ADVANCES IN MATHEMATICS 116, 263–321 (1995)

Characters of Algebras Containing a Jones Basic Construction: The Temperley–Lieb, Okada, Brauer, and Birman–Wenzl Algebras

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We begin by determining, in a general form, the characters of irreducible representations of a Jones basic construction and use this result to compute the characters of the Temperley–Lieb algebras and the Okada algebras. In the case of the Birman–Wenzl algebra some of the characters are determined by the general theorem and the others are computed by using the duality between the Birman–Wenzl algebras and the Drinfeld–Jimbo quantum groups of types B and C. The computations involve certain characters of the quantum group; these are polynomials invariant under the Weyl group of type B. We are able to decompose these Weyl group symmetric functions and obtain a combinatorial rule for computing the irreducible characters of the Birman–Wenzl algebras. The combinatorial rules for computing the irreducible characters of the Iwahori–Hecke algebras and the combinatorial rule for computing the irreducible characters of the Brauer algebras are both special cases of our rule for the Birman–Wenzl algebras. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let A, B be split semisimple algebras such that A is a subalgebra of B. Let $C = \mathrm{End}_A(B)$ be the centralizer of the action of A on B by left multiplication. The algebra C has very nice properties.

(1) The irreducible representations of C are in one-to-one correspondence with those of A, and

(2) there is an idempotent $e \in C$ such that $C = BeB$.

* Supported in part by Department of Education Fellowship P200A10014-92.
† Supported in part by a National Science Foundation postdoctoral fellowship.

0001-8708/95 $12.00

The algebra C is called the Jones basic construction for the inclusion $A \subseteq B$. We shall say that a split semisimple algebra $\mathscr{A}$ contains a basic construction if there exists an ideal $\tilde{C}$ of $\mathscr{A}$ and subalgebras $A \subseteq B \subseteq \mathscr{A}$ such that $\tilde{C}$ is isomorphic to a Jones basic construction for the inclusion $A \subseteq B$. There are several interesting examples of such algebras, in particular:

(1) The Temperley–Lieb algebras

(2) The Brauer algebras

(3) The Birman–Wenzl algebras

(4) The Okada algebras.

In Section 2 we give a general theorem which says that the characters of the Jones basic construction $C = \mathrm{End}_A(B)$ are completely determined by the characters of the subalgebra A (in fact, in some sense, they are equal). The remainder (and the difficult part) of this paper consists of computing the characters of the above algebras. Although the characters of the part of the algebra which is isomorphic to a Jones basic construction are determined by our general theorem the characters on the remaining portion of the algebra can be difficult to compute. The case of the Brauer algebra has been done previously in [R1]. The results from [R1] and the determination of the characters of the Temperley–Lieb algebras and the Okada algebras are given in the remainder of Section 2.

The determination of the characters of the Birman–Wenzl algebras involves considerably more work. To do this we have used the ideas of [R1] and [R2] to use the theory of the quantum group and the quantum Yang–Baxter equation as a tool. The Birman–Wenzl algebra maps surjectively onto the centralizer of the tensor power of the fundamental representation of the quantum enveloping algebras $U_q(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{sp}(2n))$ and one obtains a double centralizer correspondence between the Birman–Wenzl algebra and the quantum group analogous to the Schur–Weyl duality between the symmetric group and $GL(n)$. We then compute bitraces of the actions of the Birman–Wenzl algebra and the quantum group on tensor space and obtain certain polynomials, symmetric under the action of the Weyl group of type B. These polynomials can be viewed as generalizations of the power symmetric functions *and* of certain Hall–Littlewood polynomials. Upon expanding these polynomials in terms of the Weyl characters the coefficients are the characters of the Birman–Wenzl algebra.

Section 3 of this paper is concerned with finding a tractable basis of the Birman–Wenzl algebra and determining a subset of this basis (analogous to conjugacy class representatives for a finite group) such that the characters are determined by their values on this set. Many of the computations in this section are analogous to computations in knot theory and the

theory of tangles. This is not surprising as recent work on link invariants [Re], [RT], [TW] shows that any trace on the Birman–Wenzl algebras is a polynomial function of certain links and that appropriate (i.e. Markov) traces on the Birman–Wenzl algebras determine polynomial link invariants.

Section 4 gives a brief description of the duality between the Birman–Wenzl algebra and the quantum group. In Section 5 we give explicitly the action of the Birman–Wenzl algebra on tensor space and compute the bitraces necessary to determine the characters of the Birman–Wenzl algebras.

Section 6 is a study of the analogues of the power symmetric functions which we have obtained from the bitrace picture. We have developed the theory of symmetric functions for the Weyl groups of type B in a λ-ring format in order to expand these complicated symmetric functions in terms of the Weyl characters for $SO(2n+1)$. We are able to obtain generating functions for the bitrace symmetric functions and to give a combinatorial rule for computing the irreducible characters of the Birman–Wenzl algebra. This combinatorial rule uses the broken border strips which also appeared in the combinatorial rule for the characters of the Iwahori–Hecke algebras of type A. Furthermore, at $q=1$, our rule reduces to the rule given in [R1] for computing the characters of the Brauer algebras.

Section 7 contains concluding remarks. During the course of the work on this paper we have noticed several interesting subtleties, and in this section we have made some effort to describe these. We hope that by mentioning them here they will be of use to other researchers in this area.

Finally, we have included formulas for the irreducible characters of the Temperley–Lieb algebras and tables of irreducible characters for the Okada algebras and the Birman–Wenzl algebras in Section 8. Tables of the irreducible characters of the Brauer algebra can be gotten by setting $q=r=1$ in the tables for the Birman–Wenzl algebra and the character tables of the Iwahori–Hecke algebras of type A appear as the upper left portion of the tables of the Birman–Wenzl algebra characters.

We thank G. Benkart for many discussions and for her encouragement of us in our research, individually as well as together, and R. Stanley for telling us about the Okada algebra and for sending us the relevant preprint. We thank F. Goodman and S. Kerov for discussions concerning the Okada algebras and S. Kerov for graciously suggesting that we publish our proof of the Frobenius formula for the Birman–Wenzl algebra which appears without proof in [Ke].

2. Basic Construction

Let $A \subseteq B$ be split semisimple algebras such that A is a subalgebra of B. Let $\hat{A}$ and $\hat{B}$ be index sets for the irreducible representations of A and B.

Let V^λ and W^μ be the irreducible representations of A and B labeled by $\lambda \in \hat{A}$ and $\mu \in \hat{B}$, and let nonnegative integers g_{λ_μ} be defined by the restriction $W^\mu \downarrow_A^B \cong \bigoplus_{\lambda \in \hat{A}} g_{\lambda_\mu} V^\lambda$. The *Bratteli diagram* for the inclusion $A \subseteq B$ is the graph on two rows of vertices, with the vertices in the top row labeled by elements of $\hat{B}$ and the vertices in the bottom row labeled by elements of $\hat{A}$, and having g_{λ_μ} edges connecting $\mu \in \hat{B}$ to $\lambda \in \hat{A}$.

Let $A \subseteq B \subseteq C$ be split semisimple algebras such that A is subalgebra of B and B is a subalgebra of C and such that the Bratteli diagram for the tower of algebras $A \subseteq B \subseteq C$ satisfies the following property:

> *The reflections of the edges corresponding to the inclusion $A \subseteq B$ are a subset of the edges corresponding to the inclusion $B \subseteq C$.*

We say that C is an algebra *containing a Jones basic construction* for the inclusion $A \subseteq B$. Examples of such inclusions are given by the Okada algebras, the Temperley–Lieb algebras, the Brauer centralizer algebras and the Birman–Wenzl algebras. Bratteli diagrams for these examples are given later in this section.

Let us study this situation in further detail. Let A be a split semisimple algebra over a field F. A trace $\vec{t}: A \to F$ is a linear functional on A such that

$$\vec{t}(ab) = \vec{t}(ba), \qquad \text{for all} \quad a, b \in A. \tag{2.1}$$

A trace $\vec{t}$ on a split semisimple algebra A is determined by a vector $\vec{t} = (t_\lambda)_{\lambda \in \hat{A}}$ of elements of F, where $t_\lambda = \vec{t}(p_\lambda)$ for any minimal idempotent p_λ in the miniaml ideal A corresponding to $\lambda \in \hat{A}$. We say that $\vec{t}$ is nondegenerate if for every $a \in A$, $a \neq 0$, there exists an element $b \in A$ such that $\vec{t}(ba) \neq 0$. This is equivalent to saying that the bilinear form on A defined by

$$\langle a, b \rangle = \vec{t}(ab), \tag{2.2}$$

for all $a, b \in A$, is a nondegenerate bilinear form.

Let $A \subseteq B$ be split semisimple algebras such that A is a subalgebra of B. Let $\vec{t}$ be a trace on B which is nondegenerate on both A and B. Using the nondegeneracy of $\vec{t}$, we define a map $\varepsilon_A: B \to A$, called the *conditional expectation* by

$$\vec{t}(ba) = \vec{t}(\varepsilon_A(b)a), \qquad \text{for all} \quad a \in A. \tag{2.3}$$

The conditional expectation ε_A is the orthogonal projection of B onto A with respect to the bilinear form $\langle \ , \ \rangle$ defined in (2.2).

Let A, B, $\vec{t}$ and ε_A be as given above. Let $\vec{B}$ denote the left regular representation of B, i.e. as a vector space $\vec{B}$ is the same as B, and B acts on $\vec{B}$ by

$$b_1 \overrightarrow{b_2} = \overrightarrow{b_1 b_2},$$

for all $b_1 \in B$ and $\overrightarrow{b_2} \in \vec{B}$. Let $\bar{C} = \mathrm{End}_A(\vec{B})$, be the commutant of A acting on $\vec{B}$. Note that $B \cong \mathrm{End}_B(\vec{B})$ acting by right multiplication, so we have $B \subseteq \bar{C}$. Following [Jo], define $e_A \in \mathrm{End}_A(\vec{B}) = \bar{C}$ by

$$e_A \vec{b} = \overrightarrow{\varepsilon_A(b)} \tag{2.4}$$

for all $b \in B$. Viewing e_A, A, B and $\bar{C}$ inside of $\mathrm{End}(\vec{B})$, the following properties hold (see [Jo] and [Wn1]):

(1) $e_A^2 = e_A$;

(2) $e_A b e_A = \varepsilon_A(b) e_A = e_A \varepsilon_A(b)$, for all $b \in B$;

(3) The map $a \mapsto a e_A$ is an injective map from A into $\bar{C}$;

(4) $\bar{C} = B e_A B$;

(5) The irreducible components of $\bar{C}$ are in one-to-one correspondence with the irreducible components of A;

(6) In the Bratteli diagram for the tower $A \subseteq B \subseteq \bar{C}$, the edges corresponding to the inclusion $B \subseteq \bar{C}$ are the reflection of the edges corresponding to the inclusion $A \subseteq B$;

(7) If p is a minimal idempotent of A in the λth-irreducible component of A, then ep is a minimal idempotent in the corresponding irreducible component of $\bar{C}$.

In view of property (6) above the algebra $\bar{C}$ satisfies our definition of an algebra containing a Jones basic construction. The algebra is the original "basic" construction of Jones for the inclusion $A \subseteq B$.

Now assume that $A \subseteq B$ are split semisimple algebras that $\vec{t}$ is a non-degenerate trace on both B and A, and $\varepsilon_A : B \to A$ is a conditional expectation with respect to the trace $\vec{t}$. Suppose further that B is a subalgebra of a split semisimple algebra C and that there is an element $e \in C$ such that

(a) $e^2 = e$;

(b) $ebe = \varepsilon_A(b)\, e = e \varepsilon_A(b)$, for all $b \in B$; (2.5)

(c) The map $a \mapsto ae$ is an injective map from A into C.

Then we have the following theorem of Wenzl.

(2.6) THEOREM [Wn1]. *Given the above preparations view $BeB \subseteq C$ and $Be_A B = \bar{C} = \mathrm{End}_A(\vec{B})$. Then the map $BeB \to Be_A B$ defined by $e \mapsto e_A$ and $b \mapsto b$ for all $b \in B$ is well-defined and an algebra isomorphism.*

In some sense, this theorem says that properties (1), (2) and (3) of the basic construction $\bar{C}$ above are sufficient to guarantee that the algebra C contains a subspace BeB which is isomorphic to $\bar{C}$.

Now assume that BeB is an ideal of C (in all of our examples this will be the case). Then, since C is semisimple, there exists an ideal $\tilde{C}$ of C such that

$$C \cong BeB \oplus \tilde{C}. \tag{2.7}$$

By properties (3) and (4) of the Jones basic construction $\bar{C}$, and by Theorem (2.6), the irreducible representations of BeB are indexed by $\hat{A}$, and so $\hat{A} \subseteq \hat{C}$. Let χ_C^λ denote the irreducible C-character corresponding to $\lambda \in \hat{C}$, and let χ_A^μ denote the irreducible A-character corresponding to $\mu \in \hat{A} \subseteq \hat{C}$.

We shall refer to the ideal $BeB \subseteq C$ in the decomposition (2.7) as the basic construction for the inclusion $A \subseteq B$. We are interested in the irreducible characters of the ideal BeB.

(2.8) LEMMA. *Let BeB be a basic construction for $A \subseteq B$, and let χ be a character of BeB. Then χ is completely determined by the values $\chi(ae)$ where $a \in A$.*

Proof. Let $b_1 e b_2 \in BeB$, and let χ be a character of C. Then using (2.5)

$$\chi(b_1 e b_2) = \chi(b_1 e e b_2) = \chi(e b_2 b_1 e) = \chi(\varepsilon_A(b_2 b_1) e).$$

The lemma follows, since $\varepsilon_A(b_2 b_1) \in A$. ∎

Let P be a complete set of minimal orthogonal idempotents of A. This means that, in addition to being minimal idempotents, the elements of P satisfy

(1) $1 = \sum_{p \in P} p$, and

(2) $pp' = p'p = 0$, if $p, p' \in P$ and $p \neq p'$.

We shall sometimes refer to such a set of minimal idempotents as a *partition of unity*. Then, by property (7) above, the set $eP = \{ep \mid p \in P\}$ is a set of minimal orthogonal idempotents of $BeB \subseteq C$. (This is easy to check using the fact that e commutes with p since $p \in A$.) Extend the set eP to a complete set Q of minimal orthogonal idempotents of C. Then using the fact that e is idempotent and commutes with the elements of A, we have, for all $a \in A$,

$$ae = \left(\sum_{p \in P} p \right) ae \left(\sum_{p' \in P} p' \right) = \sum_{p, p' \in P} epaep'. \tag{2.9}$$

(2.10) PROPOSITION. *If $\lambda \in \hat{C}$ and $a \in A$, then*

$$\chi_C^\lambda(ae) = \begin{cases} \chi_A^\lambda(a), & \text{if } \lambda \in \hat{A}, \\ 0, & \text{if } \lambda \in \hat{C} \setminus \hat{A}. \end{cases}$$

Proof. Let Q be the set of minimal orthogonal idempotents of C described above. For $c \in C$ and $q \in Q$, let $c|_q$ denote the coefficient of c when expanded in terms of Q. Let Q^λ (resp. P^λ) be the set of minimal idempotents in Q (resp. P) in the irreducible component of C (resp. A) indexed by λ. Then,

$$\chi_C^\lambda(ae) = \sum_{q \in Q^\lambda} qaeq|_q = \sum_{q \in Q^\lambda} \sum_{p, p' \in P} q(ep)\, a(ep')q|_q,$$

where the second equality follows from (2.9). Notice that, since $ep \in Q$,

$$qep = \begin{cases} ep, & \text{if } q = ep, \\ 0, & \text{otherwise.} \end{cases}$$

Using this and the fact that $ep \in Q^\lambda$ if and only if $p \in P^\lambda$, we get

$$\chi_C^\lambda(ae) = \sum_{p \in P^\lambda} epaep|_{ep} = \sum_{p \in P^\lambda} pap|_p = \chi_A^\lambda(a). \quad \blacksquare$$

Let Ξ_C denote the character table for C, that is the table which has rows indexed by $\hat{C}$ and which has columns indexed by appropriate basis elements of C such that the character values on these elements are sufficient to determine the characters. Then Proposition (2.10) suggests that Ξ_C is of the form

$$\Xi_C = \left[\begin{array}{c|c} \Xi_A & 0 \\ \hline * & \Xi_{\tilde{C}} \end{array}\right] \tag{2.11}$$

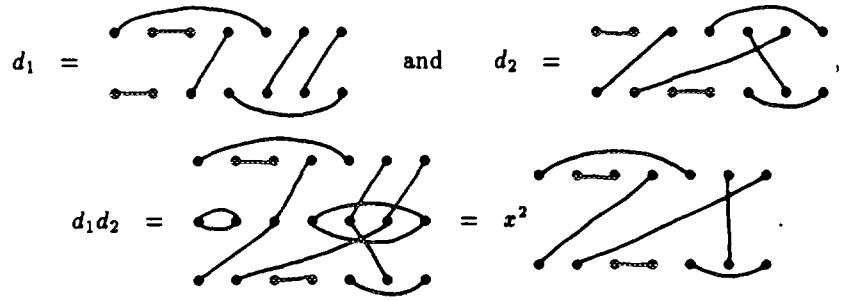
All of the character tables given in Section 8 have this form.

In the remainder of this section we describe the Brauer algebras, the Temperley–Lieb algebras, and the Okada algebras, all of which are algebras "containing a Jones basic construction" and which have a decomposition of the form (2.7).

The Brauer Algebra

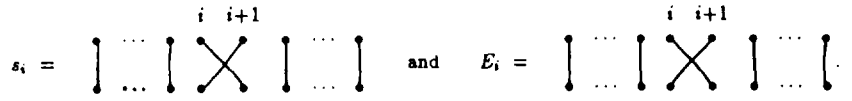
An *f-diagram* is a graph on two rows of f-vertices, one above the other, and $2f$ edges such that each vertex is incident to precisely one edge. The number of f-diagrams is $(2f)!! = (2f-1)(2f-3)\cdots 3 \cdot 1$. We multiply two f-diagrams d_1 and d_2 by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains f paths and some number γ of closed cycles. Let d be the f-diagram whose edges are the paths in this graph (with the

cycles removed). Then the product $d_1 d_2$ is given by $d_1 d_2 = x^c d$. For example, if



Let x be an indeterminate. The *Brauer algebra* $B_f(x)$ (defined originally by R. Brauer [Br]) is the $\mathbb{C}(x)$-span of the f-diagrams. Diagram multiplication makes $B_f(x)$ an associative algebra whose identity id_f is given by the diagram having each vertex in the top row connected to the vertex below it in the bottom row. By convention $B_0(x) = B_1(x) = \mathbb{C}(x)$. If d_1 and d_2 are f_1 and f_2-diagrams respectively, then $d_1 \otimes d_2$ is the $(f_1 + f_2)$-diagram obtained by placing d_2 to the right of d_1.

The group algebra $\mathbb{C}(x)[\mathscr{S}_f]$ of the symmetric group $\mathscr{S}_f$ is embedded in $B_f(x)$ as the span of the diagrams with only vertical edges. For $1 \leqslant i \leqslant f-1$, let

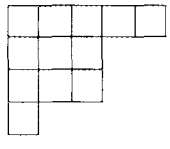


(2.12)

Then $E_i^2 = xE_i$, and the elements of the set $\{s_i, E_i \mid 1 \leqslant i \leqslant f-1\}$ generate $B_f(x)$. Note that the s_i correspond to the simple transpositions $(i, i+1)$ of $\mathscr{S}_f$ and that the s_i, $1 \leqslant i \leqslant f-1$, generate $\mathbb{C}(x)[\mathscr{S}_f]$.

A partition λ of the positive integer n, denoted $\lambda \vdash n$, is a non-decreasing sequence $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t \geqslant 0)$ of non-negative integers such that $\lambda_1 + \cdots + \lambda_t = n$. The length $l(\lambda)$ is the largest j such that $\lambda_j > 0$. The Young (or Ferrers) diagram of λ is the left-justified array of boxes with λ_i boxes in the ith row. For example,

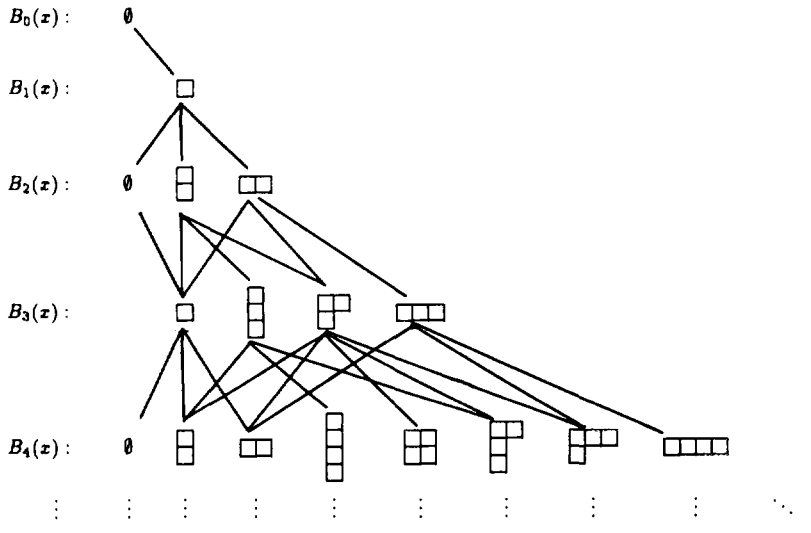
$(5, 3, 3, 1)$ =



is a partition of length 4. The algebra $B_f(x)$ is a split semisimple algebra over $\mathbb{C}(x)$ with irreducible representations labeled by partitions in the set

$$\hat{B}_f = \{\lambda \vdash (f-2k) \mid 0 \leqslant k \leqslant \lfloor f/2 \rfloor\}. \tag{2.13}$$

There is a natural inclusion of $B_{f-1}(x)$ into $B_f(x)$ given by viewing each $f-1$ diagram as an f-diagram by adding a pair of dots connected by a vertical edge to the right side. More precisely, $B_{f-1}(x) \subseteq B_f(x)$ by identifying a $(f-1)$-diagram d with the f-diagram $d \otimes id_1$. The Bratteli diagram for the inclusions $B_0(x) \subseteq B_1(x) \subseteq B_2(x) \subseteq \cdots$ is given by (see [Wn1])



There exists a nondegenerate trace tr on $B_f(x)$ defined inductively by

$$tr(1) = 1, \quad \text{and}$$

$$tr(as_{f-1}b) = tr(aE_{f-1}b) = (1/x)\, tr(ab),$$

for all $a, b \in B_{f-1}(x)$, and where s_i and E_i are the generators given by (2.12). Let

$$e_{f-1} = (1/x)E_{f-1}.$$

Then one may *define* a map $\varepsilon_{f-2}: B_{f-1}(x) \to B_{f-2}(x)$ by the relation

$$e_{f-1}be_{f-1} = \varepsilon_{f-2}(b) \otimes e_{f-1},$$

for all $b \in B_{f-1}(x)$. Wenzl [Wn1] shows (this is nontrivial) that:

(a) tr is a well defined nondegenerate trace on $B_{f-1}(x)$ which is also nondegenerate on $B_{f-2}(x)$;

(b) ε_{f-2} is a conditional expectation for the inclusion $B_{f-2}(x) \subseteq B_{f-1}(x)$ with respect to the trace tr on $B_{f-1}(x)$, and

(c) The element $e_{f-1} \in B_f(x)$ satisfies the properties (a), (b), and (c) of (2.5) for the inclusions $B_{f-2}(x) \subseteq B_{f-1}(x) \subseteq B_f(x)$.

Given these facts, one gets that $B_f(x)$ has a decomposition of the form (2.7); precisely,

$$B_f(x) \cong B_{f-1}(x)\, e_{f-1} B_{f-1}(x) \oplus \mathbb{C}(x)[\mathscr{S}_f], \tag{2.14}$$

where $B_{f-1}(x)\, e_{f-1} B_{f-1}(x)$ is a basic construction for $B_{f-2}(x) \subseteq B_{f-1}(x)$.

By Lemma (2.8), the characters of $B_{f-1}(x)\, e_{f-1} B_{f-1}(x)$ depend only on characters of elements of the form $ae_{f-1} = a \otimes e_{f-1}$ where $a \in B_{f-2}(x)$.

Let E denote the 2-diagram

$E =$  (2.15)

and define $e = (1/x)E$. Let γ_m denote the m-diagram

$\gamma_m =$  , (2.16)

and for the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$, let $\gamma_\mu = \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_t}$. The characters of elements in $\mathscr{S}_f$ depend only on the elements γ_μ, where $\mu \vdash f$, since any permutation whose cycle structure is given by μ is conjugate to γ_μ. Thus, by inductively applying Lemma (2.8), the irreducible characters of $B_f(x)$ are completely determined by their values on the diagrams of the form $\gamma_\mu \otimes e^{\otimes k}$, where $\mu \vdash (f-2k)$. By Proposition 2.10, if $\lambda \vdash (f-2l)$ then the irreducible $B_f(x)$-character χ_f^λ associated to λ is given by

$$\chi_f^\lambda(\gamma_\mu \otimes e^{\otimes k}) = \begin{cases} \chi_f^\lambda{}_{-2k}(\gamma_\mu), & \text{if } l \geqslant k, \\ 0, & \text{if } l < k. \end{cases} \tag{2.17}$$

For each complex number $\xi \in \mathbb{C}$ one defines a Brauer algebra $B_f(\xi)$ over $\mathbb{C}$ as the linear span of f-diagrams where the multiplication is given as above except with x replaced by ξ. R. Brauer [Br] originally introduced the Brauer algebra $B_f(n)$ in his study of the centralizer of the tensor representation of the complex orthogonal group $O(n) = \{ g \in M_n(\mathbb{C}) \mid gg^t = I \}$. Let $V = \mathbb{C}^n$ be the standard or fundamental representation for $O(n)$. The

tensor space $V^{\otimes f}$ is a completely reducible $O(n)$-module with irreducible summands labeled by partitions in the set

$$\hat{B}_f(n) = \{\lambda \vdash (f-2k) \mid 0 \leqslant k \leqslant \lfloor f/2 \rfloor,\ \lambda'_1 + \lambda'_2 \leqslant n\}.$$

Note that when n is sufficiently large $\hat{B}_f(n) = \hat{B}_f$ where $\hat{B}_f$ is as defined in (2.13). Brauer gives an action of $B_f(n)$ on $V^{\otimes n}$ which commutes with the action of $O(n)$ so we consider $V^{\otimes f}$ as a bimodule for $B_f(n) \times O(n)$. The decomposition of $V^{\otimes f}$ into irreducible $B_f(n) \times O(n)$-modules is (see [Wy])

$$V^{\otimes f} \cong \bigoplus_{\lambda \in \hat{B}_f(n)} D_\lambda \otimes V_\lambda, \tag{2.18}$$

where V_λ is the irreducible $O(n)$-module corresponding to λ and D_λ is an irreducible $B_f(n)$-module corresponding to λ.

For independent, commuting variables $x_1, x_2, \ldots, x_n$, define the *power symmetric functions* to be the polynomials in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ defined for each positive integer r by

$$p_r(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}) = x_1^r + x_2^r + \cdots + x_n^r + 1 + x_1^{-r} + x_2^{-r} + \cdots + x_n^{-r},$$

and for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ by $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_k}$. Define the *Weyl character for type B* by

$$sb_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}) = \frac{\det(x_i^{\lambda_j + n - j + 1/2} - x_i^{-(\lambda_j + n - j + 1/2)})}{\det(x_i^{n-j+1/2} - x_i^{-(n-j+1/2)})}.$$

By taking bitraces in (2.18), Ram [R1] proves the following Frobenius formula for $B_f(n+1)$ when n is odd

$$(2n+1)^h p_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\lambda \in \hat{B}_f(2n+1)} \chi^\lambda_{B_f(2n+1)}(d_\mu \otimes E^{\otimes h})\, sb_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}). \tag{2.19}$$

When n is even one can "essentially" replace sb_λ by sd_λ, the Weyl character of type D, and when n is negative one can "essentially" replace sb_λ by sc_λ, the Weyl character of type C (see [R1]).

Using (2.19) Ram [R1] proves the Murnaghan–Nakayama rule for computing $B_f(n)$ characters. A skew diagram λ/μ with r boxes is an *r-border strip* if it is connected and contains no 2×2 block of boxes. The weight of λ/μ is $\omega t(\lambda/\mu) = (-1)^{rows - 1}$ where *rows* is the number of rows in λ/μ. By convention $wt(\lambda/\lambda) = wt(\varnothing) = 1$. A *$\mu$-up-down border strip tableau* of shape λ is a sequence of partitions

$$T = (\varnothing = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(l(\mu))} = \lambda)$$

such that for each $1 \leqslant j \leqslant l(\mu)$, either

(1) $\lambda^{(j)} \subseteq \lambda^{(j-1)}$ and $\lambda^{(j-1)}/\lambda^{(j)}$ is a border strip with $|\lambda^{(j-1)}/\lambda^{(j)}| = \mu_j$,

(2) $\lambda^{(j-1)} \subseteq \lambda^{(j)}$ and $\lambda^{(j)}/\lambda^{(j-1)}$ is a border strip with $|\lambda^{(j)}/\lambda^{(j-1)}| = \mu_j$,

or

(3) $\lambda^{(j-1)} = \lambda^{(j)}$ and μ_j is even.

Let $wt(\mu, \nu) = wt(\mu/\nu)$, if μ/ν is a border strip, and $wt(\mu, \nu) = wt(\nu/\mu)$, if ν/μ is a border strip. Define

$$wt(T) = \prod_{j=1}^{h(\mu)} wt(\lambda^{(j-1)}, \lambda^{(j)}).$$

Then the next theorem is the Murnaghan–Nakayama rule for computing characters of the Brauer algebra.

(2.20) THEOREM [R1]. *The irreducible characters χ_f^λ, $\lambda \in \hat{B}_f$, of the Brauer algebra $B_f(x)$ are given by*

$$\chi_f^\lambda(\gamma_\mu \otimes E^{\otimes h}) = x^h \sum_T wt(T)$$

where the sum is over all μ-up-down border strip tableau of length f and shape λ.

The Birman Wenzl algebra $BW_f(r, q)$ is a q-deformation of the Brauer algebra $B_f(x)$, and in Sections 3–6 we generalize all of the above results to $BW_f(r, q)$. By setting $q = r = 1$ in Section 6 of this paper, Theorem (2.20) can be obtained from the combinatorial rule Theorem (6.15) for computing the characters of the Birman–Wenzl algebra.

The Temperley–Lieb Algebra

The Temperley–Lieb algebra TL_f (defined by [TL]) is the algebra over $\mathbb{C}(x)$ given by generators $E_1, E_2, \ldots, E_{f-1}$ and relations

(TL1) $E_i E_j = E_j E_i$, if $|i-j| > 1$,

(TL2) $E_i E_{i\pm 1} E_i = E_i$, and

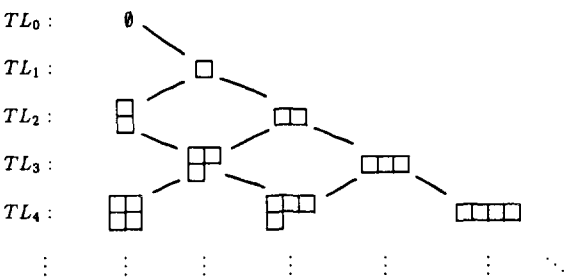
(TL3) $E_i^2 = x E_i$.

By convention $TL_0 = TL_1 = \mathbb{C}(x)$. The Temperley–Lieb algebra can be viewed as the algebra generated by the f-diagrams id_f and E_i, for $1 \leqslant i \leqslant f-1$, with multiplication as in $B_f(x)$.

The irreducible representations of TL_f are labeled by partitions in the set

$$\widehat{TL}_f = \{\lambda \vdash f \mid l(\lambda) \leqslant 2\}.$$

There is a natural embedding $TL_{f-1} \subseteq TL_f$ as the subalgebra generated by the generators E_i, $1 \leqslant i \leqslant f-2$. The Bratteli diagram for the inclusions $TL_0 \subseteq TL_1 \subseteq TL_2 \subseteq \cdots$ is given by (see [GHJ] or [Jo] for details)



In row f, the lth vertex from the right is labeled by the partition $(f-l, l)$. The dimension of the irreducible labeled by $(f-l, l)$ is $\binom{f}{l} - \binom{f}{l-1}$. Thus, letting χ_f^{λ} denote the irreducible TL_f-character labeled by the partition λ, we have

$$\chi_f^{(f-l,l)}(id_f) = \binom{f}{l} - \binom{f}{l-1}. \tag{2.21}$$

We inductively define a nondegenerate trace tr on TL_f by

$$tr(1) = 1, \quad \text{and}$$
$$tr(aE_{f-1}b) = (1/x)\, tr(ab),$$

for $a, b \in TL_{f-1}$. As in the case of Brauer algebra we define

$$e_{f-1} = (1/x)\, E_{f-1},$$

and a conditional expectation ε_{f-2}: $TL_{f-1} \to TL_{f-2}$ by

$$e_{f-1} b e_{f-1} = \varepsilon_{f-2}(b) e_{f-1},$$

for all $b \in TL_{f-1}$. Then the trace tr is nondegenerate on TL_{f-1} and TL_{f-2} and ε_{f-2} is a conditional expectation for the inclusion $TL_{f-2} \subseteq TL_{f-1}$ with respect to the trace tr. Furthermore the element $e_{f-1} = (1/x) E_{f-1}$ satisfies the relations of (2.5), and

$$TL_f \cong TL_{f-1} e_{f-1} TL_{f-1} \oplus \mathbb{C}(x),$$

where $TL_{f-1} e_{f-1} TL_{f-1}$ is a basic construction for $TL_{f-2} \subseteq TL_{f-1}$. In terms of f-diagrams, $TL_{f-1} e_{f-1} TL_{f-1}$ is the linear span of the diagrams with horizontal edges. By Lemma (2.8), the characters of $TL_{f-1} e_{f-1} TL_{f-1}$

depend only on elements of the form $ae_{f-1} = a \otimes e$, where $a \in TL_{f-2}$ and e is given by (2.15). Then, by induction, the characters of TL_f are completely determined by their values on the diagrams of the form $id_{f-2k} \otimes e^{\otimes k}$. The correspondence between the irreducibles of TL_{f-2} and TL_f is given by adding a box to each row of a partition of $f-2$ to obtain a partition of f. Therefore, from Proposition (2.10), one obtains

$$\chi_f^{(f-l,l)}(id_{f-2k} \otimes e^{\otimes k}) = \begin{cases} \chi_{f-2k}^{(f-l-k,l-k)}(id_{f-2k}) & \text{if } l \geqslant k, \\ 0 & \text{if } l < k, \end{cases} \tag{2.22}$$

and from (2.21), we get

$$\chi_f^{(f-l,l)}(id_{f-2k} \otimes e^{\otimes k}) = \begin{cases} \binom{f-2k}{l-k} - \binom{f-2k}{l-k-1}, & \text{if } l \geqslant k, \\ 0, & \text{if } l < k. \end{cases} \tag{2.23}$$

The Okada Algebra

The Okada algebra O_f (defined in [O]) is the algebra over $\mathbb{C}(x_1, \ldots, x_{f-1}, y_1, \ldots, y_{f-1})$ given by generators $E_1, E_2, \ldots, E_{f-1}$ and relations

(O1) $E_i E_j = E_j E_i$ if $|i-j| \geqslant 2$,

(O2) $E_{i+1} E_i E_{i+1} = y_i E_{i+1}$, and

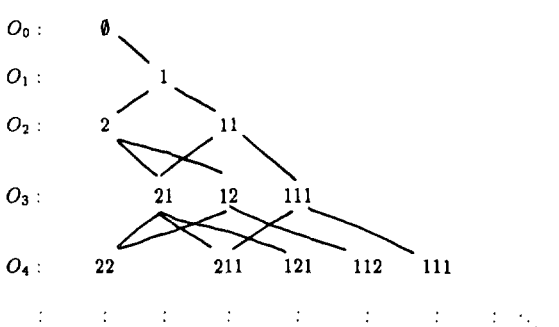
(O3) $E_i^2 = x_i E_i$.

The irreducible representations of O_f are labeled by words in the set

$$\hat{O}_f = \left\{ w = w_1 w_2 \cdots w_l \mid w_i \in \{1, 2\}, \sum_{i=1}^{l} w_i = f. \right\} \tag{2.24}$$

By convention, we let $\hat{O}_0 = \varnothing$.

There is a natural inclusion $O_{f-1} \subseteq O_f$ where O_{f-1} is the subalgebra generated by the E_i, $1 \leqslant i \leqslant f-2$. Okada [O] proves that the Bratteli diagram for $O_0 \subseteq O_1 \subseteq O_2 \subseteq \cdots$ is the Young–Fibonacci lattice:



If $w \in \hat{O}_f$, then we let $K^-(w)$ be the set of $v \in \hat{O}_{f-1}$ connected to w in the Bratteli diagram, and we let $K^+(w)$ be the set of $v \in \hat{O}_{f+1}$ connected to w in the Bratteli diagram. We use $v \lhd w$ to mean that $v \in \hat{O}_{f-1}$, $w \in \hat{O}_f$, and v is connected to w in the Bratteli diagram. If $w \in \hat{O}_f$, then $w = aw'$ where $w' \in \hat{O}_{f-1}$ and $a \in \{1, 2\}$. We have

$$\begin{aligned} K^-(2v') &= K^+(v') \\ K^-(1v') &= \{v\}. \end{aligned} \tag{2.25}$$

Okada [O] proves that the set $\mathscr{B}_f$ defined inductively by $\mathscr{B}_0 = \mathscr{B}_1 = \{1\}$ and

$$\mathscr{B}_f = \{b, bE_{f-1} \cdots E_k \mid b \in \mathscr{B}_{f-1}, k = 1, \ldots, f-1\} \tag{2.26}$$

forms a basis of O_f. In particular, $\dim(O_f) = f!$.

Define a nondegenerate trace tr on O_f, inductively, by

$$tr(1) = 1, \quad \text{and}$$

$$tr(aE_{f-1}b) = (y_{f-2}/x_{f-1})\, tr(ab),$$

for $a, b \in O_{f-1}$. Define

$$e_{f-1} = (1/x)E_{f-1},$$

and a conditional expectation $\varepsilon_{f-2}\colon O_{f-1} \to O_{f-2}$ by

$$e_{f-1}be_{f-1} = \varepsilon_{f-2}(b)e_{f-1},$$

for all $b \in O_{f-1}$. Then the trace tr is nondegenerate on O_{f-1} and O_{f-2} and ε_{f-2} is a conditional expectation for the inclusion $O_{f-2} \subseteq O_{f-1}$ with respect to the trace tr. Furthermore the element $e_{f-1} = (1/x)E_{f-1}$ satisfies the relations of (2.5), and

$$O_f \cong O_{f-1}e_{f-1}O_{f-1} \oplus O_{f-2}, \tag{2.27}$$

where $O_{f-1}e_{f-1}O_{f-1}$ is a basic construction for $O_{f-2} \subseteq O_{f-1}$.

For $w \in \hat{O}_f$, we define an element e_w of O_f by $e_\varnothing = e_1 = 1$ and

$$e_w = \begin{cases} e_{f-1}e_{w'}, & \text{if } w = 2w' \\ e_{w'} & \text{if } w = 1w'. \end{cases}$$

(2.28) PROPOSITION. *Any character χ of O_f is completely determined by its values on the elements $\{e_w \mid w \in \hat{O}_f\}$.*

Proof. The proof uses induction on f with the cases $f = 0$ and $f = 1$ being trivial. Let $f \geq 2$. The character χ is completely determined by its

values on $\mathscr{B}_f$. If $b \in \mathscr{B}_f$, then either $b \in \mathscr{B}_{f\ 1}$ or $b = b' E_{f\ 1} E_{f\ 2} \cdots E_k$. In the second case, we have

$$\begin{aligned}\chi(b) &= \chi(b' E_{f\ 1} E_{f\ 2} \cdots E_k) \\ &= \frac{1}{x_{f\ 1}} \chi(b' E_{f\ 1}^2 E_{f\ 2} \cdots E_k) \\ &= \frac{1}{x_{f\ 1}} \chi(E_{f\ 1} E_{f\ 2} \cdots E_k b' E_{f\ 1}).\end{aligned}$$

By (2.5b) we have $E_{f\ 1} E_{f\ 2} \cdots E_k b' E_{f\ 1} = E_{f\ 1} b''$ where $b'' \in O_{f\ 2} \subseteq O_{f\ 1}$. Using (2.10), we conclude that $\chi(b) = \chi(E_{f\ 1} b'') = x_{f\ 1} \chi(e_{f\ 1} b'') = x_{f\ 1} \chi(b'')$. If $a \in O_{f\ 1}$, then $\chi(a)$ is completely determined by the restriction $\chi \downarrow_{O_{f\ 1}}^{O_f}(a)$ of χ to $O_{f\ 1}$, and, by induction, $\chi \downarrow_{O_{f\ 1}}^{O_f}$ depends only on its values on the set $\{e_w \mid w \in \hat{O}_{f\ 1}\}$. Since $e_{f\ 1}$ is a scalar multiple of $E_{f\ 1}$ the proposition is proved. ∎

(2.29) THEOREM. *If* $w, v \in \hat{O}_f$, *then*

$$\chi_{O_f}^v(e_w) = \begin{cases} \chi_{O_f\ 2}^{v'}(e_{w'}) & \text{if } v = 2v' \text{ and } w = 2w', \\ 0 & \text{if } v = 1v' \text{ and } w = 2w', \\ \chi_{O_f\ 1}^{v'}(e_{w'}) & \text{if } v = 1v' \text{ and } w = 1w', \\ \sum_{v'' \rightarrow v'} \chi_{O_f\ 1}^{v''}(e_{w'}) & \text{if } v = 2v' \text{ and } w = 1w'. \end{cases}$$

Proof. If $w = 2w'$, then $e_w = e_{f\ 1} e_{w'}$ where $w' \in \hat{O}_{f\ 2}$, and the first two cases follow from (2.10). If $w = 1w'$, then $e_w = e_{w'} \in O_{f\ 1}$. Thus

$$\chi_{O_f}^v(e_w) = \chi_{O_f}^v \downarrow_{O_{f\ 1}}^{O_f}(e_{w'}) = \sum_{\tilde{v} \rightarrow v} \chi_{O_f\ 1}^{\tilde{v}}(e_{w'}).$$

When $v = 2v'$ we have $K(v) = K^+(v')$, and when $v = 1v'$ we have $K(v) = \{v'\}$, so the second two cases are proved. ∎

3. THE BIRMAN–WENZL ALGEBRA

We define the *Birman Wenzl algebra* $BW_f(r, q)$ (defined in [BW]) as the algebra generated over $\mathbb{C}(r, q)$ by $1, g_1, \ldots, g_{f\ 1}$, which are assumed to be invertible, subject to the relations

(B1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,

(B2) $g_i g_j = g_j g_i$ if $|i - j| \geq 2$,

(BW1) $(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0$,

(BW2) $E_i g_{i-1}^{\pm 1} E_i = r^{\pm 1} E_i$ and $E_i g_{i+1}^{\pm 1} E_i = r^{\pm 1} E_i$,

where E_i is defined by the equation

$$(q - q^{-1})(1 - E_i) = g_i - g_i^{-1}. \tag{3.1}$$

The element E_i is the spectral projection of the element g_i corresponding to the eigenvalue r^{-1}. Letting

$$x = \frac{r - r^{-1}}{q - q^{-1}} + 1, \tag{3.2}$$

one has the following relations

$$E_i^2 = xE_i, \tag{3.3}$$

$$E_i g_i^{\pm 1} = g_i^{\pm 1} E_i = r^{\mp 1} E_i, \tag{3.4}$$

$$g_i^2 = (q - q^{-1})(g_i - r^{-1} E_i) + 1. \tag{3.5}$$

It can be shown (see [Wn2]) that for $r = q^{n+1}$ one obtains in the limiting case $q \to 1$ the Brauer algebra $B_f(n)$. Furthermore, $BW_f(r, q)$ has the same decomposition into full matrix rings as $B_f(n)$ except possibly if q is a root of unity or if $r = q^{n+1}$ for some $n \in \mathbb{Z}$ (see [Wn2]). In particular,

$$dim_{\mathbb{C}(r,q)} BW_f(r, q) = dim_{\mathbb{C}} B_f(n) = (2f - 1)(2f - 3) \cdots 3 \cdot 1 = (2f)!!,$$

the irreducible $BW_f(r, q)$-representations are indexed by partitions in the set

$$\hat{B}_f = \{\lambda \vdash (f - 2k) \mid 0 \leqslant k \leqslant \lfloor f/2 \rfloor\}, \tag{3.6}$$

and the Bratteli diagram for $BW_f(r, q)$ is the same as the Bratteli diagram for $B_f(x)$ given in Section 1.

The *Iwahori-Hecke algebra of type A* (defined in [Iw]), denoted $H_f(q^2)$, is the algebra generated over $\mathbb{C}(q)$ by $1, g_1, \ldots, g_{f-1}$ subject to the relations

(B1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,

(B2) $g_i g_j = g_j g_i$ if $|i - j| \geqslant 2$,

(IH) $g_i^2 = (q - q^{-1}) g_i + 1$.

There exists a nondegenerate trace tr on $BW_f(r, q)$ defined inductively by

$$tr(1) = 1,$$
$$tr(ag_{f-1}^{\pm 1}b) = (r^{\pm 1}/x)\, tr(ab), \quad \text{and}$$
$$tr(aE_{f-1}b) = (1/x)\, tr(ab),$$

for all $a, b \in BW_{f-1}(r, q)$. Define

$$e_{f-1} = (1/x)E_{f-1},$$

and a conditional expectation $\varepsilon_{f-2}: BW_{f-1}(r, q) \to BW_{f-2}(r, q)$ by

$$e_{f-1}be_{f-1} = \varepsilon_{f-2}(b)e_{f-1},$$

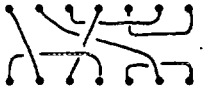
for all $b \in BW_{f-1}$. Then [Wn2] the trace tr is nondegenerate on BW_{f-1} and BW_{f-2} and ε_{f-2} is a conditional expectation for the inclusion $BW_{f-2} \subseteq BW_{f-1}$ with respect to the trace tr. Furthermore the element e_{f-1} satisfies the relations in (2.5), and

$$BW_f(r, q) \cong BW_{f-1}(r, q)\, e_{f-1}\, BW_{f-1}(r, q) \oplus H_f(q^2), \tag{3.7}$$

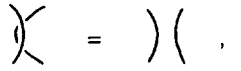
where $BW_{f-1}(r, q)\, e_{f-1}\, BW_{f-1}(r, q)$ is a basic construction for $BW_{f-2}(r, q) \subseteq BW_{f-1}(r, q)$. The fact that $H_f(q^2)$ is the complement of the basic construction follows by considering (3.5) modulo the ideal $\langle E_i \rangle$ generated by the E_i. For details, see [BW] and [Wn2].

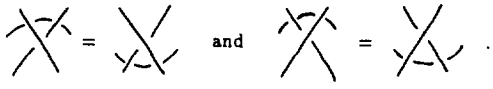
Tangles

Kauffman [Ka] has given the Birman–Wenzl algebra a diagrammatic setting which we adopt. An *f-tangle* is viewed as two rows of f vertices, one above the other, and f strands that connect vertices in such a way that each vertex is incident to precisely one strand. Strands cross over and under each other in three-space as they pass from one vertex to the next. Strands that connect vertices in the same row are horizontal, and strands that connect vertices in different rows are vertical. For example, the following are 7-tangles:

$t_1 =$  $t_2 =$  . (3.8)

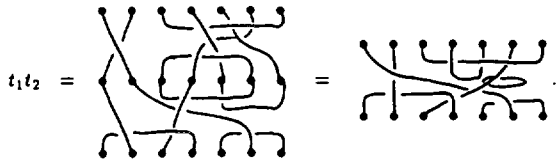
The Reidemeister moves of types II and III are (see [Ka]):

II. 

III. 

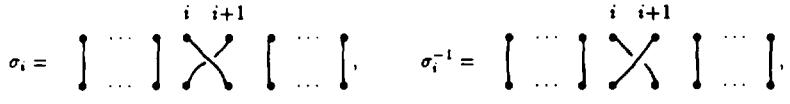
We apply these "moves" to tangles by isolating one of these crossings in an open disk in a tangle and applying the relation. These relations give an equivalence class among tangles known as regular isotopy. We will take f-tangles to be their equivalence classes under the Reidemeister moves.

We multiply f-tangles t_1 and t_2 using the concatenation product given by identifying the vertices in the top row of t_2 with the corresponding vertices in the bottom row of t_1 and then re-scaling the result to obtain the product tangle $t_1 t_2$. The concatenation product can create closed cycles, so we allow an f-tangle to contain arbitrarily many closed cycles. For example, the product of the 7-tangles in (3.8) is given by



The concatenation product makes the set $\mathfrak{T}_f$ of all f-tangles a monoid, referred to as the *tangle monoid.*

We identify special elements in $\mathfrak{T}_f$ by



The *braid monoid* M_f is the sub-monoid of $\mathfrak{T}_f$ generated by $\{id_f\} \cup \{\sigma_i^{\pm 1}, h_i \mid 1 \leqslant i \leqslant f-1\}$. We associate to M_f the free algebra $\mathscr{A}_f$ generated by M_f over $\mathbb{C}(r, q)$ subject to the relations:

(Q1) $\sigma_i = \sigma_i^{-1} + (q - q^{-1})\, id_f - (q - q^{-1}) h_i$,

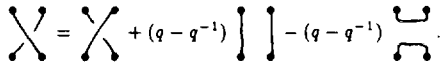
(Q2) $h_i \sigma_{i+1}^{\pm 1} h_i = r^{\pm 1} h_i$ and $h_i \sigma_i^{\pm 1} h_i = r^{\pm 1} h_i$,

(Q3) $h_i \sigma_i^{\pm 1} = \sigma_i^{\pm 1} h_i = r^{\mp 1} h_i$,

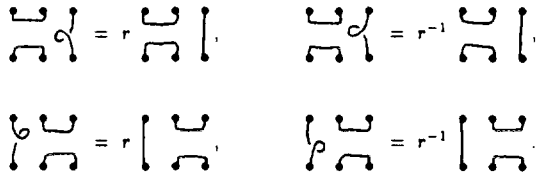
(Q4) $h_i^2 = x h_i$.

In terms of tangles, these relations give the tangle identities given below.

(Q1)

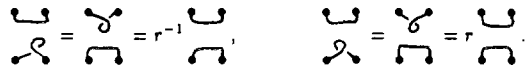


(Q2)

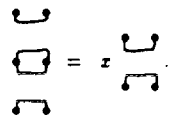


(3.9)

(Q3)



(Q4)



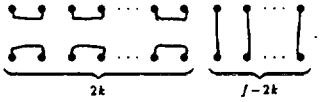
Like the Reidemeister moves, the tangle identities relate diagrams which differ in small open disks by the given relation and are the same outside the disk.

Let F_f denote the free $\mathbb{C}(r, q)$-algebra on the $BW_f(r, q)$ generators, and define a map $F_f \to \mathscr{A}_f$ by $1 \mapsto id_f$, $g_i \mapsto \sigma_i$, and $E_i \mapsto h_i$. It is easy to check that the $\mathscr{A}_f$ generators satisfy the $BW_f(r, q)$-relations, so we get an induced homomorphism $BW_f(r, q) \to \mathscr{A}_f$. Moreover, Kauffman [Ka] shows that upon setting $r = q^{n+1}$ and letting $q \to 1$, the algebra $\mathscr{A}_f$ becomes the Brauer algebra $B_f(n+2)$. In terms of diagrams, this specialization is equivalent to ignoring over and under-crossings of strands and removing curls. In particular, we have $dim_{\mathbb{C}(q,r)} \mathscr{A}_f \geqslant dim_{\mathbb{C}} B_f(n+2) = (2f)!!$. Since $dim_{\mathbb{C}(q,r)} BW_f(r, q) = (2f)!!$, we have $BW_f(r, q) \cong \mathscr{A}_f$. Thus, we identify elements of $BW_f(r, q)$ with their corresponding diagram.

We refer to the images of the tangles in $\mathscr{A}_f$ as *q-diagrams*, or sometimes, $(f; q)$*-diagrams* and to the images of the $BW_f(r, q)$-generators 1, g_i, g_i^{-1}, E_i as *q-generator diagrams*. We call the $(f; 1)$-diagrams *Brauer diagrams*, since they generate the Brauer algebra $B_f(n)$ (see [R1]), and we let D_f denote the set of Brauer diagrams. If d_1 and d_2 are $(f_1; q)$ and $(f_2; q)$-diagrams respectively, then $d_1 \otimes d_2$ denotes the $(f_1+f_2; q)$-diagram given by placing d_2 to the right of d_1, and $d_1^{\otimes k}$ denotes the diagram $d_1 \otimes d_1 \otimes \cdots \otimes d_1$ with k factors. We let E denote the $(2; q)$-diagram

$E =$  , (3.10)

and we have

$E^{\otimes k} \otimes id_{f-2k} =$ 

(3.11)

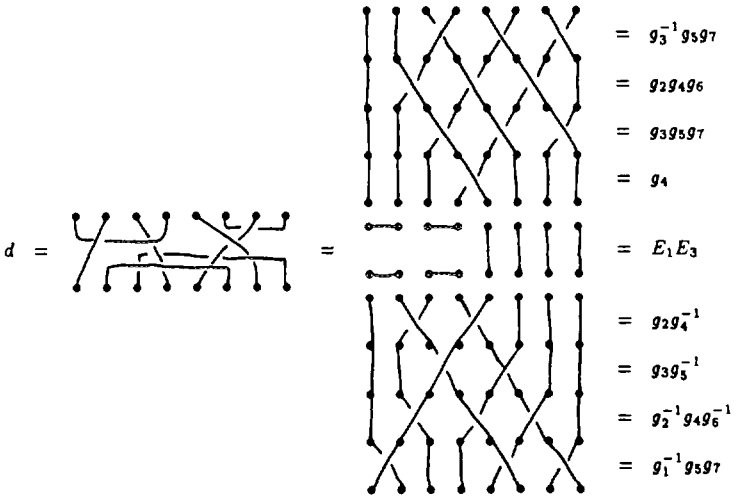
Note that $E^{\otimes k} \otimes id_{f-2k} = E_1 E_3 \cdots E_{2k}$ is a product of generator diagrams.

Not all f-tangles represent elements of $BW_f(r, q)$. For example

is a 3-tangle in $\mathfrak{T}_3$ that is not in the braid monoid M_3. To identify diagrams in $BW_f(r, q)$, we say that a q-diagram d is *standard* if (i) no two edges cross more than once, (ii) no edge crosses itself, and (iii) d contains no cycles.

(3.12) THEOREM. *Any standard $(f; q)$-diagram can be written as a product of $(f; q)$-generator diagrams and is thus an element of $BW_f(r, q)$.*

Proof. Let d be a standard $(f; q)$-diagram with k horizontal edges in each row, and let $d_0 = E^{\otimes k} \otimes id_{f-2k}$. Then there exist diagrams $\sigma, \tau \in \mathscr{S}_f$ such that, as Brauer diagrams, $d = \sigma d_0 \tau$. Write $\sigma = s_{i_1} s_{i_2} \cdots s_{i_r}$ and $\tau = s_{j_1} s_{j_2} \cdots s_{j_t}$ as products of simple transpositions. Then it is possible to choose u_i, $u_j \in \{1, -1\}$ such that $d = g_{i_1}^{u_{i_1}} g_{i_2}^{u_{i_2}} \cdots g_{i_r}^{u_{i_r}} d_0 g_{j_1}^{u_{j_1}} g_{j_2}^{u_{j_2}} \cdots g_{j_t}^{u_{j_t}}$ as q-diagrams. As the next example illustrates, choosing the u_i and u_j amounts to choosing the crossings in the diagrams. Here we "blow-up" a diagram into a product of generator diagrams.



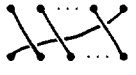
and we see that $d = g_3^{-1}g_5g_7g_2g_4g_6g_3g_5g_7g_4E_1E_3g_2g_4^{-1}g_3^{-1}g_5^{-1}g_2^{-1}g_4g_6^{-1}g_1^{-1}g_5g_7$. ∎

(3.13) THEOREM. *Any set $\tilde{D}_f$ of q-diagrams in standard form that specializes when $q \to 1$ to the set D_f of Brauer diagrams forms a basis of $BW_f(r, q)$.*

Proof. We know from the previous theorem that $\tilde{D} \subseteq BW_f(r, q)$. Since these diagrams are independent when $q \to 1$, they must be independent in $BW_f(r, q)$, and since $dim_{\mathbb{C}(r,q)} BW_f(r, q) = (2f)!!$, which is exactly $Card(D_f)$, they span $BW_f(r, q)$. ∎

Character Classes

Let $T_{\gamma f} = g_{f-1}g_{f-2}\cdots g_1$ denote the $(f; q)$-diagram

T_{γ_f} =  (3.14)

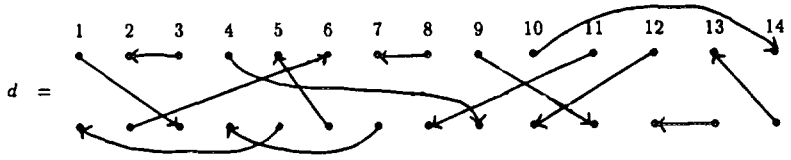
and for $\mu \in B_f$ with $\mu = (\mu_1, ..., \mu_t)$ let

$$T_{\gamma_\mu} = T_{\gamma_{\mu_1}} \otimes T_{\gamma_{\mu_2}} \otimes \cdots \otimes \cdots T_{\gamma_{\mu_t}}.$$

From Lemma (2.8) we observe that the characters of the basic construction $BW_{f-1}(r, q)\ e_{f-1}BW_{f-1}(r, q)$ depend only on their values on elements $a \otimes E$ where $a \in BW_{f-2}(r, q)$. By (3.7) and induction the characters of $BW_f(r, q)$ depend only on their values on elements $d \otimes E^{\otimes h}$ where

$d \in H_{f-2h}(q^2)$. Using the results of [R2] on Hecke algebra characters, we know that the character of d depends only on the character of T_{γ_μ} where $\mu \vdash (f-2h)$ is the cycle type of d. Thus, the characters of $BW_f(r, q)$ depend only on the characters of $T_{\gamma_\mu} \otimes E^{\otimes h}$. In the rest of the section we will explicitly construct a basis of q-diagrams in $BW_f(r, q)$ that partitions into classes, which we call *character classes*, labeled by $\mu \in \hat{B}_f$ on which $BW_f(r, q)$-characters are constant and equal to the character of $T_{\gamma_\mu} \otimes E^{\otimes h}$.

Following [R1] we associate to each Brauer diagram $d \in D_f$ a partition $\tau(d) \in B_f$ called the *cycle type* of d. To do this, we traverse the diagram d in the following way. Connect each vertex in the top row of the diagram d to the vertex just below it in the bottom row by a dotted line. Beginning with the first vertex (moving left to right) in the top row of d, follow the path determined by the edges and the dotted lines and assign to each edge the direction that it is traversed. Returning to the original vertex completes a *cycle* in d. If not all vertices in d have been visited, start with the first not-yet-visited vertex in the top row of d and traverse the cycle adjacent to it. Do this until all vertices have been visited. The diagram



(3.15)

has three cycles. The first is on vertices 1, 2, 3, 6, 5, the second on vertices 4, 7, 8, 9, 11, and the third on vertices 10, 12, 13, 14.

To each cycle c of d, let $U(c)$ denote the number of vertical edges of c directed from bottom to top and $D(c)$ denote the number of vertical edges directed from top to bottom. We distinguish the three cases:

(1) If $U(c) = D(c)$, then c is a *zero* cycle.

(2) If $U(c) > D(c)$, then c is an *up-cycle*, and

(3) If $U(c) < D(c)$, then c is a *down-cycle*.

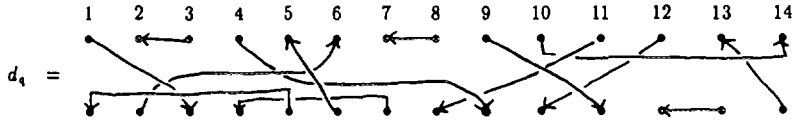
In each case, the integer

$$t(c) = |U(C) - D(c)|$$

is the called the type of the cycle c. As c runs through all non-zero cycles of d, the sequence of numbers $t(c)$ forms the partition $\tau(d)$, the *type* of the diagram d. It is not hard to check that there exists an integer $h(d)$ with $0 \leqslant h(d) \leqslant \lfloor f/2 \rfloor$ so that $\tau(d) \vdash (f-2h(d))$, and therefore $\tau(d) \in \hat{B}_f$. In

example (3.15), above, the first cycle c_1 is an up-cycle with type $t(c_1) = 1$, the second c_2 is a down-cycle with type $t(c_2) = 3$, and the third c_3 is a zero-cycle. Thus, $\tau(d) = (3, 1)$, and $h(d) = 5$.

We associate to d a standard q-diagram d_q which has the same cycle type as d and which specializes when $q \to 1$ to d. We do it by *q-traversing* the diagram d in the following manner. If the first cycle c_1 of d is an up-cycle, we start with the rightmost vertex in the bottom row of c_1, otherwise we start with the leftmost vertex in the top row of c_1. We follow the edges of d in the same way that we originally traversed d, only now, whenever we come to an edge that has already been traversed, we go under it, and whenever we come to an edge that has not been traversed, we go over it. Once a cycle is completed, we q-traverse the next (moving from left to right) cycle of d, always passing under already traversed edges and over not-yet-traversed edges, and always starting with the top-left-most vertex of down or zero-cycles and with the bottom-right-most vertex of up-cycles. In this way, the first cycle is on top of the second and the second is on top of the third, and so on. From example (3.15) above, we construct the following q-diagram



(3.16)

We let $D_f^q = \{d_q \mid d \in D_f\}$ be the set of these q-diagrams, corresponding to Brauer diagrams. Then, by Theorem (3.13), D_f^q is a basis of $BW_f(r, q)$.

Let $d_q \in D_f^q$. A horizontal edge ε in d_q is *inverted* if it is in a down-cycle or a zero-cycle (respectively up-cycle) and is directed from right to left (respectively left-to-right) *and* it is not the last edge traversed in the cycle. Define

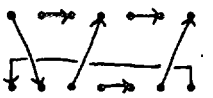
$$I_T(d_q) = (\text{number of inverted horizontal edges in the top row of } d_q),$$

$$I_B(d_q) = (\text{number of inverted horizontal edges in the bottom row of } d_q).$$

In our example (3.16), the edge connecting vertices 7 and 8 in the top row is inverted, and the edge connecting vertices 12 and 13 in the bottom row is inverted.

We say that a q-diagram is a *cycle diagram* if consists of a single cycle, and we say that it is a *straightened* cycle diagram if one vertex in each column is connected to a vertex in the next column (and a vertex in the fth

column is connected to a vertex in the 1st column). For example, the following is a straightened q-cycle diagram



We say that two elements b_1 and b_2 of $BW_f(r, q)$ are *conjugate* and write $b_1 \sim b_2$ if there exists an invertible element g of $BW_f(r, q)$ such that $b_1 = gb_2 g^{-1}$. Note that the q-diagrams that are invertible are exactly the ones which do not contain horizontal edges. Furthermore, $b_1 \sim b_2$ implies $\chi(b_1) = \chi(b_2)$ for any $BW_f(r, q)$-character χ.

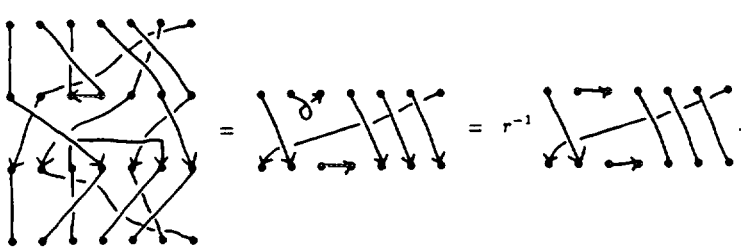
(3.17) THEOREM. *If d is a Brauer diagram and d_q is its associated q-diagram, then*

$$d_q \sim r^{(I_B(d) - I_T(d))} c_1 \otimes c_2 \otimes \cdots \otimes c_t,$$

where $\sim$ denotes conjugate elements of $BW_f(r, q)$ and where the c_i are straightened cycle diagrams.

Proof. Because the cycles of d_q are layered from top to bottom, it is easy to conjugate d_q so that $d_q \sim c_1' \otimes c_2' \otimes \cdots \otimes c_t'$, where each c_i' is a cycle diagram.

Suppose that c_i' is a down or zero-cycle on f_i. We construct an (f_i, q)-diagram g_i as follows. If when q-traversing c_i' the kth column visited is column l, then connect the lth vertex in the bottom row of g_i to the kth vertex in the top row of g_i, always passing under any edges of g_i that are already drawn. If a horizontal edge of c_i' is inverted, then we will introduce a loop when we conjugate c_i' by g_i. We remove the loop by multiplying by r^{-1} if the edge is in the top row and by r if the edge is in the bottom row (see $(Q3)$ in Figure 3.9). In this way $g_i c_i' g_i^{-1} = r^{I_T(c_i') - I_B(i)} c_i$ where c_i is the straightened version of c_i'. For example,



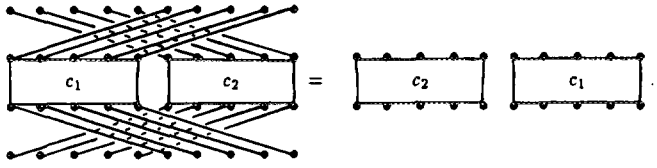
If c_1' is an up-cycle, then we rotate c_i by 180 degrees making it a down-cycle (this is why we start drawing up-cycles from the bottom right vertex). The same procedure with the rotated cycle puts c_i in straightened form. Doing this for each cycle in d_q proves the theorem. ∎

(3.18) THEOREM. *Suppose $d_q \in D_f^q$ is a q-diagram of type $\mu \in \hat{B}_f$ with $\mu \vdash (f-2h)$ and that d_q has z zero-cycles. Then if χ is any character of $BW_f(r, q)$, we have*

$$\chi(d_q) = xc^z \quad {}^h r^{I_B(d_q) - I_T(d_q) + N(d_q)} \chi(T_{\gamma_\mu} \otimes e^{\otimes h}),$$

where $N(d_q)$ is the number of non-zero cycles in d_q which have a horizontal edge.

Proof. From Theorem 3.17, we know that $d_q \sim r^{I_B(d_q) - I_T(d_q)} c_1 \otimes c_2 \otimes \cdots \otimes c_t$ where each c_i is a straightened cycle diagram on f_i-dots. Moreover $c_1 \otimes c_2 \otimes \cdots \otimes c_t \sim c_{\pi(2)} \otimes \cdots \otimes c_{\pi(t)}$ for any permutation π of the t cycles in d_q. To see this, observe that we can transpose any two cycles by



Therefore, if each non-zero cycle contains only vertical edges and each zero-cycle is E, then we can permute the cycles so that $d_q \sim r^{I_B(d_q) - I_T(d_q)} T_{\gamma_\mu}$, and we are done, since characters are the same on conjugate elements. Therefore, we assume that, for some i, c_i has a horizontal edge and c_i is not E. Moreover, we will assume, first of all, that c_i is a down or a zero-cycle.

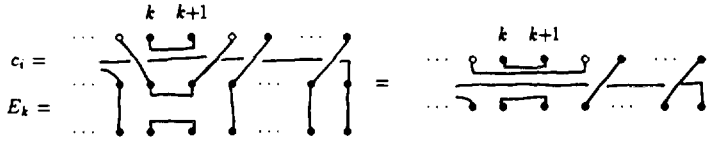
It suffices to work in $BW_{f_i}(r, q)$, since we embed an $(f_i; q)$-diagram into $BW_f(r, q)$ by placing the appropriate number of identity edges on either side of the diagram. We first note that if the edge in c_i connecting the f_ith column to the first column is vertical, then we conjugate c_i by $(g_{f_i-1} g_{f_i-2} \cdots g_1)^{-1}$ so that this edge connects the first column to the second column. We can continue to conjugate c_i in this way until the edge connecting columns 1 and f_i is a horizontal edge in the bottom row. Moreover, since c_i is a straightened down or zero-cycle, we know that this edge passes behind all the other edges.

Assume that the rightmost horizontal edge in the top row of c_i connects the kth and $(k+1)$st edge. Then $E_k c_i = x c_i$, so

$$\chi(c_i) = \frac{1}{x}\chi(E_k c_i) = \frac{1}{x}\chi(c_i E_k). \tag{3.19}$$

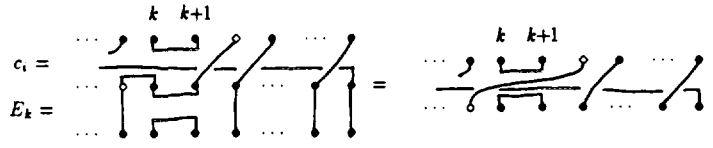
Therefore, we are interested in the product $c_i E_k$. Let ○ be the vertex in column $k-1$ of c_i which is adjacent to the edge that travels to column k, and let ◇ be the vertex in column $k+1$ of c_i that is adjacent to the edge that travels from column k. We consider in four cases the possible locations of ○ and ◇.

Case (i). ○ and ◇ are both in the top row.



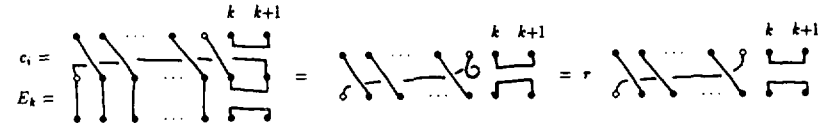
Here we see that $c_i E_k \sim c_i' \otimes E$, where c_i' is a straightened cycle diagram on $f_i - 2$ vertices with the same type as c_i.

Case (ii). ○ is in the bottom row, and ◇ is in the top row.



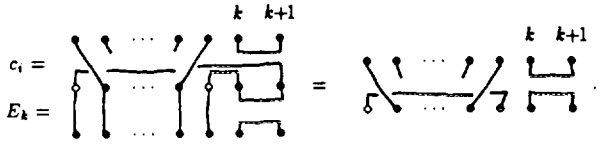
Here again $c_i E_k \sim c_i' \otimes E$, where c_i' is a straightened cycle diagram on $f_i - 2$ vertices with the same type as c_i.

Case (iii). ○ is in the top row and ◇ is in the bottom row. Notice that because we are considering the rightmost horizontal edge in the top row of c_i, this case only occurs if c_i is a not a zero-cycle and has one horizontal edge in each row and that $k = f_i - 1$.



We remove the loop using $(Q2)$ in Figure (3.9), and $c_i E_k \sim r c_i' \otimes E$, where c_i' is a straightened cycle diagram on $f_i - 2$ vertices with the same type as c_i.

Case (iv). $\bigcirc$ and $\diamond$ are both in the bottom row. As in case (iii), $k = f_i - 1$, and we have



Thus, $c_i E_k \sim c_i' \otimes E$, where c_i' is a straightened cycle diagram on $f_i - 2$ vertices with the same type as c_i.

We repeat this process with c_i' in place of c_i until either $c_i' = E$ or c_i' has no horizontal edges. If c_i is a 0-cycle, then after $h(c_i) - 1$ reductions, we get $c_i' = E$. If c_i is a down-cycle, then after $h(c_i)$ i reductions we get $c_i' = T_{\gamma(c_i)}$. At each reduction, we multiply by x, so in the end we multiply by $x^{h(c_i)i - z_i}$, where $z_i = 1$ if $c - i$ is a zero-cycle, and $z_i = 0$ otherwise. In the case where c_i is an up-cycle, we rotate c_i by 180 making it a down-cycle. Multiplying on the bottom of c_i when rotated 180 degrees corresponds to multiplying on the top of c_i. Thus (3.19) becomes $\chi(c_i) = 1/x\chi(c_i E_k) = 1/x\chi(E_k c_i)$. From $(Q2)$ in Figure (3.9) we see that we still require multiplying by r to remove the loop. Moreover, the $T_{\gamma(c_i)}$ is the same after a rotation of 180 degrees, so the result also holds for up-cycles. Repeating this process for each cycle in d_q that has a horizontal edge and is not E proves the theorem. ∎

4. The Quantum Group $\mathscr{U}_q(\mathfrak{so}(l))$ and the Universal R-Matrix

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant l}$. Then, since A is symmetrizable, there exist integers $d_i \neq 0$ such that $d_i a_{ij} = d_j a_{ji}$. Let q be an indeterminate over $\mathbb{C}$. Then $\mathscr{U}_q(\mathfrak{g})$ is the associative $\mathbb{C}(q)$-algebra with generators $\{X_i^+, X_i^-, k_i, k_i^{-1} \mid 1 \leqslant i \leqslant l\}$ and relations

(1) $k_i k_j = k_j k_i$, $k_i k_i^{-1} = k_i^{-1} k_i = 1$,

(2) $k_i X_j^{\pm} k_i^{-1} = q^{\pm d_i a_{ij}/2} X_j^{\pm}$,

(3) $[X_i^+, X_j^-] = \delta_{ij} \dfrac{k_i^2 - k_i^{-2}}{q^{d_i} - q^{-d_i}}$,

(4) $\displaystyle\sum_{v=0}^{1-a_{ij}} (-1)^v \frac{[1-a_{ij}]_{q^{d_i}}!}{[v]_{q^{d_i}}! [1-a_{ij}-v]_{q^{d_i}}!} (X_i^{\pm})^{1-a_{ij}-v} (X_j^{\pm})(X_i^{\pm})^v = 0,$ $i \neq j$.

where for any $t \in \mathbb{C}$,

$$[m]_t! = \prod_{j=1}^{m} \frac{t^j - t^{-j}}{t - t^{-1}}.$$

Upon letting $q \to 1$, one obtains the classical Serre relations for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

The algebra $\mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra whose coproduct $\Delta: \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$, antipode $S: \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$, and counit $\varepsilon: \mathcal{U}_q(\mathfrak{g}) \to \mathbb{C}(q)$ are given by

$$\begin{aligned} &\Delta(X_i^{\pm}) = k_i^{-1} \otimes X_i^{\pm} + X_i^{\pm} \otimes k_i, && \Delta(k_i) = k_i \otimes k_i, \\ &S(X_i^{\pm}) = -q^{\mp 1} X_i^{\pm}, && S(k_i) = k_i^{-1}, \\ &\varepsilon(X_i^{\pm}) = 0, && \varepsilon(k_i) = 1. \end{aligned} \tag{4.1}$$

For any invertible element $\mathscr{R} \in \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ given by $\mathscr{R} = \sum a_i \otimes b_i$, define $\mathscr{R}_{12}, \mathscr{R}_{13}, \mathscr{R}_{23} \in \mathcal{U}_q(\mathfrak{g})^{\otimes 3}$ to be the elements

$$\mathscr{R}_{12} = \sum a_i \otimes b_i \otimes 1, \qquad \mathscr{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \qquad \mathscr{R}_{23} = \sum 1 \otimes a_i \otimes b_i.$$

Then we say that $\mathscr{R}$ satisfies the quantum Yang–Baxter equation (QYBE) if

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}. \tag{4.2}$$

Let $T: \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ be given by

$$T(a \otimes b) = b \otimes a, \qquad \text{for all} \quad a, b \in \mathcal{U}_q(\mathfrak{g}). \tag{4.3}$$

Then $\mathscr{R}$ is a universal R-matrix if it satisfies the relations

$$\begin{aligned} &T\Delta(a) = \mathscr{R}\Delta(a)\,\mathscr{R}^{-1} \qquad \text{for all} \quad a \in \mathcal{U}_q(\mathfrak{g}), \\ &(\Delta \otimes id)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}, \\ &(id \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}. \end{aligned} \tag{4.4}$$

If $\mathscr{R}$ is a universal R-matrix, then $\mathscr{R}$ satisfies the QYBE.

Fix $\mathfrak{g} = \mathfrak{so}(l, \mathbb{C})$ with $l = (n+1)/2$ and n odd. The Cartan matrix of $\mathfrak{g}$ is of type B_l. To define the fundamental representation of $\mathfrak{g}$, let I be the index set

$$I = \{-n, -(n-2), \ldots, -3, -1, 0, 1, 3, \ldots, n-2, n\}. \tag{4.5}$$

We will at times let $\bar{k} = -k$ for notational convenience. Let V be the $(n+2)$-dimensional $\mathbb{C}(q^{1/2})$-vector space with basis $\{v_i \mid i \in I\}$. The

fundamental representation $\pi: \mathscr{U}_q(\mathfrak{g}) \to End(V)$ of $\mathscr{U}_q(\mathfrak{g})$ is defined in terms of matrix units $\{E_{i,j} \mid i,j \in I\}$ by

$$\begin{aligned}
\pi(X_i) &= E_{i-2,i} - E_{\bar{i},\overline{i-2}}, \qquad i = 3, 5, 7, \ldots, n-1, \\
\pi(X_1) &= \sqrt{2}\,(E_{0,1} - E_{\bar{1},0}), \\
\pi(k_j) &= q^{1/2}(E_{j,j} + E_{\overline{j-1},\overline{j-1}}) + q^{-1/2}(E_{j-1,j-1} + E_{\bar{j},\bar{j}}) \\
&\quad + \sum_{i \neq j,\bar{j}} E_{i,i}, \qquad i = 1, 3, 5, \ldots, n-1, \\
\pi(k_1) &= q^{1/2}E_{1,1} + q^{-1/2}E_{\bar{1},\bar{1}} + \sum_{i \neq 1,\bar{1}} E_{i,i}
\end{aligned}$$

and $\pi(X_i^+) = \pi(X_i)^t$, where t denotes matrix transpose. Since, $\mathscr{U}_q(\mathfrak{g})$ is a Hopf algebra, the tensor representation $(\pi^{\otimes f}, V^{\otimes f})$ is well-defined for all $f \geqslant 1$.

Note that for $1 \leqslant i \leqslant n$, i odd, the spectral projections (see [R2]) of $\pi(k_i)$ at $q^{1/2}$ and $q^{-1/2}$ are, respectively, $E_{i,i} + E_{\overline{i-1},\overline{i-1}}$ and $E_{i+1,i+1} + E_{\bar{i},\bar{i}}$. The spectral projections for k_1 at $q^{1/2}$ and $q^{-1/2}$ are $E_{1,1}$ and $E_{\bar{1},\bar{1}}$. Therefore, the set $\{E_{i,i} \mid i \in I \backslash 0\}$ is in the image of the k_i's under π. Since $\pi(1) = I$, we get all of the matrix units of the form $E_{i,i}$ in $\pi(\mathfrak{g})$. In particular, for $\alpha_i \in \mathbb{C}(q)$, the matrix

$$d = \sum_{i \in I} \alpha_i E_{i,i} \tag{4.6}$$

is in $\pi(\mathscr{U}_q(\mathfrak{g}))$. Moreover, the k_i are group-like elements i.e., $\Delta(k_i) = k_i \otimes k_i$, so the matrix $d^{\otimes f} = d \otimes d \otimes \cdots \otimes d$ (with f factors) is in $\pi^{\otimes f}(\mathscr{U}_q(\mathfrak{g}))$.

Let $\check{R} = \pi^{\otimes 2}(T\mathscr{R})$ where $\mathscr{R}$ is the universal R-matrix and T is as given in 4.4. In terms of matrix units, we have (see [Wn2])

$$\begin{aligned}
\check{R} &= q \sum_{i \neq 0} E_{i,i} \otimes E_{i,i} + E_{0,0} \otimes E_{0,0} + \sum_{i \neq j,\bar{j}} E_{j,i} \otimes E_{i,j} + q^{-1} \sum_{i \neq 0} E_{\bar{i},i} \otimes E_{i,\bar{i}} \\
&\quad + (q - q^{-1}) \sum_{j<i} E_{j,j} \otimes E_{i,i} - (q - q^{-1}) \sum_{i>j} q^{(j-i)/2} E_{\bar{i},j} \otimes E_{i,\bar{j}}.
\end{aligned} \tag{4.7}$$

One can check by explicit computation that

$$\begin{aligned}
\check{R}^{-1} &= q^{-1} \sum_{i \neq 0} E_{i,i} \otimes E_{i,i} + E_{0,0} \otimes E_{0,0} + \sum_{i \neq j,\bar{j}} E_{j,i} \otimes E_{i,j} \\
&\quad + q \sum_{i \neq 0} E_{\bar{i},i} \otimes E_{i,\bar{i}} - (q - q^{-1}) \sum_{j>i} E_{j,j} \otimes E_{i,i} \\
&\quad + (q - q^{-1}) \sum_{i<j} q^{(j-i)/2} E_{\bar{i},j} \otimes E_{i,\bar{j}}.
\end{aligned} \tag{4.8}$$

Let $x \in \mathbb{C}(q)$ be defined by

$$x = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + 1 = \sum_{i \in I} q^i. \tag{4.9}$$

Then x corresponds to the parameter x in $BW_f(q^{n+1}, q)$ (defined in (3.2)) for the specialization $r = q^{n+1}$. If we define $F \in End(V \otimes V)$ by

$$F = \sum_{i,j} q^{(i+j)/2} E_{i,j} \otimes E_{i,j}, \tag{4.10}$$

then F satisfies

$$F^2 = xF, \tag{4.11}$$

and $(1/x)F$ is a projection of $V \otimes V$ onto the subspace spanned by the vector $\sum_{i \in I} q^{i/2} v_i v_{-i}$. Furthermore, one can check that

$$\check{R} - \check{R}^{-1} = (q - q^{-1})(1 - F). \tag{4.12}$$

Define elements $\check{R}_i$, $F_i \in End(V^{\otimes f})$ by

$$\check{R}_i = \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{i-1} \otimes \check{R} \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{f-(i+1)} \quad \text{and}$$

$$F_i = \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{i-1} \otimes F \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{f-(i+1)}.$$

It follows from the QYBE or by very lengthy direct computation that

$$\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1}$$

which implies that the $\check{R}_i$ satisfy the braid relation (B2) in the definition of $BW_f(q^{n+1}, q)$. One can check directly that all the $BW_f(q^{n+1}, q)$ relations are satisfied. Thus, defining a map $\phi: BW_f(q^{n+1}, q) \to End(V^{\otimes f})$ by $\phi(g_i) = \check{R}_i$ and $\phi(e_i) = F_i$ gives a representation of $BW_f(q^{n+1}, q)$ on $V^{\otimes f}$ (see [Wn2]). Through quite tedious computations, one can check that the matrices $\check{R}_i$ and F_i each commute with the action of $\mathcal{U}_q(\mathfrak{so}(l))$ on $V^{\otimes f}$. The results of the next sections prove that the algebras $\phi(BW_f(q^{n+1}, q))$ and $\pi^{\otimes f}(\mathcal{U}_q(\mathfrak{so}(l)))$ are mutual commutants of one another (in analogy with the duality between $H_f(q^2)$ and $\mathcal{U}_q(\mathfrak{sl}(n))$ [Ji], [R2]) and that

(4.13) THEOREM. *Let $n+2 \geqslant 2f$. Then as a $BW_f(q^{n+1}, q) \otimes \mathcal{U}_q(\mathfrak{so}(l))$-representation*

$$V^{\otimes f} = \bigoplus_{\mu \in \hat{B}_f} M^\mu \otimes V^\mu,$$

where $l=(n+1)/2$, M^{μ} *is an irreducible* $BW_f(q^{n+1}, q)$*-representation,* V^{μ} *is an irreducible* $\mathcal{U}_q(\mathfrak{so}(l))$*-representation, and* $\hat{B}_f$ *is the set of partitions given by* (2.13).

5. Weighted Traces of $BW_f(q^{n+1}, q)$ Acting on $V^{\otimes f}$

The Action of $BW_f(q^{n+1}, q)$ *on* $V^{\otimes f}$

Let n be a positive odd integer and let $I=\{-n, -(n-2), \ldots, -3, -1, 0, 1, 3, \ldots, n-2, n\}$. Let $\{v_i \mid i\in I\}$ be a set of independent noncommuting variables. Define V to be the vector space over $\mathbb{C}(q^{1/2})$ with basis $\{v_i \mid i\in I\}$, and define

$$V^{\otimes f}=\mathbb{C}(q^{1/2})\text{-span}\,\{v_{i_1}v_{i_2}\cdots v_{i_f} \mid i_k\in I\},$$

so that the words (simple tensors) $v_{i_1}v_{i_2}\cdots v_{i_f}$ are a basis of $V^{\otimes f}$.

The symmetric group $\mathcal{S}_f$ of permutations on $\{1, 2, \ldots, f\}$ acts on words $v=v_{i_1}v_{i_2}\cdots v_{i_f}\in V^{\otimes f}$ by place permutations. That is, if $\sigma\in\mathcal{S}_f$, then

$$(v_{i_1}v_{i_2}\cdots v_{i_f})\,\sigma=v_{i_{\sigma(1)}}v_{i_{\sigma(2)}}\cdots v_{i_{\sigma(f)}}.$$

For $1\leqslant i\leqslant f-1$, let $s_i=(i, i+1)$ denote the simple transposition that switches i and $i+1$.

Let $v=v_{i_1}v_{i_2}\cdots v_{i_f}$. For $k=1, 2, \ldots, f-1$, define an action of the generator g_k of $BW_f(q^{n+1}, q)$ on v by

$$vg_k=\begin{cases} vs_k, & \text{if } i_k>i_{k+1},\ i_k\neq -i_{k+1},\\ qv, & \text{if } i_k=i_{k+1}\neq 0,\\ vs_k+(q-q^{-1})\,v, & \\ & \text{if } i_k<i_{k+1},\ i_k\neq -i_{k+1},\\ q^{-1}vs_k-(q-q^{-1})\sum_{j>i_k}q^{(i_k-j)/2}v_{i_1}\cdots v_{i_{k-1}}v_{-j}v_jv_{i_{k+2}}\cdots v_{i_f} & \\ & \text{if } i_k>i_{k+1},\ i_k=-i_{k+1},\\ v-(q-q^{-1})\sum_{j>0}q^{(i_k-j)/2}v_{i_1}\cdots v_{i_{k-1}}v_{-j}v_jv_{i_{k+2}}\cdots v_{i_f} & \\ & \text{if } i_k=-i_{k+1}=0,\\ q^{-1}vs_k+(q-q^{-1})\,v-(q-q^{-1})\sum_{j>i_k}q^{(i_k-j)/2}v_{i_1}\cdots v_{i_{k-1}}v_{-j}v_jv_{i_{k+2}}\cdots v_{i_f} & \\ & \text{if } i_k<i_{k+1},\ i_k=-i_{k+1}, \end{cases} \tag{5.1}$$

and define the action of the generator e_k of $BW_f(q^{n+1}, q)$ on v by

$$ve_k=\delta_{i_k,\,-i_{k+1}}\sum_{j\in I}q^{(j+i_k)/2}v_{i_1}\cdots v_{i_{k-1}}v_jv_{-j}v_{i_{k+2}}\cdots v_{i_f}. \tag{5.2}$$

(5.3) PROPOSITION. *The action defined above extends to a well-defined action of* $BW_f(q^{n+1}, q)$ *on* $V^{\otimes f}$.

Proof. One can either check this directly through very tedious computation or note that the action of g_i is that of the R-matrix $\check{R}_i$ in (4.7) and that the action of e_i is that of F_i in (4.10). ∎

The Weighted Trace

Let $x_1, x_3, x_5, \ldots, x_n$ be commuting, independent variables. Define $x_0 = 1$ and $x_{-i} = x_i^{-1}$ for $i = 1, 3, 5, \ldots, n$, so that x_i is defined for each $i \in I$. Define the weight of each word $v_{i_1} \cdots v_{i_f}$ of $V^{\otimes f}$ to be $wt(v_{i_1} \cdots v_{i_f}) = x_{i_1} \cdots x_{i_f}$, and define a weighted trace of $BW_f(q^{n+1}, q)$ acting on $V^{\otimes f}$ by

$$wtr(b) = \sum_{i_1, \ldots, i_f} v_{i_1} \cdots v_{i_f} b \,|_{v_{i_1} \cdots v_{i_f}} wt(v_{i_1} \cdots v_{i_f}) \tag{5.4}$$

for all $b \in BW_f(q^{n+1}, q)$, where the sum is over all sequences $i_1, i_2, \ldots, i_f$ with $i_j \in I$, and where $v_{i_1} \cdots v_{i_f} b \,|_{v_{i_1} \cdots v_{i_f}}$ is the coefficient of $v_{i_1} \cdots v_{i_f}$ in $v_{i_1} \cdots v_{i_f} b$. Since the action of $BW_f(q^{n+1}, q)$ on words w of $V^{\otimes f}$ preserves the weight of w (see (5.1) and (5.2)), the weighted trace satisfies the trace property $wtr(b_1 b_2) = wtr(b_2 b_1)$ for all $b_1, b_2 \in BW_f(q^{n+1}, q)$.

For each positive integer r, define $\mathcal{C}_r(n)$ to be the set of words $v_{i_1} v_{i_2} \cdots v_{i_r}$ such that

(1) $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$, $i_j \in I$, and

(2) the number of $i_j = 0$ is either 0 or a positive odd integer.

Define the ω-weight of a word $v_{i_1} \cdots v_{i_r} \in \mathcal{C}_r(n)$ by

$$wt_\omega(v_{i_1} \cdots v_{i_r}) = (q - q^{-1})^{\#\{i_j < i_{j+1}\}} q^{\#\{i_j = i_{j+1} \neq 0\}} x_{i_1} \cdots x_{i_r},$$

and let

$$Q_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q) = \sum_{w \in \mathcal{C}_r(n)} wt_\omega(w). \tag{5.5}$$

For each positive integer r, define $\mathcal{E}_r(n)$ to be the set of words $v_{i_1} v_{i_2} \cdots v_{i_r}$ such that

(1) $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$, $i_j \in I$, and

(2) one of the following holds:

(i) the number of $i_j = 0$ is a positive even integer, or

(ii) w contains the subword $v_{-k} v_k$ for some odd integer k with $1 \leqslant k \leqslant n$.

Define the ε-weight of a word $v_{i_1} \cdots v_{i_r} \in \mathcal{S}_r(n)$ by

$$wt_\varepsilon(v_{i_1} \cdots v_{i_r}) = \begin{cases} (q-q^{-1})^{\#\{i_j < i_{j+1}\}} q^{\#\{i_j = i_{j+1} \neq 0\}} (-q^{-k}) x_{i_1} \cdots x_{i_r}, \\ \quad \text{if } w \text{ contains the subword } v_{-k} v_k, \\ (q-q^{-1})^{\#\{i_j < i_{j+1}\}} q^{\#\{i_j = i_{j+1} \neq 0\}} x_{i_1} \cdots x_{i_r}, \\ \quad \text{otherwise.} \end{cases}$$

and let

$$E_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q) = \sum_{w \in \mathcal{S}_r(n)} wt_\varepsilon(w). \tag{5.6}$$

(5.7) PROPOSITION. *Let* $T_{\gamma_r} = g_{r-1} g_{r-2} \cdots g_1$. *Then the weighted trace of the element* $T_{\gamma_r} \in BW_r(q^{n+1}, q)$ *acting on* $V^{\otimes r}$ *is*

$$wtr(T_{\gamma_r}) = Q_r(X_n^{\pm 1}; q) + E_r(X_n^{\pm 1}; q).$$

Proof. The proof is by induction on r. Let $w = v_{i_1} v_{i_2} \cdots v_{i_r}$, $w' = v_{i_1} v_{i_2} \cdots v_{i_{r-1}}$, and $w'' = v_{i_1} v_{i_2} \cdots v_{i_{r-2}}$.

Case 1. $i_{r-1} > i_r$ and $i_{r-1} \neq -i_r$.

$$wT_{\gamma_r}|_w = w'' v_{i_r} v_{i_{r-1}} T_{\gamma_{r-1}}|_w = 0,$$

since $T_{\gamma_{r-1}}$ acts only on $w'' v_{i_r}$ and $v_{i_{r-1}} \neq v_{i_r}$.

Case 2. $i_{r-1} = i_r \neq 0$.

$$wT_{\gamma_r}|_w = qwT_{\gamma_{r-1}}|_w = qw' T_{\gamma_{r-1}}|_{w'}.$$

Case 3. $i_{r-1} < i_r$ and $i_{r-1} \neq -i_r$.

$$wT_{\gamma_r}|_w = w'' v_{i_r} v_{i_{r-1}} T_{\gamma_{r-1}}|_w + qwT_{\gamma_{r-1}}|_w = 0 + (q-q^{-1}) w' T_{\gamma_{r-1}}|_{w'}.$$

Case 4. $i_{r-1} > i_r$ and $i_{r-1} = -i_r$.

$$wT_{\gamma_r}|_w = q^{-1} w'' v_{i_r} v_{i_{r-1}} T_{\gamma_{r-1}}|_w - (q-q^{-1}) \sum_{j > i_{r-1}} w'' v_{-j} v_j T_{\gamma_{r-1}}|_w = 0,$$

since $j > i_{r-1} > i_r$, so that we never have $j = i_r$.

Case 5. $i_{r-1} = i_r = 0$.

$$wT_{\gamma_r}|_w = w' v_{i_r} T_{\gamma_{r-1}}|_w - (q-q^{-1}) \sum_{j>0} q^{-j/2} w'' v_{-j} v_j T_{\gamma_{r-1}}|_w = w' T_{\gamma_{r-1}}|_{w'},$$

since $j > i_r = 0$, so that we never have $j = i_r$.

Case 6. $i_{r-1} < i_r$ and $i_{r-1} = -i_r$.

$$\begin{aligned} wT_{\gamma_r}|_w &= w'' v_{i_r} v_{i_{r-1}} T_{\gamma_{r-1}}|_w + (q-q^{-1})\, w' v_{i_r} T_{\gamma_{r-1}}|_w \\ &\quad -(q-q^{-1}) \sum_{j > -i_r} q^{-i_r} w'' v_{-j} v_j T_{\gamma_{r-1}}|_w \\ &= 0 + (q-q^{-1})(1-q^{-i_r})\, w' T_{\gamma_{r-1}}|_{w'}. \end{aligned}$$

The result now follows by induction on r. ∎

(5.8) LEMMA. *Let* $\gamma = (\gamma_1, \gamma_3, \gamma_5, \ldots, \gamma_n)$ *be a sequence of integers such that* $\gamma_1 \geqslant \cdots \geqslant \gamma_n \geqslant 0$ *and* $\gamma_1 \neq 0$. *Then*

(a) $E_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q)|_{x_1^{\gamma_1} x_3^{\gamma_3} \cdots x_n^{\gamma_n}} = 0$,

and

(b) $E_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q)|_{x_1^0 x_3^0 \cdots x_n^0} = \begin{cases} q^{-(n+1)}, & \text{if } r \text{ is even}, \\ 0, & \text{if } r \text{ is odd}, \end{cases}$

where $E_r|_{x_1^{\gamma_1} x_3^{\gamma_3} \cdots x_n^{\gamma_n}}$ *denotes the coefficient of* $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ *in* E_r *as a polynomial in* $\mathbb{C}(q^{1/2})[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Proof. (a) Let $\mathscr{E}_{r,\gamma}(n) = \{w = v_{i_1} \cdots v_{i_r} \in \mathscr{E}_r(n) \mid x_{i_1} \cdots x_{i_r} = x_1^{\gamma_1} \cdots x_n^{\gamma_n}\}$. Let $2k = r - |\gamma| = r - \gamma_1 - \cdots - \gamma_n$ ($r - |\gamma|$ must be even), and note that every word in $\mathscr{E}_{r,\gamma}(n)$ can be written in the form

$$\begin{aligned} w = &\underbrace{v_{-n} \cdots v_{-n}}_{\mu_n} \underbrace{v_{-(n-2)} \cdots v_{-(n-2)}}_{\mu_{n-2}} \cdots \\ &\cdots \underbrace{v_{-1} \cdots v_{-1}}_{\mu_1} \underbrace{v_0 \cdots v_0}_{2k-2\mu_1} \underbrace{v_1 \cdots v_1}_{\mu_1} \underbrace{v_1 \cdots v_1}_{\gamma_1} \underbrace{v_3 \cdots v_3}_{\gamma_3} \cdots \underbrace{v_n \cdots v_n}_{\gamma_n} \\ = &\, v_{-n}^{\mu_n} \cdots v_1^{\mu_1} v_0^{2k-2\mu_1} v_1^{\mu_1} v_1^{\gamma_1} v_3^{\gamma_3} \cdots v_n^{\gamma_n} \end{aligned}$$

for nonnegative integers k, μ_i, ν_i such that $\nu_i - \mu_i = \gamma_i$ and $0 \leqslant \mu_1 \leqslant k$. Define $w_{\hat{\nu}} = v_3^{\nu_3} \cdots v_n^{\nu_n}$ and $w_{\hat{\mu}} = v_{-n}^{\mu_n} \cdots v_{-3}^{\mu_3}$ so that $w = w_{\hat{\mu}} v_{-1}^{\mu_1} v_0^{2k-2\mu_1} v_1^{\mu_1} v_1^{\gamma_1} w_{\hat{\nu}}$. Note that $\hat{\mu}$, $\hat{\nu}$, and μ_1 uniquely determine w, and

$$\begin{aligned} E_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q)|_{x_1^{\gamma_1} \cdots x_n^{\gamma_n}} &= \sum_{w \in \mathscr{E}_{r,\gamma}(n)} wt_\varepsilon(w) \\ &= \sum_{\hat{\mu}, \hat{\nu}} \sum_{\mu_1 = 0}^{k} wt_\varepsilon(w_{\hat{\mu}} v_{-1}^{\mu_1} v_0^{2k-2\mu_1} v_1^{\mu_1} v_1^{\gamma_1} w_{\hat{\nu}}). \end{aligned}$$

We show that the inner sum is equal to 0:

$$\sum_{\mu_1=0}^{k} wt_\varepsilon(w_{\hat{\mu}} v_{-1}^{\mu_1} v_0^{2k-2\mu_1} v_1^{\mu_1} v_1^{\gamma_1} w_{\hat{\gamma}}) = wt_\varepsilon(w_{\hat{\mu}} v_{-1}^{\mu_1} v_0^{2k} v_1^{\mu_1} v_1^{\gamma_1} w_{\hat{\gamma}}) + \sum_{\mu_1=1}^{k} wt_\varepsilon(w_{\hat{\mu}} v_{-1}^{\mu_1} v_0^{2k-2\mu_1} v_1^{\mu_1} v_1^{\gamma_1} w_{\hat{\gamma}} + wt_\varepsilon(w_{\hat{\mu}} v^{k}{}_{1} v_1^{\gamma_1} w_{\hat{\gamma}}).$$

Then, using the definition of wt_ε we have that

$$\begin{aligned}
&\sum_{\mu_1=0}^{k} wt_\varepsilon(w_{\hat{\mu}} v_{-1}^{\mu_1} v_0^{2k-2\mu_1} v_1^{\mu_1} v_1^{\gamma_1} w_{\hat{\gamma}}) \\
&\quad = wt_\varepsilon(w_{\hat{\mu}})(q-q^{-1})(q-q^{-1})\, wt_\varepsilon(v_1^{\gamma_1} w_{\hat{\gamma}}) \\
&\qquad + \sum_{\mu_1=1}^{k-1} wt_\varepsilon(w_{\hat{\mu}})(q-q^{-1})\, q^{\mu_1-1}(q-q^{-1})(q-q^{-1})\, q^{\mu_1-1} q\, wt_\varepsilon(v_1^{\gamma_1} w_{\hat{\gamma}}) \\
&\qquad + wt_\varepsilon(w_{\hat{\mu}})(q-q^{-1})\, q^{k-1}(q-q^{-1})(-q^{-1})\, q\, wt_\varepsilon(v_1^{\gamma_1} w_{\hat{\gamma}}) \\
&\quad = wt_\varepsilon(w_{\hat{\mu}})(q-q^{-1})^2\, wt_\varepsilon(v_1^{\gamma_1} w_{\hat{\gamma}}) \left(1 + \sum_{\mu_1=1}^{k-1} (q-q^{-1})\, q^{2\mu_1-1} - q^{2k-2}\right) \\
&\quad = wt_\varepsilon(w_{\hat{\mu}})(q-q^{-1})^2\, wt_\varepsilon(v_1^{\gamma_1} w_{\hat{\gamma}})(1 + (-1+q^{2k-2}) - q^{2k-2}) \\
&\quad = 0.
\end{aligned}$$

(b) Note that if $w = v_{i_1} \cdots v_{i_r} \in \mathscr{E}_r(n)$ and $x_{i_1} \cdots x_{i_r} = x_1^0 \cdots x_n^0$ then r must be even. Thus

$$E_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q)|_{x_1^0 \cdots x_n^0} = 0, \qquad \text{if } r \text{ is odd.}$$

Let $\mathscr{E}_{2r,0}(n) = \{w = v_{i_1} \cdots v_{i_{2r}} \in \mathscr{E}_r(n) \mid x_{i_1} \cdots x_{i_{2r}} = x_1^0 \cdots x_n^0 = 1\}$, and let

$$e(2r, n) = E_{2r}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q)\,|_{x_1^0 \cdots x_n^0} = \sum_{w \in \mathscr{E}_{2r,0}(n)} wt_\varepsilon(w).$$

We shall show that $e(2r, n) = q^{-(n+1)}$ for all $r \geqslant 1$ and $n \geqslant 1$, n odd. The proof is by induction on n. If $n = 1$, then

$$\begin{aligned}
e(2r, 1) &= \sum_{w \in \mathscr{E}_{2r,0}(1)} wt_\varepsilon(w) \\
&= wt_\varepsilon(v_0^{2r}) + \sum_{k=1}^{r-1} wt_\varepsilon(v^{k}{}_{1} v_0^{2r-2k} v_1^{r}) + wt_\varepsilon(v^{r}{}_{-1} v_1^{r}) \\
&= 1 + \sum_{k=1}^{r-1} (q-q^{-1})^2\, q^{2k-2} + q^{2r-2}(-q^{-1})(q-q^{-1})
\end{aligned}$$

$$= 1 + (q - q^{-1}) \left(\sum_{k=1}^{r-1} (q - q^{-1}) q^{2k-2} - q^{2r-3} \right)$$
$$= 1 + (q - q^{-1})(-q^{-1} + q^{2r-3} - q^{2r-3})$$
$$= q^{-2}.$$

Now let $n > 1$. Then every word $w \in \mathscr{E}_{2r,0}(n)$ can be written in the form

$$w = \underbrace{v_{-n} \cdots v_{-n}}_{\mu_n} w' \underbrace{v_n \cdots v_n}_{\mu_n} = v_{-n}^{\mu_n} w' v_n^{\mu_n}$$

where $w' \in \mathscr{E}_{2r-2\mu_n,0}(n-2)$ and $0 \leqslant \mu_n \leqslant r$. Then

$$\begin{aligned} e(2r, n) &= \sum_{w \in \mathscr{E}_{2r,0}(n)} wt_\varepsilon(w) \\ &= \sum_{\mu_n=0}^{r} \left(\sum_{w' \in \mathscr{E}_{2r-2\mu_n,0}(n-2)} wt_\varepsilon(v_{-n}^{\mu_n} w' v_n^{\mu_n}) \right) \\ &= \sum_{w' \in \mathscr{E}_{2r,0}(n-2)} wt_\varepsilon(w') + \sum_{\mu_n=1}^{r} q^{\mu_n - 1}(q - q^{-1}) \\ &\quad \times \left(\sum_{w' \in \mathscr{E}_{2r-2\mu_n,0}(n-2)} wt_\varepsilon(w') \right) (q - q^{-1}) q^{\mu_n - 1} \\ &\quad + q^{r-1}(q - q^{-1})(-q^{-n}) q^{r-1} \\ &= e(2r, n-2) + \sum_{\mu_n=1}^{r} e(2r - 2\mu_n, n-2) \\ &\quad \times (q - q^{-1})^2 q^{2\mu_n - 2} + (q - q^{-1})(-q^{2r-2-n}) \end{aligned}$$

By induction, $e(k, n-2) = q^{-(n-1)}$ for all k, so

$$\begin{aligned} e(2r, n) &= q^{-(n-1)} + (q - q^{-1}) \left(\sum_{\mu_n=1}^{r} (q - q^{-1}) q^{-(n-1)} q^{2\mu_n - 2} - q^{2r-2-n} \right) \\ &= q^{-(n-1)} + (q - q^{-1})(q^{2-2-(n-1)-1} + q^{2(r-1)-2-(n-1)+1} - q^{2r-2-n}) \\ &= q^{-n-1)} + (q - q^{-1})(-q^{-n}) = q^{-(n+1)}. \quad \blacksquare \end{aligned}$$

(5.9) PROPOSITION. *Define* $Q_0(x_1^{\pm 1}, \ldots, x_n^{\pm 1}), q) = 1/(q - q^{-1})$. *Then the generating function for* Q_r *is given by*

$$\begin{aligned} &(q - q^{-1}) \sum_{r \geqslant 0} Q_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q) z^r \\ &\quad = \left(\prod_{i>0} \frac{1 - q^{-1} x_i^{-1} z}{1 - q x_i^{-1} z} \frac{1 - q^{-1} x_i z}{1 - q x_i z} \right) \left(\frac{(q - q^{-1}) z^2}{1 - z^2} + 1 \right). \end{aligned} \tag{5.10}$$

Proof. Let $\mathcal{C}(n) = \bigcup_{r \geq 0} \mathcal{C}_r(n)$ where $\mathcal{C}_0(n)$ is the set consisting of the empty word $\varnothing$. We will say that $wt_\omega(\varnothing) = 1/(q - q^{-1})$. Then

$$\sum_{r \geq 0} Q_r(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q)\, z^r = \sum_{r \geq 0} \sum_{w \in \mathcal{C}_r(n)} wt_\omega(w)\, z^r = \sum_{w \in \mathcal{C}(n)} wt_\omega(w)\, z^{l(w)}, \qquad (*)$$

where $l(w)$ is the length of the word w, i.e., the number of letters in w. Every word $w \in \mathcal{C}(n)$ can be written in the form

$$\begin{aligned} w &= \underbrace{v_{-n} \cdots v_{-n}}_{\mu_n} \underbrace{v_{-1} \cdots v_{-1}}_{\mu_1} \underbrace{v_0 \cdots v_0}_{2\nu_0} \underbrace{v_1 \cdots v_1}_{\nu_1} \cdots \underbrace{v_n \cdots v_n}_{\nu_n} \\ &= v_{-n}^{\mu_n} \cdots v_{-1}^{\mu_1} v_0^{2\nu_0} v_1^{\nu_1} \cdots v_n^{\nu_n} \end{aligned}$$

for some nonnegative integers ν_i, μ_i. The right hand side of equation $(*)$ can be re-written in the form:

$$\begin{aligned} &\frac{1 - q^{-1} x_n^{-1} z}{1 - q x_n^{-1} z} \cdots \frac{1 - q^{-1} x_3^{-1} z}{1 - q x_3^{-1} z} \frac{1 - q^{-1} x_1^{-1} z}{1 - q x_1^{-1} z} \left(\frac{(q - q^{-1}) z^2}{1 - z^2} + 1 \right) \\ &\quad \times \frac{1 - q^{-1} x_1 z}{1 - q x_1 z} \cdots \frac{1 - q^{-1} x_n z}{1 - q x_n z}. \end{aligned} \qquad (**)$$

The result follows by substituting

$$\frac{1 - q^{-1} x_i z}{1 - q x_i z} = 1 + (q - q^{-1}) x_i z \sum_{\nu_i \geq 0} (q x_i z)^{\nu_i} \qquad \text{and}$$

$$\frac{q - q^{-1} z^2}{1 - z^2} + 1 = 1 + (q - q^{-1}) \sum_{\nu_0 \geq 1} z^{2\nu_0},$$

in $(**)$ and comparing terms with $(*)$. ∎

A Frobenius Formula for the Birman–Wenzl Algebras

The Weyl character formula for type B gives an expression for the characters of the irreducible representations of the orthogonal group $SO(n+2)$, $n+2$ odd. The irreducible characters are indexed by partitions λ such that $\lambda'_1 + \lambda'_2 \leq n+2$ (the total length of the first two columns is less than or equal to $n+2$) and are given by

$$sb_\lambda(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}) = \frac{\det(x_i^{\lambda_j + l - j + 1/2} - x_i^{-(\lambda_j + l - j + 1/2)})}{\det(x_i^{l - j + 1/2} - x_i^{-(l - j + 1/2)})}. \qquad (5.11)$$

where $l=(n+1)/2$. We shall refer to the sb_λ as the Weyl characters of type B. Let us denote

$$\hat{B}_f(n+2)=\{\lambda \vdash (f-2k) \mid 0 \leqslant k \leqslant \lfloor f/2 \rfloor, \lambda'_1+\lambda'_2 \leqslant n+2\}.$$

Note that when n is sufficiently large $\hat{B}_f(n+2)=\hat{B}_f$ where $\hat{B}_f$ is as defined in (2.13).

(5.12) LEMMA. (a) *For any idempotent* $p \in BW_f(q^{n+1}, q)$, $wtr(p)$ *is independent of* q.

(b) *If* p_λ, $\lambda \in \hat{B}_f(n+2)$, *is a minimal idempotent of the Brauer algebra* $B_f(n+2)$ *then*

$$wtr(p_\lambda)=sb_\lambda(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}),$$

where sb_λ *is the Weyl character of type B.*

(c) *For any* $b \in BW_f(q^{n+1}, q)$,

$$wtr(b)=\sum_{\lambda \in \hat{B}_f(n+2)} \chi^\lambda_{f,n}(b)\, sb\lambda(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}),$$

where, for each $\lambda \in \hat{B}_f(n+2)$, $\chi^\lambda_{f,n}$ *is the corresponding irreducible character of* $BW_f(q^{n+1}, q)$ *and* sb_λ *is the Weyl character for type B given by* (5.12).

Proof. (a) Recall that the weight of the word $v_{i_1} \cdots v_{i_f}$ is $wt(v_{i_1} \cdots v_{i_f})=x_{i_1} \cdots x_{i_f}$. Let $X=\{x_1^{i_1} x_3^{i_3} \cdots x_n^{i_n} \mid |i_1|+|i_3|+\cdots+|i_n|=f-2k\}$. Then X is the set of all possible weights of words in $\otimes^f V$. For $x \in X$, let P_x be the projection operator defined on words $w \in \otimes^f V$ by

$$P_x w=\begin{cases} w, & \text{if } wt(w)=x, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(P_x)^2=P_x$, and $P_x P_y=0$ if $x \neq y$, so P_x is an idempotent operator. Let $P=\sum_{x \in X} xP_x$. The action of $BW_f(q^{n+1}, q)$ on $\otimes^f V$ preserves weight, so if $b \in BW_f(q^{n+1}, q)$, then

$$wtr(b)=\sum_{i_1, \ldots, i_f} v_{i_1} \cdots v_{i_f} b \mid_{v_{i_1} \cdots v_{i_f}} x_{i_1} \cdots x_{i_f}=tr(bP),$$

where $tr(bP)$ is the trace of the operator bP on $\otimes^f V$. Now if p is an idempotent, then pP_x is an idempotent operator on $\otimes^f V$. The trace of an idempotent operator is the rank of that operator, so $tr(pP)=\sum_{x \in X} x\, tr(pP_x) \in \mathbb{Z}$. Since $tr(pP)$ is also rational function in q it must be independent of q.

(b) If d is an element of the Brauer algebra $B_f(n+2)$, then one has the following Frobenius formula for the characters of the Brauer algebra [R1]:

$$wtr(d) = \sum_{\mu \in \hat{B}_f(n+2)} \eta^{\mu}(d)\, sb_{\lambda}(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}),$$

where $\eta^{\mu}(d)$ is the irreducible character of $B_f(n+2)$ evaluated at the element d. If $\lambda \in \hat{B}_f(n+2)$ and p_{λ} is a minimal idempotent of $B_f(n+2)$ in the minimal corresponding to $\lambda \in \hat{B}_f(n+2)$, then $\eta^{\mu}(p_{\lambda}) = \delta_{\lambda_{\mu}}$. Thus,

$$wtr(p_{\lambda}) = \sum_{\mu \in \hat{B}_f(n+2)} \delta_{\lambda_{\mu}} sb_{\lambda}(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}) = sb_{\lambda}(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}).$$

(c) Let d_{λ} denote the dimension of the irreducible $BW_f(q^{n+1}, q)$-module labeled by λ, and let $\{p_i^{\lambda} \mid \lambda \in B_f(n+2),\ 1 \leqslant i \leqslant d_{\lambda}\}$, where be a partition of unity in $BW_f(q^{n+1}, q)$ with the property that when we specialize $q=1$ each p_i^{λ} is well defined and that, at $q=1$ $\{p_i^{\lambda}|_{q=1}\}$ is a partition of unity for $B_f(n+2)$. Such a partition of unity is given by [RW] Corollary 2.5. For each $\lambda \in \hat{B}_f(n+2)$ and each $1 \leqslant i \leqslant d_{\lambda}$ let b_{ii}^{λ} be the constant in $\mathbb{C}(q)$ such that $p_i^{\lambda} b p_{ii}^{\lambda} = b_{ii}^{\lambda} p_i^{\lambda}$. Note that the b_{ii}^{λ} are the diagonal elements of the matrix of b in the irreducible representation corresponding to λ determined by this partition of unity. Thus, for each $\lambda \in \hat{B}_f(n+2)$, we have $\sum_i b_{ii}^{\lambda} = \chi_{f,n}^{\lambda}(b)$.

From the trace property of wtr we have $wtr(p_i^{\lambda} b p_j^{\mu}) = wtr(p_j^{\mu} p_i^{\lambda} b) = 0$ unless $\lambda = \mu$ and $i = j$. Thus

$$\begin{aligned} wtr(b) &= \sum_{\lambda, \mu \in \hat{B}_f(n+2)} \sum_{i=1}^{d_{\lambda}} \sum_{j=1}^{d_{\mu}} wtr(p_i^{\lambda} b p_j^{\mu}) = \sum_{\lambda \in \hat{B}_f(n+2)} \sum_{i=1}^{d_{\lambda}} wtr(p_i^{\lambda} b p_i^{\lambda}) \\ &= \sum_{\lambda \in \hat{B}_f(n+2)} \sum_{i=1}^{d_{\lambda}} b_{ii}^{\lambda} wtr(p_i^{\lambda}). \end{aligned}$$

By part (a), $wtr(p_i^{\lambda})$ is independent of q. By part (b) $wtr(p_i^{\lambda}) = sb_{\lambda}(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1})$ for each i. Part (c) follows, since $\sum_i b_{ii}^{\lambda} = \chi_{f,n}^{\lambda}(b)$. ∎

(5.13) THEOREM. *Let $\mu \vdash f - 2h$. The Frobenius formula for $BW_f(q^{n+1}, q)$ is*

$$wtr(T_{\gamma_{\mu}} \otimes E^{\otimes h}) = \sum_{\lambda \in \hat{B}_f(n+2)} \chi_{f,n}^{\lambda}(T_{\gamma_{\mu}} \otimes E^{\otimes h})\, sb_{\lambda}(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}),$$

where sb_{λ} is the Weyl character of type B corresponding to $\lambda \in \hat{B}_f(n+2)$ and $\chi_{f,n}^{\lambda}$ is the irreducible character of $BW_f(q^{n+1}, q)$ corresponding to λ.

Proof. The theorem follows immediately from Lemma (5.12)(b). ∎

(5.14) THEOREM. (a) *The weighted trace of* $E = E_1 \in BW_2(q^{n+1}, q)$ *acting on* $V^{\otimes 2}$ is

$$wtr(E) = x = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + 1,$$

where x *is as defined in* (3.2).

(b) *Let* $T_{\gamma_r} = g_{r-1} g_{r-2} \cdots g_1$. *Then the weighted trace of the element* $T_{\gamma_r} \in BW_r(q^{n+1}, q)$ *acting on* $V^{\otimes r}$ *is*

$$wtr(T_{\gamma_r}) = Q_r(X_n^{\pm 1}; q) + \begin{cases} 0, & \text{if } r \text{ is odd,} \\ q^{-(n+1)}, & \text{if } r \text{ is even.} \end{cases}$$

(c) *Let* $\mu = (\mu_1, \mu_2, \ldots, \mu_t) \vdash f - 2h$ *and* $T_{\gamma_\mu} = T_{\gamma_1} \otimes T_{\gamma_2} \otimes \cdots \otimes T_{\gamma_{\mu_t}}$. *Then the weighted trace of* $T_{\gamma_\mu} \otimes E^{\otimes h}$ *on* $V^{\otimes f}$ *is*

$$wtr(T_{\gamma_\mu} \otimes E^{\otimes h}) = x^h wtr(T_{\gamma_{\mu_1}})\, wtr(T_{\gamma_{\mu_2}}) \cdots wtr(T_{\gamma_{\mu_t}}).$$

Proof. (a) from (5.2), we have

$$\begin{aligned} wtr(E) &= \sum_{i,j \in I} v_i v_j E \,|_{v_i v_j} x_i x_j \\ &= \sum_{i,j \in I} \delta_{i,-j} \sum_{k \in I} q^{(k+i)/2} v_k v_{-k} \,|_{v_i v_j} x_i x_j \\ &= \sum_{i \in I} q^i = x. \end{aligned}$$

(b) From (5.9) is evident that Q_r is a symmetric function. By the Frobenius Formula, Theorem (5.13), it is evident that $wtr(T_{\gamma_r})$ is a symmetric function. Thus, by (5.7), it is sufficient to determine the coefficient of $x_1^{\gamma_1} x_3^{\gamma_3} \cdots x_n^{\gamma_n}$ in E_r. The result now follows from Proposition (5.7) and Lemma (5.8).

For part (c) let $d = d_1 \otimes d_2 \in BW_f(q^{n+1}, q)$ with $d_1 \in BW_{f_1}(q^{n+1}, q)$ $d_2 \in BW_{f_2}(q^{n+1}, q)$, and $f_1 + f_2 = f$. Then since d_1 only acts on the first f_1 letters of a word $v_{i_1} \cdots v_{i_f}$ and d_2 acts only on the remaining letters of $v_{i_1} \cdots v_{i_f}$,

$$\begin{aligned} &\sum_{i_1, \ldots, i_f} v_{i_1} \cdots v_{i_f} d_1 \otimes d_2 \,|_{v_{i_1} \cdots v_{i_f}} x_{i_1} \cdots x_{i_f} \\ &\quad = \sum_{i_1, \ldots, i_{f_1}, j_1, \ldots, j_{f_2}} v_{i_1} \cdots v_{i_{f_1}} v_{j_1} \cdots v_{j_{f_2}} d_1 \otimes d_2 \,|_{v_{i_1} \cdots v_{i_{f_1}} v_{j_1} \cdots v_{j_{f_1}}} \\ &\qquad \times x_{i_1} \cdots x_{i_{f_2}} x_{j_1} \cdots x_{j_{f_2}} \end{aligned}$$

$$= \left(\sum_{i_1, \ldots, i_{f_1}} v_{i_1} \cdots v_{i_{f_1}} d_1 \,|_{v_{i_1} \cdots v_{i_{f_1}}} x_{i_1} \cdots x_{i_{f_1}} \right)$$
$$\times \left(\sum_{j_1, \ldots, j_{f_2}} v_{j_1} \cdots v_{j_{f_2}} d_2 \,|_{v_{j_1} \cdots v_{j_{f_2}}} x_{j_1} \cdots x_{j_{f_2}} \right).$$

Therefore, $wtr(d_1 \otimes d_2) = wtr(d_1)\, wtr(d_2)$. Note this is simply a proof of the fact that the trace of the action of $BW_{f_1}(q^{n+1}, q) \otimes BW_{f_2}(q^{n+1}, q)$ on $V^{\otimes f_1} \otimes V^{\otimes f_2}$ is the product of the traces of the action of BW_f, (q^{n+1}, q) on $V^{\otimes f}$. Part (c) follows immediately. ∎

6. Symmetric Functions

We think of an alphabet as a sum of commuting variables, so that, for example, $X = x_1 + x_2 + \cdots + x_n$ is the set of commuting variables $\{x_1, x_2, \ldots, x_n\}$. From this point of view one may use the following notations:

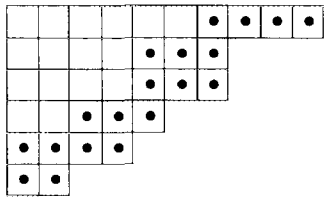
$$\{x_1, x_2, \ldots, x_n\} = X,$$
$$\{y_1, y_2, \ldots, y_n\} = Y,$$
$$\{x_i y_j\}_{1 \leq i,j \leq n} = XY,$$

and

$$\{x_1, \ldots, x_n, y_1, \ldots, y_n\} = X + Y.$$

Extending this idea, let $-X$ denote a formal (anti-)alphabet such that $X + (-X) = 0$.

If λ and μ are partitions such that $\mu \subseteq \lambda$, then λ/μ shall denote the *skew diagram* determined by the set theoretic difference of the Ferrers diagrams λ and μ. In the following diagram the filled boxes form the skew diagram $(10, 7, 7, 5, 4, 2)/(6, 4, 4, 2)$.



Every partition can be expressed as a skew diagram in the form $\lambda = \lambda/\varnothing$. A column strict tableau of shape λ/μ is a filling of the boxes of the skew diagram λ/μ such that each box is filled with an element of the set $\{1, \ldots, n\}$

and such that the numbers are strictly increasing down the columns of λ/μ and weakly increasing across the rows of λ/μ. For partitions λ, μ with $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}(X)$ in the alphabet X is defined by

$$s_{\lambda/\mu}(X) = \sum_T X^T \tag{6.1}$$

where the sum is taken over all column strict tableau T of shape λ/μ, and $X^T = x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$ where t_i is the number of i's in T. The set of $s_\lambda(X)$ as λ runs over all partitions forms a basis of the ring of symmetric functions in X. As in [Mac] Section 2 we shall let $n \to \infty$ and assume that X is an alphabet of infinitely many variables $x_1, x_2, \ldots$.

We have the following properties of Schur functions:

$$\begin{aligned} s_{\lambda/\mu}(X+Y) &= \sum_{\mu \subseteq \gamma \subseteq \lambda} s_{\lambda/\gamma}(X)\, s_{\gamma/\mu}(Y), && \text{(sum rule)} \\ s_{\lambda/\mu}(-X) &= (-1)^{|\lambda/\mu|}\, s_{\lambda'/\mu'}(X), && \text{(duality rule)} \\ s_{\lambda/\mu}(zX) &= z^{|\lambda/\mu|} s_{\lambda/\mu}(X), && \text{(homogeneity)} \end{aligned} \tag{6.2}$$

where $|\lambda/\mu|$ denotes the number of boxes in the skew diagram λ/μ and λ' denotes the conjugate of the partition λ. For proofs of the first two, see [Mac] Chapter I, (5.10), p. 46, and (3.10), p. 26. The third property follows immediately from the definition of the Schur function.

For an alphabet X, define the *Cauchy* kernel

$$\Omega(X) = \prod_{x_i \in X} \frac{1}{1 - x_i}. \tag{6.3}$$

We have the following properties of the Cauchy kernel:

$$\begin{aligned} \Omega(X+Y) &= \Omega(X)\, \Omega(Y), \\ \Omega(-X) &= \frac{1}{\Omega(X)}, \\ \Omega(XY) &= \sum_\lambda s_\lambda(X)\, s_\lambda(Y), \end{aligned} \tag{6.4}$$

where the sum is over all partitions λ (see [Mac], I (4.3), p. 33).

Define the orthogonal Schur function $sb_\lambda(X)$ by the equation

$$\Omega(XY - s_{(2)}(Y)) = \sum_\lambda s_\lambda(Y)\, sb_\lambda(X), \tag{6.5}$$

where the sum is over all partitions λ.

Broken Border Strips

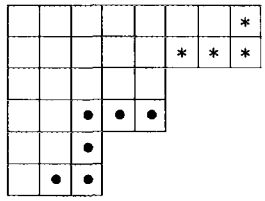
A skew diagram λ/μ is a *vertical strip* if each row contains at most one box. A skew diagram λ/μ is a *horizontal strip* if each column contains at most one box. A skew diagram λ/μ is a border strip if it is connected and contains no 2×2 block of boxes (see [Mac] I. Section 3 Ex. 11). A skew diagram is a *broken border strip* if it contains no 2×2 block of boxes. Any broken border strip is a union of its connected components, each of which is a border strip. Define the *weight* of a border strip λ/μ by

$$wt(\lambda/\mu; q) = q^{c-1}(-q^{-1})^{r-1},$$

where c is the number of columns and r is the number of rows in the border strip λ/μ. Define the *weight* of a broken border strip λ/μ by

$$wt(\lambda/\mu; q) = (q-q^{-1})^{cc-1} \prod_{bs\in\lambda/\mu} wt(bs; q), \tag{6.6}$$

where cc is the number of connected components (border strips) in λ/μ and the product is over the border strips bs in the broken border strip λ/μ. For convenience let us define $wt(\lambda/\mu; q) = 0$ if λ/μ is not a broken border strip, and $wt(\lambda/\lambda) = wt(\varnothing) = 1$. The following is a broken border strip of weight $(q-q^{-1})\, q^2(-q^{-1})\, q^3(-q^{-1})^2$.



(6.7) LEMMA. *Let q be a variable. Then in λ-ring notation,*

(a) $$s_{\mu/\nu}(q-q^{-1}) = \begin{cases} (q-q^{-1})\, wt(\mu/\nu; q), & \text{if } \mu/\nu \text{ is a broken border strip;} \\ 0, & \text{if } \mu/\nu \text{ is not a broken border strip.} \end{cases}$$

(b) $$s_\mu((q-q^{-1})z + Y) = \sum_{\nu\subseteq\mu} z^{|\mu/\nu|}(q-q^{-1})\, wt(\mu/\nu; q)\, s_\nu(Y).$$

Proof. (a) By the sum rule,

$$s_{\mu/\nu}(q-q^{-1}) = \sum_{\nu\subseteq\gamma\subseteq\mu} s_{\mu/\gamma}(q)\, s_{\mu/\gamma}(q)\, s_{\gamma/\nu}(-q^{-1}).$$

By the definition of the Schur function,

$$s_{\mu/\gamma}(q) = \begin{cases} q^k, & \text{if } \mu/\nu \text{ is a horizontal strip of length } k; \\ 0, & \text{otherwise.} \end{cases}$$

By duality, and the definition of the Schur function,

$$s_{\gamma/\nu}(-q^{-1}) = (-1)^{|\gamma/\nu|}\, s_{\gamma'/\nu'}(q^{-1})$$
$$= \begin{cases} (-q^{-1})^m, & \text{if } \mu/\nu \text{ is a vertical strip of length } m; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have that μ is "gotten" from ν by adding a vertical strip (to get γ) and then adding a horizontal strip (to γ). Then μ/ν is a broken strip (see the picture below). We have

$$s_{\mu/\nu}(q-q^{-1}) = \sum_{\substack{hs,\ vs \\ \nu + hs + vs = \mu}} q^{hs}(-q^{-1})^{vs},$$

where the sum is over all horizontal strips hs and all vertical strips vs such that μ is obtained from ν by first placing hs and then placing vs.

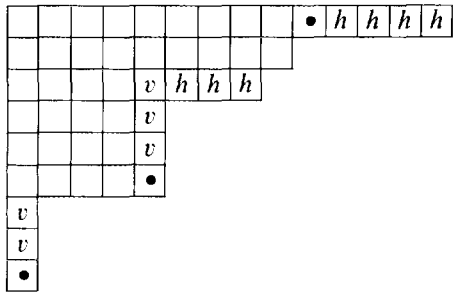
Suppose that bs is a border strip appearing in μ/ν. Each box in bs satisfies one of the following:

(1) There is a box of bs immediately below it,

(2) There is a box of bs immediately to its left,

(3) Neither (1) nor (2) holds.

In the picture below the boxes satisfying (1), (2), (3) are labeled with v, h, and • respectively, In case (1) the box must have come from the application of the vertical strip to μ and thus this box has weight $-q^{-1}$. In case (2) the box must have come from the application of the horizontal strip to μ and thus this box has weight q. In case (3) the box could have come from either the application of the horizontal strip or the vertical strip and thus this box has weight $q-q^{-1}$. Each bs in μ/ν contains exactly one box of this type.



The result follows by noting that the product of these weights is exactly $(q-q^{-1})\, wt(\mu/\nu; q)$ where $wt(\mu/\nu; q)$ is as defined in (6.6).

(b) By using the sum rule, and homogeneity,

$$s_\mu((q-q^{-1})z + Y) = \sum_{\nu\subseteq\mu} s_{\mu/\nu}((q-q^{-1})z)\, s_\nu(Y)$$

$$= \sum_{\nu\subseteq\mu} z^{|\mu/\nu|} s_{\mu/\nu}((q-q^{-1}))\, s_\nu(Y).$$

Part (b) now follows by application of part (a). ∎

Define symmetric functions $q_r(X; q)$ and $Q_r(X; q)$ by the following generating functions.

$$(q-q^{-1}) \sum_{r\geq 0} q_r(X; q)\, z^r = \Omega(X(q-q^{-1})z), \qquad \text{and} \tag{6.8}$$

$$(q-q^{-1}) \sum_{r\geq 0} Q_r(X; q)\, z^r = \Omega(X(q-q^{-1})z - s_{(2)}((q-q^{-1})z)) \tag{6.9}$$

(6.10) PROPOSITION.

(a) $$q_r(X; q) = \sum_{m=1}^{r} (-q^{-1})^{r-m} q^{m-1} s_{(m,1^{r-m})}(X).$$

(b) $$Q_r(X; q) = \sum_{m=1}^{r} (-q^{-1})^{r-m} q^{m-1} sb_{(m,1^{r-m})}(X).$$

(c) $$Q_r(X; q) = q_r(X; q) + (1-q^2) \sum_{m=1}^{\lfloor r/2 \rfloor} q_{r-2m}(X; q).$$

Proof. (a) It follows from the definition of $q_r(X; q)$ and the product rule for the Cauchy kernel that

$$(q-q^{-1}) \sum_{r\geq 0} q_r(X; q)\, z^r = \Omega(X(q-q^{-1})z) = \sum_\lambda s_\lambda((q-q^{-1})z)\, s_\lambda(X).$$

Applying Lemma (6.7a) gives

$$(q-q^{-1}) \sum_{r\geq 0} q_r(X; q)\, z^r = \sum_{r\geq 0} (q-q^{-1}) z^r \sum_{m=1}^{r} (-q^{-1})^{r-m} q^{m-1} s_{(m,1^{r-m})}(X).$$

The result follows by comparing coefficients of z^r.

(b) By the definitions of $Q_r(X; q)$ and the orthogonal Schur functions

$$(q-q^{-1}) \sum_{r \geq 0} Q_r(X; q)\, z^r = \sum_{\lambda} s_\lambda((q-q^{-1})\, z)\, sb_\lambda(X).$$

By applying Lemma (6.7a) we have

$$(q-q^{-1}) \sum_{r \geq 0} Q_r(X; q)\, z^r = \sum_{r \geq 0} (q-q^{-1})\, z^r \sum_{m=1}^{r} (-q^{-1})^{r-m} q^{r-m} sb_{(m, 1^{r-m})}(X),$$

and the result follows by comparing coefficients of z^r.

(c) It follows from homogeneity and Lemma (6.7a), that

$$s_{(2)}((q-q^{-1})\, z) = z^2(q-q^{-1})\, q = -z^2 + q^2 z^2.$$

Therefore,

$$\begin{aligned}(q-q^{-1}) \sum_{r \geq 0} Q_r(X; q)\, z^r &= \Omega(X(q-q^{-1})\, z + z^2 - q^2 z^2) \\ &= \Omega(X((q-q^{-1})\, z)) \frac{1-q^2 z^2}{1-z^2} \\ &= \left((q-q^{-1}) \sum_{k \geq 0} q_k z^k\right)(1 + (1-q^2)(z^2 + z^4 + \cdots)).\end{aligned}$$

The result follows by equating coefficients of z^r. ∎

(6.11) PROPOSITION. (a) *For each $r > 0$ and each partition λ,*

$$q_r(X; q)\, s_\lambda(X) = \sum_{\mu \supseteq \lambda} s_\mu(X)\, wt(\mu/\lambda; q),$$

where the sum is over partitions μ such that μ/λ is a broken border strip and $|\mu/\lambda| = r$.

(b) *For each $r > 0$ and each partition λ,*

$$Q_r(X; q)\, sb_\lambda(X) = \sum_{\nu \subseteq \lambda} \sum_{\mu \supseteq \nu} wt(\lambda/\nu; q)\, wt(\mu/\nu; q)\, sb_\mu(X),$$

where the sum is over all partitions μ and ν such that λ/ν and μ/ν are broken border strips and $|\lambda/\nu| + |\mu/\nu| = r$.

Proof. (a) Let GF be a short notation for the following generating function

$$GF=(q-q^{-1})\sum_{r\geq 0} q_r(X;q)\,z^r \sum_{\lambda} s_\lambda(X)\,s_\lambda(Y).$$

It follows from the product rule for the Cauchy kernel and the definition of the $q_r(X;q)$ that

$$GF=\Omega(X(q-q^{-1})\,z)\,\Omega(XY).$$

By the addition rule for the Cauchy kernel,

$$GF=\Omega(X((q-q^{-1})\,z+Y)).$$

Using the product rule for the Cauchy kernel to reexpand, we have

$$GF=\sum_{\mu} s_\mu(X)\,s_\mu((q-q^{-1})\,z+Y).$$

Now use Lemma (6.7b) to rewrite the Schur function $s_\mu((q-q^{-1})\,z+Y)$ and get

$$GF=\sum_{\mu} s_\mu(X)\sum_{\lambda\subseteq\mu} z^{|\mu/\lambda|}(q-q^{-1})\,wt(\mu/\lambda;q)\,s_\lambda(Y).$$

Summarizing, we have obtained

$$\sum_{r\geq 0} q_r(X;q)\,z^r\sum_{\lambda} s_\lambda(X)\,s_\lambda(Y)=\sum_{\lambda}\sum_{\mu\supseteq\lambda} wt(\mu/\lambda;q)\,s_\mu(X)\,z^{|\mu/\lambda|}s_\lambda(Y).$$

The result now follows by taking the coefficient of $z^r s_\lambda(Y)$ on each side of this equation.

(b) Let GF be a short notation for the following generating function:

$$GF=(q-q^{-1})\sum_{r\geq 0} Q_r(X;q)\,z^r\sum_{\lambda} sh_\lambda(X)\,s_\lambda(Y).$$

It follows from the definitions of the orthogonal Schur functions and $Q_r(X;q)$ that

$$GF=\Omega(X(q-q^{-1})\,z-s_{(2)}((q-q^{-1})\,z))\,\Omega(XY-s_{(2)}(Y)).$$

By the addition rule for the Cauchy kernel this is equal to

$$GF=\Omega(X((q-q^{-1})\,z+Y)-(s_{(2)}(Y)+s_{(2)}((q-q^{-1})\,z))-s_{(1)}((q-q^{-1})\,z)$$
$$+s_{(1)}(Y)\,s_{(1)}((q-q^{-1})\,z)).$$

The expressions

$$s_{(2)}(Y+(q-q^{-1})z)=s_{(2)}(Y)+s_{(1)}(Y)\,s_{(1)}((q-q^{-1})z)+s_{(2)}((q-q^{-1})z),$$
$$\text{and}\quad s_{(1)}(Y)\,s_{(1)}((q-q^{-1})z)=Y(q-q^{-1})z,$$

follow from the addition rule for Schur functions and the definitions of the Schur functions respectively. Substituting these expressions gives

$$GF=\Omega(X((q-q^{-1})z+Y)-(s_{(2)}((q-q^{-1})z+Y)+Y(q-q^{-1})z).$$

Using the definitions of the orthogonal Schur functions and the definition of the $q_r(X;q)$ and reexpanding,

$$GF=\Big(\sum_{\mu} sb_{\mu}(X)\,s_{\mu}((q-q^{-1})z+Y\Big)\Big((q-q^{-1})\sum_{k\geqslant 0} z^k q_k(Y;q)\Big).$$

Now use Lemma (6.7b) to rewrite the Schur function $s_{\mu}((q-q^{-1})z+Y)$ and get

$$GF=(q-q^{-1})\sum_{\mu} sb_{\mu}(X)\sum_{\nu\subseteq\mu} z^{|\mu/\nu|}wt(\mu/\nu;q)\,s_{\nu}(Y)\sum_{k\geqslant 0} z^k q_k(Y;0).$$

Using part (a) of this proposition to expand the products $q_k(Y;q)\,s_{\nu}(Y)$ this is

$$GF=(q-q^{-1})\sum_{\mu} sb_{\mu}(X)\sum_{\nu\subseteq\mu} z^{|\mu/\nu|}wt(\mu/\nu;q)\sum_{k\geqslant 0} z^k \sum_{\lambda\supseteq\nu} s_{\lambda}(Y)\,wt(\lambda/\nu;q),$$

where the second sum is over all $\lambda\supseteq\nu$ such that $|\lambda/\nu|=k$. Summarizing, we have obtained

$$\sum_{r\geqslant 0} Q_r(X;q)\,z^r\sum_{\lambda} sb_{\lambda}(X)\,s_{\lambda}(Y)$$
$$=\sum_{\lambda}\sum_{\nu\subseteq\lambda}\sum_{\mu\supseteq\nu} wt(\lambda/\nu;q)\,wt(\mu/\nu;q)\,sb_{\mu}(X)\,z^{|\lambda/\nu|+|\lambda/\mu|}s_{\lambda}(Y).$$

The result now follows by taking the coefficient of $z^r s_{\lambda}(Y)$ on each side of this equation. ∎

The Characters of the Birman–Wenzl Algebra

Define a μ-up-down broken border strip tableau of length f and shape λ to be a sequence of partitions

$$T=(\varnothing=\lambda^{(0)}\supseteq\lambda^{(1)}\subseteq\lambda^{(2)}\supseteq,\ldots,\supseteq\lambda^{(2f-1)}\subseteq\lambda^{(2f)}=\lambda)$$

such that for each $1 \leqslant j \leqslant f$, either

(1) (a) $\lambda^{(2j-2)}/\lambda^{(2j-1)}$ is a broken border strip,

(b) $\lambda^{(2j)}/\lambda^{(2j-1)}$ is a broken border strip, and

(c) $|\lambda^{(2j-2)}/\lambda^{(2j-1)}| + |\lambda^{(2j)}/\lambda^{(2j-1)}| = \mu_j$,

or

(2) $\lambda^{(2j-2)} = \lambda^{(2j-1)} = \lambda^{(2j)}$ and μ_j is even.

In case (1) define

$$wt(\lambda^{(2j)}, \lambda^{(2j-1)}, \lambda^{(2j-2)}) = wt(\lambda^{(2j-2)}/\lambda^{(2j-1)}; q)\, wt(\lambda^{(2j)}/\lambda^{(2j-1)}; q), \quad (6.12)$$

where the weights $wt(\mu/\nu; q)$ of broken border strip are given by (6.6). In case (2) define

$$wt(\lambda^{(2j)}, \lambda^{(2j-1)}, \lambda^{(2j-2)}) = r^{-1}. \quad (6.13)$$

Define the weight of a μ-up-down broken border strip tableau of length f to be

$$wt(T; r, q) = \prod_{j=1}^{f} wt(\lambda^{(2j)}, \lambda^{(2j-1)}, \lambda^{(2j-2)}). \quad (6.14)$$

(6.15) THEOREM. *The irreducible characters χ_f^λ, $\lambda \in \hat{B}_f$, of the Birman–Wenzl algebra $BW_f(r, q)$ are given by*

$$\chi_f^\lambda(T_{\gamma\mu} \otimes E^{\otimes k}) = x^k \sum_T wt(T; r, q),$$

where the sum is over all μ-up-down broken border strip tableaux of length f and shape λ and

$$x = \frac{r - r^{-1}}{q - q^{-1}} + 1.$$

Proof. We first prove the formula for all cases such that $r = q^{n+1}$, and n is a large ($n > 2l(\lambda)$) odd integer. Let $\chi_{f,n}^\lambda$ denote the irreducible character of $BW_f(q^{n+1}, q)$ corresponding to a partition $\lambda \in \hat{B}_f(n+2)$. By the Frobenius formula, Theorem (5.13), we have that

$$x^h wtr(T_{\gamma\mu}) = \sum_{\lambda \in \hat{B}_f} \chi_{f,n}^\lambda(T_{\gamma\mu} \otimes E^{\otimes h})\, sb_\lambda(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1})$$

and by Theorem (5.14), that

$$wtr(T_{\gamma_\mu}) = \prod_i wtr(T_{\gamma_{\mu_i}}).$$

Let $X_n^{\pm 1} + 1$ denote the alphabet $X_n^{\pm 1} + 1 = \{x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}, 1\}$. It is well known (see [Wey], [Li], or [KT] Proposition 2.2.1) that if n is sufficiently large ($n > 2l(\lambda)$), then

$$sb_\lambda(X_n^{\pm 1} + 1) = sb_\lambda(x_1^{\pm 1}, x_3^{\pm 1}, \ldots, x_n^{\pm 1}) = \frac{\det(x_i^{\lambda_j + l - j + 1/2} - x_i^{-(\lambda_j + l - j + 1/2)})}{\det(x_i^{l-j+1/2} - x_i^{-(l-j+1/2)})},$$

where $l = (n+1)/2$, which is the Weyl character for type B as given in (5.11). It follows from Theorem (5.14) and Proposition (6.11) that if n is sufficiently large ($n \geqslant 2(\mu_i + l(\lambda))$) then

$$\begin{aligned}
&wtr(T_{\gamma_{\mu_i}})\, sb_\lambda(X_n^{\pm 1} + 1) \\
&\quad = Q_{u_i}(X_n^{\pm 1} + 1)\, sb_\lambda(X_n^{\pm 1} + 1) \\
&\qquad + \begin{cases} q^{-(n+1)} sb_\lambda(X_n^{\pm 1} + 1), & \text{if } \mu_i \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \\
&\quad = \sum_{\nu \subseteq \lambda} \sum_{\pi \supseteq \nu} wt(\lambda/\nu; q)\, wt(\pi/\nu; q)\, sb_\pi(X_n^{\pm 1} + 1) \\
&\qquad + \begin{cases} q^{-(n+1)} sb_\lambda(X_n^{\pm 1} + 1), & \text{if } \mu_i \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where the sum is over all partitions ν and π such that λ/ν and π/ν are broken border strips and $|\lambda/\nu| + |\pi/\nu| = \mu_i$. Assuming n is sufficiently large, we have, by induction,

$$wtr(T_{\gamma_\mu}) = \sum_{\lambda \in \hat{B}_f} \sum_T wt(T; q^{n+1}, q),$$

where the inner sum is over all μ-up-down broken border strip tableaux T of length f and shape λ. Thus,

$$x^h \sum_{\lambda \in \hat{B}_f} \sum_T wt(T; q^{n+1}, q)\, sb_\lambda(X_n^{\pm 1} + 1) = \sum_{\lambda \in \hat{B}_f} \chi^\lambda(T_{\gamma_\mu} \otimes E^{\otimes h})\, sb_\lambda(X_n^{\pm 1} + 1).$$

If we take the coefficient of $sb_\lambda(X_n^{\pm 1} + 1)$ on both sides (the sb_λ are linearly independent for n large), we get

$$\chi^\lambda_{f,n}(T_{\gamma_\mu}) = x^h \sum_T wt(T; q^{n+1}, q). \tag{6.16}$$

where the sum is over all μ-up-down broken border strip tableaux of length f and shape λ. This proves the result for all cases where $r = q^{n+1}$ and n is a large odd integer ($n > 2l(\lambda)$).

We know that for a given $\lambda \in \hat{B}_f$ and given $\mu \vdash f - 2k$, the value $\chi_f^\lambda(T_{\gamma_\mu})$ is a rational function of r and q. Similarly, the value

$$x^k \sum_T wt(T; r, q),$$

where the sum is over all μ-up-down broken border strip tableaux of length f and shape λ, is a rational function of r and q. Furthermore, (6.16) shows that these two functions are equal for all cases where $r = q^{n+1}$ and n is a large odd integer. In particular, for fixed $q \in \mathbb{C}$ these two functions are equal for infinitely many different values of r. Therefore they must be equal everywhere and the theorem follows. ∎

(6.17) COROLLARY. *For each positive integer f and each $\lambda \in \hat{B}_f$ let χ_f^λ denote the the irreducible character of $BW_f(r, q)$. Let $\mu, \lambda \in \hat{B}_f$ with $\mu \vdash f - 2h$. Suppose that $\mu = (\mu_1, \ldots, \mu_t)$ with $\mu_t > 0$, and define $\hat{\mu} = (\mu_1, \ldots, \mu_{t-1})$. Then*

$$\chi_f^\lambda(T_{\gamma_\mu} \otimes E^{\otimes h}) = x^h \sum_{\nu \subseteq \lambda} \sum_{\nu \subseteq \pi} wt(\lambda/\nu; q)\, wt(\pi/\nu; q)\, \chi_f^{\pi}{}_{\mu_t}(T_{\gamma_{\hat{\mu}}}) + \begin{cases} r^{-1} \chi_f^\lambda{}_{\mu_t}(T_{\gamma_{\hat{\mu}}}), & \text{if } \mu_t \text{ is even,} \\ 0, & \text{if } \mu_t \text{ is odd,} \end{cases}$$

where the sum is over all partitions ν and π such that λ/ν and π/ν are broken border strips and $|\lambda/\nu| + |\pi/\nu| = \mu_t$. The weights $wt(\lambda/\nu; q)$ and $wt(\pi/\nu; q)$ are given by (6.6).

7. CONCLUDING REMARKS

1. In Section 2 we have shown that the character table of any algebra containing a Jones basic construction should have a certain form with a block of zeros in one corner. We also remarked that for our examples we are able to choose a set of elements of the algebra that determine the characters and that with respect to these elements the character tables are square. This second property holds in any semisimple algebra and is proved in Chapter 1 of [R3]. Many other facts concerning characters are presented in [R3] in the generality of split semisimple algebras, including general Frobenius characteristic maps, and general formulas for induced characters analogous to those for characters of finite groups.

2. The character formula (2.22) for the Okada algebra is the same as the recurrence formula for Okada's Young–Fibonacci analogues of the Kostka numbers ([O], Proposition 4.2), and the character table for O_f is precisely the table of "Kostka numbers" for the Okada algebra. It is not immediately obvious whether this fact is coincidence or not. Is there a good reason why the values of the irreducible characters on the special set of elements that we have chosen are the same as the multiplicities of irreducibles in certain special induced representations of the Okada algebras? If so, does this fact generalize to algebras which have Bratteli diagrams determined by other r-differentiable posets?

3. Halverson [H1], [H2] studies a another example of an algebra containing a Jones basic construction that we should have included here. He computes the characters of the centralizer algebra $H^r_{m,n}(q)$ of the representation of the quantum general linear group $\mathscr{U}_q(\mathfrak{g}(r))$ on the mixed tensor space $(V\otimes\mathbb{C}(q))^{\otimes m}\otimes(V\otimes\mathbb{C}(q))^{\otimes n}$. The algebra $H^r_{m,n}(q)$ specializes at $q=1$ to a subalgebra $B_{m,n}(r)$ of the Brauer algebra $B_{m+n}(r)$, and it is isomorphic to the direct sum of the tensor product of Hecke algebras $H_m(q)\otimes H_n(q)$ and a Jones basic construction for $H^r_{m-2,n-2}\subseteq H^r_{m-1,n-1}(q)$. In [H3] the Murnaghan–Nakayama rule for computing the irreducible characters of $H^r_{m,n}(q)$ is given.

4. In the definition of the Birman–Wenzl algebra $BW_f(r,q)$ given in section 3, relation (BW2) does not follow from relations (B1), (B2), and (BW1). The algebra subject only to (B1), (B2), and (BW1) has a 1-dimensional representation given by sending g_i to r^{-1} that does not satisfy (BW2). It is an open problem to determine the structure of the algebra generated by g_i, $1\leqslant i\leqslant f-1$ which are subject only to the relations (B1), (B2), and (BW1). A priori, there is no reason to assume that such an algebra is even finite dimensional. It will be true however that any representation of such an algebra is also a representation of the braid group.

5. The basis given in section 3 for $BW_f(r,q)$, when restricted to q-diagrams with only vertical edges, gives a basis of the Hecke algebra $H_f(q^2)$. This is a special basis for the Iwahori–Hecke algebras of type A which divides into "character classes." The transition matrix from this special basis to the usual basis T_π, $\pi\in\mathscr{S}_f$ is a triangular matrix (with respect to the Bruhat order on the symmetric group) with powers of q on the diagonal. Furthermore, all of the entries in this matrix are in $\mathbb{Z}[q]$. Theorem (3.18) provides another proof of the result of [R2] Theorem that any character of $H_f(q^2)$ is determined by its values on the elements T_{γ_μ} with $\mu\vdash f$. The problem of determining such a basis for the other types of Iwahori-Hecke algebras to our knowledge is still open.

6. We have, for the most part, in section 6, avoided the difficulties which arise when working the Weyl characters with n small. In these "low rank" cases one would have to use the modification rules for the Weyl characters of type B which are given by King [Ki] and Koike–Terada [KT]. If one uses these modification rules the same methods will determine the characters of $BW_f(q^{n+1}, q)$ in the cases where n is small.

We should also mention that our general construct $sb_\lambda(X)$, the "orthogonal Schur function", which we view as a function on an "infinite alphabet" X, is equivalent to the corresponding object in the universal character ring which Koike and Terada consider [KT]. We do feel, however, that our "λ-ring" approach is much easier, without it we would not have been able to derive the combinatorial rule for the characters of the Birman–Wenzl algebra. Furthermore, we would like to point out that we have used the theory of Weyl group symmetric functions for type B in a crucial way to arrive at our results. The theory of symmetric functions for Weyl groups other than the symmetric group is still in its infancy, and we expect that as it develops it will become a useful tool in representation theory.

7. The group algebra of the symmetric group is contained in the Brauer algebra in a very natural way. This is equivalent, by taking centralizers, to the fact that the orthogonal group $O(n)$ is a natural subgroup of the general linear group $Gl(n)$. This fact was used in a crucial way in [R1] and is evidence by the fact that the bitrace on $V^{\otimes f}$ is the power symmetric function in both the general linear group case and the orthogonal group case. In analogy, we are expecting that the weighted trace of the Birman–Wenzl algebra on $V^{\otimes f}$ (which is our function Q_r in Section 5) would be the same as the weighted trace of the Iwahori–Hecke algebra acting on $V^{\otimes f}$ (the function q_r of section 6) as derived in [R2]. However, it is not true that the Iwahori-Hecke algebra is contained in the Birman–Wenzl algebra in an analogous way. This was surprising to us at first because, by taking centralizers, it implies that the quantum group corresponding to $\mathfrak{so}(n)$ is not contained in a natural way inside the quantum group corresponding to $\mathfrak{sl}(n)$. A consequence of this is that there is no branching rule for $H_f(q^2) \subseteq BW_f(r, q)$ analogous to the branching rule between the Brauer algebra and the group algebra of the symmetric group given in [R1].

8. In [Ke], S. Kerov has given the Frobenius formula for the Birman–Wenzl algebras, our Theorem (5.14). There is no proof of this formula in his paper and very few clues. It seems plausible that he arrived at this formula by considering a double centralizer correspondence between the Birman–Wenzl algebra and the Iwahori–Hecke algebra acting on a special set of tangles. This would constitute an approach significantly different

from the one we have used in this paper. Such an approach could be interesting in itself. Other approaches to the same Frobenius formula might be by considering the Markov trace tr on the Birman–Wenzl algebra defined in Section 3. Wenzl [Wn2] has decomposed these traces into irreducibles (by using q-traces on the quantum group), and such a decomposition could be viewed as a specialization of the Frobenius formula.

8. Tables and Formulas for Irreducible Characters

The Temperley–Lieb Algebra

The irreducible character $\chi_f^{(f-l,l)}$ of TL_f evaluated on the element $id_{f-2k} \otimes e^{\otimes k}$ is given by

$$\chi_f^{(f-l,l)}(id_{f-2k} \otimes e^{\otimes k}) = \begin{cases} \binom{f-2k}{l-k} - \binom{f-2k}{l-k-1}, & \text{if } l \geqslant k, \\ 0, & \text{if } l < k. \end{cases}$$

The Okada Algebra

The irreducible characters of the Okada algebra O_f are given for $f \leqslant 5$ in the following tables. The (v, w)-entry is $\chi_{O_f}^v(e_w)$.

Character Table for O_2

$v \setminus w$	2	11
2	1	1
11	0	1

Character Table for O_3

$v \setminus w$	21	12	111
21	1	1	2
12	0	1	1
111	0	0	1

Character Table for O_4

$v \setminus w$	22	211	121	112	1111
22	1	1	1	2	3
211	0	1	1	1	3
121	0	0	1	1	2
112	0	0	0	1	1
1111	0	0	0	0	1

Character Table for O_5

$v \setminus w$	221	212	2111	122	1211	1121	1112	11111
221	1	1	2	1	2	3	4	8
212	0	1	1	1	1	1	3	4
2111	0	0	1	0	1	1	1	4
122	0	0	0	1	1	1	2	3
1211	0	0	0	0	1	1	1	3
1121	0	0	0	0	0	1	1	2
1112	0	0	0	0	0	0	1	1
11111	0	0	0	0	0	0	0	1

The Birman–Wenzl Algebra

We have written a Maple program to inductively compute the character tables for $BW_f(r, q)$ using (6.17). The (λ, μ)-entry of the table is $\chi^{\lambda}_{BW_f(r,q)}(T_{y_\mu} \otimes E^{\otimes h})$ where $\mu \vdash f - 2h$. We give tables for $f \leqslant 5$. Setting $q = r = 1$ (and leaving x as a parameter) in these tables gives the corresponding character value for the Brauer algebra $B_f(x)$. For each integer n, setting $q = r = 1$ and $x = n$ in these tables gives the correspoding character for the Brauer algebras $B_f(n)$ (assuming that $|n| \geqslant f$).

Character Table for $BW_2(r, q)$.

$\lambda \setminus \mu$	(2)	(1^2)	$\emptyset$
(2)	q	1	0
(1^2)	$-q^{-1}$	1	0
$\emptyset$	r^{-1}	1	x

Character Table for $BW_3(r, q)$.

$\lambda \setminus \mu$	(3)	(2, 1)	(1^3)	(1)
(3)	q^2	q	1	0
(2, 1)	-1	$q - q^{-1}$	2	0
(1^3)	q^{-2}	$-q^{-1}$	1	0
(1)	0	$q - q^{-1} + r^{-1}$	3	x

Character Table for $BW_4(r,q)$

$\lambda \setminus \mu$	(4)	(3,1)	(2^2)	$(2,1^2)$	(1^4)	(2)	(1^2)	$\emptyset$
(4)	q^3	q^2	q^2	q	1	0	0	0
(3,1)	$-q$	q^2-1	q^2-2	$2q-q^{-1}$	3	0	0	0
(2^2)	0	-1	q^2+q^{-2}	$q-q^{-1}$	2	0	0	0
$(2,1^2)$	q^{-1}	$-1+q^{-2}$	$-2+q^{-2}$	$q-2q^{-1}$	3	0	0	0
(1^4)	$-q^{-3}$	q^{-2}	q^{-2}	$-q^{-1}$	1	0	0	0
(2)	0	q^2-1	$2qr^{-1}+q-q^{-1}$	$r^{-1}+3q-2q^{-1}$	6	xq	x	0
(1^2)	0	$-1+q^{-2}$	$-2q^{-1}r^{-1}+q-q^{-1}$	$r^{-1}+2q-3q^{-1}$	6	$-xq^{-1}$	x	0
$\emptyset$	r^{-1}	0	$q^2+q^{-2}+r^{-2}$	$r^{-1}+q-q^{-1}$	3	xr^{-1}	x	x^2

Character Table for $BW_5(r,q)$

$\lambda \setminus \mu$	(5)	(4,1)	(3,2)	$(3,1^2)$	$(2^2,1)$
(5)	q^4	q^3	q^3	q^2	q^2
(4,1)	$-q^2$	q^3-q	q^3-2q	$2q^2-1$	$2q^2-2$
(3,2)	0	$-q$	q^3-q+q^{-1}	q^2-2	$2q^2-2+q^{-2}$
$(3,1^2)$	1	$-q+q^{-1}$	$-2q+2q^{-1}$	q^2-2+q^{-2}	q^2-4+q^{-2}
$(2^2,1)$	0	q^{-1}	$-q+q^{-1}-q^{-3}$	$-2+q^{-2}$	$q^2+2q^{-2}-2$
$(2,1^3)$	$-q^{-2}$	$q^{-1}-q^{-3}$	$2q^{-1}-q^{-3}$	$-1+2q^{-2}$	$-2+2q^{-2}$
(1^5)	q^{-4}	$-q^{-3}$	$-q^{-3}$	q^{-2}	q^{-2}
(3)	0	q^3-q	$q^2-1+q^2r^{-1}$	$3q^2-2$	$+2qr^{-1}+q-q^{-1}+2q^2-2$
(2,1)	0	$-q+q^{-1}$	q^2-1-r^{-1}	$2q^2-5+2q^{-2}$	$2qr^{-1}+2q-2q^{-1}-2q^{-1}r^{-1}+2q^2+2q^{-2}-4$
(1^3)	0	$q^{-1}-q^{-3}$	$-1+q^{-2}+r^{-1}q^{-2}$	$-2+3q^{-2}$	$-2q^{-1}r^{-1}+q-q^{-1}-2+2q^{-2}$
(1)	0	r^{-1}	$-q+q^{-1}-q^{-3}+q^3$	q^2-2+q^{-2}	$q^2+q^{-2}+r^{-2}-2q^{-1}r^{-1}+2q-2q^{-1}+2qr^{-1}$

$BW_5(r, q)$ **continued.**

$\lambda \setminus \mu$	$(2,1^3)$	(1^5)	(3)	$(2,1)$	(1^3)	(1)
(5)	q	1	0	0	0	0
$(4,1)$	$3q - q^{-1}$	4	0	0	0	0
$(3,2)$	$3q - 2q^{-1}$	5	0	0	0	0
$(3,1^2)$	$3q - 3q^{-1}$	6	0	0	0	0
$(2^2,1)$	$2q - 3q^{-1}$	5	0	0	0	0
$(2,1^3)$	$q - 3q^{-1}$	4	0	0	0	0
(1^5)	$-q^{-1}$	1	0	0	0	0
(3)	$r^{-1} + 6q - 3q^{-1}$	10	xq^2	xq	x	0
$(2,1)$	$2r^{-1} + 9q - 9q^{-1}$	20	$-x$	$x(q - q^{-1})$	$2x$	0
(1^3)	$r^{-1} + 3q - 6q^{-1}$	10	$-xq^{-2}$	$-xq^{-1}$	x	0
(1)	$3r^{-1} + 6q - 6q^{-1}$	15	0	$x(q - q^{-1} + r^{-1})$	$3x$	x^2

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