_r(X(1-q^{-1}))q^r y^r\\ &= \frac{1}{q-1}\sum_{r\geqq 0}\sum_{\mu\vdash r}\frac{\prod_i(1-q^{-\mu_i})}{\mu?}\,p_\mu(X)q^r y^r\,.\end{aligned}$$

(c) and (d) follow from the last two expressions respectively by comparing coefficients of y^r with the left hand side. Using Lemma 4.12,

$$\begin{aligned}\sum_\mu s_\mu(X)s_\mu(qy(1-q^{-1})) &= \sum_{r\geqq 0}\sum_{\mu\vdash r} s_\mu(X)q^{|\mu|}y^{|\mu|}s_\mu(1-q^{-1})\\ &= \sum_{r\geqq 0} q^r y^r \sum_{m=1}^{r} s_{(1^{r-m}m)}(X)(1-q^{-1})(-q^{-1})^{r-m}\\ &= \sum_{r\geqq 0} y^r(q-1)\sum_{m=1}^{r}(-1)^{r-m}q^{m-1}s_{(1^{r-m}m)}(X)\,,\end{aligned}$$

giving (e). (f) follows immediately from (c). □

The Frobenius formula

Let $m_\mu(x)$ and $s_\lambda(x)$ denote the monomial symmetric function and the Schur function respectively, in the variables $x_1, x_2, \ldots, x_n$. Recall that $T_{\gamma_r} = g_{r-1}g_{r-2}\ldots g_1$ and that for any partition $\mu = (\mu_1, \ldots, \mu_k)$, $T_{\gamma_\mu} = T_{\gamma_{\mu_1}} \times \ldots \times T_{\gamma_{\mu_k}}$.

For each $r \geqq 1$, and each partition μ define

$$\tilde{q}_r(x;q) = \sum_{\mu\vdash r} q^{r-\ell(\nu)}(q-1)^{\ell(\nu)-1}m_\nu(x)\,,$$

$$\tilde{q}_\mu(x_1,\ldots,x_n;q) = \prod_{i=1}^{\ell(\mu)}\tilde{q}_{\mu_i}(x;q)\,,$$

respectively.

We have the following Frobenius type formula for the characters of H_f.

(4.14) Theorem. *For each* $\mu\vdash f$,

$$\tilde{q}_\mu(x;q) = \sum_{\lambda\vdash f}\chi^\lambda(T_{\gamma_\mu})s_\lambda(x)\,,$$

where χ^λ *denotes the irreducible character of* H_f *corresponding to* λ *and* $s_\lambda(x)$ *denotes the Schur function in the variables* $x_1, \ldots, x_n$.

Proof. The theorem follows immediately from Theorem 3.8 and Theorem 4.13(a). □

5 Formulas for the irreducible characters of H_f

(5.1) Theorem. *For each T_σ, $\sigma \in S_f$, there exists a $\mathbb{Z}[q]$ linear combination*

$$c_\sigma = \sum_{\mu \vdash f} a_{\sigma\mu} T_{\gamma_\mu},$$

$a_{\sigma\mu} \in \mathbb{Z}[q]$, such that $\chi(T_\sigma) = \chi(c_\sigma)$ for all characters χ of H_f.

Proof. Let c be a composition of f and let μ be the partition of f determined by rearranging the parts of c in decreasing order. Then by Theorem 4.2 we know that $\operatorname{tr}(DT_{\gamma_c}) = \operatorname{tr}(DT_{\gamma_\mu})$. Since the Schur functions $s_\lambda(x_1, x_2, \ldots, x_n)$ are linearly independent, it follows from Theorem 3.8 that $\chi^\lambda(T_{\gamma_c}) = \chi^\lambda(T_{\gamma_\mu})$ for all irreducible characters χ^λ of H_f. This shows that it is sufficient to prove that there exists a $\mathbb{Z}[q]$ linear combination

$$d_\sigma = \sum_{c \models f} b_{\sigma c} T_{\gamma_c},$$

$b_{\sigma c} \in \mathbb{C}(q)$, such that $\chi(T_\sigma) = \chi(d_\sigma)$ for all characters χ of H_f.

Let χ be a character of H_f and let $\sigma \in S_f$. Let i be the first i such that $\sigma(i) > i + 1$. The proof is by induction on i and reverse induction on $\sigma(i)$. Note that any σ that does not have such an i is of the form γ_c for some $c \models f$.

Let $j = \sigma(i) - 1$. Since $\sigma^{-1}(j) > i$ and $\sigma^{-1}(j+1) = i$ we have that $\ell(\sigma s_j) < \ell(\sigma)$ and thus $T_\sigma = T_{\sigma s_j} g_j$.

Case 1 $\ell(s_j \sigma s_j) > \ell(\sigma s_j)$:

Then

$$\begin{aligned}\chi(T_\sigma) &= \chi(T_{\sigma s_j} g_j)\\ &= \chi(g_j T_{\sigma s_j})\\ &= \chi(T_{s_j \sigma s_j}).\end{aligned}$$

Note now that for the permutation $\sigma' = s_j \sigma s_j$ we have $\sigma'(i) = j = \sigma(i) - 1$.

Case 2 $\ell(s_j \sigma s_j) < \ell(\sigma s_j)$:

Then $T_{\sigma s_j} = g_j T_{s_j \sigma s_j}$ and thus $T_\sigma = g_j T_{s_j \sigma s_j} g_j$. So

$$\begin{aligned}\chi(T_\sigma) &= \chi(g_j T_{s_j \sigma s_j} g_j)\\ &= \chi(g_j^2 T_{s_j \sigma s_j})\\ &= (q-1)\chi(g_j T_{s_j \sigma s_j}) + q\chi(T_{s_j \sigma s_j})\\ &= (q-1)\chi(T_{\sigma s_j}) + q\chi(T_{s_j \sigma s_j}).\end{aligned}$$

Here again, for each of the permutations $\sigma' = \sigma s_j$ and $\sigma' = s_j \sigma s_j$ we have that $\sigma'(i) = j = \sigma(i) - 1$.

Thus $\chi(T_\sigma)$ can be written as a linear combination of $\chi(T_{\sigma'})$ where for each σ' we have that $\sigma'(i) = \sigma(i) - 1$. This completes the induction. □

The following corollary is an immediate consequence of Theorem 5.1.

(5.2) Corollary. *Any character of H_f is determined by its values on the elements T_{γ_μ}, $\mu \vdash f$.*

Since H_f is semisimple and its irreducible components are indexed by partitions of f, every character of H_f must be a linear combination of the irreducible characters χ^λ, $\lambda \vdash f$, of H_f. This implies that the set of values $\chi(T_{\gamma_\lambda})$, $\lambda \vdash f$, is actually a minimal set of values necessary to determine the character χ of H_f.

The following is an example to illustrate the algorithm used for finding the linear combination c_σ in Theorem 5.1. The notation is such that 3421 denotes the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ in two line notation.

$$\begin{aligned}\chi(T_{3421}) &= (q-1)\chi(T_{2431}) + q\chi(T_{2341}) \\ &= (q-1)((q-1)\chi(T_{2341}) + q\chi(T_{2314})) + q\chi(T_{2341}) \\ &= ((q-1)^2 + q)T_{\gamma_3} + (q-1)qT_{\gamma_2 \times \gamma_1}\,.\end{aligned}$$

A formula in terms of characters of the symmetric group

For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of f let $\phi^\lambda\colon \mathbb{C}S_f \to \mathbb{C}$ be the character of the representation of S_f given by inducing the trivial representation of the group $S_{\lambda_1} \times \ldots \times S_{\lambda_k}$ to S_f. Let $\phi^\lambda(\mu)$ denote the value ϕ^λ on elements of S_f of cycle type μ. It is well known that the $\phi^\lambda(\mu)$ are the coefficients in the expansion,

$$p_\mu(x) = \sum_\lambda \phi^\lambda(\mu) m_\lambda(x)\,, \tag{5.3}$$

of the power symmetric functions $p_\mu(x)$ in terms of the monomial symmetric functions $m_\lambda(x)$ in the variables $x_1, \ldots, x_n$.

(5.4) Theorem. *Let $\lambda \vdash f$ and let χ^λ be the irreducible character of the Hecke algebra H_f corresponding to λ. Then*

$$\chi^\lambda(T_{\gamma_\mu}) = \frac{1}{(q-1)^{\ell(\mu)}} \sum_{\nu \vdash f} \frac{\prod_i (q^{\nu_i} - 1)}{\nu?} \chi^\lambda_{S_f}(\nu)\phi^\mu(\nu)\,,$$

where $\nu?$ is given by (1.1) and $\chi^\mu_{S_f}$ is the irreducible character of S_f corresponding to the partition μ. The ν_i are the parts of the partition ν.

Proof. Using the expansions in (4.10),

$$\begin{aligned}h_\mu(X(1-t))|_{s_\lambda(X)} &= \sum_\mu h_\mu(X(1-t))m_\mu(Y)|_{s_\lambda(X)m_\mu(Y)} \\ &= \sum_\nu \frac{\prod_i (1-t^{\nu_i})}{\nu?} p_\nu(X)p_\nu(Y)|_{s_\lambda(X)m_\mu(Y)}\,.\end{aligned}$$

By the expansion

$$p_\mu(X) = \sum_\lambda \chi^\lambda_{S_f}(\mu) s_\lambda(X)$$

and (5.3),

$$\begin{aligned} h_\mu(X(1-t))|_{s_\lambda(X)} &= \sum_\nu \frac{\prod_i (1-t^{\nu_i})}{\nu?} \left(\sum_\lambda \chi^\lambda_{S_f}(\nu) s_\lambda(X) \right) \left(\sum_\mu \phi^\mu(\nu) m_\mu(Y) \right) \Bigg|_{s_\lambda(X) m_\mu(Y)} \\ &= \sum_\nu \frac{\prod_i (1-t^{\nu_i})}{\nu?} \chi^\lambda_{S_f}(\nu) \phi^\mu(\nu) \,. \end{aligned}$$

The result follows from Theorem 4.13(f) and the Frobenius formula. □

Remark. For each permutation $\pi \in S_f$ define $\tau(\pi) = (\tau_1, \ldots, \tau_k)$ to be the cycle type of π and for each $\mu \vdash f$ define a class function $\Phi^\mu : S_f \to \mathbb{Z}[q]$ by

$$\Phi^\mu(\pi) = \frac{\phi^\mu(\pi)}{(q-1)^{\ell(\mu)}} \cdot \frac{\prod_i (q^{\tau_i} - 1)}{\tau(\pi)?} \,.$$

The usual inner product on class functions is given by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{f!} \sum_{\pi \in S_f} \chi_1(\pi) \chi_2(\pi)$$

for class functions χ_1, χ_2 on S_f. Using this one can restate Theorem 5.4 in the form

$$\chi^\lambda(T_{\gamma_\mu}) = \langle \chi^\lambda_{S_f}, \Phi^\mu \rangle \,,$$

where, as before, $\chi^\lambda_{S_f}$ denotes the irreducible character of S_f corresponding to the partition λ.

A formula in terms of multiplicities

(5.5) Theorem. *Let $c_{\lambda\mu\nu}$ denote the Kronecker coefficient and let $K_{\lambda\mu}$ be the Kostka number. Then the irreducible character of the Hecke algebra corresponding to the partition λ is given by the formula*

$$\chi^\lambda(T_{\gamma_\mu}) = \frac{1}{(q-1)^{\ell(\mu)-1}} \sum_{m=1}^{f} (-1)^{f-m} q^{m-1} \sum_\nu c_{\lambda\nu(1^{f-m}m)} K_{\nu\mu} \,.$$

Proof. Using the product formula for Schur functions and then Lemma 4.12 we have that

$$\begin{aligned} h_\mu(X(1-t)) &= \sum_\nu s_\nu(X(1-t)) K_{\nu\mu} \\ &= \sum_\nu \sum_\lambda \sum_\rho c_{\nu\lambda\rho} s_\lambda(X) s_\rho(1-t) K_{\nu\mu} \\ &= \sum_\lambda s_\lambda(X) \sum_\nu \sum_{m=1}^{f} c_{\nu\lambda(1^{f-m}m)} s_{(1^{f-m}m)}(1-t) K_{\nu\mu} \\ &= \sum_\lambda s_\lambda(X) \sum_{m=1}^{f} (1-t)(-t)^{f-m} \sum_\nu c_{\nu\lambda(1^{f-m}m)} K_{\nu\mu} \,. \end{aligned}$$

Then, by Theorem 4.13(f) and the Frobenius formula,

$$\chi^\lambda(T_{\gamma_\mu}) = \frac{q^f}{(q-1)^{\ell(\mu)}} \sum_{m=1}^{f} (1-q^{-1})(-q^{-1})^{f-m} \sum_\nu c_{\nu\lambda(1^{f-m}m)} K_{\nu\mu} . \ \square$$

6 A combinatorial rule for computing the irreducible characters of the Hecke algebras

In this section we rewrite the formula

$$\chi^\lambda(T_{\gamma_\mu}) = \frac{1}{(q-1)^{\ell(\mu)}} \sum_{m=1}^{f} (-1)^{f-m} q^{m-1}(q-1) \sum_\nu c_{\lambda\nu(1^{f-m}m)} K_{\nu\mu} \tag{6.1}$$

from Theorem 5.5 to derive a combinatorial rule for computing the irreducible characters of the Hecke algebra. For the remainder of this section we will assume that all symmetric functions are in the variables $x_1, x_2, \ldots, x_n$, so that we may write s_λ for $s_\lambda(x_1, \ldots, x_n)$.

For any two partitions $\lambda, \mu \vdash f$ define

$$s_\lambda \otimes s_\mu = \sum_{\nu \vdash f} c_{\lambda\mu\nu} s_\nu ,$$

where, as in (4.7), $c_{\lambda\mu\nu}$ denotes the Kronecker coefficient. Extend this definition linearly to all symmetric functions.

(6.2) Lemma.

$$\sum_\nu c_{\nu\lambda\rho} K_{\nu\mu} = (h_\mu \otimes s_\rho)|_{s_\lambda}$$

Proof.

$$\begin{aligned}\sum_\nu c_{\nu\lambda\rho} K_{\nu\mu} &= \sum_\nu (s_\nu \otimes s_\rho)|_{s_\lambda} K_{\nu\mu} \\ &= \left(\sum_\nu s_\nu K_{\nu\mu} \otimes s_\rho \right)\Bigg|_{s_\lambda} \\ &= (h_\mu \otimes s_\rho)|_{s_\lambda} . \ \square\end{aligned}$$

Evaluation of the product $h_\mu \otimes s_\rho$ is facilitated by the following theorem of Garsia and Remmel [G-R].

(6.3) Theorem.

$$h_\mu \otimes s_\rho = \sum_{(\rho)} \prod_{i=1}^{k} s_{\rho^{(i)}/\rho^{(i-1)}}$$

where the sum is over all sequences of partitions $(\rho) = (\emptyset = \rho^{(0)} \subseteq p^{(1)} \subseteq \ldots \subseteq \rho^{(k)} = \rho)$ *such that* $|\rho^{(i)} - \rho^{(i-1)}| = \mu_i$.

Applying Theorem 6.3 when $\rho = (1^{f-m}m)$ gives

$$h_\mu \otimes s_{(1^{f-m}m)} = \sum_{(m)} s_{(1^{\mu_1 - m_1} m_1)} \prod_{i=2}^{k} h_{m_i} e_{\mu_i - m_i} ,$$

where the sum is over all sequences of nonnegative integers $(m) = (m_1, m_2, \ldots, m_k)$ such that $m_1 > 0$, $\sum_i m_i = m$ and $m_i \leqq \mu_i$. Then

$$\sum_{m=1}^{f} (-1)^{f-m} q^{m-1}(q-1)(h_\mu \otimes s_{(1^{f-m}m)})$$
$$= \sum_{m=1}^{f} \sum_{(m)} (-1)^{f-m} q^{m-1}(q-1) s_{(1^{\mu_1-m_1}m_1)} \prod_{i=2}^{k} h_{m_i} e_{\mu_i - m_i} .$$

Since $|\mu| = f$ we can write this in the form

$$\sum_{m=1}^{f} \sum_{(m)} (-1)^{\mu_1-m_1} q^{m_1-1}(q-1) s_{(1^{\mu_1-m_1}m_1)} \prod_{i=2}^{k} (-1)^{\mu_i-m_i} q^{m_i} h_{m_i} e_{\mu_i-m_i}$$
$$= \sum_{(m)} (-1)^{\mu_1-m_1} q^{m_1-1}(q-1) s_{(1^{\mu_1-m_1}m_1)} \prod_{i=2}^{k} (-1)^{\mu_i-m_i} q_{m_i} h_{m_i} e_{\mu_i-m_i} ,$$

where the last sum is over all sequences of nonnegative integers $(m) = (m_1, m_2, \ldots, m_k)$ such that $m_1 > 0$ and $m_i \leqq \mu_i$. Summarizing, we may write (6.1) in the form

$$(q-1)^{\ell(\mu)} \chi^\lambda(T_{\gamma_\mu}) = \sum_{(m)} (-1)^{\mu_1-m_1} q^{m_1-1}(q-1) s_{(1^{\mu_1-m_1}m_1)}$$
$$\times \prod_{i=2}^{k} (-1)^{\mu_i-m_i} q^{m_i} h_{m_i} e_{\mu_i-m_i}|_{s_\lambda} . \tag{6.4}$$

In the standard fashion (see [Mac]), to each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we associate a diagram of $\ell(\lambda)$ rows of boxes such that each row i contains λ_i boxes. If λ, μ are partitions, $\mu \subseteq \lambda$, then the skew diagram λ/μ, also denoted $\lambda - \mu$, is the set theoretic difference of the diagrams λ and μ. A horizontal (resp. vertical) strip is a skew diagram with at most one box in each column (row). A strip is a skew diagram that does not contain any 2×2 block of boxes. Two boxes are connected if they have an edge in common. A connected strip is called a hook. Any strip is a union of connected components, each of which is a hook. As an example, in the diagram (6.5), the diagonally hatched boxes form a horizontal strip, the cross hatched boxes form a vertical strip, and together the diagonally hatched boxes and the vertically hatched boxes form a strip.

Pieri's rule states that

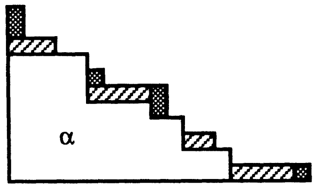
$$s_\alpha h_{m_i} = \sum_\beta s_\beta ,$$

where the sum is over all partitions β such that β/α is a horizontal strip of length m_i. Pictorially the shape of β is produced by adding a horizontal strip of m_i boxes to the shape of α. Similarly,

$$s_\alpha e_{\mu_i - m_i} = \sum_\beta s_\beta ,$$

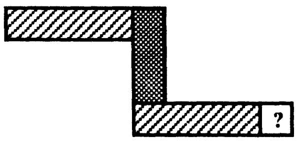
where this time the sum is over all partitions β such that β/α is a vertical strip of length $\mu_i - m_i$.

Thus one can view the product $s_\alpha h_{m_i} e_{\mu_i - m_i}$ as the sum of Schur functions of shapes obtained by an application of m_i boxes in a horizontal strip followed by an application of $\mu_i - m_i$ boxes in a vertical strip. In the same fashion the product $s_\alpha \cdot (-1)^{\mu_i - m_i} q^{m_i} h_{m_i} e_{\mu_i - m_i}$ is a sum of Schur functions obtained similarly except that each horizontal box contributes a weight of q and each vertical box contributes a weight of -1 to the resulting Schur function. Observe that the result of an application of m_i boxes in a horizontal strip and $\mu_i - m_i$ boxes in a vertical strip to a shape α produces a shape β such that β/α is a strip of length μ_i.



(6.5)

Now, consider a strip of length μ_i and assume that it arose from an application of a horizontal strip followed by an application of a vertical strip. The strip is a union of components each of which is a hook. Any box with a box to its right must have come from the horizontal application. Any box with a box under it must have come from the vertical application. In any component only the rightmost bottommost (end of the hook) box could have been placed by either the horizontal or the vertical application of boxes.



Not including the ? box, the number of boxes resulting from a vertical application is equal to the number of rows in the hook minus 1. Similarly, the number of boxes resulting from a horizontal application is the number of columns in the hook minus 1. If we weight horizontal boxes by q and vertical boxes by -1 then the weight of the hook over all applications (first horizontal boxes and then vertical boxes) which produce this hook is

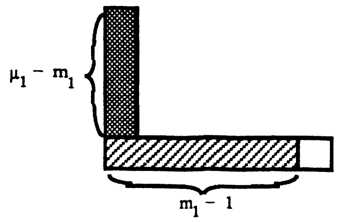
$$q^{\#\text{ cols} - 1}(-1)^{\#\text{ rows} - 1}(q-1)\,.$$

The q in the $(q-1)$ factor arises from the application in which the ? box is a horizontal box and the -1 in the $(q-1)$ factor from the application in which the ? box is a vertical box. The weight of the whole strip is the product of the above weight over all components in the strip.

In a similar fashion we can view the factor

$$(-1)^{\mu_1 - m_1} q^{m_1 - 1}(q-1) s_{(1^{\mu_1 - m_1} m_1)}$$

in (6.4) as having arisen from an application of a strip of length μ_1 to the empty partition $\varnothing$.



In this case the strip has exactly one component receiving a weight of $q^{\#\,\text{cols}\,-\,1}(-1)^{\#\,\text{rows}\,-\,1}(q-1)$.

In this way we rewrite the sum in (6.1) in the form

$$\frac{1}{(q-1)^{\ell(\mu)}}\sum_T wt(T)s_\lambda|_{s_\lambda}\,,$$

where the sum is over all tableaux T of shape λ arising from applications of strips of lengths μ_i and $wt(T)$ is given by

$$wt(T)=\prod_{\substack{\text{hook}\\ \text{components of } T}} q^{\#\,\text{cols in hook}\,-\,1}(-1)^{\#\,\text{rows in hook}\,-\,1}(q-1)\,. \tag{6.6}$$

Define a diagonally strict tableau of shape λ and weight μ to be a filling of the shape λ with μ_1 1's, μ_2 2's, etc., such that

(1) the rows of T are weakly increasing from left to right,
(2) the columns of T are weakly increasing from bottom to top,
(3) the diagonals are strictly increasing in the northeast direction.

For each $1 \leqq i \leqq \ell(\mu)$ the i's in T form a strip. A hook component of T is a component of one of these strips. We have proved the following theorem.

(6.7) Theorem. *For any two partitions $\lambda, \mu \vdash f$ the value of the irreducible character χ^λ at the element $T_{\gamma_\mu} \in H_f$ is given by*

$$\chi^\lambda(T_{\gamma_\mu})=\frac{1}{(q-1)^{\ell(\mu)}}\sum_T wt(T)\,,$$

where the sum is over all diagonally strict tableaux T of shape λ and weight μ and $wt(T)$ is given by (6.6).

As an example to illustrate Theorem 6.7 we calculate $\chi^\lambda(T_{\gamma_\mu})$ for the partitions $\lambda=(42)$ and $\mu=(33)$. The following gives the diagonally strict tableaux of shape λ and weight μ. $\chi^\lambda(T_{\gamma_\mu})$ is given by summing the weights ((6.6)) of these tableau and dividing by $(q-1)^{\ell(\mu)}$.

$$\begin{array}{|c|c|cc} \hline 1 & 2 & & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|cc} \hline 2 & 2 & & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$\chi^{(42)}(T_{\gamma(33)})=\frac{1}{(q-1)^{\ell(\mu)}}((-q)q(q-1)^3+q^2q(q-1)^3)=q^4-2q^3+q^2\,.$$

7 Work of King-Wybourne and Vershik-Kerov

After the preparation of the preliminary version of this manuscript other work concerning characters of the Hecke algebra has come to our attention. It is clear that the Frobenius formula for the characters of the Hecke algebra, our Theorem 4.14, has also been discovered independently by Vershik-Kerov and King-Wybourne.

The work of King and Wybourne is written up in a letter [KW1] and another more complete preprint [KW2]. Their approach to the Frobenius formula is by a formula relating the Ocneanu trace on the infinite Hecke algebra H_∞, the inductive limit of H_f as f goes to infinity, to the irreducible characters. The following gives their derivation of the Frobenius formula.

The Ocneanu trace tr is given as follows (see [O, FYHLMO, Jo]).

(7.1) Theorem. ([Jo] Theorem 5.1) *To each* $z \in \mathbb{C}$ *there is a linear trace* tr *on* $\bigcup_{f=1}^{\infty} H_f$ *uniquely defined by*

$$\mathrm{tr}(ab) = \mathrm{tr}(ba)\,,$$

$$\mathrm{tr}(1) = 1\,,$$

$$\mathrm{tr}(hg_f) = z\,\mathrm{tr}(h), \quad \textit{for } h \in H_f\,.$$

Any trace in H_f can be given as a linear combination of the irreducible characters of H_f. In particular for the Ocneanu trace we have

$$\mathrm{tr}(h) = \sum_{\lambda} W_\lambda(q, z)\chi^\lambda(h)\,, \tag{7.2}$$

where χ^λ is the irreducible character of H_f and $W_\lambda(q, z)$ is given by

$$W_\lambda(w, z) = s_\lambda\left(\frac{w - z}{1 - q}\right) = \prod_{(i,j)\in\lambda} \frac{wq^{i-1} - zq^{j-1}}{1 - q^{h(i,j)}}\,,$$

where $w = 1 - q + z$. Here $h(i,j) = \lambda_i - i + \lambda_j' - j + 1$ denotes the hook length at the position (i,j) in λ. The coefficients $W_\lambda(q, z)$ appearing in this decomposition are called the weights of the trace. The weights of tr were computed by Ocneanu [O]. A derivation of these weights is given in [Wz1]. The $W_\lambda(q, z)$ can be written as Schur functions, see [Mac, §3, Example 3]. In λ-ring notation we have

$$W_\lambda(q, z) = s_\lambda\left(\frac{w - z}{1 - q}\right).$$

This given, King and Wybourne prove the following lemma.

(7.3) Lemma. ([KW2, Lemma 2]) *For each* T_{γ_μ}, $\mu \vdash f$,

$$\mathrm{tr}(T_{\gamma_\mu}) = z^{f-\ell(\mu)} = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} h_\mu\left(\frac{w - z}{1 - q}(1 - q^{-1})\right).$$

Proof. Recall that if $\mu = (\mu_1, \ldots, \mu_k)$, $T_{\gamma_\mu} = T_{\mu_1} \times \ldots \times T_{\mu_k}$, and that $T_{\gamma_r} = g_{r-1} \ldots g_1$. Thus, from the definition of tr we have that $\mathrm{tr}(T_{\gamma_r}) = z^{r-1}$ and that $\mathrm{tr}(T_{\gamma_\mu}) = z^{\mu_1 - 1} \ldots z^{\mu_k - 1} = z^{f-k}$.

Now,

$$h_r\left(\frac{w-z}{1-q}(1-q^{-1})\right) = h_r\left(\frac{w-z}{1-q}\cdot\frac{q-1}{q}\right) = q^{-r}h_r(z-w) = q^{-r}s_{(r)}(z-w)\,.$$

Then using the addition formula for Schur functions and duality we have that

$$h_r(z-w) = \sum_{k=0}^{r} s_{(r-k)}(z)(-1)^k s_{(1^k)}(w)\,.$$

$s(w) = 1$, $s_{(1)}(w) = 1$, and $s_{(1^k)}(w) = 0$ for $k > 1$ giving

$$h_r(z-w) = s_{(r)}(z) - w s_{(r-1)}(z) = z^r - wz^{r-1} = (z-w)z^{r-1}\,.$$

Substituting $w = 1 - q + z$ we have shown that

$$\frac{q^r}{1-q}h_r\left(\frac{w-z}{1-q}(1-q^{-1})\right) = z^{r-1}\,.$$

The lemma follows since $h_\mu = h_{\mu_1}h_{\mu_2}\ldots h_{\mu_k}$. □

Combining the lemma with formula (7.2) we have

$$\frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}h_\mu\left(\frac{w-z}{1-q}(1-q^{-1})\right) = \sum_\lambda \chi^\lambda(h)s_\lambda\left(\frac{w-z}{1-q}\right).$$

Then, extending to an arbitrary alphabet, one gets the Frobenius formula

$$\frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}h_\mu(X(1-q^{-1})) = \sum_\lambda \chi^\lambda(h)s_\lambda(X)\,,$$

for any alphabet X.

Work of Vershik and Kerov

The Frobenius formula appears explicitly on p. 36 of a recent preprint of Kerov [K] with a reference to [VK1] for a proof. We have not been able to find any reference to or proof of the Frobenius formula in [VK1]. There is, however, another paper of Vershik-Kerov [VK2] which gives the relationship of the function $\tilde{q}_r$ to the characters of the infinite dimensional Hecke algebra $H_\infty(q)$. In [VK2] they give the generating function of the $\tilde{q}_r$ and remark that this is a Hall-Littlewood polynomial. They also remark that the characters of $H_\infty(q)$ are related to the solutions of the quantum Yang-Baxter equation (our §2) and give the action of the R-matrix on tensor space (our (3.2)). Lastly, they give necessary and sufficient conditions that a character of $H_\infty(q)$ is semi-Markov, i.e. whether the character is an Ocneanu trace. Despite the fact that there are no proofs given, these remarks give an outline of the mechanisms involved in obtaining the Frobenius formula.

8 Tables and formulas for special cases

The following special cases follow immediately from Theorem 6.7. For all $f \geqq 1$ and all $\mu \vdash f$,

$$\chi^{(1^f)}(T_{\gamma_\mu}) = (-1)^{f-\ell(\mu)}, \tag{8.1}$$

$$\chi^{(f)}(T_{\gamma_\mu}) = q^{f-\ell(\mu)}. \tag{8.2}$$

For all $f \geqq 1$ and all $\lambda \vdash f$,

$$\chi^\lambda((f)) = \begin{cases} 0 & \text{if } \lambda \text{ is not a hook}, \\ (-1)^{f-m} q^{m-1}, & \text{if } \lambda = (1^{f-m} m). \end{cases} \tag{8.3}$$

The following are tables of $\chi^\lambda(T_{\gamma_\mu})$ for H_f, up to $f = 6$.

$\mathbf{H_2(q)}$:

$\lambda \backslash \mu$	(1^2)	(2)
(1^2)	1	-1
(2)	1	q

$\mathbf{H_3(q)}$:

$\lambda \backslash \mu$	(1^3)	(21)	(3)
(1^3)	1	-1	1
(21)	2	$q-1$	$-q$
(3)	1	q	q^2

$\mathbf{H_4(q)}$:

$\lambda \backslash \mu$	(1^4)	(21^2)	(2^2)	(31)	(4)
(1^4)	1	-1	1	1	-1
(21^2)	3	$q-2$	$1-2q$	$1-q$	q
(2^2)	2	$q-1$	q^2+1	$-q$	0
(31)	3	$2q-1$	q^2-2q	q^2-q	$-q^2$
(4)	1	q	q^2	q^2	q^3

$H_5(q)$:

$\lambda\backslash\mu$	(1^5)	(21^3)	(2^21)	(31^2)	(32)	(41)	(5)
(1^5)	1	-1	1	1	-1	-1	1
(21^3)	4	$q-3$	$2-2q$	$2-q$	$2q-1$	$q-1$	$-q$
(2^21)	5	$2q-3$	q^2-2q+2	$1-2q$	$-q^2+q-1$	q	0
(31^2)	6	$3q-3$	q^2-4q+1	q^2-2q+1	$-2q^2+2q$	$-q^2+q$	q^2
(32)	5	$3q-2$	$2q^2-2q+1$	q^2-2q	q^3-q^2+q	$-q^2$	0
(41)	4	$3q-1$	$2q^2-2q$	$2q^2-q$	q^3-2q^2	q^3-q^2	$-q^3$
(5)	1	q	q^2	q^2	q^3	q^3	q^4

$H_6(q)$:

$\lambda\backslash\mu$	(1^6)	(21^4)	(2^21^2)	(2^3)
(1^6)	1	-1	1	-1
(21^4)	5	$q-4$	$-2q+3$	$3q-2$
(2^21^2)	9	$3q-6$	q^2-4q+4	$-3q^2+3q-3$
(2^3)	5	$2q-3$	q^2-2q+2	q^2+3q-1
(31^3)	10	$4q-6$	q^2-6q+3	$-3q^2+6q-1$
(321)	16	$8q-8$	$4q^2-8q+4$	q^3-5q^2+5q-1
(41^2)	10	$6q-4$	$3q^2-6q+1$	q^3-6q^2+3q
(3^2)	5	$3q-2$	$2q^2-2q+1$	q^3-3q^2-1
(42)	9	$6q-3$	$4q^2-4q+1$	$3q^3-3q^2+3q$
(51)	5	$4q-1$	$3q^2-2q$	$2q^3-3q^2$
(6)	1	q	q^2	q^3

$\lambda\backslash\mu$	(31^3)	(321)	(41^2)
(1^6)	1	-1	-1
(21^4)	$3-q$	$2q-2$	$q-2$
(2^21^2)	$3-3q$	$-q^2+3q-2$	$2q-1$
(2^3)	$1-2q$	$-q^2+q-1$	q
(31^3)	q^2-3q+3	$-2q^2+4q-1$	$-q^2+2q-1$
(321)	$2q^2-6q+2$	$2q^3-6q^2+6q-2$	$-2q^2+2q$
(41^2)	$3q^2-3q+1$	q^3-4q^2+2q	q^3-2q^2+q
(3^2)	q^2-2q	q^3-q^2+q	$-q^2$
(42)	$3q^2-3q$	$2q^3-3q^2+q$	q^3-2q^2
(51)	$3q^2-q$	$2q^3-2q^2$	$2q^3-q^2$
(6)	q^2	q^3	q^3

$\lambda\backslash\mu$	(3^2)	(42)	(51)	(6)
(1^6)	1	1	1	-1
(21^4)	$-2q+1$	$-2q+1$	$-q+1$	q
(2^21^2)	q^2-2q+1	q^2-q+1	$-q$	0
(2^3)	q^2+1	$-q$	0	0
(31^3)	$3q^2-2q$	$2q^2-2q$	q^2-q	$-q^2$
(321)	$-2q^3+2q^2-2q$	$-q^3+2q^2-q$	q^2	0
(41^2)	$-2q^3+3q^2$	$-2q^3+2q^2$	$-q^3+q^2$	q^3
(3^2)	q^4+q^2	$-q^3$	0	0
(42)	$q^4-2q^3+q^2$	$q^4-q^3+q^2$	$-q^3$	0
(51)	q^4-2q^3	q^4-2q^3	q^4-q^3	$-q^4$
(6)	q^4	q^4	q^4	q^5

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