ADVANCES IN MATHEMATICS **125**, 1–94 (1997)
ARTICLE NO. AI971602

A Ribbon Hopf Algebra Approach to the Irreducible Representations of Centralizer Algebras: The Brauer, Birman–Wenzl, and Type A Iwahori–Hecke Algebras

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Received June 24, 1994; accepted September 12, 1994

We show how the ribbon Hopf algebra structure on the Drinfel'd–Jimbo quantum groups of Types A, B, C, and D can be used to derive formulas giving explicit realizations of the irreducible representations of the Iwahori–Hecke algebras of type A and the Birman–Wenzl algebras. We use this derivation to give explicit realizations of the irreducible representations of the Brauer algebras as well. The derivation is accomplished by way of a combination of techniques from operator algebras, quantum groups, and the theory of 3-manifold invariants. Although our applications are in the cases of the quantum groups of Types A, B, C, and D, most of the aspects of our approach apply in the general setting of ribbon Hopf algebras.

0. INTRODUCTION

The Iwahori–Hecke algebras of Type A and the Birman–Wenzl–Murakami algebras arise naturally in the following setting: Let $\mathfrak{U}$ be a quantum group corresponding to a finite dimensional complex simple Lie algebra of Type A, B, C, or D, and let V be the irreducible representation of $\mathfrak{U}$ corresponding to the fundamental weight ω_1. Then the centralizer

* Supported in part by National Science Foundation Grant DMS-9300523 to the University of Wisconsin.

† Supported in part by National Science Foundation Postdoctoral Fellowship DMS-9107863.

0001-8708/97 $25.00

algebra $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ is isomorphic to a quotient of either the Iwahori–Hecke algebra or the Birman–Wenzl algebra.

The purpose of this paper is to give a unified approach for determining explicit realizations of the irreducible representations of the Iwahori–Hecke algebras of Type A and the Birman–Wenzl–Murakami algebras. Indeed, the formulas for the irreducible representations which we find are equivalent to those in [H] and [W2] in the case of the Iwahori–Hecke algebras of type A and to those in [M2] for the case of the Birman–Wenzl–Murakami algebras. However, we have found that in all three of these previous works the appropriate formulas are stated without derivation and then proved to be correct. In this paper we show that there is indeed a consistent method by which one may actually derive the appropriate formulas.

Our method is motivated strongly by the machinery which has developed in the context of operator algebras, quantum groups, and link invariants, in particular the work of Reshetikhin [Re], Drinfel'd [D], Wenzl [W3], and Turaev [7]. See also the papers [RT, RT2, TW, W4, and BW]. Although we have applied our methods in the particular case of the quantum groups corresponding to finite dimensional simple Lie algebras of types A, B, C, it is clear that main aspects of our approach hold in the setting of quasi-triangular Hopf algebras and ribbon Hopf algebras. The following list describes the central features in our approach.

(1) From operator algebras: We have used the path model approach for towers of algebras in [GHJ] in order to work with infinite families of centralizer algebras all at once. In some sense the path algebra mechanism reduces all of the "difficult" parts of the derivation to simple computations with matrix units in direct sums of ordinary $n \times n$ matrix algebras.

(2) From quantum groups: The Drinfel'd–Jimbo quantum groups carry the structure of quasitriangular Hopf algebras and ribbon Hopf algebras [D]. We have been able to use this structure to get very specific information about certain elements in the centralizer algebra. The quasitriangular structure guarantees that the product $\mathscr{R}_{21}\mathscr{R}_{12}$, where $\mathscr{R}$ is the $\mathscr{R}$-matrix, is always an element of the centralizer algebra and the ribbon structure allows us to determine the eigenvalues of this element. These eigenvalues turn out to be determined by the Casimir element from the corresponding Lie algebra. This idea is the central idea in [Re].

(3) Combining tools from 3-manifold invariants and operator algebras: We show that the Markov traces used to derive link invariants and 3-manifold invariants are equivalent to certain traces on towers of algebras that arise from Wenzl's approach to the Jones basic construction. This was observed in [W3] for the case of quantum groups of type B using the explicit form of the $\check{R}$ matrix. In our approach we have obtained this

result for any ribbon Hopf algebra. This idea allows one to give an easy derivation of the framing anomalies for the Reshetikhin–Turaev 3-manifold invariants.

In the first three sections of this paper we develop these tools in the context of centralizer algebras. Although the main objects have all appeared in previous work ([Re, T, D, W3]), we have felt it necessary to give a consistent presentation in the context of centralizer algebras since it is not necessarily clear from the previous work how these techniques apply to our situation.

Our paper is organized as follows:

In Section 1 we review the path algebra setup. In the second half of Section 1 we show that if $\mathfrak{U}$ is a Hopf algebra such that all finite dimensional representations of $\mathfrak{U}$ are completely reducible and if V is a $\mathfrak{U}$-module then the centralizer algebras $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ can be identified with path algebras in a natural way.

In Section 2 we begin by reviewing the definitions of quasitriangular Hopf algebras, ribbon Hopf algebras, and the Drinfel'd–Jimbo quantum groups. Then, letting $\mathfrak{U}$ be a quasitriangular Hopf algebra and letting V be a $\mathfrak{U}$-module, we show how to determine explicitly the image of the element $\mathscr{R}_{21}\mathscr{R}_{12}$ both as an element of the centralizer algebra $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ and as an element of the corresponding path algebra.

In Section 3 we let $\mathfrak{U}$ be a ribbon Hopf algebra and let V be a $\mathfrak{U}$-module. Then there is a natural projection $\check{e} \in \mathrm{End}_{\mathfrak{U}}(V \otimes V^*)$ onto the invariants in the $\mathfrak{U}$-module $V \otimes V^*$. This projection gives rise to a natural trace on the centralizer algebras $\mathscr{Z}_m$, and it turns out that this trace is always a Markov trace with respect to the corresponding $\mathscr{R}$-matrix. We are able to determine explicit formulas for the image of the element $\check{e}$ in the path algebras corresponding to the centralizer algebras $\mathscr{Z}_m$.

In Section 4 we apply the results of the first three sections to compute the irreducible representations, in terms of path algebras, of the centralizer algebras corresponding to the quantum groups $\mathfrak{U}_h(\mathfrak{sl}(r+1))$ and the fundamental representation.

In Section 5 we apply the results of the first two sections to compute the irreducible representations, in terms of path algebras, of the centralizer algebras corresponding to the quantum groups corresponding to complex simple Lie algebras of Types B, C, D, and the fundamental representation. This derivation is only slightly more complex than that for the Type A case given in Section 4.

We finish in Section 6 by deriving, explicitly, irreducible representations of the Iwahori–Hecke algebras, the Birman–Wenzl–Murakami algebras, and the Brauer algebras.

Some further remarks on the results in this paper:

(1) All of the representations obtained in this paper are, in some sense, analogues of Young's orthogonal representations for the symmetric group. This is due to the way that we inductively identify the centralizer algebras $\mathscr{Z}_m$ with path algebras.

(2) Hoefsmit determined explicit irreducible representations of the Iwahori–Hecke algebras of Type A in [H]. One of the consequences of our approach is that the mysterious axial distances which have appeared in the work of Hoefsmit are completely explained in terms of the values of the Casimir element of the complex simple Lie algebras of type A acting on irreducible representations. Similarly, some of the constants appearing in the formulas for the irreducible representations of the Birman–Wenzl–Murakami algebras are obtained from the values of the Casimir elements of the complex simple Lie algebras of type B or C acting on irreducible representations. In fact, the only other values that are needed in order to give closed form formulas for the irreducible representations are the "quantum dimensions" of the irreducible representations of the corresponding Lie algebra. These are determined by the Weyl character formula.

(3) Although our formulas for the irreducible representations of the Birman–Wenzl–Murakami algebra are equivalent to those in [M2] we have found ours to be more tractable, in particular, it is a trivial matter to specialize appropriately to give formulas, to our knowledge new ones, for the irreducible representations of the Brauer algebras [Br, W1].

(4) We have found that it is quite easy to derive the formulas for the basic construction element (which was obtained by various authors [RW, Theorem 1.4; GHJ, (2.6.5.4); Su]) by simple path algebra (matrix algebra) computations and thus we give an alternate and elementary proof of some of the results in [W1, Section 1]. This result appears in our Theorem (3.12).

(5) In Sections 4 and 5 we give formulas for matrix units in the centralizer algebras corresponding to quantum groups of types A, B, C, and D. Similar formulas have been given in [RW]. The formulas we give here, in the cases of types B, C, and D, are new formulas for the same matrix units that were given in [RW].

1. PATH ALGEBRAS AND TENSOR POWER CENTRALIZER ALGEBRAS

Bratteli Diagrams

A *Bratteli diagram A* is a graph with vertices from a collection of sets $\hat{A}_m$, $m \geqslant 0$, and edges that connect vertices in $\hat{A}_m$ to vertices in $\hat{A}_{m+1}$. We assume that the set $\hat{A}_0$ contains a unique vertex denoted $\varnothing$. It is possible that there are multiple edges connecting any two vertices. We shall call the vertices *shapes*. The set $\hat{A}_m$ is the set of shapes on *level m*. If $\lambda \in \hat{A}_m$ is connected by an edge to a shape $\mu \in \hat{A}_{m+1}$ we write $\lambda \leqslant \mu$.

A *multiplicity free Bratteli diagram* is a Bratteli diagram such that there is at most one edge connecting any two vertices. Alternatively we could define a multiplicity free Bratteli diagram to be a ranked poset A which is ranked by the nonnegative integers and such that there is a unique vertex on level 0 called $\varnothing$. Identifying the poset A with its Hasse diagram we see that these two definitions are the same since the poset condition implies that the resulting Bratteli diagram is multiplicity free. In order to make sure that we do not make careless statements in this paper.

> *Assume throughout this paper that all Bratteli diagrams are multiplicity free.*

We make this assumption to simplify our proofs and our notation. See [GHJ] for the more general setting.

The Bratteli diagrams which we will be most interested in, see Figures 1 and 2, are multiplicity free and arise naturally in the representation theory of centralizer algebras. Other examples of Bratteli diagrams arise from differential posets [St] and towers of C^* algebras [GHJ]. The Bratteli diagrams in Figures 1 and 2 are described further in Sections 4 and 5 respectively.

Paths and Tableaux

Let A be a multiplicity free Bratteli diagram and let $\lambda \in \hat{A}_m$ and $\mu \in \hat{A}_n$ where $m < n$. A *path* from λ to μ is a sequence of shapes $\lambda^{(i)}$, $m \leqslant i \leqslant n$,

$$P = (\lambda^{(m)}, \lambda^{(m+1)}, ..., \lambda^{(n)})$$

such that $\lambda = \lambda^{(m)} \leqslant \lambda^{(m+1)} \leqslant \cdots \leqslant \lambda^{(n)} = \mu$ and $\lambda^{(i)} \in \hat{A}_i$. In the poset sense the path P is a saturated chain from λ to μ. (If we are working in the non-multiplicity free setting we must distinguish paths which "travel" from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ along different edges.) A *tableau* T of shape λ is a path from $\varnothing$ to λ

$$T = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(m)})$$

such that $\varnothing = \lambda^{(0)} \leqslant \lambda^{(1)} \leqslant \cdots \leqslant \lambda^{(m)} = \lambda$ and $\lambda^{(i)} \in \hat{A}_i$ for each $1 \leqslant i \leqslant m$. We write $\operatorname{shp}(T) = \lambda$ if T is a tableau of shape λ. We say that the length of T is m if $\operatorname{shp}(T) \in \hat{A}_m$.

Let us make the following (hopefully suggestive) notations.

$\mathscr{T}^{\lambda}$ is the set of tableaux of shape λ,
$\mathscr{T}^{m}$ is the set of tableaux of length m,
$\mathscr{T}^{\mu}_{\lambda}$ is the set of paths from λ to μ,
$\mathscr{T}^{m}_{\lambda}$ is the set of paths from λ to any shape on level m,
$\mathscr{T}^{m}_{T}$ is the set of paths from $\mathrm{shp}(T)$ to any shape on level m.

Similarly, we define

Ω^{λ} is the set of pairs (S, T) of paths $S, T \in \mathscr{T}^{\lambda}$,
Ω^{m} is the set of pairs (S, T) of paths $S, T \in \mathscr{T}^{m}$ such that $\mathrm{shp}(S) = \mathrm{shp}(T)$,
Ω^{μ}_{λ} is the set of pairs (S, T) of paths $S, T \in \mathscr{T}^{\mu}_{\lambda}$,
Ω^{m}_{λ} is the set of pairs (S, T) of paths $S, T \in \mathscr{T}^{m}_{\lambda}$ such that $\mathrm{shp}(S) = \mathrm{shp}(T)$.

Path Algebras

For each m define an algebra A_m over a field k with basis E_{ST}, $(S, T) \in \Omega^m$ and multiplication given by

$$E_{ST} E_{PQ} = \delta_{TP} E_{SQ}. \tag{1.1}$$

Note that $A_0 \simeq k$. Every element $a \in A_m$ can be written in the form

$$a = \sum_{(S,\, T) \in \Omega^m} a_{ST} E_{ST},$$

for some constants $a_{ST} \in k$. In this way we can view each element $a \in A_m$ as weighted sum of pairs of paths, where the weight of a pair of paths $(S, T) \in \Omega^m$ is the constant a_{ST}. We shall refer to the collection of algebras A_m as the *tower of path algebras corresponding to the Bratteli diagram A.*

Each of the algebras A_m is isomorphic to a direct sum of matrix algebras

$$A_m \simeq \bigoplus_{\lambda \in \hat{A}_m} M_{d_\lambda}(k),$$

where $M_d(k)$ denotes the algebra of $d \times d$ matrices with entries from k and $d_\lambda = \mathrm{Card}(\mathscr{T}^{\lambda})$. Thus, the irreducible representations of A_m are indexed by the elements of $\hat{A}_m$. Furthermore, the dimensions of these irreducible representations are equal to $\mathrm{Card}(\mathscr{T}^{\lambda})$, and thus, the set of tableaux $\mathscr{T}^{\lambda}$ is a natural index set for a basis of the irreducible A_m-module indexed by $\lambda \in \hat{A}_m$.

The Inclusions $A_m \subseteq A_n$, $m \leqslant n$

Given a path $T=(\lambda, ..., \mu)$ from λ to μ and a path $S=(\mu, ..., \nu)$ from μ to ν define

$$T * S=(\lambda, ..., \mu, ..., \nu) \tag{1.2}$$

to be the concatenation of the two paths (the shape μ is not repeated since that would not produce a path).

Let $0 \leqslant m < n$. Define an inclusion of $A_m \subseteq A_n$ as follows: For each $(P, Q) \in \Omega^m$ view E_{PQ} as an element of A_n by the formula

$$E_{PQ}=\sum_{T \in \mathscr{T}^n_\lambda} E_{P*T,\, Q*T}, \qquad \text{where} \quad \lambda=\operatorname{shp}(P)=\operatorname{shp}(Q). \tag{1.3}$$

In particular we have an inclusion of A_{m-1} into A_m for every $m>0$. Let $\lambda \in \hat{A}_m$ and let V^λ be the irreducible representation of A_m corresponding to λ. Then the restriction of V^λ to A_{m-1} decomposes as

$$V^\lambda \downarrow^{A_m}_{A_{m-1}} \simeq \bigoplus_{\mu \in \lambda^-} V^\mu,$$

where $\lambda^- = \{\mu \in \hat{A}_{m-1} \mid \mu \leqslant \lambda\}$. The multiplicity free condition on the Bratteli diagram guarantees that this decomposition is *multiplicity free.*

The Centralizer of A_m Contained in A_n, $0 \leqslant m < n$

Define

$$\mathscr{Z}(A_m \subseteq A_n)=\{a \in A_n \mid ab=ba \text{ for all } b \in A_m\}.$$

Let us extend the notation in (1.3) and define

$$E_{ST}=\sum_{P \in \mathscr{T}^\lambda} E_{P*S,\, P*T},$$

for each pair $(S, T) \in \Omega^\mu_\lambda$, $\lambda \in \hat{A}_m$, $\mu \in \hat{A}_n$, the following result appears in [GHJ, Proposition 2.3.12].

(1.4) PROPOSITION. *The elements E_{ST}, $(S, T) \in \Omega^\mu_\lambda$, $\lambda \in \hat{A}_m$, $\mu \in \hat{A}_n$, are a basis of $\mathscr{Z}(A_m \subseteq A_n)$.*

Proof. First let us show that the elements $E_{ST} \in \mathscr{Z}(A_m \subseteq A_n)$. Let $\gamma \in \hat{A}_m$ and let $Q, R \in \mathscr{T}^\gamma$. Then

$$
\begin{aligned}
E_{ST}E_{QR} &= \Bigg(\sum_{P\in\mathscr{T}^{\lambda}} E_{P*S,\,P*T}\Bigg)\Bigg(\sum_{U\in\mathscr{T}^{n}_{\gamma}} E_{Q*U,\,R*U}\Bigg)\\
&= E_{Q*S,\,Q*T}\Bigg(\sum_{U\in\mathscr{T}^{n}_{\gamma}} E_{Q*U,\,R*U}\Bigg)\\
&= E_{Q*S,\,Q*T}E_{Q*T,\,R*T} = E_{Q*S,\,R*T}.
\end{aligned}
$$

Similarly one shows that $E_{QR}E_{ST} = E_{Q*S,\,R*T}$, giving that $E_{ST}\in\mathscr{Z}(A_m\subseteq A_n)$.

Now we show that if $a\in\mathscr{Z}(A_m\subseteq A_n)$ then a is a linear combination of E_{ST}. Suppose

$$
a = \sum_{(M,N)\in\Omega^n} a_{MN}E_{MN}\in\mathscr{Z}(A_m\subseteq A_n).
$$

Let $\lambda\in\hat{A}_m$ and let $P\in\mathscr{T}^{\lambda}$. Then

$$
\begin{aligned}
aE_{PP} &= \Bigg(\sum_{(M,N)\in\Omega^n} a_{MN}E_{MN}\Bigg)\Bigg(\sum_{T\in\mathscr{T}^{n}_{\lambda}} E_{P*T,\,P*T}\Bigg)\\
&= \sum_{(M,\,P*T)\in\Omega^n} a_{M,\,P*T}E_{M,\,P*T}\\
E_{PP}a &= \Bigg(\sum_{S\in\mathscr{T}^{n}_{\lambda}} E_{P*S,\,P*S}\Bigg)\Bigg(\sum_{(M,N)\in\Omega^n} a_{MN}E_{MN}\Bigg)\\
&= \sum_{(P*S,\,N)\in\Omega^n} a_{P*S,\,N}E_{P*S,\,N}
\end{aligned}
$$

This implies that $a_{M,\,P*T}=0$ unless $M=P*S$ for some $S\in\mathscr{T}^{n}_{\lambda}$ and $a_{P*S,\,N}=0$ unless $N=P*T$, for some $T\in\mathscr{T}^{n}_{\lambda}$. Thus, a must be of the form

$$
a = \sum_{\substack{P\in\mathscr{T}^{m}\\ (S,T)\in\Omega^{n}_{\lambda}}} a_{P*S,\,P*T}E_{P*S,\,P*T}.
$$

If $\lambda\in\hat{A}_m$ and $(P,Q)\in\Omega^{\lambda}$ then

$$
\begin{aligned}
E_{PQ}a &= \Bigg(\sum_{S\in\mathscr{T}^{n}_{\lambda}} E_{P*S,\,Q*S}\Bigg)\Bigg(\sum_{\substack{R\in\mathscr{T}^{m}\\ (S,T)\in\Omega^{n}_{\lambda}}} a_{R*S,\,R*T}E_{R*S,\,R*T}E_{R*S,\,R*T}\Bigg)\\
&= \sum_{(S,T)\in\Omega^{n}_{\lambda}} a_{Q*S,\,Q*T}E_{P*S,\,Q*T},\\
aE_{PQ} &= \Bigg(\sum_{\substack{R\in\mathscr{T}^{m}\\ (S,T)\in\Omega^{n}_{\lambda}}} a_{R*S,\,R*T}E_{R*S,\,R*T}\Bigg)\Bigg(\sum_{T\in\mathscr{T}^{n}_{\lambda}} E_{P*T,\,Q*T}\Bigg)\\
&= \sum_{(S,T)\in\Omega^{n}_{\lambda}} a_{P*S,\,P*T}E_{P*S,\,Q*T}.
\end{aligned}
$$

This implies that $a_{Q*S,\,Q*T}=a_{P*S,\,P*T}$ for all $(P,Q)\in\Omega^{\lambda}$. Let us denote this coefficient by a_{ST}. Then

$$a=\sum_{\substack{\lambda\in\hat{A}_m\\(S,T)\in\Omega^m_\lambda}} a_{ST}\sum_{P\in\mathcal{T}^\lambda} E_{P*S,\,P*T}=\sum_{\substack{\lambda\in\hat{A}_m\\(S,T)\in\Omega^m_\lambda}} a_{ST}E_{ST}.$$

Thus, if $a\in\mathscr{Z}(A_m\subseteq A_n)$ then a is a linear combination of E_{ST}. The elements E_{ST}, $(S,T)\in\Omega^n_m$ are independent since the elements E_{MN}, $(M,N)\in\Omega^n$ are. ∎

(1.5) Corollary. *Let A_m, $m\geqslant 0$, be the tower of path algebras corresponding to a multiplicity free Bratteli diagram A and suppose that $g_i\in A_{i+1}$, $i\geqslant 1$, are elements such that*

(1) *For each m, the elements $g_1, g_2, ..., g_{m-1}$ generate A_m,*

(2) *$g_ig_j=g_jg_i$ for all i,j such that $|i-j|>1$.*

Then

$$g_{m-1}=\sum_{(P,Q)\in\Omega^m_{m-2}}(g_{m-1})_{PQ}\,E_{PQ}$$

for some constants $(g_{m-1})_{PQ}\in k$.

Proof. It follows from the relations on the g_i that g_{m-1} commutes with A_{m-2}. The result then follows from Proposition (1.4). ∎

(1.6) Corollary. *Let A_m, $m\geqslant 0$, be the tower of path algebras corresponding to a multiplicity free Bratteli diagram A and suppose that $g_i\in A_{i+1}$, $i\geqslant 1$, are elements such that*

(1) *For each m, the elements $g_1, g_2, ..., g_{m-1}$ generate A_m,*

(2) *$g_ig_j=g_jg_i$ for all i,j such that $|i-j|>1$.*

(3) *$g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1}$ for all $i\geqslant 1$.*

Define $D_m=g_{m-1}g_{m-2}\cdots g_2g_1g_1g_2\cdots g_{m-3}g_{m-2}g_{m-1}\in A_m$. Then

$$D_m=\sum_{S\in\mathcal{T}^m_{m-1}} D_{SS}E_{SS},$$

for some constants $D_{SS}\in k$.

Proof. Using the braid relation (3) for the elements g_i we have

$$\begin{aligned}g_{m-2}D_m &= g_{m-1}g_{m-2}g_{m-1}g_{m-3}\cdots g_1g_1\cdots g_{m-1}\\ &= g_{m-1}g_{m-2}g_{m-3}\cdots g_1g_1\cdots g_{m-1}g_{m-2}g_{m-1}\\ &= D_mg_{m-2}.\end{aligned}$$

It follows that D_m commutes with g_{m-2}. By induction we have that

$$\begin{aligned}g_jD_m &= g_{m-1}\cdots g_{j+2}g_jD_{j+2}g_{j+2}g_{j+3}\cdots g_{m-1}\\ &= g_{m-1}\cdots g_{j+2}D_{j+2}g_jg_{j+2}g_{j+3}\cdots g_{m-1}\\ &= D_mg_j,\end{aligned}$$

for all $1\leqslant j<m-2$. Thus, D_m commutes with A_{m-1}. The result now follows from Proposition (1.4). ∎

Remark. All of the above results hold even if the Bratteli diagram is not multiplicity free since the main result Proposition (1.4) holds in that case. We have stated these results only for the multiplicity free case in order to simplify our notation. See [GHJ] for the more general setting.

Centralizers of Tensor Power Representations

Let k be a field. We shall assume that k is characteristic zero and algebraically closed. Let $\mathfrak{U}$ be a Hopf algebra over k such that all finite dimensional representations of $\mathfrak{U}$ are completely reducible. Let V be a finite dimensional representation of $\mathfrak{U}$ and define

$$\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m}). \tag{1.7}$$

Let $\hat{\mathfrak{U}}$ be an index set for the finite dimensional irreducible representations of $\mathfrak{U}$. Let $\hat{\mathscr{Z}}_m$ be an index set for the finite dimensional representations of $\mathscr{Z}_m$. It is natural to view $\hat{\mathscr{Z}}_m$ as a subset of $\hat{\mathfrak{U}}$ since, by Schur–Weyl duality, the $(\mathscr{Z}_m\otimes\mathfrak{U})$-module $V^{\otimes m}$ has a decomposition

$$V^{\otimes m}\cong\bigoplus_{\lambda\in\hat{\mathscr{Z}}_m}\mathscr{Z}^\lambda\otimes A_\lambda,$$

where $\mathscr{Z}^\lambda$ is the irreducible $\mathscr{Z}_m$-module indexed by λ and A_λ is the irreducible $\mathfrak{U}$-module indexed by λ.

For $0<m<n$ there is a natural inclusion $\mathscr{Z}_m\subseteq\mathscr{Z}_n$ given by

$$\begin{aligned}\mathscr{Z}_m &\hookrightarrow \mathscr{Z}_n\\ a &\mapsto a\otimes\mathrm{id}^{\otimes(n-m)}\end{aligned}$$

where $a \otimes \mathrm{id}^{\otimes(n-m)}$ acts as a on the first m factors of $V^{\otimes n}$ and as the identity on the last $m-n$ tensor factors. By convention we shall set $\mathscr{Z}_0 = k$. If V is an irreducible $\mathfrak{U}$-module then, by Schur's lemma, $\mathscr{Z}_1 \cong k$.

The Bratteli Diagram for Tensor Powers of V

Assume that V is an irreducible $\mathfrak{U}$-module. Let $\lambda \in \hat{\mathscr{Z}}_m$ for some m. Then there is a *branching rule for tensoring by V* which describes the decomposition

$$\Lambda_\lambda \otimes V = \bigoplus_{\mu \in \hat{\mathscr{Z}}_{m+1}} c^{\mu}_{\lambda V} \Lambda_\mu, \tag{1.8}$$

as $\mathfrak{U}$-modules. The multiplicities $c^{\nu}_{\lambda V}$ are nonnegative integers. This decomposition is *multiplicity free* if all the multiplicities $c^{\nu}_{\lambda V} \leqslant 1$. Let $\nu \in \hat{\mathscr{Z}}_{m+1}$. Then the *branching rule for inclusion* $\mathscr{Z}_m \subseteq \mathscr{Z}_{m+1}$ describes the decomposition

$$\mathscr{Z}^{\nu} = \bigoplus_{\lambda \in \hat{\mathscr{Z}}_m} c^{\nu}_{\lambda V} \mathscr{Z}^{\lambda}, \tag{1.9}$$

as $\mathscr{Z}_m$-modules. There is a standard reciprocity result for branching rules ([Bou] Chpt. VIII §5 Ex. 17, see also [R] Theorem 5.9 for a simple proof), that states that the constants $c^{\nu}_{\lambda V}$ appearing in (1.8) and (1.9) are the same.

We define a *Bratteli diagram for tensor powers of V*, or equivalently, a *Bratteli diagram for the tower of algebras $\mathscr{Z}_m$*, as follows. Let the elements of the set $\hat{\mathscr{Z}}_m$ be the vertices on level m. A vertex $\lambda \in \hat{\mathscr{Z}}_m$ is connected to a vertex $\mu \in \hat{\mathscr{Z}}_{m+1}$ by $c^{\mu}_{\lambda V}$ edges. This Bratteli diagram is multiplicity free if the corresponding branching rule for tensoring by V is multiplicity free.

Identification of the Centralizer Algebras $\mathscr{Z}_m$ with Path Algebras

By working inductively, we can view the algebras $\mathscr{Z}_m$ as path algebras for the Bratteli diagram for tensor powers of V. Let us denote this Bratteli diagram by A and denote the corresponding path algebras by A_m. Clearly $\mathscr{Z}_0 \cong k$ can be identified with the corresponding path algebra A_0. For each $\lambda \in \hat{\mathfrak{U}}$ let Λ_λ denote the irreducible $\mathfrak{U}$ module corresponding to λ. Suppose that there is an identification of $\mathscr{Z}_m$ with the path algebra A_m so that

$$V^{\otimes m} = \bigoplus_{\lambda \in \hat{\mathscr{Z}}_m} \left(\bigoplus_{T \in \mathscr{T}^{\lambda}} E_{TT} V^{\otimes m} \right),$$

is a decomposition of $V^{\otimes m}$ so that the $\mathfrak{U}$-submodule $E_{TT} V^{\otimes m} \cong \Lambda_\lambda$. The element E_{TT} is a $\mathfrak{U}$-invariant projection onto the irreducible $\mathfrak{U}$-module $E_{TT} V^{\otimes m}$.

Given a tableau $T = (\tau^{(0)}, ..., \tau^{(m-1)}, \lambda) \in \mathscr{T}^{\lambda}$ and a shape $\nu \in \mathscr{Z}_{m+1}$ such that $\nu \geqslant \lambda$ let $T * \nu$ be the path given by $T * \nu = (\tau^{(0)}, ..., \tau^{(m-1)}, \lambda, \nu)$. Since the branching rule for tensoring by V is multiplicity free, there is a unique decomposition

$$(E_{TT}V^{\otimes m})\otimes V=\bigoplus_{\substack{\nu\in\hat{\mathscr{Z}}_{m+1}\\ \nu\geqslant\lambda}} V_{T*\nu}, \tag{1.10}$$

into nonisomorphic irreducible $\mathfrak{U}$-modules $V_{T*\nu}\cong A_\nu$. Define $E_{T*\nu,T*\nu}\in\mathscr{Z}_{m+1}$ to be the unique $\mathfrak{U}$-invariant projection onto the irreducible $V_{T*\nu}$ in the decomposition (1.10). In this way we can define elements E_{SS} for every $S\in\mathscr{T}^{m+1}$ and we have that

$$V^{\otimes(m+1)}=\bigoplus_{\nu\in\hat{\mathscr{Z}}_{m+1}}\left(\bigoplus_{S\in\mathscr{T}^\nu} E_{SS}V^{\otimes m}\right),$$

is a decomposition of $V^{\otimes(m+1)}$ into irreducible $\mathfrak{U}$-modules $E_{SS}V^{\otimes(m+1)}\cong A_\nu$, $S\in\mathscr{T}^\nu$. This makes an identification of each basis element E_{SS}, $S\in\mathscr{T}^{m+1}$, of the path algebra A_{m+1} with a transformation in $\mathscr{Z}_{m+1}$. Now, for each pair of paths $(P,Q)\in\Omega^{m+1}$ choose nonzero transformations

$$E_{PQ}\in E_{PP}\mathscr{Z}_{m+1}E_{QQ} \qquad\text{and}\qquad E_{QP}\in E_{QQ}\mathscr{Z}_{m+1}E_{PP}$$

and normalize them so that

$$E_{PQ}E_{QP}=E_{PP}, \tag{1.11}$$

as transformations in $\mathscr{Z}_{m+1}$. In this way, one can identify the path algebra A_{m+1} with the algebra $\mathscr{Z}_{m+1}$. This identification is not canonical, there is the following freedom in the choice of the normalization of the transformations E_{PQ} and E_{QP}: For any nonzero constant $\alpha\in k$, one may

$$\text{replace } E_{PQ} \text{ and } E_{QP} \text{ by } \alpha E_{PQ} \text{ and } (1/\alpha)\,E_{PQ} \text{ respectively,} \tag{1.12}$$

to get another solution.

Suppose that an identification of the centralizer algebras $\mathscr{Z}_m$ with the path algebras is given. This identification determines a choice of the irreducible representations of $\mathscr{Z}_m$ in the following way. If $a\in\mathscr{Z}_m$, and

$$a=\sum_{\lambda\in\hat{\mathscr{Z}}_m}\sum_{(S,T)\in\Omega^\lambda}(a)_{ST}E_{ST},$$

then the maps

$$\begin{aligned}\pi^\lambda\colon \mathscr{Z}_m &\to M_{d_\lambda}(k)\\ a &\mapsto ((a)_{ST})_{(S,T)\in\Omega^\lambda}\end{aligned}$$

for $\lambda \in \hat{\mathscr{Z}}_m$, determine a complete set of nonisomorphic irreducible representations of $\mathscr{Z}_m$. In this paper we shall find path algebra formulas for the generators of tensor power centralizer algebras, $\mathscr{Z}_m$, and thus, in essence, we are finding the irreducible representations.

2. QUASITRIANGULAR HOPF ALGEBRAS, RIBBON HOPF ALGEBRAS AND QUANTUM GROUPS

If $\mathfrak{U}$ is a Hopf algebra, we shall denote the coproduct by Δ, the counit by ε and the antipode by S. We shall always assume that both the antipode S and the skew antipode S^{-1} exist. If $a \in \mathfrak{U}$ and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$, then the opposite coproduct is defined by $\Delta^{op}(a) = \sum_a a_{(2)} \otimes a_{(1)}$. Recall that if V and W are $\mathfrak{U}$ modules, then $\mathfrak{U}$ acts on the tensor product $V \otimes W$ by

$$a(v \otimes w) = \Delta(a)(v \otimes w) = \sum_a a_{(1)} v \otimes a_{(2)} w,$$

for all $a \in \mathfrak{U}$, $v \in V$, and $w \in W$.

A *quasitriangular Hopf algebra* is a pair $(\mathfrak{U}, \mathscr{R})$ consisting of a Hopf algebra $\mathfrak{U}$, and an invertible element $\mathscr{R} \in \mathfrak{U} \otimes \mathfrak{U}$ such that

$$\mathscr{R}\Delta(a)\,\mathscr{R}^{-1} = \Delta^{op}(a), \qquad \text{for all} \quad a \in U, \tag{2.1}$$

$$(\Delta \otimes id)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}, \tag{2.2}$$

$$(id \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}, \tag{2.3}$$

where, if $\mathscr{R} = \sum a_i \otimes b_i$, then

$$\mathscr{R}_{12} = \sum a_i \otimes b_i \otimes 1, \qquad \mathscr{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \qquad \mathscr{R}_{23} = \sum 1 \otimes a_i \otimes b_i.$$

Let $(\mathfrak{U}, \mathscr{R})$ be a quasitriangular Hopf algebra, let $\mathscr{R} = \sum a_i \otimes b_i \in \mathfrak{U} \otimes \mathfrak{U}$, $\mathscr{R}_{21} = \sum b_i \otimes a_i$, and define

$$u = \sum S(b_i)\, a_i \in \mathfrak{U} \qquad \text{and} \qquad z = uS(u). \tag{2.4}$$

Then, we have the following facts:

$$(S \otimes id)(\mathscr{R}) = \mathscr{R}^{-1}, \tag{2.5}$$

$$(S \otimes S)(\mathscr{R}) = \mathscr{R}, \tag{2.6}$$

$$u^{-1} = \sum_j S^{-1}(d_j)\, c_j, \text{ where } \mathscr{R}^{-1} = \sum_j c_j \otimes d_j, \tag{2.7}$$

$$uau^{-1} = S^2(a), \qquad \text{for all} \quad a \in \mathfrak{U}, \tag{2.8}$$

$$\Delta(u) = (\mathcal{R}_{21}\mathcal{R})^{-1}(u \otimes u), \tag{2.9}$$

$$z \text{ is an invertible central element of } \mathfrak{U}, \tag{2.10}$$

$$\Delta(z) = (\mathcal{R}_{21}\mathcal{R})^{-2}(z \otimes z), \tag{2.11}$$

These facts are proved in [D, Propositions 2.1, 3.1, 3.2, and the remarks immediately preceding Proposition 3.2]. The proofs are calculations involving only the definition of a quasitriangular Hopf algebra.

A *ribbon Hopf algebra* is a triple $(\mathfrak{U}, \mathcal{R}, v)$ consisting of a quasitriangular Hopf algebra $(\mathfrak{U}, \mathcal{R})$, and an invertible element v in the center of $\mathfrak{U}$, such that

$$\begin{aligned} &v^2 = uS(u), \qquad S(v) = v, \qquad \varepsilon(v) = 1, \\ &\Delta(v) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(v \otimes v). \end{aligned} \tag{2.12}$$

It is important to note that the element $v^{-1}u \in \mathfrak{U}$ is grouplike, i.e., $\Delta(v^{-1}u) = v^{-1}u \otimes v^{-1}u$.

Quantum Groups

Let $\mathbb{C}[[h]]$ be the ring of formal power series in an indeterminate h. The notation e^x shall always denote the formal exponential

$$e^x = \sum_{k \geqslant 0} \frac{x^k}{k!},$$

and define $q = e^{h/2}$. For each positive integer n define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1]\cdots[2][1], \quad [0]! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}, \qquad \text{for} \quad 0 \leqslant k \leqslant n.$$

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Let $\alpha_i \in \mathfrak{h}^*$ be the simple roots and let $H_i = \alpha_i^\vee \in \mathfrak{h}$ be the simple coroots so that the Cartan matrix is given by

$$(\langle \alpha_i, \alpha_j^\vee \rangle) = (a_{ij}) = A.$$

Let $\mathfrak{U}_h(\mathfrak{g})$ be the associative algebra with 1 over $\mathbb{C}[[h]]$ generated (as an algebra complete in the h-adic topology) by the space $\mathfrak{h}$ and the elements $X_1, ..., X_r, Y_1, ..., Y_r$ with relations

$$[a_1, a_2] = 0, \qquad \text{for all} \quad a_1, a_2 \in \mathfrak{h},$$

$$[a, X_j] = \langle \alpha_j, a \rangle X_j, \quad [a, Y_j] = \langle -\alpha_j, a \rangle Y_j, \qquad \text{for all} \quad a \in \mathfrak{h},$$

$$X_i Y_j - Y_j X_i = \delta_{ij} \frac{e^{(h/2) H_i} - e^{-(h/2) H_i}}{h},$$

$$\sum_{s+t=1-a_{ji}} (-1)^t \begin{bmatrix} 1-a_{ji} \\ s \end{bmatrix} X_i^s X_j X_i^t = 0, \qquad i \neq j,$$

$$\sum_{s+t=1-a_{ji}} (-1)^t \begin{bmatrix} 1-a_{ji} \\ s \end{bmatrix} Y_i^s Y_j Y_i^t = 0, \qquad i \neq j.$$

There is a Hopf algebra structure on $\mathfrak{U}_h(\mathfrak{g})$ given by

$$\Delta(X_i) = X_i \otimes e^{(h/4) H_i} + e^{-(h/4) H_i} \otimes X_i,$$

$$\Delta(Y_i) = Y_i \otimes e^{(h/4) H_i} + e^{-(h/4) H_i} \otimes Y_i,$$

$$\varepsilon(X_i) = \varepsilon(Y_i) = \varepsilon(a) = 0, \qquad \text{for all} \quad a \in \mathfrak{h},$$

$$S(X_i) = -e^{h/2} X_i, \quad S(Y_i) = -e^{-h/2} Y_i, \quad S(a) = -a, \qquad \text{for all} \quad a \in \mathfrak{h}.$$

Given the definition of the coproduct Δ one can easily show that the formulas for the counit ε and the antipode S are forced by the definitions of a Hopf algebra.

There is a $\mathbb{Z}$ grading on the algebra $\mathfrak{U}_h(\mathfrak{g})$ determined by defining

$$\deg(h) = 0, \qquad \text{for all} \quad h \in \mathfrak{h},$$

$$\deg(E_i) = 1, \quad \deg(F_i) = -1, \qquad \text{for all} \quad 1 \leqslant i \leqslant r.$$

Let $\mathfrak{U}_h(\mathfrak{g})^{\geqslant 0}$ be the subalgebra of $\mathfrak{U}_h(\mathfrak{g})$ generated by $\mathfrak{h}$ and the elements X_i, $1 \leqslant i \leqslant r$. Similarly let $\mathfrak{U}_h(\mathfrak{g})^{\leqslant 0}$ be the subalgebra generated by $\mathfrak{h}$ and the elements Y_i, $1 \leqslant i \leqslant r$. Let $\tilde{H}_1, ..., \tilde{H}_r$ be an orthonormal basis of $\mathfrak{h}$ and let $t_0 = \sum_{i=1}^r \tilde{H}_i \otimes \tilde{H}_i$. The algebra $\mathfrak{U}_h(\mathfrak{g})$ is a quasitriangular Hopf algebra and the element $\mathcal{R}$ can be written in the form, See [D, Sect. 4],

$$\mathcal{R} = \exp\left(\frac{h}{2} t_0\right) + \sum a_i^+ \otimes b_i^-, \tag{2.13}$$

where the elements $a_i^+ \in \mathfrak{U}_h(\mathfrak{g})^{\geqslant 0}$, $b_i^- \in \mathfrak{U}_h(\mathfrak{g})^{\leqslant 0}$ are homogeneous elements of degrees $\geqslant 1$ and $\leqslant -1$ respectively.

As in the classical case, each finite dimensional $\mathfrak{U}_h(\mathfrak{g})$-module, M, is a direct sum of its weight spaces, i.e.,

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda, \qquad \text{where} \quad M^\lambda = \{m \in M \mid am = \langle \lambda, a \rangle m, \text{ for all } a \in \mathfrak{h}.\}$$

Every finite dimensional module is completely reducible and the finite dimensional irreducible modules L_λ of $\mathfrak{U}_h(\mathfrak{g})$ are labeled by the dominant integral weights λ. Each of these modules is a highest weight module of highest weight λ, i.e., there is a unique vector $m \in L_\lambda$ (up to constant multiples) such that

$$am = \langle \lambda, a \rangle m, \qquad \text{for all} \quad a \in \mathfrak{h}, \qquad \text{and}$$
$$X_i m = 0, \qquad \text{for all } i.$$

All of the facts in this paragraph can be proved, see [D, remarks after Proposition 4.2], by showing that since $H^2(\mathfrak{g}, \mathfrak{U}\mathfrak{g}) = 0$, the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of a finite dimensional complex simple Lie algebra $\mathfrak{g}$ has no nontrivial deformations as an algebra and thus there must be an algebra isomorphism $\mathfrak{U}_h(\mathfrak{g}) \simeq \mathfrak{U}\mathfrak{g}$. Note that this is only on the level of algebras, $\mathfrak{U}_h(\mathfrak{g})$ and $\mathfrak{U}\mathfrak{g}$ are not isomorphic as Hopf algebras. Thus, the representation theory of $\mathfrak{U}_h(\mathfrak{g})$, provided we are not considering questions of tensor products of representations, depends only on its structure as an algebra and is the same as the representation theory of $\mathfrak{U}\mathfrak{g}$.

Quantum Groups are Ribbon Hopf Algebras

(2.14) PROPOSITION [D]. *Let $\mathfrak{U}_h(\mathfrak{g})$ be a Drinfel'd–Jimbo quantum group and let ρ be the element of $\mathfrak{h}$ such that $\langle \alpha_i, \rho \rangle = 1$ for all simple roots α_i. Let u be as given in* (2.4). *Then*

(1) $e^{h\rho} a e^{-h\rho} = S^2(a)$ *for all* $a \in \mathfrak{U}_h(\mathfrak{g})$.

(2) $e^{-h\rho} u = u e^{-h\rho}$ *is a central element in* $\mathfrak{U}_h(\mathfrak{g})$.

(3) $(e^{-h\rho})^2 = uS(u) = S(u)u$.

(4) $e^{-h\rho} u$ *acts in an irreducible representation L_λ of $\mathfrak{U}_h(\mathfrak{g})$ of highest weight λ by the constant* $\exp(-(h/2)\langle \lambda, \lambda + 2\rho \rangle) = q^{-\langle \lambda, \lambda + 2\rho \rangle}$.

(5) $\Delta(e^{-h\rho} u) = (\mathcal{R}_{21} \mathcal{R})^{-1} (e^{-h\rho} u \otimes e^{-h\rho} u)$.

(6) $S(e^{-h\rho} u) = e^{-h\rho} u$.

(7) $\varepsilon(e^{-h\rho} u) = 1$.

Proof. (1) Since both S^2 and conjugation by $e^{h\rho}$ are algebra homomorphisms it is sufficient to check this on generators. We shall show how this is done for the generator X_j. It follows from the fact $[\rho, X_j] = \rho X_j - X_j \rho = \langle \alpha_j, \rho \rangle X_j$, that

$$e^{h\rho} X_j e^{-h\rho} = e^{h\rho} e^{-h(\rho - \langle \alpha_j, \rho \rangle)} X_j = e^{h\langle \alpha_j, \rho \rangle} X_j = e^h X_j = q^2 X_j = S^2(X_j).$$

(2) This follows from (1) and (2.8), since $e^{-h\rho} u a u^{-1} e^{h\rho} = S^{-2}(S^2(a)) = a$.

(4) Let $\tilde{H}_1, ..., \tilde{H}_r$ be an orthonormal basis of $\mathfrak{h}$. For each element $\lambda \in \mathfrak{h}^*$ let $\lambda_i = \langle \lambda, \tilde{H}_i \rangle$. Note that if m is a weight vector of weight λ in a $\mathfrak{U}_h(\mathfrak{g})$-module then $\tilde{H}_i m = \lambda_i m$. Let Λ_λ be an irreducible $\mathfrak{U}_h(\mathfrak{g})$-module of highest weight λ and let v_λ be a highest weight vector in Λ_λ. Since elements of $\mathfrak{U}_h(\mathfrak{g})^{\geqslant 0}$ which are of degree $\geqslant 1$ annihilate v_λ it follows that

$$\begin{aligned} uv_\lambda &= \exp\left(\frac{h}{2} \sum_{i=1}^{r} S(\tilde{H}_i)\, \tilde{H}_i\right) v_\lambda \\ &= \prod_{i=1}^{r} \left(\sum_{k \geqslant 0} \left(\frac{h}{2}\right)^k \frac{S(\tilde{H}_i)^k\, \tilde{H}_i^k}{k!} \right) v_\lambda \\ &= \prod_{i=1}^{r} \left(\sum_{k \geqslant 0} \left(-\frac{h}{2}\right)^k \frac{\tilde{H}_i^k \tilde{H}_i^k}{k!} \right) v_\lambda \\ &= \prod_{i=1}^{r} \left(\sum_{k \geqslant 0} \left(-\frac{h}{2}\right)^k \frac{\lambda_i^{2k}}{k!} \right) v_\lambda \\ &= \exp\left(-\frac{h}{2} \sum_{i=1}^{r} \lambda_i^2\right) v_\lambda \\ &= \exp\left(-\frac{h}{2} \langle \lambda, \lambda \rangle\right) v_\lambda \end{aligned}$$

The result follows since $e^{-h\rho} v_\lambda = e^{-h\langle \lambda, \rho \rangle} v_\lambda$.

(5) This follows from (2.9), since

$$\begin{aligned} \Delta(e^{-h\rho}u) = \Delta(ue^{-h\rho}) = \Delta(u)\, \Delta(e^{-h\rho}) &= (\mathcal{R}_{21} \mathcal{R})^{-1}\, (u \otimes u)(e^{-h\rho} \otimes e^{-h\rho}) \\ &= (\mathcal{R}_{21} \mathcal{R})^{-1}\, (e^{-h\rho} u \otimes e^{-h\rho} u). \end{aligned}$$

(3) and (6) and (7) follow from equality $e^{h\rho} S(u) = e^{-h\rho} u$ which is proved as follows. Clearly, $e^{h\rho} S(u) = S(ue^{-h\rho})$ is a central element of $\mathfrak{U}_h(\mathfrak{g})$, so it is sufficient to check that both $e^{h\rho} S(u)$ and $ue^{-h\rho}$ act by the same constant on an irreducible representation Λ_λ of $\mathfrak{U}_h(\mathfrak{g})$. But $e^{h\rho} S(u) = S(ue^{-h\rho})$ acts on the representation Λ_λ in the same way that $ue^{-h\rho}$ acts on the irreducible module Λ_λ^* which has highest weight $-w_0 \lambda$ where w_0 is the longest element of the Weyl group. Thus, $ue^{-h\rho}$ acts on the irreducible module Λ_λ^* by the constant

$$e^{-(h/2)\langle -w_0\lambda,\, -w_0\lambda \rangle} e^{-h\langle -w_0 \lambda,\, \rho \rangle} = e^{-(h/2)\langle \lambda,\, \lambda + 2\rho \rangle} = q^{-\langle \lambda,\, \lambda + \rho \rangle}$$

since $w_0 \rho = -\rho$ and the inner product is invariant under the action of w_0. ∎

(2.15) COROLLARY. *The Drinfel'd–Jimbo quantum group* $(\mathfrak{U}_h(\mathfrak{g}), \mathcal{R}, e^{-h\rho} u)$ *is a ribbon Hopf algebra.*

Centralizer Algebras of Tensor Power Representations and the $\mathcal{R}_{21}\mathcal{R}_{12}$ Matrix

Let $(\mathfrak{U}, \mathcal{R})$ be a quasitriangular Hopf algebra. Let V be a $\mathfrak{U}$-module and let $R \in \mathrm{End}(V \otimes V)$ be the linear transformation induced by the action of $\mathcal{R}$ on $V \otimes V$. Let

$$\check{R} = \sigma R, \tag{2.16}$$

where σ: $V \otimes V \to V \otimes V$ is the linear transformation given by $\sigma(v \otimes w) = w \otimes v$. For each $1 \leqslant i \leqslant m-1$ define

$$\check{R}_i = 1 \otimes \cdots \otimes 1 \otimes \check{R} \otimes 1 \otimes \cdots \otimes 1 \in \mathrm{End}(V^{\otimes m}) \tag{2.17}$$

where the $\check{R}$ appears as a transformation on the ith and $(i+1)$st tensor factors.

(2.18) PROPOSITION. *The transformations $\check{R}_i$ are elements of the centralizer $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ and satisfy the following relations*

$$\begin{aligned} \check{R}_i \check{R}_j &= \check{R}_j \check{R}_i, && |i-j| > 1, \\ \check{R}_i \check{R}_{i+1} \check{R}_i &= \check{R}_{i+1} \check{R}_i \check{R}_{i+1}, && 1 \leqslant i \leqslant m-2 \end{aligned}$$

Proof. Let $(\pi^{\otimes 2}, V \otimes V)$ be the representation of $\mathfrak{U}$ on $V \otimes V$. Let us abuse notation and denote the transformation on $V \otimes V$ induced by the action of $\Delta(a)$, $a \in \mathfrak{U}$, also by $\Delta(a)$. It follows from the equation

$$\check{R}\pi^{\otimes 2}(a) = \sigma R \Delta(a) = \sigma \Delta^{op}(a)\, R = \sigma \Delta^{op}(a)\, \sigma^{-1} \sigma R = \Delta(a)\, \check{R} = \pi^{\otimes 2}(a)\, \check{R},$$

that $\check{R} \in \mathrm{End}_{\mathfrak{U}}(V \otimes V)$. It follows that each $\check{R}_i \in \mathscr{Z}_m$ and that the algebra of transformations generated by the $\check{R}_i$ is contained in the centralizer $\mathscr{Z}_m$.

The fact that the $\check{R}_i$ satisfy the first relation follows immediately from the definition of the $\check{R}_i$. The second relation is derived from the relations (2.1) and (2.2) as follows. In the following calculations we abuse notations so that all factors in the computation are viewed as elements of $\mathrm{End}(V^{\otimes 3})$. We shall let R_{ij} denote the transformation of $V^{\otimes 3}$ induced by the action of $\mathcal{R}_{ij}$. We shall let σ_{ij} denote the transformation of $V^{\otimes 3}$ which transposes the ith and the jth tensor factors of $V^{\otimes 3}$. Then, using the equation

$$\begin{aligned} R_{12} R_{13} R_{23} &= R_{12} (\Delta \otimes id)(\mathcal{R}_{12}) && \text{by (2.2)} \\ &= (\Delta^{op} \otimes id)(\mathcal{R}_{12}) R_{12} && \text{by (2.1)} \\ &= R_{23} R_{13} R_{12}, \end{aligned}$$

we have

$$\begin{aligned}\check{R}_1\check{R}_2\check{R}_1 &= \sigma_{12}R_{12}\sigma_{23}R_{23}\sigma_{12}R_{12}\\ &= \underbrace{\sigma_{12}\sigma_{23}\sigma_{12}}\,\underbrace{\sigma_{12}\sigma_{23}R_{12}\sigma_{23}\sigma_{12}}\,\underbrace{\sigma_{12}R_{23}\sigma_{12}}\,R_{12}\\ &= \sigma_{13}R_{23}R_{13}R_{12},\end{aligned}$$

and

$$\begin{aligned}\check{R}_2\check{R}_1\check{R}_2 &= \sigma_{23}R_{23}\sigma_{12}R_{12}\sigma_{23}R_{23}\\ &= \underbrace{\sigma_{23}\sigma_{12}\sigma_{23}}\,\underbrace{\sigma_{23}\sigma_{12}R_{23}\sigma_{12}\sigma_{23}}\,\underbrace{\sigma_{23}R_{12}\sigma_{23}}\,R_{23}\\ &= \sigma_{13}R_{12}R_{13}R_{23},\end{aligned}$$

It follows that $\check{R}_1\check{R}_2\check{R}_1=\check{R}_2\check{R}_1\check{R}_2$. ∎

The proof of the following proposition is similar to the proof of Lemma 3.3.1 in [W4].

(2.19) PROPOSITION. (1) *If* (π_W, W) *and* (π_V, V) *are two representations of* $\mathfrak{U}$, *then* $(\pi_W\otimes\pi_V)(\mathscr{R}_{21}\mathscr{R}_{12})\in\mathrm{End}_{\mathfrak{U}}(W\otimes V)$.

(2) *Let* (π, V) *be a representation of* $\mathfrak{U}$. *Then*

$$(\pi^{\otimes(m-1)}\otimes\pi)(\mathscr{R}_{21}\mathscr{R}_{12})=\check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_1\check{R}_1\check{R}_2\cdots\check{R}_{m-1}\in\mathrm{End}_{\mathfrak{U}}(V^{\otimes m}).$$

Proof. (1) The equality $\mathscr{R}\Delta(a)\mathscr{R}^{-1}=\Delta^{op}(a)$ is equivalent to $\mathscr{R}_{21}\Delta^{op}(a)\mathscr{R}_{21}^{-1}=\Delta(a)$ which in turn implies $\Delta^{op}(a)=\mathscr{R}_{21}^{-1}\Delta(a)\mathscr{R}_{21}$. Thus, we have $\mathscr{R}_{21}^{-1}\Delta(a)\mathscr{R}_{21}=\mathscr{R}\Delta(a)\mathscr{R}^{-1}$, which is the same as

$$\mathscr{R}_{21}\mathscr{R}\Delta(a)=\Delta(a)\mathscr{R}_{21}\mathscr{R}.$$

(2) Using (2.2), we have, by induction,

$$\begin{aligned}(\Delta^{(m-2)}\otimes\mathrm{id})(\mathscr{R}_{12}) &= (\Delta\otimes\mathrm{id}^{\otimes(m-2)})(\Delta^{(m-3)}\otimes\mathrm{id})(\mathscr{R})\\ &= (\Delta\otimes\mathrm{id}^{\otimes(m-2)})(\mathscr{R}_{1(m-1)}\mathscr{R}_{2(m-1)}\cdots\mathscr{R}_{(m-2)(m-1)})\\ &= \mathscr{R}_{1m}\mathscr{R}_{2m}\cdots\mathscr{R}_{(m-1)m}.\end{aligned}\tag{2.20}$$

Similarly, we get that $(\Delta^{(m-2)}\otimes\mathrm{id})(\mathscr{R}_{21})=\mathscr{R}_{m(m-1)}\mathscr{R}_{m(m-2)}\cdots\mathscr{R}_{m2}\mathscr{R}_{m1}$.

Let $\sigma\colon V^{\otimes(m-1)}\otimes V\to V^{\otimes(m-1)}$ be the transformation which transposes the tensor factors $V^{\otimes(m-1)}$ and V. As a transformation in the symmetric group S_m acting on $V^{\otimes m}$ we have $\sigma=\sigma_{1\cdots m}=\sigma_{12}\sigma_{23}\cdots\sigma_{(m-1)m}$ where $\sigma_{i(i+1)}$ is the permutation in S_m that switches the ith and the $(i+1)$st tensor factors of $V^{\otimes m}$. Let R_{ij} denote the endomorphism of $V^{\otimes m}$ induced by multiplying by $\mathscr{R}_{ij}\in\mathfrak{U}^{\otimes m}$. Then, viewing $(\Delta^{(m-2)}\otimes \mathrm{id})(\mathscr{R})$ as a transformation on $V^{\otimes m}$, we have

$$\begin{aligned}
\sigma(\Delta^{(m-2)}\otimes \mathrm{id})(\mathscr{R}) &= \sigma_{1\cdots m}R_{1m}R_{2m}R_{3m}\cdots R_{(m-1)m}\\
&= \sigma_{12}\sigma_{2\cdots m}R_{1m}\sigma_{2\cdots m}^{-1}\sigma_{2\cdots m}R_{2m}\sigma_{3\cdots m}^{-1}\sigma_{3\cdots m}\\
&\quad\times R_{3m}\cdots R_{(m-2)m}\sigma_{(m-1)m}^{-1}\sigma_{(m-1)m}R_{(m-1)m}\\
&= \sigma_{12}\sigma_{2\cdots m}R_{1m}\sigma_{2\cdots m}^{-1}\sigma_{23}\sigma_{3\cdots m}R_{2m}\sigma_{3\cdots m}^{-1}\\
&\quad\times \sigma_{34}\sigma_{4\cdots m}R_{3m}\cdots\sigma_{(m-1)m}R_{(m-1)m}\\
&= \sigma_{12}R_{12}\sigma_{23}R_{23}\sigma_{34}R_{34}\cdots\sigma_{(m-1)m}R_{(m-1)m}\\
&= \check{R}_1\check{R}_2\cdots\check{R}_{m-1}.
\end{aligned}$$

In a similar fashion one shows that

$$\begin{aligned}
(\Delta^{(m-2)}\otimes \mathrm{id})(\mathscr{R}_{21})\,\sigma^{-1} &= R_{m1}R_{m2}R_{m3}\cdots R_{m(m-1)}\sigma_{m\cdots 1}\\
&= \check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_1,
\end{aligned}$$

where $\sigma^{-1}=\sigma_{m\cdots 1}=\sigma_{(m-1)m}\cdots\sigma_{23}\sigma_{12}$. Thus, it follows that

$$(\Delta^{(m-2)}\otimes \mathrm{id})(\mathscr{R}_{21}\mathscr{R}) = \check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_1\check{R}_1\check{R}_2\cdots\check{R}_{m-1}. \quad \blacksquare$$

(2.21) Proposition. (1) *Let $(\mathfrak{U},\mathscr{R})$ be a quasitriangular Hopf algebra and let $z=uS(u)$ be as given in* (2.4). *The element z acts on each irreducible representation Λ_λ of $\mathfrak{U}$ by a scalar. Denote this scalar by $z(\lambda)$. Then the element $(\mathscr{R}_{21}\mathscr{R}_{12})^2$ acts on the irreducible component of Λ_ν of $\Lambda_\lambda\otimes\Lambda_\mu$ by the scalar*

$$\frac{z(\lambda)\,z(\mu)}{z(\nu)}.$$

(2) *Let $(\mathfrak{U},\mathscr{R},v)$ be a ribbon Hopf algebra. The element v acts on each irreducible representation Λ_λ of $\mathfrak{U}$ by a scalar. Denote this scalar by $v(\lambda)$. Then the element $\mathscr{R}_{21}\mathscr{R}_{12}$ acts on the irreducible component Λ_ν of $\Lambda_\lambda\otimes\Lambda_\mu$ by the scalar*

$$\frac{v(\lambda)\,v(\mu)}{v(\nu)}.$$

(3) *Let $\mathfrak{U}_h(\mathfrak{g})$ be a Drinfel'd–Jimbo quantum group. The element $\mathscr{R}_{21}\mathscr{R}_{12}$ acts on the irreducible component Λ_ν of $\Lambda_\lambda\otimes\Lambda_\mu$ by the scalar*

$$q^{\langle \nu,\,\nu+2\rho\rangle-\langle\lambda,\,\lambda+2\rho\rangle-\langle\mu,\,\mu+2\rho\rangle}.$$

Proof. (1) Since z is in the center of $\mathfrak{U}$, the element z acts on each irreducible representation Λ_λ of $\mathfrak{U}$ by a scalar. The element $(z \otimes z)$ acts on $\Lambda_\lambda \otimes \Lambda_\mu$ by the constant $z(\lambda)\, z(\mu)$. Similarly, $\Delta(z)$ acts on the irreducible component Λ_ν of $\Lambda_\lambda \otimes \Lambda_\mu$ by the scalar $z(\nu)$. The result now follows from the identity $\Delta(z) = (\mathcal{R}_{21}\mathcal{R})^{-2}\,(z \otimes z)$.

The proof of (2) is entirely similar to the proof of (1). Now, (3) follows from (2) by noting that the quantum group is a ribbon Hopf algebra with $v = e^{-h\rho}u$ and that the element $e^{-h\rho}u$ acts on each irreducible representation Λ_λ of $\mathfrak{U}_h(\mathfrak{g})$ by the scalar $q^{-\langle \lambda, \lambda + 2\rho\rangle}$. ∎

(2.22) Corollary. (1) *Let $(\mathfrak{U}, \mathcal{R})$ be a quasitriangular Hopf algebra and denote the constant given by the action of $z = uS(u)$ on an irreducible representation Λ_ν by $z(\nu)$. Suppose that $V = \Lambda_\omega$ is an irreducible representation of $\mathfrak{U}$. Let $\hat{\mathcal{L}}_2$ be an index set for the irreducible $\mathfrak{U}$-modules appearing the decomposition of $V^{\otimes 2}$. Then $\check{R}_i$ satisfies the equation*

$$\prod_{\nu \in \hat{\mathcal{L}}_2} \left(\check{R}_i^4 - \frac{z(\omega)^2}{z(\nu)} \right) = 0.$$

(2) *Let $(\mathfrak{U}, \mathcal{R}, v)$ be a ribbon Hopf algebra and denote the constant given by the action of v on an irreducible representation Λ_ν by $v(\nu)$. Suppose that $V = \Lambda_\omega$ is an irreducible representation of $\mathfrak{U}$. Then $\check{R}_i$ satisfies the equation*

$$\prod_{\nu \in \hat{\mathcal{L}}_2} \left(\check{R}_i^2 - \frac{v(\omega)^2}{v(\nu)} \right) = 0.$$

(3) ([Re], *formula* (1.38)) *Suppose that $V = \Lambda_\omega$ is an irreducible representation of a Drinfel'd–Jimbo quantum group $\mathfrak{U}_h(\mathfrak{g})$ and that the Bratteli diagram for tensoring by V is multiplicity free. Then $\check{R}_i$ satisfies the equation*

$$\prod_{\nu \in \hat{\mathcal{L}}_2} \left(\check{R}_i \pm q^{(1/2)\langle \nu, \nu + 2\rho\rangle - \langle \omega, \omega + 2\rho\rangle} \right) = 0,$$

where the sign in the factor $(\check{R}_i \pm q^{(1/2)\langle \nu, \nu + 2\rho\rangle - \langle \omega, \omega + 2\rho\rangle})$ is negative if Λ_ν is an irreducible component of the symmetric part of $V^{\otimes 2}$ and positive if Λ_ν is an irreducible component of the antisymmetric part $\bigwedge^2(V)$ of $V^{\otimes 2}$.

Proof. (1) By Proposition (2.19) part (2), $\check{R}_1^2 = \pi^{\otimes 2}(\mathcal{R}_{21}\mathcal{R})$. Suppose that $V \otimes V = \bigoplus_{T \in \mathcal{T}^2} V_T$, is a decomposition of $V^{\otimes 2}$ into irreducibles. Then, by Proposition (2.21), $\check{R}_1^4$ acts on the irreducible V_T by the constant $z(\omega)^2/z(\nu)$ if $V_T \cong \Lambda_\nu$. It follows that $\check{R}_1^4$ is a central element of $\mathcal{L}_2$ and that the minimal polynomial of $\check{R}_1^4$ is

$$\prod_{\nu \in \tilde{\mathcal{L}}_2} \left(t - \frac{z(\nu)}{z(\omega)^2} \right).$$

The proof of (2) is similar to the proof of (1). Let us complete the proof of (3). It follows from (2) that $\check{R}_1$ satisfies the polynomial $\prod_{\nu \in \hat{\mathscr{T}}_2} (R_1^2 - q^{\langle \nu, \nu+2\rho\rangle - 2\langle \omega, \omega+2\rho\rangle}) = 0$. Given that $\check{R}_1$ is a central element of $\mathrm{End}_{\mathfrak{U}_h(\mathfrak{g})}(V^{\otimes 2})$ since the Bratteli diagram is multiplicity free, it follows that the eigenvalues of $\check{R}_1$ are $\pm q^{(1/2)\langle \nu, \nu+2\rho\rangle - \langle \omega, \omega+2\rho\rangle}$. Since, $\check{R}_1$ is a deformation of the transposition which switches the two factors of $V^{\otimes 2}$ we know that if we specialize $q=1$ the eigenvalues of $\check{R}_1$ are $+1$ if Λ_ν is an irreducible component of the symmetric part of $V^{\otimes 2}$ and -1 if Λ_ν is an irreducible component of the antisymmetric part $\bigwedge^2(V)$ of $V^{\otimes 2}$. This observation determines the signs of the eigenvalues of $\check{R}_1$. ∎

Let $V = \Lambda_\omega$ be an irreducible representation of $\mathfrak{U}$ and let $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$. Recall, from Section 1, that there is a natural way of identifying the path algebras corresponding to the Bratteli diagram for tensor powers of V with the centralizer algebras $\mathscr{Z}_m$. As stated in Section 1 we shall always assume that the Bratteli diagram for tensor powers of V is multiplicity free. This is probably not necessary for part (1) of the following corollary but it is certainly necessary for part (2).

(2.23) Corollary. *Let* $(\mathfrak{U}, \mathscr{R})$ *be a quasitriangular Hopf algebra and let* $V = \Lambda_\omega$ *be an irreducible representation of* $\mathfrak{U}$. *Identify the path algebras corresponding to the Bratteli diagram for tensor powers of V with the centralizer algebras* $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ *as in Section* 1.

(1) *Let* $D_m = \check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_1\check{R}_1\check{R}_2\cdots\check{R}_{m-1} \in \mathscr{Z}_m$ *be the element given in Proposition* (2.19). *Then*

$$D_m^2 = \sum_{T \in \mathscr{T}^m} (D_m^2)_{TT} E_{TT}, \qquad \textit{where} \quad (D_m^2)_{TT} = \frac{z(\tau^{(m-1)})\, z(\omega)}{z(\tau^{(m)})},$$

for each $T = (\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)}) \in \mathscr{T}^m$.

(2) *Fix* $T = (\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)}) \in \mathscr{T}^m$ *and let* $T' = (\tau^{(0)}, ..., \tau^{(m-1)}) \in \mathscr{T}^{m-1}$. *Let* $(T')^+$ *be the set of tableaux that are extensions of* T', *i.e. the set of* $S = (\tau^{(0)}, ..., \tau^{(m-1)}, \sigma^{(m)}) \in \mathscr{T}^m$. *If the values* $(D_m^2)_{SS}$ *are all different as S runs over all elements of* $(T')^+$ *then*

$$E_{TT} = \prod_{\substack{S \in (T')^+ \\ S \neq T}} \frac{E_{T'T'} D_m^2 E_{T'T'} - (D_m^2)_{SS} E_{T'T'}}{(D_m^2)_{TT} - (D_m^2)_{SS}}$$

Proof. (1) Recall that the identification of the path algebras with the centralizer algebras $\mathscr{Z}_m$ is done so that for each $T = (\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)}) \in \mathscr{T}^m$ we have that $E_{TT} V^{\otimes m}$ is an irreducible $\mathfrak{U}$ module isomorphic to $\Lambda_{\tau^{(m)}}$. Furthermore, if we let $T' = (\tau^{(0)}, ..., \tau^{(m-1)}) \in \mathscr{T}^{m-1}$ we know that

$$E_{T'T'} V^{\otimes m} = E_{T'T'} V^{\otimes (m-1)} \otimes V \cong \Lambda_{\tau^{(m-1)}} \otimes \Lambda_\omega$$

and that, by Proposition (2.21), $D_m^2 = (\mathcal{R}_{12}\mathcal{R}_{21})^2$ acts on each irreducible component $\Lambda_{\tau^{(m)}}$ of the tensor product $E_{T'T'}V^{\otimes(m-1)} \otimes V$ by the constant $z(\tau^{(m-1)})\, z(\omega)/z(\tau^{(m)})$. It follows that

$$\begin{aligned}
D_m^2 V^{\otimes m} &= D_m^2 \sum_{T' \in \mathcal{T}^{m-1}} E_{T'T'}(V^{\otimes(m-1)} \otimes V)\\
&= \sum_{T' \in T^{m-1}} D_m^2(E_{T'T'}V^{\otimes(m-1)} \otimes V)\\
&= \sum_{T' \in \mathcal{T}^{m-1}} (\mathcal{R}_{21}\mathcal{R}_{12})^2\, (E_{T'T'}V^{\otimes(m-1)} \otimes V)\\
&= \sum_{T' \in \mathcal{T}^{m-1}} (\mathcal{R}_{21}\mathcal{R})^2 \left(\sum_{T \in (T')^+} E_{TT} V^{\otimes m} \right)\\
&= \sum_{T \in T^m} \frac{z(\tau^{(m-1)})\, z(\omega)}{z(\tau^{(m)})} E_{TT} V^{\otimes m}.
\end{aligned}$$

The result follows as D_m^2 is determined by its action on $V^{\otimes m}$.

(2) It follows from part (1) that

$$E_{T'T'} D_m^2 E_{T'T'} = \sum_{T \in (T')^+} \frac{z(\tau^{(m-1)})\, z(\omega)}{z(\tau^{(m)})} E_{TT}.$$

If the Bratteli diagram is multiplicity free and the eigenvalues $z(\tau^{(m-1)})\, z(\omega)/z(\tau^{(m)})$ are all different, then the result follows by taking the spectral projection of $E_{T'T'} D_m^2 E_{T'T'}$ with respect to a particular eigenvalue. ∎

The following corollaries follow in exactly the same fashion.

(2.24) Corollary. *Let $(\mathfrak{U}, \mathcal{R}, v)$ be a ribbon Hopf algebra and let $V = \Lambda_\omega$ be an irreducible representation of $\mathfrak{U}$. Identify the path algebras corresponding to the Bratteli diagram for tensor powers of V with the centralizer algebras $\mathcal{Z}_m$ as in Section* 1.

(1) *Let $D_m = \check{R}_{m-1}\check{R}_{m-2} \cdots \check{R}_1 \check{R}_1 \check{R}_2 \cdots \check{R}_{m-1} \in \mathcal{Z}_m$ be the element given in Proposition* (2.19). *Then*

$$D_m = \sum_{T \in \mathcal{T}^m} (D_m)_{TT}\, E_{TT}, \qquad \textit{where} \quad (D_m)_{TT} = \frac{v(\tau^{(m-1)})\, v(\omega)}{v(\tau^{(m)})},$$

for $T = (\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)}) \in \mathcal{T}^m$.

(2) *Fix* $T=(\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)}) \in \mathscr{T}^m$ *and let* $T'=(\tau^{(0)}, ..., \tau^{(m-1)}) \in \mathscr{T}^{m-1}$. *Let* $(T')^+$ *be the set of tableaux that are extensions of* T', *i.e. the set of* $S=(\tau^{(0)}, ..., \tau^{(m-1)}, \sigma^{(m)}) \in \mathscr{T}^m$. *If the values* $(D_m)_{SS}$ *are all different as S runs over all elements of* $(T')^+$ *then*

$$E_{TT}=\prod_{\substack{S\in(T')^+\\ S\neq T}} \frac{E_{T'T'}D_mE_{T'T'}-(D_m)_{SS}\,E_{T'T'}}{(D_m)_{TT}-(D_m)_{SS}}$$

(2.25) Corollary [Re, formula (3.19)]. *Let* $(\mathfrak{U}_h(\mathfrak{g}), \mathscr{R}, e^{-h\rho}u)$ *be a Drinfel'd–Jimbo quantum group and let* $V=\Lambda_\omega$ *be an irreducible representation of* $\mathfrak{U}$. *Identifying the path algebras* A_m *corresponding to the Bratteli diagram for tensor powers of V with the centralizer algebras* $\mathscr{Z}_m$ *as in Section* 1.

(1) *Let* $D_m=\check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_1\check{R}_1\check{R}_2\cdots\check{R}_{m-1}\in\mathscr{Z}_m$ *be the element given in Proposition* (2.19). *Then*

$$D_m=\sum_{T\in\mathscr{T}^m}(D_m)_{TT}\,E_{TT},$$

$$\text{where}\quad (D_m)_{TT}=q^{\langle\tau^{(m)},\,\tau^{(m)}+2\rho\rangle-\langle\tau^{(m-1)},\,\tau^{(m-1)}+2\rho\rangle-\langle\omega,\,\omega+2\rho\rangle},$$

and $\tau^{(m)}$ *and* $\tau^{(m-1)}$ *are determined from T by* $T=(\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)})\in\mathscr{T}^m$.

(2) *Fix* $T=(\tau^{(0)}, ..., \tau^{(m-1)}, \tau^{(m)}) \in \mathscr{T}^m$ *and let* $T'=(\tau^{(0)}, ..., \tau^{(m-1)}) \in \mathscr{T}^{m-1}$. *Let* $(T')^+$ *be the set of tableaux that are extensions of* T', *i.e. the set of* $S=(\tau^{(0)}, ..., \tau^{(m-1)}, \sigma^{(m)}) \in \mathscr{T}^m$. *If the values* $(D_m)_{SS}$ *are all different as S runs over all elements of* $(T')^+$ *then*

$$E_{TT}=\prod_{\substack{S\in(T')^+\\ S\neq T}} \frac{E_{T'T'}D_mE_{T'T'}-(D_m)_{SS}\,E_{T'T'}}{(D_m)_{TT}-(D_m)_{SS}}.$$

3. RIBBON HOPF ALGEBRAS, CONDITIONAL EXPECTATIONS, AND MARKOV TRACES ON CENTRALIZER ALGEBRAS

Let $(\mathfrak{U}, \mathscr{R}, v)$ be a ribbon Hopf algebra. Let W be a finite dimensional $\mathfrak{U}$-module. Let $\{w_i\}$ be a basis of W and let $\{w^i\}$ be the dual basis in W^*. Let $\langle\ ,\ \rangle$ be the ordinary pairing between W and W^* so that $\langle\phi, w\rangle=\langle w, \phi\rangle=\phi(w)$ for elements $\phi\in W^*$ and $w\in W$. Using this notation, the action of an element $b\in\mathrm{End}(W)$ can be given in the form

$$bw_i=\sum_j\langle bw_i, w^j\rangle\,w_j.$$

The Hopf algebra $\mathfrak{U}$ acts on W^* via the antipode S in the standard way,

$$a\phi = \sum_j \langle a\phi, w_j \rangle w^j = \sum_j \langle \phi, S(a)\, w_j \rangle w^j,$$

for all $a \in \mathfrak{U}$ and $\phi \in W^*$. We shall often use the relation $\langle a\phi, w_j \rangle = \langle \phi, S(a)\, w_j \rangle$, which follows from this definition. The material in this section is very much motivated by [W1, Section 1] and [W3].

Quantum Trace and Quantum Dimension

Define the quantum trace of an element $b \in \mathrm{End}_{\mathfrak{U}}(W)$ by

$$\mathrm{tr}_q(b) = \mathrm{Tr}(v^{-1}ub) = \sum_i \langle v^{-1}ubw_i, w^i \rangle,$$

where the sum is over the basis W_i of W. If $a, b \in \mathrm{End}_{\mathfrak{U}}(W)$ then both a and b commute with $v^{-1}u$; thus, $\mathrm{tr}_q(ab) = \mathrm{tr}_q(ba)$ for all $a, b \in \mathrm{End}_{\mathfrak{U}}(W)$. Define the quantum dimension of the $\mathfrak{U}$-module W to be

$$\dim_q(W) = \mathrm{tr}_q(\mathrm{id}),$$

where id denotes the identity operator on W.

(3.1) Lemma. *Let $\hat{W}$ be the subset of $\hat{\mathfrak{U}}$ that indexes the irreducible modules Λ_μ appearing in the decomposition of W. As a trace on* $\mathrm{End}_{\mathfrak{U}}(W)$, *the quantum trace* tr_q *has weights given by*

$$\mathrm{wt}(\mu) = \dim_q(\Lambda_\mu), \qquad \mu \in \hat{W},$$

where Λ_μ are the irreducible $\mathfrak{U}$-modules appearing in the decomposition of W.

Proof. By the double centralizer theory we know that as $\mathrm{End}_{\mathfrak{U}}(W) \otimes \mathfrak{U}$ modules, $W \cong \bigoplus_\lambda \mathscr{Z}^\lambda \otimes \Lambda_\lambda$, where $\mathscr{Z}^\lambda$ are irreducible modules for $\mathrm{End}_{\mathfrak{U}}(W)$ and Λ_λ are irreducible modules for $\mathfrak{U}$. By taking traces on both sides of this isomorphism we have

$$\mathrm{tr}_q(b) = \mathrm{Tr}(v^{-1}ub) = \sum_{\lambda \in \hat{W}} \eta_\lambda(b)\, \chi^\lambda(v^{-1}u) = \sum_{\lambda \in \hat{W}} \eta_\lambda(b) \dim_q(\Lambda_\lambda),$$

where η_λ is the irreducible character of $\mathrm{End}_{\mathfrak{U}}(W)$ on the module $\mathscr{Z}^\lambda$ and χ^λ is the irreducible character of the irreducible $\mathfrak{U}$-module Λ_λ. Thus the trace of a minimal idempotent p_μ in the minimal ideal corresponding to μ is

$$\mathrm{wt}(\mu) = \mathrm{tr}_q(p_\mu) = \sum_{\lambda \in \hat{W}} \eta_\lambda(p_\mu) \dim_q(\Lambda_\lambda) = \sum_{\lambda \in \hat{W}} \delta_{\lambda\mu} \dim_q(\Lambda_\lambda) = \dim_q(\Lambda_\mu). \quad \blacksquare$$

The Projection onto the Invariants

Let V be a $\mathfrak{U}$-module and let V^* be the dual module to V. Let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be the dual basis in V^*. Define

$$\begin{aligned}\check{e}\colon V\otimes V^* &\to V\otimes V^*\\ x\otimes\phi &\mapsto \langle(\dim_q(V))^{-1}\phi, v^{-1}ux\rangle \sum_i e_i\otimes e^i\end{aligned} \tag{3.2}$$

Where $\langle\phi, v^{-1}ux\rangle = \phi(v^{-1}ux)$ denotes the evaluation of the functional $\phi\in V^*$ at the element $v^{-1}ux\in V$. It follows from (a) and (b) of the following proposition that

(1) $\check{e}\in \mathrm{End}_{\mathfrak{U}}(V\otimes V^*)$, and

(2) $\check{e}$ is the $\mathfrak{U}$-invariant projection onto the invariants in $V\otimes V^*$.

(3.3) PROPOSITION. (a) *For every $g\in\mathfrak{U}$ we have $g\check{e}=\check{e}g=\varepsilon(g)\check{e}$,*

(b) $\check{e}^2=\check{e}$.

Proof. (a) Let $g\in\mathfrak{U}$, $x\in V$, $\phi\in V^*$. Then, by direct computation,

$$\begin{aligned}
g\check{e}(x\otimes\phi) &= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle g\left(\sum_i e_i\otimes e^i\right)\\
&= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle\, \Delta(g)\left(\sum_i e_i\otimes e^i\right)\\
&= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle \sum_{g,i} g_{(1)}e_i\otimes g_{(2)}e^i\\
&= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle \sum_{i,j,k}\sum_g \langle g_{(1)}e_i, e^j\rangle\\
&\quad\times\langle g_{(2)}e^i, e_k\rangle(e_j\otimes e^k)\\
&= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle \sum_{i,j,k}\sum_g \langle g_{(1)}e_i, e^j\rangle\\
&\quad\times\langle e^i, S(g_{(2)})e_k\rangle(e_j\otimes e^k)\\
&= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle \sum_{j,k}\sum_g \langle g_{(1)}S(g_{(2)})\, e_k, e^j\rangle(e_j\otimes e^k)\\
&= (\dim_q(V))^{-1}\langle\phi, v^{-1}ux\rangle \sum_{j,k} \langle\varepsilon(g)\, e_k, e^j\rangle\, e_j\otimes e^k\\
&= (\dim_q(V))^{-1}\,\varepsilon(g)\langle\phi, v^{-1}ux\rangle \sum_j e_j\otimes e^j\\
&= \varepsilon(g)\,\check{e}(x\otimes\phi),
\end{aligned}$$

where we are using the identity $\sum_g g_{(1)} S(g_{(2)}) = \varepsilon(g)$ which follows from the definition of the antipode in a Hopf algebra. On the other hand, since v is in the center of $\mathfrak{U}$ and $u^{-1} x u = S^{-2}(x)$ for all $x \in \mathfrak{U}$,

$$\begin{aligned}
\check{e} g(x \otimes \phi) &= \check{e}\left(\sum_g g_{(1)} x \otimes g_{(2)} \phi\right) \\
&= (\dim_q(V))^{-1} \sum_{g,\, i} \langle g_{(2)} \phi,\, v^{-1} u g_{(1)} x \rangle\, e_i \otimes e^i \\
&= (\dim_q(V))^{-1} \sum_{g,\, i} \langle \phi,\, u u^{-1}\, S(g_{(2)})\, v^{-1} u g_{(1)} x \rangle\, e_i \otimes e^i \\
&= (\dim_q(V))^{-1} \sum_{g,\, i} \langle \phi,\, v^{-1} u S^{-1}(g_{(2)})\, g_{(1)} x \rangle\, e_i \otimes e^i \\
&= (\dim_q(V))^{-1} \sum_{i} \langle \phi,\, v^{-1} u \varepsilon(g) x \rangle\, e_i \otimes e^i \\
&= \varepsilon(g)\, \check{e}(x \otimes \phi),
\end{aligned}$$

where we are using the identity $\sum_g S^{-1}(g_{(2)})\, g_{(1)} = \varepsilon(g)$ which follows from the definition of the skew antipode in a Hopf algebra.

(b) This follows from the following easy computation.

$$\begin{aligned}
\check{e}^2(x \otimes \phi) &= (\dim_q(V))^{-1}\, \check{e}\left(\langle \phi,\, v^{-1} u x \rangle \sum_i e_i \otimes e^i\right) \\
&= (\dim_q(V))^{-2} \langle \phi,\, v^{-1} u x \rangle \sum_i \langle e^i,\, v^{-1} u e_i \rangle \sum_j e_j \otimes e^j \\
&= (\dim_q(V))^{-2} \langle \phi,\, v^{-1} u x \rangle \dim_q(V) \sum_j e_j \otimes e^j \\
&= \check{e}(x \otimes \phi). \quad \blacksquare
\end{aligned}$$

The Conditional Expectation

Let V be a $\mathfrak{U}$-module and let V^* be the dual $\mathfrak{U}$-module to V. For each m,

$$\text{let } \mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m}) \text{ and define } \mathscr{C}_{m+1} = \mathrm{End}_{\mathfrak{U}}(V^{\otimes (m-1)} \otimes V^*). \quad (3.4)$$

Let $\{w_s\}$ be a basis of $V^{\otimes (m-1)}$ and let $\{w^s\}$ be a dual basis in $(V^{\otimes (m-1)})^*$. Let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be a dual basis in V^*. Then define an operator $\varepsilon_{m-1}\colon \mathscr{Z}_m \to \mathrm{End}(V^{\otimes (m-1)})$ by

$$\varepsilon_{m-1}(b)\, w_j = (\dim_q(V))^{-1} \sum_{k,\, p} \langle (\mathrm{id} \otimes v^{-1} u)\, b(w_j \otimes e_k),\, w^p \otimes e^k \rangle\, w_p. \quad (3.5)$$

for each $b \in \mathscr{Z}_m$. The map ε_{m-1} is called the *conditional expectation.* Let

$$\check{e}_m = \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \check{e} \in \mathscr{C}_{m+1}. \tag{3.6}$$

(3.7) PROPOSITION.

(a) $\check{e}_m b \check{e}_m = \varepsilon_{m-1}(b)\, \check{e}_m = \check{e}_m \varepsilon_{m-1}(b)$ *for all* $b \in \mathscr{Z}_m$.

(b) $\varepsilon_{m-1}(a_1 b a_2) = a_1 \varepsilon_{m-1}(b) a_2$, *for all* $a_1, a_2 \in \mathscr{Z}_{m-1}$ *and* $b \in \mathscr{Z}_m$. *In particular*, $\varepsilon_{m-1}(a) = a$ *for all* $a \in \mathscr{Z}_{m-1}$.

(c) $\varepsilon_{m-1}(b) \in \mathscr{Z}_{m-1}$ *for all* $b \in \mathscr{Z}_m$.

Proof. Let $W = V^{\otimes(m-1)}$. Let $\{w_t\}$ and $\{e_i\}$ be bases of W and V respectively and let $\{w^s\}$ and $\{e^i\}$ be dual bases in W^* and V^* respectively.

(a) Then

$$\begin{aligned}
&\check{e}_m b \check{e}_m (w_s \otimes e_i \otimes e^j) \\
&\quad = (\dim_q(V))^{-1} \langle e^j, v^{-1} u e_i \rangle \sum_k \check{e}_m b (w_s \otimes e_k \otimes e^k) \\
&\quad = (\dim_q(V))^{-1} \langle e^j, v^{-1} u e_i \rangle \sum_{k,t,l} \langle b(w_s \otimes e_k), w^t \otimes e^l \rangle\, \check{e}_m (w_t \otimes e_l \otimes e^k) \\
&\quad = (\dim_q(V))^{-2} \langle e^j, v^{-1} u e_i \rangle \sum_{k,t,l,p} \langle b(w_s \otimes e_k), w^t \otimes e^l \rangle \langle e^k, v^{-1} u e_l \rangle \\
&\qquad \times (w_t \otimes e_p \otimes e^p) \\
&\quad = (\dim_q(V))^{-2} \langle e^j, v^{-1} u e_i \rangle \sum_{k,t,p} \langle (\mathrm{id} \otimes v^{-1} u)\, b(w_s \otimes e_k), w^t \otimes e^k \rangle \\
&\qquad \times (w_t \otimes e_p \otimes e^p) \\
&\quad = \varepsilon_{m-1}(b)\, \check{e}_m (w_s \otimes e_i \otimes e^j).
\end{aligned}$$

The remaining assertion follows since $\check{e}_m$ commutes with elements of $\mathrm{End}(W) \subseteq \mathrm{End}(W \otimes V \otimes V^*)$.

(b) The action of $\varepsilon_{m-1}(a_1 b a_2)$ on a basis element w_j of W satisfies

$$\begin{aligned}
\varepsilon_{m-1}(a_1 b a_2) w_j &= (\dim_q(V))^{-1} \sum_{k,p} \langle (\mathrm{id} \otimes v^{-1} u)(a_1 \otimes \mathrm{id}) \\
&\qquad \times b(a_2 \otimes \mathrm{id})(w_j \otimes e_k), w^p \otimes e^k \rangle\, w_p \\
&= (\dim_q(V))^{-1} a_1 \sum_{k,p} \langle (\mathrm{id} \otimes v^{-1} u)\, b(a_2 w_j \otimes e_k), w^p \otimes e^k \rangle\, w_p \\
&= a_1 \varepsilon_{m-1}(b)\, a_2 w_j.
\end{aligned}$$

(c) Let $x \in \mathfrak{U}$ and let $\bar{x} \in \mathrm{End}(W)$ be the endomorphism of W determined by the action of x on W. Then, since $b \in \mathrm{End}_{\mathfrak{U}}(W \otimes V)$, $\bar{x}\check{e}_m b \check{e}_m = \check{e}_m \bar{x} b \check{e}_m = \check{e}_m b \bar{x} \check{e}_m = \check{e}_m b \check{e}_m \bar{x}$. This implies that $\bar{x}\varepsilon_{m-1}(b)\check{e}_m = \varepsilon_{m-1}(b)\,\bar{x}\check{e}_m$. Since the map $\mathrm{End}(W) \to \mathrm{End}(W \otimes V \otimes V^*)$ given by $a \mapsto a\check{e}_m$ is injective, it follows that $\bar{x}\varepsilon_{m-1}(b) = \varepsilon_{m-1}(b)\bar{x}$. ▮

Markov Traces and Framing Anomalies

Assume that V is an irreducible $\mathfrak{U}$-module and let $\mathcal{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$. Define traces $\mathrm{mt}_m\colon \mathcal{Z}_m \to k$ by

$$\mathrm{mt}_m(b) = \frac{\mathrm{tr}_q(b)}{\dim_q(V)^m}. \tag{3.8}$$

The traces mt_m are called *Markov traces.*

Let $\check{R}$ be the element of $\mathcal{Z}_2$ given in (2.16). Since V is irreducible it follows from Schur's lemma that $\mathcal{Z}_1 \cong k$. Thus, $\varepsilon_1\colon \mathcal{Z}_2 \to k$ and

$$\varepsilon_1(\check{R}) = \frac{\alpha}{\dim_q(V)}, \tag{3.9}$$

for some constant $\alpha \in k$. The constant α is called the *framing anomaly* of $\check{R}$.

(3.10) THEOREM.

(a) *If* $a \in \mathcal{Z}_{m-1}$ *then* $\mathrm{mt}_{m-1}(a) = \mathrm{mt}_m(a)$. *In particular* $\mathrm{mt}_m(1) = 1$ *for all m.*

(b) *For each* $b \in \mathcal{Z}_m$, $\mathrm{mt}_m(b) = \mathrm{mt}_{m-1}(\varepsilon_{m-1}(b))$.

(c) *For each* $a \in \mathcal{Z}_{m-1}$, $\mathrm{mt}_m(a\check{R}_{m-1}) = \dim_q(V)^{-1}\,\alpha\,\mathrm{mt}_{m-1}(a)$, *where* α *is the framing anomaly of* $\check{R}$.

(d) *The Markov traces* mt_m *have weights given by*

$$\mathrm{wt}_m(\lambda) = \frac{\dim_q(\Lambda_\lambda)}{\dim_q(V)^m}, \qquad \lambda \in \hat{\mathcal{Z}}_m,$$

where Λ_λ *denotes the irreducible* $\mathfrak{U}$*-module corresponding to* λ.

Proof. (a) By the definition of the Markov trace and the fact that $v^{-1}u$ is a grouplike element of $\mathfrak{U}$,

$$\mathrm{mt}_m(a) = \frac{\mathrm{Tr}_q((v^{-1}u \otimes v^{-1}u)(a \otimes \mathrm{id}))}{\dim_q(V)^m},$$

by the definition of the quantum trace on $V^{\otimes m}$. Since traces on tensor products of modules are the products of the individual traces we may write

$$\mathrm{mt}_m(a) = \frac{\mathrm{Tr}(v^{-1}ua)\,\mathrm{Tr}(v^{-1}u\,\mathrm{id})}{\dim_q(V)^m},$$

where the first Tr in the numerator is on $V^{\otimes(m-1)}$ and the second is on V. Then, by the definition of quantum dimension, we get

$$\mathrm{mt}_m(a) = \frac{\mathrm{Tr}(v^{-1}ua)\dim_q(V)}{\dim_q(V)^m} = \mathrm{mt}_{m-1}(a).$$

In particular, $\mathrm{mt}_m(1) = \mathrm{tr}_q(\mathrm{id}^{\otimes m})/\dim_q(V)^m = 1$.

(b) Let $W = V^{\otimes(m-1)}$. Let $\{w_s\}$ be a basis of W and let $\{w^s\}$ be a dual basis in W^*. Let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be a dual basis of V^*. Let $b \in \mathscr{Z}_m$. Since the element $v^{-1}u$ is a grouplike element of $\mathfrak{U}$ we have

$$\begin{aligned}
\dim_q(V)\,\mathrm{tr}_q(\varepsilon_{m-1}(b)) &= \dim_q(V) \sum_s \langle v^{-1}u\varepsilon_{m-1}(b)\,w_s, w^s\rangle \\
&= \sum_{s,k} \langle (v^{-1}u \otimes \mathrm{id})(\mathrm{id} \otimes v^{-1}u)\,b(w_s \otimes e_k), w^s \otimes e^k\rangle \\
&= \sum_{s,k} \langle (v^{-1}u \otimes v^{-1}u)\,b(w_s \otimes e_k), w^s \otimes e^k\rangle \\
&= \sum_{s,k} \langle v^{-1}ub(w_s \otimes e_k), w^s \otimes e^k\rangle \\
&= \mathrm{tr}_q(b),
\end{aligned}$$

where the quantum trace on the left hand side of equation is the quantum trace on $V^{\otimes(m-1)}$ and the quantum trace on the right side of the equation is the quantum trace on $V^{\otimes m}$. The statement follows by converting to Markov traces.

(c) Let $\check{e}_m$ be the element of $\mathrm{End}_{\mathfrak{U}}(V^{\otimes m} \otimes V^*)$ given by $\check{e}_m = \mathrm{id}^{\otimes(m-1)} \otimes \check{e}$, where $\check{e}$ is as in (3.2). Then, since $a \in \mathrm{End}_{\mathfrak{U}}(V^{\otimes(m-1)})$, a commutes with $\check{e}_m$ and

$$\begin{aligned}
\check{e}_m a\check{R}_{m-1}\check{e}_m &= a\check{e}_m\check{R}_{m-1}\check{e}_m \\
&= a(\mathrm{id}^{\otimes(m-1)} \otimes (\check{e}\check{R}\check{e})) \\
&= a(\mathrm{id}^{\otimes(m-1)} \otimes \varepsilon_1(\check{R})\check{e}) \\
&= \dim_q(V)^{-1}\,\alpha a\check{e}_m.
\end{aligned}$$

It follows that $\varepsilon_{m-1}(a\check{R}_{m-1}) = \alpha a$ and thus that

$$\mathrm{mt}_m(a\check{R}_{m-1}) = \mathrm{mt}_{m-1}(\varepsilon_{m-1}(a\check{R}_{m-1})) = \dim_q(V)^{-1}\,\alpha\,\mathrm{mt}_{m-1}(a).$$

(d) This follows immediately from Lemma (3.1) and the definition of the Markov traces. ∎

(3.11) PROPOSITION. (1) *Let $\mathfrak{U} = (\mathfrak{U}, \mathcal{R}, v)$ be a ribbon Hopf algebra and let $V = \Lambda_\lambda$ be a irreducible $\mathfrak{U}$-module. Since v is a central element of $\mathfrak{U}$, the element v acts by a constant $v(\lambda)$ on $V = \Lambda_\lambda$. Then the framing anomaly α of $\check{R}$ is given by $\alpha = v(\lambda)^{-1}$.*

(2) *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U} = \mathfrak{U}_h(\mathfrak{g})$ be the corresponding Drinfel'd–Jimbo quantum group. Suppose that $V = \Lambda_\lambda$ is an irreducible representation of highest weight λ. Then the framing anomaly α of $\check{R}$ is given by $\alpha = q^{\langle \lambda, \lambda + 2\rho \rangle}$.*

Proof. (1) By Proposition (3.7)(a) it is enough to show that $\check{e}_2 \check{R} \check{e}_2 = (\dim_q(V))^{-1}\, v(\lambda)^{-1}\, \check{e}_2$ as elements of $\mathrm{End}_{\mathfrak{U}}(V \otimes V \otimes V^*)$. Let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be a dual basis in V^*. It follows from the identities (2.5), (2.6) and (2.7) that if $\mathcal{R} = \sum_i a_i \otimes b_i$ and $(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = \sum_j c_j \otimes d_j$, then

$$\sum_i b_i S^2(a_i) = \sum_j d_j S(c_j) = \sum_j S^{-1}(d_j) c_j = u^{-1}.$$

Let $x, y \in V$ and let $\phi \in V^*$. Then,

$$\begin{aligned}
\check{e}_2 \check{R} \check{e}_2(x \otimes y \otimes \phi) &= (\dim_q(V))^{-1}\, \langle \phi, v^{-1} u y \rangle\, \check{e}_2 \check{R} \sum_k x \otimes e_k \otimes e^k \\
&= (\dim_q(V))^{-1}\, \langle \phi, v^{-1} u y \rangle\, \check{e}_2 \sum_{k,i} b_i e_k \otimes a_i x \otimes e^k \\
&= (\dim_q(V))^{-2}\, \langle \phi, v^{-1} u y \rangle \sum_{k,i,l} \langle e^k, v^{-1} u a_i x \rangle\, b_i e_k \\
&\quad \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-2}\, \langle \phi, v^{-1} u y \rangle \sum_{i,l} (b_i v^{-1} u a_i x) \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-2}\, \langle \phi, v^{-1} u y \rangle \sum_{i,l} b_i S^2(a_i)\, v^{-1} u x \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-2}\, \langle \phi, v^{-1} u y \rangle \sum_{l} u^{-1} v^{-1} u x \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-1}\, \check{e}_2(v^{-1} x \otimes y \otimes \phi) \\
&= (\dim_q(V))^{-1}\, v(\lambda)^{-1}\, \check{e}_2(x \otimes y \otimes \phi).
\end{aligned}$$

(2) follows immediately, since, by Proposition (2.14), the element $v=e^{-h\rho}u$ acts on an irreducible module A_λ of highest weight λ by the constant $q^{-\langle\lambda, \lambda+2\rho\rangle}$. ∎

A Path Algebra Formula for $\check{e}_m$

Assume that V is an irreducible $\mathfrak{U}$ module and that the branching rule for tensoring by V is multiplicity free. Let $\mathscr{Z}_m=\mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ and $\mathscr{C}_{m+1}=\mathrm{End}_{\mathfrak{U}}(V^{\otimes m}\otimes V^*)$ as in (3.4). Identify the centralizer algebras $\mathscr{Z}_k$, $1\leqslant k\leqslant m$, with path algebras as in Section 1. It can be shown that if the branching rule for tensoring by V is multiplicity free, then the branching rule for tensoring by V^* is also multiplicity free. It follows that the sequence of centralizer algebras $\mathscr{Z}_0\subseteq\cdots\subseteq\mathscr{Z}_{m-1}\subseteq\mathscr{Z}_m\subseteq\mathscr{C}_{m+1}$ can be identified with a sequence of path algebras corresponding to a multiplicity free Bratteli diagram. Let us review the notation.

(1) $\hat{\mathfrak{U}}$ is a set indexing the irreducible representations of $\mathfrak{U}$.

(2) $\hat{\mathscr{Z}}_k$ is a set indexing the irreducible representations of the algebra $\mathscr{Z}_k$.

(3) By the double centralizer theory $\hat{\mathscr{Z}}_k$ is naturally identified with the subset of $\hat{\mathfrak{U}}$ containing the indexes of representations that appear in the decomposition of $V^{\otimes k}$ into irreducible $\mathfrak{U}$-modules.

(4) Let $\hat{\mathscr{C}}_{m+1}$ be an index set for the irreducible representations of $\mathscr{C}_{m+1}$ which is naturally identified with the subset of $\hat{\mathfrak{U}}$ containing indexes of representations that appear in the decomposition of $V^{\otimes m}\otimes V^*$ into irreducible $\mathfrak{U}$-modules.

The notation for paths and tableaux will be as in Section 1. Let mt_m denote the Markov trace on $\mathscr{Z}_m$ and let wt_m denote the weights of the Markov trace.

(3.12) THEOREM. (a) *Viewing* $\hat{\mathscr{Z}}_{m-1}$ *and* $\hat{\mathscr{C}}_{m+1}$ *as sets with elements in* $\hat{\mathfrak{U}}$, *we have* $\hat{\mathscr{Z}}_{m-1}\subseteq\hat{\mathscr{C}}_{m+1}$.

(b) *One can identify the centralizer algebra* $\mathscr{C}_{m+1}$ *with a path algebra in such way that* $\check{e}_m$ *is given by the formula*

$$\check{e}_m=\sum_{(S,\,T)\in\Omega_{m-1}^{m+1}}(\check{e}_m)_{ST}\,E_{ST},$$

where, if $S=(\sigma^{(m-1)},\sigma^{(m)},\sigma^{(m+1)})$, *and* $T=(\sigma^{(m-1)},\tau^{(m)},\sigma^{(m+1)})$, *then*

$$(\check{e}_m)_{ST}=\begin{cases}\dfrac{\sqrt{\mathrm{wt}_m(\tau^{(m)})\,\mathrm{wt}_m(\sigma^{(m)})}}{\mathrm{wt}_{m-1}(\sigma^{(m-1)})}, & \text{if}\quad \sigma^{(m+1)}=\sigma^{(m-1)} \text{ as elements of } \hat{\mathfrak{U}},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. *Step* 1. Let $M'=(\mu^{(0)}, ..., \mu^{(m)}) \in \mathscr{T}^m$ and let $M''=(\mu^{(0)}, ..., \mu^{(m-1)})$. Then

$$\varepsilon_{m-1}(E_{M'M'}) = \frac{\mathrm{wt}_m(\mu^{(m)})}{\mathrm{wt}_{m-1}(\mu^{(m-1)})} E_{M''M''} \neq 0.$$

Proof. Suppose that

$$\varepsilon_{m-1}(E_{M'M'}) = \sum_{(U'', R'') \in \Omega^{m-1}} a_{U''R''} E_{U''R''} \in \mathscr{Z}_{m-1},$$

for some constants $a_{U''R''} \in k$. Suppose that $(S'', T'') \in \Omega^{m-1}$ and that $S''=(\sigma^{(0)}, ..., \sigma^{(m-1)})$. Then

$$\begin{aligned} \mathrm{mt}_m(E_{S''T''} E_{M'M'}) &= \mathrm{mt}_m(E_{S''S''} E_{S''T''} E_{M'M'}) \\ &= \mathrm{mt}_m(E_{S''T''} E_{M'M'} E_{S''S''}) \\ &= \delta_{S''M''} \delta_{T''M''} wt_m(\mu^{(m)}). \end{aligned}$$

On the other hand, by Proposition (3.7b)

$$\begin{aligned} \mathrm{mt}_{m-1}(\varepsilon_{m-1}(E_{S''T''} E_{M'M'})) &= \mathrm{mt}_{m-1}(E_{S''T''} \varepsilon_{m-1}(E_{M'M'})) \\ &= \mathrm{mt}_{m-1}(E_{S''T''} \varepsilon_{m-1}(E_{M'M'})\, E_{S''S''}) \\ &= \mathrm{mt}_{m-1}(E_{S''T''} a_{S''T''} E_{T''S''} E_{S''S''}) \\ &= a_{S''T''} \mathrm{wt}_{m-1}(\sigma^{(m-1)}). \end{aligned}$$

By Proposition (3.7a), these two expressions are equal. Since the weights of the Markov trace are nonzero, it follows that

$$a_{S''T''} = \begin{cases} \dfrac{\mathrm{wt}_m(\mu^{(m)})}{\mathrm{wt}_{m-1}(\mu^{(m-1)})}, & \text{if} \quad S''=T''=M'', \\ 0, & \text{otherwise,} \end{cases}$$

and, if $S''=T''=M''$ then $a_{S''T''} \neq 0$. The formula for $\varepsilon_{m-1}(E_{M'M'})$ follows. ∎

Step 2. It follows from Proposition (1.4) that $\check{e}_m$ has the form

$$\check{e}_m = \sum_{(S, T) \in \Omega_{m-1}^{m+1}} (\check{e}_m)_{ST} E_{ST}, \tag{3.13}$$

since, by its definition, $\check{e}_m$ commutes with all elements of $\mathscr{Z}_{m-1}$.

Step 3. Let $(S, T) \in \Omega^{m+1}$. Suppose $S = (\sigma^{(0)}, ..., \sigma^{(m+1)})$ and define $S' = (\sigma^{(0)}, ..., \sigma^{(m)})$ and $S'' = (\sigma^{(0)}, ..., \sigma^{(m-1)})$. Define T' and T'' analogously. Let $M' = (\mu^{(0)}, ..., \mu^{(m)}) \in \mathscr{T}^m$ and let $M'' = (\mu^{(0)}, ..., \mu^{(m-1)})$. Then, if $(\check{e}_m)_{SS} \neq 0$, then

$$(\check{e}_m)_{SS} = \frac{\mathrm{wt}_m(\sigma^{(m)})}{\mathrm{wt}_{m-1}(\sigma^{(m-1)})}, \tag{3.14}$$

and

$$(\check{e}_m)_{SM}\,(\check{e}_m)_{MS} = \frac{\mathrm{wt}_m(\mu^{(m)})\,\mathrm{wt}_m(\sigma^{(m)})}{\mathrm{wt}_{m-1}(\sigma^{(m-1)})^2}. \tag{3.15}$$

Proof. It follows from the path algebra definitions and (3.13) that

$$E_{SS}\check{e}_m E_{M'M'}\check{e}_m E_{TT} = \delta_{S''M''}\delta_{M''T''}\sum_M (\check{e}_m)_{SM}\,(\check{e}_m)_{MT}\,E_{ST},$$

where the sum is over all tableaux M such that $M = (\mu^{(0)}, ..., \mu^{(m)}, \sigma^{(m+1)})$. Since the Bratteli diagram is multiplicity free there is at most one such M. Thus

$$E_{SS}\check{e}_m E_{M'M'}\check{e}_m E_{TT} = \delta_{S''M''}\delta_{M''T''}(\check{e}_m)_{SM}\,(\check{e}_m)_{MT}\,E_{ST}. \tag{3.16}$$

Let S and M'' be as above. Then

$$\begin{aligned} E_{SS}\varepsilon_{m-1}(E_{M'M'})\,\check{e}_m E_{TT} &= \frac{\mathrm{wt}_m(\mu^{(m)})}{\mathrm{wt}_{m-1}(\mu^{(m-1)})}\,E_{SS}E_{M''M''}\check{e}_m E_{TT} \\ &= \frac{\mathrm{wt}_m(\mu^{(m)})}{\mathrm{wt}_{m-1}(\mu^{(m-1)})}\,\delta_{S''M''}\delta_{M''T''}(\check{e}_m)_{ST}\,E_{ST}. \end{aligned} \tag{3.17}$$

Since $\check{e}_m E_{M'M'}\check{e}_m = \varepsilon_{m-1}(E_{M'M'})\check{e}_m$, it follows that (3.16) and (3.17) are equal. Assuming that $S = T$ and that $S'' = M''$, i.e. $\sigma^{(i)} = \mu^{(i)}$, for all $i \leqslant m-1$, this gives the following equation.

$$(\check{e}_m)_{SM}\,(\check{e}_m)_{MS} = \frac{\mathrm{wt}_m(\mu^{(m)})}{\mathrm{wt}_{m-1}(\mu^{(m-1)})}\,(\check{e}_m)_{SS}. \tag{3.18}$$

The formula in (3.14) follows by setting $M = S$. The formula in (3.15) now follows from (3.14) and (3.18) (recall that $\mu^{(m-1)} = \sigma^{(m-1)}$). ∎

Step 4. For each $\lambda \in \hat{\mathscr{Z}}_{m-1}$ there exist S such that $(S, S) \in \Omega_\lambda^{m+1}$ and $(\check{e}_m)_{SS} \neq 0$.

Proof. Fix $\lambda \in \hat{\mathscr{Z}}_{m-1}$ and let M be such that $\mu^{(m-1)} = \lambda$. Assume that $(\check{e}_m)_{SS} = 0$ for all S such that $(S, S) \in \Omega_\lambda^{m+1}$. Then $(\check{e}_m)_{SM}\,(\check{e}_m)_{MS} = 0$ for all S. So, by (3.16), $E_{SS}\check{e}_m E_{M'M'}\check{e}_m E_{SS}=0$ for all S. This implies that $\varepsilon_{m-1}(E_{M'M'})=0$ which is a contradiction to Step 1. ∎

Step 5. If S is such that $(\check{e}_m)_{SS}\neq 0$ then $\sigma^{(m-1)}=\sigma^{(m+1)}$.

Proof. Let $S\in\mathscr{T}^{m+1}$ be a tableau such that $(\check{e}_m)_{SS}\neq 0$. Then, as $\mathfrak{U}$-modules, $\Lambda_{\sigma^{(m+1)}}\cong E_{SS}b(V^{\otimes m}\otimes V^*)$ for all $b\in\mathscr{C}_{m+1}$ such that $E_{SS}b\neq 0$. In particular, since $(\check{e}_m)_{SS}\neq 0$,

$$E_{SS}\check{e}_m E_{S'S'}\check{e}_m=c_{S'}E_{SS}E_{S''S''}\check{e}_m\neq 0$$

and we have that

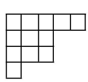
$$\begin{aligned}\Lambda_{\sigma^{(m+1)}}&\cong E_{SS}\check{e}m E_{S'S'}\check{e}_m(V^{\otimes m}\otimes V^*)\\ &=c_{S'}E_{SS}E_{S''S''}\check{e}_m(V^{\otimes m}\otimes V^*)\\ &\cong E_{SS}(E_{S''S''}V^{\otimes(m-1)}\otimes\check{e}_m(V\otimes V^*)).\end{aligned}$$

Since $E_{S''S''}V^{\otimes(m-1)}\cong\Lambda_{\sigma^{(m-1)}}$ and $e_m(V\otimes V^*)\cong\Lambda_{\varnothing}$ it follows that $\Lambda_{\sigma^{(m+1)}}$ is isomorphic to an irreducible component in the tensor product $\Lambda_{\sigma^{(m-1)}}\otimes\Lambda_{\varnothing}$. Thus $\Lambda_{\sigma^{(m+1)}}\cong\Lambda_{\sigma^{(m-1)}}$, and so $\sigma^{(m+1)}=\sigma^{(m-1)}$ as elements of $\hat{\mathfrak{U}}$. ∎

Let us complete the proof of the theorem. Part (a) follows from step 5. Recall from Section 1 that there is some freedom in the choice of the matrix units E_{SM} and E_{MS} when $M\neq S$. This freedom allows us to normalize the matrix units E_{SM} and E_{MS} in any way such that (3.15) holds. In particular, we can choose that normalization so that the formula is as in the theorem. The fact that $(\check{e}_m)_{SM}=0$ if $\sigma^{(m-1)}\neq\sigma^{(m+1)}$ follows from steps 3, 4, and 5. ∎

4. CENTRALIZER ALGEBRAS OF TENSOR POWERS OF V^{ω_1}, TYPE A_r

We shall use the notations for partitions given in [Mac]. In particular, a partition λ of the positive integer m, denoted $\lambda\vdash m$, is a decreasing sequence $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_t\geqslant 0)$ of non-negative integers such that $\lambda_1+\cdots+\lambda_t=m$. The length $l(\lambda)$ is the largest j such that $\lambda_j>0$. The Ferrers diagram of λ is the left-justified array of boxes with λ_i boxes in the ith row. For example,

$(5, 3, 3, 1) =$ 

is a partition of length 4. Given two partitions λ, μ we write $\lambda \subseteq \mu$ if $\lambda_i \leqslant \mu_i$ for all i. We have that $\lambda \subseteq \mu$ if the Ferrers diagram of λ is a subset of the Ferrers diagram of μ.

The Bratteli diagram given in Fig. 1 is called the Young lattice. The shapes $\lambda \in \hat{Y}_m$ of Y which are on level m are the partitions of m;

$$\hat{Y}_m = \{\lambda \vdash m\}.$$

A partition $\lambda \in \hat{Y}_m$ is connected by an edge to a partition $\mu \in \hat{Y}_{m+1}$ if μ can be obtained by adding a box to λ. The Young lattice Y is a multiplicity free Bratteli diagram.

Classically, a *standard tableau* of shape $\lambda \vdash m$ is a filling of the boxes in the Ferrers diagram of λ with the numbers $1, 2, ..., m$ such that the numbers are increasing left to right in the rows and increasing down the columns. Each tableau $T \in \mathscr{T}^\lambda$ in the Bratteli diagram Y can be identified in a natural way with a standard tableau of shape λ. Let P be a standard tableau of shape λ and let $T = (\tau^{(0)}, \tau^{(1)}, ..., \tau^{(m)}) \in \mathscr{T}^\lambda$ be the tableau such that $\tau^{(i)}$ is the partition given by the set of boxes of P which contain the numbers $1, 2, ..., i$. One easily shows that this identification is a bijection between the standard tableaux P of shape λ and the tableaux $T \in \mathscr{T}^\lambda$.

The r-truncated Young lattice is the Bratteli diagram $Y(r)$ which is given by the sets

$$\hat{Y}_m(r) = \{\lambda \vdash m \mid l(\lambda) \leqslant r\}.$$

A partition $\lambda \in \hat{Y}_m(r)$ is connected by an edge to a partition $\mu \in \hat{Y}_{m+1}(r)$ if $\lambda \subseteq \mu$, or equivalently, if μ can be obtained by adding a box to λ. The r-truncated Young lattice can be obtained by removing all the partitions with more than r rows (and the edges connected to them) from the full Young lattice Y. It is easy to see that tableaux in the r-truncated Young lattice correspond to standard tableaux of shapes $\lambda \in \hat{Y}(r)$, in exactly the same way as tableaux in Y correspond to standard tableaux. Note also that the full Young lattice can be viewed as the limit of the r-truncated Young lattices as r goes to infinity.

For the remainder of this section let us fix r, and, unless otherwise specified, all paths and tableaux shall be from the Bratteli diagram $Y(r)$.

Fix $S = (\sigma^{(m-2)}, \sigma^{(m-1)}, \sigma^{(m)}) \in \mathscr{T}^m_{m-2}$. Suppose that $\sigma^{(m-1)}$ is obtained by adding a box to the kth row of $\sigma^{(m-2)}$ and that $\sigma^{(m)}$ is obtained by adding a box to the lth row of $\sigma^{(m-1)}$. Now suppose that $T = (\sigma^{(m-2)}, \tau^{(m-1)}, \sigma^{(m)})$ is such that $(S, T) \in \Omega^m_{m-2}$. If $k = l$ then we must have that

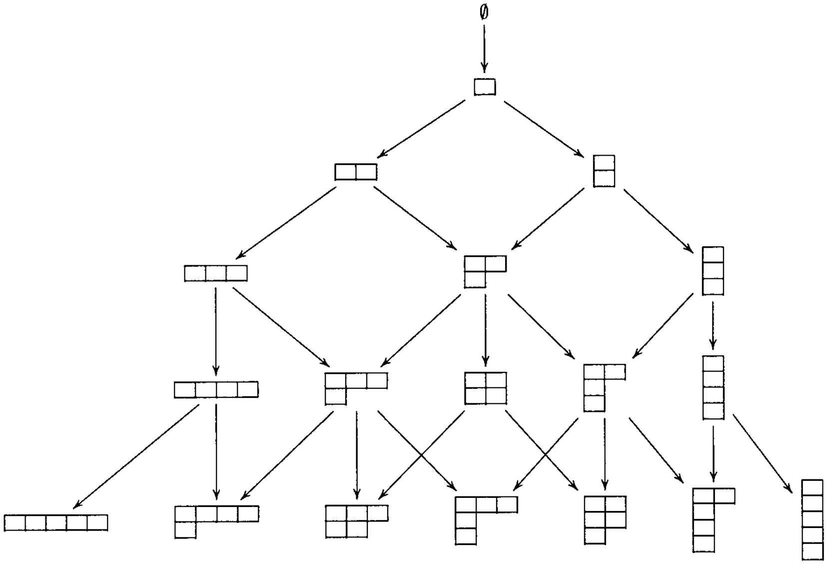


FIGURE 1

$\tau^{(m-1)} = \sigma^{(m-1)}$. If $k \neq l$ then either $\tau^{(m-1)} = \sigma^{(m-1)}$, or $\tau^{(m-1)}$ is the shape obtained by adding a box to the lth row of $\sigma^{(m-2)}$. Thus,

$$\text{there is at most one } T \neq S \text{ such that } (S, T) \in \Omega^m_{m-2}. \tag{4.1}$$

The Centralizer Algebras $\mathscr{Z}_m$

For the remainder of this section fix $\mathfrak{U} = \mathfrak{U}_h(\mathfrak{sl}(r+1))$. Let $\varepsilon_1, ..., \varepsilon_{r+1}$ be an orthonormal basis of $\mathbb{R}^{r+1}$. Then $\mathfrak{h}^*$, the simple roots, α_i, the fundamental weights, ω_i, and the element 2ρ are given by

$$\mathfrak{h}^* = \left\{ = \lambda_1 \varepsilon_1 + \cdots + \lambda_{r+1} \varepsilon_{r+1}, \;\middle|\; \sum_{i=1}^{r+1} \lambda_i = 0 \right\},$$

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \qquad 1 \leqslant i \leqslant r,$$

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{r+1} \sum_{i=1}^{r+1} \varepsilon_j, \qquad 1 \leqslant i \leqslant r,$$

$$2\rho = \sum_i 2\rho_i \varepsilon_i = r\varepsilon_1 + (r-2)\,\varepsilon_2 + \cdots - (r-2)\,\varepsilon_r - r\varepsilon_{r+1}.$$

The finite dimensional irreducible modules Λ_λ of $\mathfrak{U}_h(\mathfrak{sl}(r+1))$ are indexed by the dominant integral weights,

$$\hat{\mathfrak{U}} = \left\{ \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r - \frac{|\lambda|}{r+1} \sum_{j=1}^{r+1} \varepsilon_j, \;\middle|\; \lambda_i \in \mathbb{Z}, \lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0 \right\},$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_r$. It is sometimes helpful to identify each dominant integral weight λ with the partition $\lambda = (\lambda_1, ..., \lambda_r)$. Note that all partitions in $\hat{\mathfrak{U}}$ have at most r rows. It will also be helpful to note that, if $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r - (|\lambda|/(r+1)) \sum_j \varepsilon_j \in \hat{\mathfrak{U}}$ then

$$\langle \lambda, \lambda + 2\rho \rangle = \sum_{i=1}^{r} \lambda_i^2 - \frac{|\lambda|^2}{r+1} + \sum_{i=1}^{r} 2\rho_i \lambda_i. \tag{4.2}$$

Let $V = \Lambda_{\omega_1}$ the irreducible $\mathfrak{U}$-module of highest weight ω_1. The decomposition rule for tensoring by V is given by

$$\Lambda_\lambda \otimes V \cong \bigoplus_{\mu \in \lambda^+} \Lambda_\mu, \tag{4.3}$$

where the sum is over all partitions $\mu \in \hat{\mathfrak{U}}$ that are gotten by adding a box to the partition λ. It follows that the Bratteli diagram for tensor powers of $V = V^{\omega_1}$ is the r-truncated Young lattice $Y(r)$.

(4.4) Proposition. *Let $V = V^{\omega_1}$ be the irreducible $\mathfrak{U} = \mathfrak{U}_h(\mathfrak{sl}(r+1))$-module indexed by the fundamental weight ω_1. The matrices $\check{R}_i \in \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ satisfy the relations*

$$\begin{aligned} \check{R}_i \check{R}_j &= \check{R}_j \check{R}_i, && |i-j| > 1, \\ \check{R}_i \check{R}_{i+1} \check{R}_i &= \check{R}_{i+1} \check{R}_i \check{R}_{i+1}, && 1 \leqslant i \leqslant m-2, \\ (q^{(1/r+1)} \check{R}_i - q)(q^{(1/r+1)} \check{R}_i &+ q^{-1}) = 0, && 1 \leqslant i \leqslant m-1. \end{aligned}$$

Proof. The first two relations follow from Proposition (2.18). From (4.3), we have

$$V \otimes V \cong \Lambda_{(1^2)} \oplus \Lambda_{(2)} = \Lambda_{2\omega_1} \oplus \Lambda_{\omega_2}.$$

Use (4.2) to show that

$$\begin{aligned} \langle \omega_2, \omega_2 + 2\rho \rangle &= 2 - (4/(r+1)) + 2\rho_1 + 2\rho_2 = 2r - 4/(r+1), \\ \langle 2\omega_1, 2\omega_1 + 2\rho \rangle &= 4 - (4/(r+1)) + 4\rho_1 = 4 + 2r - 4/(r+1), \\ \langle \omega_1, \omega_1 + 2\rho \rangle &= 1 - (1/(r+1)) + 2\rho_1 = r + 1 - 1/(r+1). \end{aligned}$$

It follows that

$$\begin{aligned} q^{(1/2)\langle \omega_2, \omega_2 + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle} &= q^{-1-(1/(r+1))}, \\ q^{(1/2)\langle 2\omega_1, 2\omega_1 + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle} &= q^{1-(1/(r+1))}. \end{aligned}$$

The result now follows from Proposition (2.22) part (3) and the standard fact that $\Lambda_{(1^2)}$ is antisymmetric part of $V^{\otimes 2}$ and $\Lambda_{(2)}$ is the symmetric part of $V^{\otimes 2}$. ▮

A Path Algebra Formula for $\check{R}_i$

Let $Y = Y(r)$ be the Bratteli diagram for tensor powers of $V = V^{\omega_1}$. Identify the centralizer algebras $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ with the path algebras corresponding to the Bratteli diagram $Y(r)$. Recall that the path algebras have a natural basis E_{ST}, $(S, T) \in \Omega^m$ of matrix units.

For each tableau $S = (\sigma^{(0)}, ..., \sigma^{(m)}) \in \mathscr{T}^m$ define

$$\nabla_i(S) = \langle \sigma^{(i)}, \sigma^{(i)} + 2\rho \rangle - \langle \sigma^{(i-1)}, \sigma^{(i-1)} + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle. \tag{4.5}$$

Let (S, T) be a pair of tableaux

$$S = (\sigma^{(0)}, ..., \sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)}, ..., \sigma^{(m)}),$$

and

$$T = (\sigma^{(0)}, ..., \sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)}, ..., \sigma^{(m)}),$$

in $\mathscr{T}^m$ such that S and T are the same except possibly at the shape at level $i-1$. In other words the pair $(S, T) \in \Omega^{i+1}_{i-1}$. Define

$$\diamondsuit_i(S, T) = \frac{1}{2}(\nabla_{i+1}(S) - \nabla_i(T)) + \frac{1}{r+1}. \tag{4.6}$$

These constants are defined so that, if $D_m = \check{R}_{m-1}\check{R}_{m-2} \cdots \check{R}_2\check{R}_1\check{R}_1\check{R}_2 \cdots \check{R}_{m-2}\check{R}_{m-1}$, then

$$D_m = \sum_{S \in \mathscr{T}^m} (D_m)_{SS} E_{SS}, \qquad \text{where} \quad (D_m)_{SS} = q^{\nabla_m(S)},$$

and

$$q^{-2\diamondsuit_{m-1}(S, T)} = q^{-(2/r+1)}(D_m^{-1})_{SS}(D_{m-1})_{TT}.$$

The first of these formulas is a consequence of Corollary (2.25).

(4.7) Proposition. *Let* $S = (\sigma^{(0)}, ..., \sigma^{(m)})$ *be a tableau in the Bratteli diagram* $Y(r)$. *Then*

$$\nabla_m(S) = 2(\sigma_k^{(m-1)} - k + 1) + \frac{-2m+2}{r+1},$$

$$\diamondsuit_{m-1}(S, S) = \sigma_k^{(m)} - \sigma_l^{(m-1)} - k + l,$$

where $\sigma^{(m)}$ is obtained by adding a box to the kth row of $\sigma^{(m-1)}$ and $\sigma^{(m-1)}$ is obtained by adding a box to the lth row of $\sigma^{(m-2)}$.

Proof. Let $S=(\sigma^{(0)}, ..., \sigma^{(m)}$ be a tableau in the Bratteli diagram $Y(r)$. Then, since $\sigma^{(m)}$ differs from $\sigma^{(m-1)}$ by adding a box in the kth row we have

$$\sigma^{(m-1)}=\sum_{i=1}^{r} \sigma_i^{(m-1)}\varepsilon_i-\frac{m-1}{r+1}\sum_{j=1}^{r+1}\varepsilon_j,$$

and

$$\sigma^{(m)}=(\sigma_k^{(m-1)}+1)\,\varepsilon_k+\sum_{\substack{1\leqslant i\leqslant r\\ i\neq k}}\sigma_i^{(m)}\varepsilon_i-\frac{m}{r+1}\sum_{j=1}^{r+1}\varepsilon_j.$$

Using (4.2) to compute $\nabla_m(S)$ we get

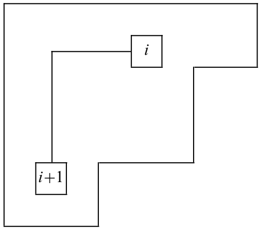
$$\begin{aligned}
\nabla_m(S)&=\langle \sigma^{(m)}, \sigma^{(m)}+2\rho\rangle-\langle\sigma^{(m-1)}, \sigma^{(m-1)}+2\rho\rangle-\langle\varepsilon_1, \varepsilon_1+2\rho\rangle\\
&=\left(\sum_{i\neq k}(\sigma_i^{(m-1)})^2\right)+(\sigma_k^{(m-1)}+1)^2-\frac{m^2}{r+1}\\
&\quad+\left(\sum_{i\neq k}2\sigma_i^{(m-1)}\rho_i\right)+2(\sigma_k^{(m-1)}+1)\,\rho_k\\
&\quad-\left(\sum_{i\neq k}(\sigma_i^{(m-1)})^2\right)-(\sigma_k^{(m-1)})^2+\frac{(m-1)^2}{r+1}\\
&\quad-\left(\sum_{i\neq k}2\sigma_i^{(m-1)}\rho_i\right)-2\sigma_k^{(m-1)}\rho_k-1+\frac{1}{r+1}-2\rho_1\\
&=2\sigma_k^{(m-1)}+2(\rho_k-\rho_1)+\frac{-2m+2}{r+1}.
\end{aligned}$$

The formula for $\nabla_m(S)$ follows since $2(\rho_k-\rho_1)=(r-(2k-1))-(r-1)=-2k-2=2(-k+1)$.

The formula for $\diamond_{m-1}(S, S)$ now follows easily since

$$\begin{aligned}
\diamond_{m-1}(S, S)&=\frac{1}{2}(\nabla_m(S)-\nabla_{m-1}(S))+\frac{1}{r+1}\\
&=(\sigma_k^{(m-1)}-k+1)+\frac{-m+1}{r+1}-(\sigma_l^{(m-2)}-l+1)\\
&\quad-\frac{-(m-1)+1}{r+1}+\frac{1}{r+1}\\
&=\sigma_k^{(m)}-k-\sigma_l^{(m-1)}+l-\frac{1}{r+1}+\frac{1}{r+1}. \quad\blacksquare
\end{aligned}$$

Remark. In terms of standard tableaux, the value of $\diamond_i(S, S)$ is the "axial distance" between the box containing i and the box containing $i+1$.



(4.8) THEOREM. *One can choose the identification* (*Section* 1) *of the centralizer algebras* $\mathscr{Z}_m$ *with the path algebras corresponding to the Bratteli diagram* $Y(r)$ *so that the matrices* $\check{R}_i$ *are given by the formula*

$$\check{R}_i = \sum_{(S,\,T) = \Omega^{i+1}_{i-1}} (\check{R}_i)_{ST}\, E_{ST},$$

where for each $S \in \mathscr{T}^m$ *we have*

$$q^{1/(r+1)}(\check{R}_i)_{SS} = \frac{q^{\diamond_i(S,\,S)}}{[\diamond_i(S, S)]}$$

and for each pair $(S, T) \in \Omega^{i+1}_{i-1}$ *such that* $S \neq T$ *we have*

$$q^{1/(r+1)}(\check{R}_i)_{ST} = \frac{\sqrt{[\diamond_i(S, S) - 1][\diamond_i(S, S) + 1]}}{[|\diamond_i(S, S)|]}$$

Proof. Since $\check{R}_i \in \mathscr{Z}_i = \mathrm{End}_{\mathfrak{U}}(V^{\otimes(i+1)})$ commutes with all elements of $\mathscr{Z}_{i-1} = \mathrm{End}_{\mathfrak{U}}(V^{\otimes(i-1)})$ it follows from Corollary (1.5) that

$$\check{R}_i = \sum_{(S,\,T) \in \Omega^{m}_{m-2}} (\check{R}_i)_{ST}\, E_{ST},$$

for some constants $(\check{R}_i)_{ST}$. In view of the imbeddings $\mathscr{Z}_0 \subseteq \mathscr{Z}_1 \subseteq \cdots \subseteq \mathscr{Z}_m$, it is sufficient to show that the formulas for $\check{R}_i$ hold for $i = m-1$.

By definition $D_m = \check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_2\check{R}_1\check{R}_1\check{R}_2\cdots\check{R}_{m-2}\check{R}_{m-1}$ and it follows that

$$\check{R}^{-1}_{m-1} = D_m^{-1}\check{R}_{m-1}D_{m-1}.$$

The relation $(q^{1/(r+1)}\check{R}_i - q)(q^{1/(r+1)}\check{R}_i + q^{-1}) = 0$ from Proposition (4.4) can be written in the form $q^{1/(r+1)}\check{R}_{m-1} - q^{-1/(r+1)}\check{R}^{-1}_{m-1} = (q - q^{-1})$ or, equivalently, in the form

$$q^{1/(r+1)}\check{R}_{m-1} - q^{-2/(r+1)}D_m^{-1}q^{1/(r+1)}\check{R}_{m-1}D_{m-1} = (q - q^{-1}). \tag{4.9}$$

Let $S \in \mathscr{T}^m$ and view (4.9) as an equation in the path algebra. Since the matrices D_m and D_{m-1} are diagonal, taking the E_{SS}-entry of this equation yields

$$q^{1/(r+1)}(\check{R}_{m-1})_{SS} - q^{-2/(r+1)}(D_m^{-1})_{SS}\, q^{1/(r+1)}(\check{R}_{m-1})_{SS}\,(D_{m-1})_{SS}$$
$$= (q-q^{-1})\,\delta_{SS},$$

or, equivalently,

$$(1-q^{-2/(r+1)}(D_m^{-1})_{SS}\,(D_{m-1})_{SS})\, q^{1/(r+1)}\,(\check{R}_{m-1})_{SS} = (q-q^{-1}), \tag{4.10}$$

Since the right hand side of this equation is nonzero, the left hand side is also nonzero and we may write

$$q^{1/(r+1)}(\check{R}_{m-1})_{SS} = \frac{(q-q^{-1})\,\delta_{SS}}{1-q^{-2/(r+1)}(D_m^{-1})_{SS}\,(D_{m-1})_{SS}}.$$

Plugging in the following

$$\frac{q-q^{-1}}{1-q^{-2/(r+1)}(D_m^{-1})_{SS}\,(D_{m-1})_{SS}} = \frac{q-q^{-1}}{1-q^{-2\diamond_{m-1}(S,\,S)}}$$
$$= \frac{q^{\diamond_{m-1}(S,\,S)}(q-q^{-1})}{q^{\diamond_{m-1}(S,\,S)} - q^{-\diamond_{m-1}(S,\,S)}}$$
$$= \frac{q^{\diamond_{m-1}(S,\,S)}}{[\diamond_{m-1}(S,\,S)]}$$

gives the first formula in Theorem (4.8).

Now let us prove the second formula in Theorem (4.8). Let $S \in \mathscr{T}^m_{m-2}$ and suppose that $T \in \mathscr{T}^m_{m-2}$ is such that $(S, T) \in \Omega^m_{m-2}$ and $T \neq S$. By the remark in (4.1), T is unique. It follows that

$$q^{2/(r+1)}(\check{R}^2_{m-1})_{SS} = q^{2/(r+1)}((\check{R}_{m-1})_{SS})^2 + q^{2/(r+1)}(\check{R}_{m-1})_{ST}\,(\check{R}_{m-1})_{TS}. \tag{4.11}$$

On the other hand, the relation $(q^{1/(r+1)}\check{R}_{m-1} - q)(q^{1/(r+1)}\check{R}_{m-1} + q^{-1}) = 0$ from Proposition (4.4) can be written in the form $q^{2/(r+1)}\check{R}^2_{m-1} - 1 = q^{1/(r+1)}\check{R}_{m-1}(q-q^{-1})$, giving that

$$q^{2/(r+1)}(\check{R}^2_{m-1})_{SS} = (q-q^{-1})\, q^{1/(r+1)}(\check{R}_{m-1})_{SS} + 1 \tag{4.12}$$

Equating (4.11) and (4.12) and using the formula for $q^{1/(r+1)}(\check{R}_{m-1})_{SS}$ gives

$$\begin{aligned}
&(q^{1/(r+1)}\check{R}_{m-1})_{ST}\,(q^{1/(r+1)}\check{R}_{m-1})_{TS}\\
&\quad = 1+(q-q^{-1})(q^{1/(r+1)}\check{R}_{m-1})_{SS}-(q^{1/(r+1)}\check{R}_{m-1})^2_{SS}\\
&\quad = (q-(q^{1/(r+1)}\check{R}_{m-1})_{SS})(q^{-1}+(q^{1/(r+1)}\check{R}_{m-1})_{SS})\\
&\quad = \left(q-\frac{q^{\diamond_i(S,S)}}{[\diamond_i(S,S)]}\right)\left(q^{-1}+\frac{q^{\diamond_i(S,S)}}{[\diamond_i(S,S)]}\right)\\
&\quad = \frac{q(q^{\diamond_i(S,S)}-q^{-\diamond_i(S,S)})-q^{\diamond_i(S,S)}(q-q^{-1})}{q^{\diamond_i(S,S)}-q^{-\diamond_i(S,S)}}\\
&\qquad \cdot\frac{q^{-1}(q^{\diamond_i(S,S)}-q^{-\diamond_i(S,S)})+q^{\diamond_i(S,S)}(q-q^{-1})}{q^{\diamond_i(S,S)}-q^{-\diamond_i(S,S)}}\\
&\quad = \frac{(q^{\diamond_i(S,S)+1}-q^{-\diamond_i(S,S)-1})(q^{-\diamond_i(S,S)+1}-q^{\diamond_i(S,S)-1})}{q^{\diamond_i(S,S)}-q^{-\diamond_i(S,S)}}\\
&\quad = \frac{[\diamond_i(S,S)+1][\diamond_i(S,S)-1]}{[\diamond_i(S,S)]^2}
\end{aligned}$$

It follows from the remarks at the end of Section 1 that we can choose the normalization of the elements E_{ST} so that $(\check{R}_{m-1})_{ST}$ and $(\check{R}_{m-1})_{TS}$ are as given in the theorem. ▮

Remark. If $(S,T)\in\Omega^{i+1}_{i-1}$ such that $S\neq T$ then $\diamond_i(S,S)=-\diamond_i(T,T)$. Thus, the formula for $q^{1/(r+1)}(\check{R}_{m-1})_{ST}$ given in Theorem (4.8) is actually symmetric in S and T.

Matrix Units

Given a tableau $T=(\tau^{(0)},...,\tau^{(m)})\in\mathcal{T}^m$ let T' denote the tableau $T'=(\tau^{(0)},...,\tau^{(m-1)})\in\mathcal{T}^{m-1}$. Let $(T')^+$ denote the set of all extensions of T';

$$(T')^+=\{S\in\mathcal{T}^m \mid S'=T'\}.$$

Given tableaux $S=(\sigma^{(0)},...,\sigma^{(m)})$ and $T=(\tau^{(0)},...,\tau^{(m)})$ in $\mathcal{T}^m$ let $(\check{R}_{m-1})_{ST}$ be the constant given by Theorem (4.8) in the case that $((\sigma^{(m-2)},\sigma^{(m-1)},\sigma^{(m)}),(\tau^{(m-2)},\tau^{(m-1)},\tau^{(m)}))\in\Omega^m_{m-2}$ and let $(\check{R}_{m-1})_{ST}=0$ otherwise.

(4.13) Lemma. *Let $T'=(\tau^{(0)},...,\tau^{(m-1)})\in\mathcal{T}^{m-1}$ and let $(T')^+$ be the set of extensions of T'. Then the values $\nabla_m(S)$ are all different as S ranges over all elements of $(T')^+$.*

Proof. Let $S=(\tau^{(0)}, ..., \tau^{(m-1)}, \sigma^{(m)}) \in (T')^+$. By the previous lemma, $\nabla_m(S)=2(\tau_k^{(m-1)}-k+1-(m-1)/(r+1))$ if $\sigma^{(m)}$ is obtained by adding a box to the kth row of $\tau^{(m-1)}$. Since

$$\tau_1^{(m-1)} \geqslant \cdots \geqslant \tau_k^{(m-1)} \geqslant \cdots \geqslant \tau_r^{(m-1)}$$

it follows that

$$\tau_1^{(m-1)}-\frac{m-1}{r+1} > \cdots > \tau_k^{(m-1)}-k+1-\frac{m-1}{r+1}$$
$$> \cdots > \tau_r^{(m-1)}-r+1-\frac{m-1}{r+1}. \quad \blacksquare$$

(4.14) THEOREM [RW]. *The matrix units $E_{ST} \in \mathscr{Z}_m$, $(S, T) \in \Omega^m$ are given in terms of the $\check{R}_i$, $1 \leqslant i \leqslant m-1$, inductively, by the following formulas.*

(1) *Let $T \in \mathscr{T}^m$. Then $E_{TT}=\prod_{S \in \mathscr{T}^m,\, S \neq T,\, S'=T'} (E_{T'T'}\check{R}_{m-1}E_{T'T'}-(\check{R}_{m-1})_{SS}\, E_{T'T'})/((\check{R}_{m-1})_{TT}-(\check{R}_{m-1})_{SS}$*

(2) *Let $(S, T) \in \Omega^m$. If $shp(S')=shp(T')$ then $E_{ST}=E_{S'T'}E_{TT}$ where E_{TT} is given by* (1).

(3) *Let $(S, T) \in \Omega^m$. If $shp(S') \neq shp(T')$ then*

$$E_{ST}=\frac{1}{(\check{R}_{m-1})_{MN}} E_{S'M'}\check{R}_{m-1}E_{N'T'}E_{TT},$$

where $M, N \in \mathscr{T}^m$ are of the form $M=(\mu^{(0)}, ..., \mu^{(m-2)}, shp(S'), shp(S))$ and $N=(\mu^{(0)}, ..., \mu^{(m-2)}, shp(T'), shp(S))$.

Proof. (1) Let $T' \in \mathscr{T}^{m-1}$. It follows from the formula for $\check{R}_{m-1}$ that

$$E_{T'T'}\check{R}_{m-1}E_{T'T'}=\sum_{S \in (T')^+} (\check{R}_{m-1})_{SS}\, E_{SS}.$$

The identity (1) follows if we show that the values $(\check{R}_{m-1})_{SS}$ are all different as S runs over all tableaux in $(T')^+$. Since $\diamond_{m-1}(S, S)=\frac{1}{2}(\nabla_m(S)-\nabla_{m-1}(S))+1/(r+1)=\frac{1}{2}(\nabla_m(S)-\nabla_{m-1}(T'))+1/(r+1)$ it follows that the values $(\check{R}_{m-1})_{SS}$ are all different as S runs over all tableaux in $(T')^+$ if and only if the values $\nabla_m(S)$ are all different as S runs over all tableaux in $(T')^+$. Statement (1) now follows from Lemma (4.13).

(2) follows from the definition (1.3) of the embedding of path algebras.

(3) We must show two things:

(a) For each possible choice of M and N the formula determines E_{ST}.

(b) There exist tableaux M and N in $\mathcal{T}^m$ of the form $M = (\mu^{(0)}, ..., \mu^{(m-2)}, shp(S'), shp(S))$ and $N = (\mu^{(0)}, ..., \mu^{(m-2)}, shp(T'), shp(S))$.

Suppose that M and N are given. Since $shp(S') \neq shp(T')$, it follows from (4.1) that M and N are the unique extensions of M' and N' respectively, such that $shp(M) = shp(N) = shp(S)$. By Theorem (4.8) we know that the values $(\check{R}_{m-1})_{MN}$ are nonzero. It follows that

$$\frac{1}{(\check{R}_{m-1})_{MN}} E_{S'M'} \check{R}_{m-1} E_{N'T'} = \frac{1}{(\check{R}_{m-1})_{MN}} E_{S'M'} \sum_{(U,V) \in \Omega^m} (\check{R}_{m-1})_{UV} E_{UV} E_{N'T'}$$

$$= \frac{1}{(\check{R}_{m-1})_{MN}} E_{S'M'} (\check{R}_{m-1})_{MN} E_{NM} E_{N'T'} = E_{ST},$$

proving (a). To see that (b) is true we reason as follows. Suppose that $shp(S')$ is a partition that is the same as $shp(S)$ except that there is a box missing from the kth row. Suppose that $shp(T')$ is a partition that is the same as $shp(S)$ except that there is a box missing from the lth row. Since $shp(S') \neq shp(T')$ we know that $k \neq l$. Then there is a unique partition $\mu^{(m-2)}$ that is the same as $shp(S)$ except that there is a box missing from the lth row and a box missing from the kth row. The partition $\mu^{(m-2)}$ is uniquely determined by S and T, and M and N can be determined by fixing some tableau $(\mu^{(0)}, ..., \mu^{(m-2)}) \in \mathcal{T}^{m-2}$ of shape $\mu^{(m-2)}$. ∎

(4.15) Corollary. *The centralizer* $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ *is generated by the matrices* $\check{R}_i$, $1 \leqslant i \leqslant m-1$.

Proof. It follows from the identification of the centralizer algebras $\mathscr{Z}_m$ with the path algebras that the matrix units E_{ST}, $(S, T) \in \Omega^m$ span the centralizer algebras $\mathscr{Z}_m$. In view of Theorem (4.14), the matrix units E_{ST}, $(S, T) \in \Omega^m$ can be written in terms of the $\check{R}_i$ matrices. The statement follows. ∎

5. CENTRALIZER ALGEBRAS OF TENSOR POWERS OF $V = \Lambda_{\omega_1}$, TYPE B_r

The Bratteli diagram is given in Fig. 2. The shapes $\lambda \in \hat{B}_m$ of B which are on level m are the partitions of $m - 2k$, $0 \leqslant k \leqslant \lfloor m/2 \rfloor$;

$$\hat{B}_m = \{\lambda \vdash m - 2k, 0 \leqslant k \leqslant \lfloor m/2 \rfloor\}.$$

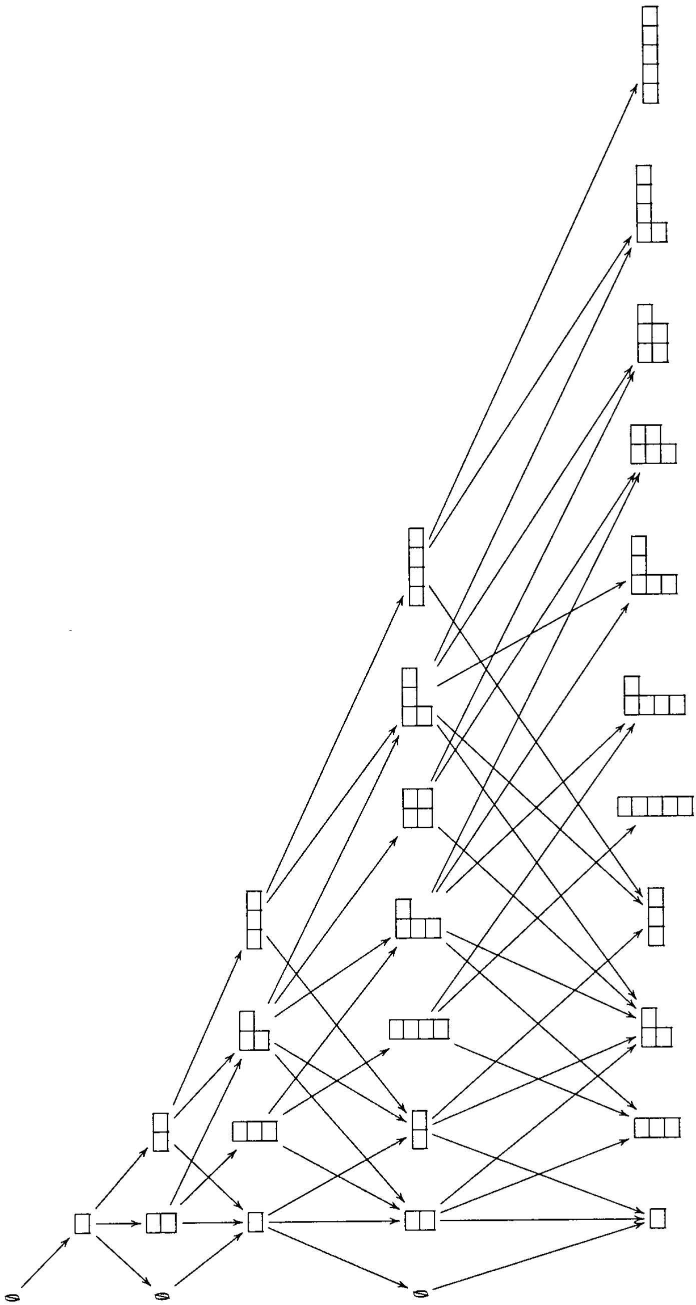


FIGURE 2

A partition $\lambda \in \hat{B}_m$ is connected by an edge to a partition $\mu \in \hat{B}_{m+1}$ if μ can be obtained from λ by adding a box to λ or by removing a box from λ. The diagram B is a multiplicity free Bratteli diagram. The tableaux $T \in \mathscr{T}^{\lambda}$ in the Bratteli diagram B are called *up-down tableaux* since they are sequences of partitions in which each partition differs from the previous one by either adding or removing a box.

The r-truncated Bratteli diagram $B(r)$ is given by the sets

$$\hat{B}_m(r) = \{\lambda \vdash m - 2k,\ 0 \leqslant k \leqslant \lfloor m/2 \rfloor \mid l(\lambda) \leqslant r\}.$$

A partition $\lambda \in \hat{B}_m(r)$ is connected by an edge to a partition $\mu \in \hat{B}_{m+1}(r)$ if μ can be obtained by adding or removing a box from λ. The Bratteli diagram $B(r)$ is a multiplicity free Bratteli diagram. It can be obtained from the Bratteli diagram B by removing all the partitions with more than r rows (and the edges connected to them). It is easy to see that tableaux in the r-truncated Bratteli diagram $B(r)$ are up-down tableaux that never pass through a partition of length greater than r. The Bratteli diagram B can be viewed as the limit of the Bratteli diagrams $B(r)$, as r goes to infinity.

For the remainder of this section let us fix r, and, unless otherwise specified, all paths and tableaux shall be from the Bratteli diagram $B(r)$.

(5.1) Lemma. *Fix $S = (\sigma^{m-2}, \sigma^{(m-1)}, \sigma^{(m)}) \in \mathscr{T}^{m}_{m-2}$ and assume that $\sigma^{(m-2)} \neq \sigma^{(m)}$ as partitions. Then there is at most one $T \neq S$ such that $(S, T) \in \Omega^{m}_{m-2}$.*

Proof. Given a partition λ let us write $\mu = \lambda + \varepsilon_k$ (resp. $\mu = \lambda - \varepsilon_k$) to denote that μ is obtained by adding (resp. removing) a box to (resp. from) the kth row of λ. Fix $S = (\sigma^{m-2}, \sigma^{(m-1)}, \sigma^{(m)}) \in \mathscr{T}^{m}_{m-2}$ and assume that $\sigma^{(m-2)} \neq \sigma^{(m)}$ as partitions. Suppose that $\sigma^{(m-1)} = \sigma^{(m-2)} + \delta_1 \varepsilon_k$ and that $\sigma^{(m)} = \sigma^{(m-1)} + \delta_2 \varepsilon_l$, where δ_1 and δ_2 are either ± 1. If T exists then $T = (\sigma^{m-2}, \tau^{(m-1)}, \sigma^{(m)})$ is given by $\tau^{(m-1)} = \sigma^{(m-2)} + \delta_2 \varepsilon_l$ and $\sigma^{(m)} = \tau^{(m-1)} + \delta_1 \varepsilon_k$. The path T exists when $\tau^{(m-1)} = \sigma^{(m-2)} + \delta_2 \varepsilon_l$ is a partition and not equal to $\sigma^{(m-1)}$. ∎

The Centralizer Algebras $\mathscr{Z}_m$

For the remainder of this section fix $\mathfrak{g}$ to be a complex simple Lie algebra of type B_r, C_r or D_r and let $\mathfrak{U} = \mathfrak{U}_h(\mathfrak{g})$ be the corresponding quantum group. We shall use the standard notations ([Bou], pp. 252–258) for the root systems of Types B_r, C_r, and D_r so that $\varepsilon_1, ..., \varepsilon_r$ are an orthonormal basis of $\mathfrak{h}^*$ and the element 2ρ is given by

$$2\rho = \sum_{k=1}^{r} 2\rho_k \varepsilon_j = \sum_{k=1}^{r} (y - 2k + 1)\varepsilon_k, \tag{5.2}$$

where

$$y = \begin{cases} 2r, & \text{in Type } B_r, \\ 2r+1, & \text{in Type } C_r, \\ 2r-1, & \text{in Type } D_r. \end{cases} \tag{5.3}$$

The finite dimensional irreducible representations of $\mathfrak{U}_h(\mathfrak{g})$ which appear as irreducible summands in the tensor powers of $V = \Lambda_{\omega_1}$ of $\mathfrak{U}_h(\mathfrak{g})$ are indexed by the dominant integral weights in the set

$$\hat{\mathfrak{U}} = \{\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r, \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0\}.$$

We shall identify each dominant integral weight $\lambda \in \hat{\mathfrak{U}}$ with the partition $\lambda = (\lambda_1, ..., \lambda_r)$. It will be helpful to note that, if $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r \in \hat{\mathfrak{U}}$ then

$$\langle \lambda, \lambda + 2\rho \rangle = \sum_{i=1}^{r} \lambda_i^2 + \sum_{i=1}^{r} 2\rho_i \lambda_i. \tag{5.4}$$

Let $V = \Lambda_{\omega_1}$ the irreducible $\mathfrak{U}$-module of highest weight ω_1. In type C_r, the decomposition rule for tensoring by V is given by

$$\Lambda_\lambda \otimes V \cong \bigoplus_{\mu \in \lambda^{\pm}} \Lambda_\mu, \tag{5.5}$$

where the sum is over all partitions $\mu \in \hat{\mathfrak{U}}$ that are gotten by adding or removing a box from the partition λ. It follows that in Type C_r the Bratteli diagram for tensor powers of $V = \Lambda_{\omega_1}$ is given by $B(r)$. In Types B_r and D_r the tensor product rule given in (5.5) holds whenever $|\lambda| < r-1$ but must be modified slightly when $|\lambda| \geqslant r-1$. In order to avoid this complication

(5.6) *For the remainder of this section, we shall assume that in Types B_r and D_r we have that $r \gg 0$; in particular, $m < r$ and $i < r$ whenever the constants m and i are used,*

The Elements $\check{E}_i$

The weights of the Markov traces on $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ are given by

$$\mathrm{wt}_m(\lambda) = \frac{\dim_q \Lambda_\lambda}{(\dim_q(V))^m}, \qquad \lambda \in \hat{\mathfrak{U}}. \tag{5.7}$$

where the quantum dimension of $V = \Lambda_{\omega_1}$ is given by

$$\dim_q(V) = \begin{cases} [2r]+1, & \text{in Type } B_r, \\ [2r+1]-1, & \text{in Type } C_r, \\ [2r-1]+1, & \text{in Type } D_r. \end{cases}$$

Since all automorphisms of the Dynkin diagram corresponding to $\mathfrak{g}$ fix the node corresponding to the fundamental weight ω_1 it follows that $V = \Lambda_{\omega_1} \cong V^* = (\Lambda_{\omega_1})^*$. As in (3.2), define $\check{e} \in \mathrm{End}_{\mathfrak{U}}(V \otimes V)$ to be the $\mathfrak{U}$-invariant projection onto the invariants, $\Lambda_{(0)} \subseteq V \otimes V$. Define

$$\check{E}_i = \delta \dim_q(V)(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \check{e} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \in \mathrm{End}_{\mathfrak{U}}(V^{\otimes m}),$$

where the factor $\check{e}$ appears as a transformation on the ith and the $(i+1)$st tensor factors and

$$\delta = \begin{cases} 1, & \text{in Types } B_r \text{ and } D_r, \\ -1, & \text{in Type } C_r. \end{cases} \tag{5.8}$$

By Theorem (3.12), there is a natural identification of the centralizer algebras $\mathscr{Z}_m$ with the path algebras corresponding to the Bratteli diagram $B(r)$ so that

$$\check{E}_{m-1} = \sum_{(S,T) \in \Omega_{m-1}^m} (\check{E}_{m-1})_{ST} E_{ST}$$

where, if $S = (\sigma^{(m-2)}, \sigma^{(m-1)}, \sigma^{(m)})$ and $T = (\sigma^{(m-2)}, \tau^{(m-1)}, \sigma^{(m)})$, then

$$(\check{E}_{m-1})_{ST} = \begin{cases} \dfrac{\delta\sqrt{\dim_q(\Lambda_{\sigma^{(m-1)}}) \cdot \dim_q(\Lambda_{\tau^{(m-1)}})}}{\dim_q(\Lambda_{\sigma^{(m-2)}})} & \text{if} \quad \sigma^{(m-2)} = \sigma^{(m)}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

where we have replaced the weights of the Markov trace by q-dimensions.

(5.10) PROPOSITION. *Let $V = \Lambda_{\omega_1}$ be the irreducible $\mathfrak{U} = \mathfrak{U}_h(\mathfrak{g})$-module indexed by the fundamental weight ω_1. The matrices $\check{R}_i$ and $\check{E}_i$ in* $\mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ *satisfy the relations*

(a) $\check{R}_i\check{R}_j = \check{R}_j\check{R}_i, \qquad |i-j|>1,$

(b) $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}, \qquad 1 \leqslant i \leqslant m-2,$

(c) $(\check{R}_i - z^{-1})(\check{R}_i - q)(\check{R}_i + q^{-1}) = 0, \qquad 1 \leqslant i \leqslant m-1.$

(d) $\check{E}_i\check{R}_{i-1}^{\pm 1}\check{E}_i = z^{\pm 1}\check{E}_i$ *and* $\check{E}_i\check{R}_{i+1}^{\pm 1}\check{E}_i = z^{\pm 1}\check{E}_i,$

(e) $\check{R}_i - \check{R}_i^{-1} = (q-q^{-1})(1-\check{E}_i),$

(f) $\check{E}_i\check{R}_i^{\pm 1} = \check{R}_i^{\pm 1}\check{E}_i = z^{\mp 1}\check{E}_i,$

where

$$z=\delta q^{y}=\begin{cases} q^{2r}, & \text{in Type } B_r,\\ -q^{2r+1}, & \text{in Type } C_r,\\ q^{2r-1}, & \text{in Type } D_r,\end{cases}$$

Proof. (a) and (b) follow from Proposition (2.18). From (5.5), we have

$$V\otimes V\cong \Lambda_{\varnothing}\oplus\Lambda_{(1^2)}\oplus\Lambda_{(2)}=\Lambda_0\oplus\Lambda_{2\omega_1}\oplus\Lambda_{\omega_2}.$$

Use (5.4) to show that

$$\langle 0, 0+2\rho\rangle=0,\qquad \langle\varepsilon_1+\varepsilon_2,\varepsilon_1+\varepsilon_2+2\rho\rangle=2y-2,$$

$$\langle 2\varepsilon_1, 2\varepsilon_1+2\rho\rangle=2y+2,\qquad\text{and}\qquad\langle\varepsilon_1,\varepsilon_1+2\rho\rangle=y.$$

It follows that

$$q^{(1/2)\langle 0,0+2\rho\rangle-\langle\varepsilon_1,\varepsilon_1+2\rho\rangle}=q^{-y},$$

$$q^{(1/2)\langle\varepsilon_1+\varepsilon_2,\varepsilon_1+\varepsilon_2+2\rho\rangle-\langle\varepsilon_1,\varepsilon_1+2\rho\rangle}=q^{-1},$$

$$\text{and}\qquad q^{(1/2)\langle 2\varepsilon_1, 2\varepsilon_1+2\rho\rangle-\langle\varepsilon_1,\varepsilon_1+2\rho\rangle}=q.$$

Relation (c) now follows from Corollary (2.22); the signs of the eigenvalues of $\check{R}_i$ are determined by which summands are in $\bigwedge^2(V)$,

$$\bigwedge^2(V)=\begin{cases}\Lambda_{(1^2)}, & \text{in Types } B_r \text{ and } D_r,\\ \Lambda_{(1^2)}\oplus\Lambda_0, & \text{in Type } C_r.\end{cases}$$

(d) follows from Proposition (3.11) part (2) and the fact that $q^{\langle\varepsilon_1,\varepsilon_1+2\rho\rangle}=q^{y}$.

Let us prove (e). By Corollary (2.22), $\check{R}_1$ acts by the eigenvalue z^{-1} on the irreducible summand Λ_0 in $V\otimes V$. Thus, it follows from relation (c) that

$$\check{E}_i=\delta\dim_q(V)\,\frac{(\check{R}_i-q)(\check{R}_i+q^{-1})}{(z^{-1}-q)(z^{-1}+q^{-1})}.$$

Using this formula and the relation

$$\delta\dim_q(V)=\frac{z-z^{-1}}{q-q^{-1}}+1,$$

it can be easily checked that relation (c) is equivalent to relation (e).

The relation $\check{R}_1\check{E}_1=z\check{E}_1$, follows by noting that, except for the constant $\delta\dim_q(V)$, $\check{E}_i$ is the projection onto the invariants $\Lambda_0\subseteq V\otimes V$ and that $\check{R}_i$ acts by constant z^{-1} on Λ_0. All of the relations in (f) follow silimlarly. ∎

A Path Algebra Formula for $\check{R}_i$

Let $B(r)$ be the Bratteli diagram for tensor powers of $V = \Lambda_{\omega_1}$ (with the assumptions in (5.6)). Identify the centralizer algebras $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ with the path algebras corresponding to the Bratteli diagram $B(r)$. Recall that the path algebras have a natural basis E_{ST}, $(S, T) \in \Omega^m$ of matrix units.

For each tableau $S = (\sigma^{(0)}, ..., \sigma^{(m)}) \in \mathscr{T}^m$ define

$$\nabla_i(S) = \langle \sigma^{(i)}, \sigma^{(i)} + 2\rho \rangle - \langle \sigma^{(i-1)}, \sigma^{(i-1)} + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle. \quad (5.11)$$

Let (S, T) be a pair of tableaux

$$S = (\sigma^{(0)}, ..., \sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)}, ..., \sigma^{(m)}),$$

and

$$T = (\sigma^{(0)}, ..., \sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)}, ..., \sigma^{(m)}),$$

in $\mathscr{T}^m$ such that S and T are the same except possibly at the shape at level i. In other words the pair $(S, T) \in \Omega_{i-1}^{i+1}$. Define

$$\diamond_i(S, T) = \tfrac{1}{2}(\nabla_{i+1}(S) - \nabla_i(T)). \quad (5.12)$$

These constants are defined so that, if $D_m = \check{R}_{m-1} \check{R}_{m-2} \cdots \check{R}_2 \check{R}_1 \check{R}_1 \check{R}_2 \cdots \check{R}_{m-2} \check{R}_{m-1}$, then

$$D_m = \sum_{S \in \mathscr{T}^m} (D_m)_{SS} E_{SS}, \qquad \text{where} \quad (D_m)_{SS} = q^{\nabla_m(S)},$$

and

$$q^{-2\diamond_{m-1}(S, T)} = (D_m^{-1})_{SS} (D_{m-1})_{TT}.$$

The first of these formulas is a consequence of Corollary (2.25).

(5.13) Proposition. *Let y be as given in* (5.3).

(a) *Let $S = (\sigma^{(0)}, ..., \sigma^{(m)})$ be a tableau in the Bratteli diagram $B(r)$. Then*

$$\nabla_m(S) = \begin{cases} 2(\sigma_k^{(m-1)} - k + 1), & \textit{when} \quad \sigma^{(m)} = \sigma^{(m-1)} + \varepsilon_k, \\ 2(-\sigma_k^{(m-1)} - y + k), & \textit{when} \quad \sigma^{(m)} = \sigma^{(m-1)} - \varepsilon_k. \end{cases}$$

(b) *Let* $S=(\sigma^{(m-2)}, \sigma^{(m-1)}, \sigma^{(m)})$ *and* $T=(\sigma^{(m-2)}, \tau^{(m-1)}, \sigma^{(m)})$ *be such that* $(S, T)\in \Omega^m_{m-2}$. *Then*

$$\diamond_{m-1}(S,T)=\begin{cases} \pm(\sigma^{(m)}_k - k - \tau^{(m-1)}_l + l), \\ \qquad \text{if} \quad \tau^{(m-1)}=\tau^{(m-2)}\pm\varepsilon_l \text{ and } \sigma^{(m)}=\sigma^{(m-1)}\pm\varepsilon_k, \\ \pm(\tau^{(m-1)}_l - l + \sigma^{(m)}_k - k + y + 1), \\ \qquad \text{if} \quad \sigma^{(m)}=\sigma^{(m-1)}\pm\varepsilon_k \text{ and } \tau^{(m-1)}=\tau^{(m-2)}\mp\varepsilon_l. \end{cases}$$

Proof. Let $S=(\sigma^{(0)}, ..., \sigma^{(m)})$ be a tableau in the Bratteli diagram $B(r)$. Then, since $\sigma^{(m)}$ differs from $\sigma^{(m-1)}$ by either adding or removing a box in the kth row,

$$\sigma^{(m-1)}=\sum_{j=1}^{r} \sigma^{(m-1)}_j \varepsilon_j \qquad \text{and} \qquad \sigma^{(m)}=(\sigma^{(m-1)}_k \pm 1)\,\varepsilon_k + \sum_{\substack{1\leqslant j\leqslant r\\ j\neq k}} \sigma^{(m-1)}_j \varepsilon_j.$$

Using (5.4) to compute $\nabla_m(S)$ we get

$$\begin{aligned}\nabla_m(S) &= \langle \sigma^{(m)}, \sigma^{(m)}+2\rho\rangle - \langle \sigma^{(m-1)}, \sigma^{(m-1)}+2\rho\rangle - \langle \varepsilon_1, \varepsilon_1+2\rho\rangle \\ &= \left(\sum_{j\neq k} (\sigma^{(m-1)}_j)^2\right) + (\sigma^{(m-1)}_k \pm 1)^2 + \left(\sum_{j\neq k} 2\sigma^{(m-1)}_j \rho_j\right) \\ &\quad + 2(\sigma^{(m-1)}_k \pm 1)\,\rho_k - \left(\sum_{j\neq k} (\sigma^{(m-1)}_j)^2\right) - (\sigma^{(m-1)}_k)^2 \\ &\quad - \left(\sum_{j\neq k} 2\sigma^{(m-1)}_j \rho_j\right) - 2\sigma^{(m-1)}_k \rho_k - 1 - 2\rho_1 \\ &= \pm 2\sigma^{(m-1)}_k + 2(\pm\rho_k - \rho_1).\end{aligned}$$

The formula for $\nabla_m(S)$ follows since $\rho_k-\rho_1=(y-2k+1)-(y-1)=2(-k+1)$ and $-\rho_k-\rho_1=-(y-2k+1)-(y-1)=2(-y+k)$.

(b) The formulas for $\diamond_{m-1}(S, T)$ now follow from the definition of $\diamond_{m-1}$ and the formula for ∇_m in (a). ∎

(5.14) THEOREM. *One can choose the identification* (*Section* 1) *of the centralizer algebras* $\mathscr{Z}_m$ *with the path algebras corresponding to the Bratteli diagram* $B(r)$ (*with the assumption in* (5.6)) *so that the matrices* $\check{R}_i$ *are given by the formula*

$$\check{R}_i = \sum_{(S,T)\in\Omega^{i+1}_{i-1}} (\check{R}_i)_{ST}\, E_{ST},$$

where, for each $S=(\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)})$,

$$(\check{R}_i)_{SS}=\begin{cases}\dfrac{q^{\diamond_i(S,S)}}{[\diamond_i(S,S)]}, & \text{if} \quad \sigma^{(i-1)}\neq\sigma^{(i+1)},\\ \dfrac{q^{\diamond_i(S,S)}}{[\diamond_i(S,S)]}\left(1-\dfrac{\delta\dim_q(\Lambda_{\sigma^{(i)}})}{\dim_q(\Lambda_{\sigma^{(i-1)}})}\right), & \text{if} \quad \sigma^{(i-1)}=\sigma^{(i+1)},\end{cases}$$

and for each pair $(S,T)=((\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)}), (\sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)}))\in\Omega_{i-1}^{i+1}$ *such that* $S\neq T$,

$$(\check{R}_i)_{ST}=\begin{cases}\dfrac{\sqrt{[\diamond_i(S,S)-1][\diamond_i(S,S)+1]}}{[|\diamond_i(S,S)|]}, & \text{if} \quad \sigma^{(i-1)}\neq\sigma^{(i+1)},\\ -\dfrac{q^{\diamond_i(S,T)}}{[\diamond_i(S,T)]}\dfrac{\delta\sqrt{\dim_q(\Lambda_{\sigma^{(i)}})\cdot\dim_q(\Lambda_{\tau^{(i)}})}}{\dim_q(\Lambda_{\sigma^{(i-1)}})}, & \text{if} \quad \sigma^{(i-1)}=\sigma^{(i+1)},\end{cases}$$

where $\diamond_i(S,T)$ *and* δ *are given by* (5.11) *and* (5.12) *respectively.*

Proof. Since $\check{R}_i\in\mathscr{Z}_i=\mathrm{End}_{\mathfrak{U}}(V^{\otimes(i+1)})$ commutes with all elements of $\mathscr{Z}_{i-1}=\mathrm{End}_{\mathfrak{U}}(V^{\otimes(i-1)})$ it follows from Proposition Corollary (1.5) that

$$\check{R}_i=\sum_{(S,T)\in\Omega_{i-1}^{i+1}}(\check{R}_i)_{ST}E_{ST},$$

for some constants $(\check{R}_i)_{ST}$. In view of the imbeddings $\mathscr{Z}_0\subseteq\mathscr{Z}_1\subseteq\cdots\subseteq\mathscr{Z}_m$, it is sufficient to show that the formulas for $\check{R}_i$ hold for $i=m-1$.

By definition $D_m=\check{R}_{m-1}\check{R}_{m-2}\cdots\check{R}_2\check{R}_1\check{R}_1\check{R}_2\cdots\check{R}_{m-2}\check{R}_{m-1}$ and it follows that $\check{R}_{m-1}^{-1}=D_m^{-1}\check{R}_{m-1}D_{m-1}$. Thus, we may rewrite the relation (5.10e) in the form

$$\check{R}_{m-1}-D_m^{-1}\check{R}_{m-1}D_{m-1}=(q-q^{-1})(1-\check{E}_{m-1}). \tag{5.15}$$

Let $(S,T)\in\Omega^m$ and view (5.15) as an equation in the path algebra. Since the matrices D_m and D_{m-1} are diagonal, taking the E_{ST}-entry of this equation yields

$$(\check{R}_{m-1})_{ST}-(D_m^{-1})_{SS}(\check{R}_{m-1})_{ST}(D_{m-1})_{TT}=(q-q^{-1})(\delta_{ST}-(\check{E}_{m-1})_{ST}),$$

or, equivalently,

$$(1-(D_m^{-1})_{SS}(D_{m-1})_{TT})(\check{R}_{m-1})_{ST}=(q-q^{-1})(\delta_{ST}-(\check{E}_{m-1})_{ST}), \tag{5.16}$$

Hence,

$$(\check{R}_{m-1})_{ST}=\frac{(q-q^{-1})(\delta_{ST}-(\check{E}_{m-1})_{ST})}{1-(D_m^{-1})_{SS}(D_{m-1})_{TT}}, \qquad \text{if} \quad 1-(D_m^{-1})_{SS}(D_{m-1})_{TT}\neq 0.$$

Plugging in the following

$$\begin{aligned}\frac{q-q^{-1}}{1-(D_m^{-1})_{SS}\,(D_{m-1})_{TT}} &= \frac{q-q^{-1}}{1-q^{-2\diamond_{m-1}(S,\,T)}}\\ &= \frac{q^{\diamond_{m-1}(S,\,T)}(q-q^{-1})}{q^{\diamond_{m-1}(S,\,T)}-q^{-\diamond_{m-1}(S,\,T)}}\\ &= \frac{q^{\diamond_{m-1}(S,\,T)}}{[\diamond_{m-1}(S,T)]}\end{aligned}$$

we get

$$(\check{R}_{m-1})_{ST} = \frac{q^{\diamond_{m-1}(S,\,T)}}{[\diamond_{m-1}(S,T)]}\,(\delta_{ST}-(\check{E}_{m-1})_{ST}),$$
$$\text{if} \quad 1-(D_m^{-1})_{SS}\,(D_{m-1})_{TT}\neq 0.$$

All except the last of the formulas in Theorem (5.14) now follow immediately from (5.9) and the following lemma.

(5.17) LEMMA. *Let* $(S,T)\in\Omega^m_{m-2}m$. *If* $S=T$ *or if* $\sigma^{(m-2)}\neq\sigma^{(m)}$ *then* $1-(D_m^{-1})_{SS}\,(D_{m-1})_{TT}\neq 0$.

Proof. Consider the equation (5.16).

Case 1. If $S\neq T$ and $\sigma^{(m-2)}=\sigma^{(m)}$ then $\delta_{ST}=0$ and $(\check{E}_i)_{ST}\neq 0$ since the weights $wt_k(\mu)$ are all nonzero. Thus the right hand side of (5.16) is nonzero. This implies that $1-(D_m^{-1})_{SS}\,(D_{m-1})_{TT}$ is nonzero.

Case 2. If $S=T$ and $\sigma^{(m-2)}\neq\sigma^{(m)}$ then $(\check{E}_i)_{ST}=0$ and $\delta_{ST}\neq 0$. Thus the right hand side of (5.16) is nonzero. This implies that $1-(D_m^{-1})_{SS}\,(D_{m-1})_{TT}$ is nonzero.

Case 3. Suppose $S=T$ and $\sigma^{(m-2)}=\sigma^{(m)}$. Clearly $1-(D_m^{-1})_{SS}$ $(D_{m-1})_{SS}$ is nonzero if and only if $\diamond_m(S,S)\neq 0$. Then, by Proposition (5.13), there is some k such that $\diamond_m(S,S)=\pm(2\sigma_k^{(m-1)}-2k+y)$. This value is nonzero in Types C_r and D_r since y is odd, and is nonzero in type B_r since, by the assumption in (5.6), $2k<y$ and $\sigma_k^{(m-1)}\geqslant 0$. ∎

Now let us prove the last formula in Theorem (5.14). Let $S\in\mathscr{T}^m_{m-2}$ and suppose that $T\in\mathscr{T}^m_{m-2}$ is such that $(S,T)\in\Omega^m_{m-2}$ and $T\neq S$. By Lemma (5.1), T is unique. It follows that

$$\begin{aligned}(\check{R}^2_{m-1})_{SS} &= \sum_{L\in\mathscr{T}_m}(\check{R}_{m-1})_{SL}\,(\check{R}_{m-1})_{LS}\\ &= ((\check{R}_{m-1})_{SS})^2+(\check{R}_{m-1})_{ST}\,(\check{R}_{m-1})_{TS} \qquad (5.18)\end{aligned}$$

On the other hand

$$\begin{aligned}\check{R}^2_{m-1} &= \check{R}_{m-1}(\check{R}_{m-1} - \check{R}^{-1}_{m-1})\\ &= \check{R}_{m-1}(q-q^{-1})(1-\check{E}_{m-1})\\ &= (q-q^{-1})(\check{R}_{m-1} - z^{-1}\check{E}_{m-1}),\end{aligned}$$

since $\check{E}_{m-1}\check{R}_{m-1} = z^{-1}\check{E}_{m-1}$. Since $\sigma^{(m-2)} \neq \sigma^{(m)}$, it follows that $(\check{E}_{m-1})_{SS} = 0$ and thus that

$$(\check{R}^2_{m-1})_{SS} = (q-q^{-1})(\check{R}_{m-1})_{SS} + 1. \tag{5.19}$$

Equating (5.18) and (5.19) and using the formula for $(\check{R}_{m-1})_{SS}$ gives

$$\begin{aligned}(\check{R}_{m-1})_{ST}(\check{R}_{m-1})_{TS} &= (q-q^{-1})(\check{R}_{m-1})_{SS} + 1 - (\check{R}^2_{m-1})_{SS}\\ &= \frac{[\diamondsuit_{m-1}(S,S)-1][\diamondsuit_{m-1}(S,S)+1]}{[\diamondsuit_{m-1}(S,S)]^2},\end{aligned}$$

exactly as in the proof of Theorem (4.8). It follows from the remarks at the end of Section 1 that we can choose the normalization of the elements E_{ST} so that $(\check{R}_{m-1})_{ST}$ and $(\check{R}_{m-1})_{TS}$ are as given in the theorem. ▮

Matrix Units

Given a tableau $T = (\tau^{(0)}, ..., \tau^{(m)}) \in \mathscr{T}^m$ let T' denote the tableau $T' = (\tau^{(0)}, ..., \tau^{(m-1)}) \in \mathscr{T}^{m-1}$). Let $(T')^+$ denote the set of all extensions of T';

$$(T')^+ = \{S \in \mathscr{T}^m \mid S' = T'\}.$$

Given tableaux $S = (\sigma^{(0)}, ..., \sigma^{(m)})$ and $T = (\tau^{(0)}, ..., \tau^{(m)})$ in $\mathscr{T}^m$ let $(\check{R}_{m-1})_{ST}$ be the constant given by Theorem (5.14) in the case that $((\sigma^{(m-2)}, \sigma^{(m-1)}, \sigma^{(m)}), (\tau^{(m-2)}, \tau^{(m-1)}, \tau^{(m)})) \in \Omega^m_{m-2}$ and let $(\check{R}_{m-1})_{ST} = 0$ otherwise.

(5.20) Lemma. *Let $T' = (\tau^{(0)}, ..., \tau^{(m-1)}) \in \mathscr{T}^{m-1}$ and let $(T')^+$ be the set of extensions of T'. Then the values $\nabla_m(S)$ are different as S ranges over all elements of $(T')^+$.*

Proof. Let $S = (\tau^{(0)}, ..., \tau^{(m-1)}, \sigma^{(m)}) \in (T')^+$. By Proposition (5.13),

$$\nabla_m(S) = 2(\tau^{(m-1)}_k - k + 1) \qquad \text{or} \qquad \nabla_m(S) = 2(-\tau^{(m-1)}_k - y + k),$$

for some positive integer k. Let l be the largest value of k such that $S \in (T')^+$. By the assumption in (5.6), $2l-1<y$. Since

$$\tau_1^{(m-1)} \geqslant \cdots \geqslant \tau_k^{(m-1)} \geqslant \cdots \geqslant \tau_l^{(m-1)}, \qquad \text{and}$$
$$-\tau_l^{(m-1)} \geqslant \cdots \geqslant -\tau_k^{(m-1)} \geqslant \cdots \geqslant -\tau_1^{(m-1)},$$

it follows that

$$\tau_1^{(m-1)} > \cdots > \tau_k^{(m-1)} - k + 1 > \cdots > \tau_l^{(m-1)} - l + 1, \qquad \text{and}$$
$$-\tau_l^{(m-1)} - y + l > \cdots > -\tau_k^{(m-1)} - y + k > \cdots > -\tau_1^{(m-1)} - y + 1.$$

Since $\tau_l^{(m-1)} \geqslant -\tau_l^{(m-1)}$ and $-l+1>-y+l$, it follows that

$$\tau_l^{(m-1)} - l + 1 > -\tau_l^{(m-1)} - y + l.$$

The result follows. ∎

The proofs of the following results are essentially the same as the proofs of Theorem (4.14) and Corollary (4.15).

(5.21) THEOREM. *The matrix units $E_{ST} \in \mathscr{Z}_m$, $(S, T) \in \Omega^m$ are given in terms of the $\check{R}_i$, $1 \leqslant i \leqslant m-1$, inductively, by the following formulas.*

(1) *Let $T \in \mathscr{T}^m$. Then $E_{TT} = \prod_{S \in \mathscr{T}_m,\, S \neq T,\, S' = T'} (E_{T'T'} \check{R}_{m-1} E_{T'T'} - (\check{R}_{m-1})_{SS} E_{T'T'})/((\check{R}_{m-1})_{TT} - (\check{R}_{m-1})_{SS})$*

(2) *Let $(S, T) \in \Omega^m$. If $shp(S') = shp(T')$ then $E_{ST} = E_{S'T'} E_{TT}$ where E_{TT} is given by (1).*

(3) *Let $(S, T) \in \Omega^m$. If $shp(S') \neq shp(T')$ then*

$$E_{ST} = \frac{1}{(\check{R}_{m-1})_{MN}} E_{S'M'} \check{R}_{m-1} E_{N'T'} E_{TT},$$

where M and N are tableaux in $\mathscr{T}^m$ of the form $M = (\mu^{(0)}, ..., \mu^{(m-2)}, shp(S'), shp(S))$ and $N = (\mu^{(0)}, ..., \mu^{(m-2)}, shp(T'), shp(S))$.

(5.22) COROLLARY. *The centralizer $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ is generated by the matrices $\check{R}_i$, $1 \leqslant i \leqslant m-1$.*

6. IRREDUCIBLE REPRESENTATIONS OF THE IWAHORI–HECKE ALGEBRAS OF TYPE A, THE BIRMAN–WENZL ALGEBRAS AND THE BRAUER ALGEBRAS

The Iwahori–Hecke Algebras of Type A, $H_m(q^2)$

The *Iwahori–Hecke algebra of type A*, denoted $H_m(q^2)$, is the algebra generated over $\mathbb{C}(q)$ by $1, g_1, ..., g_{m-1}$ subject to the relations

(B1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,

(B2) $g_i g_j = g_j g_i \quad$ if $\quad |i-j| \geqslant 2$,

(IH) $g_i^2 = (q - q^{-1})\, g_i + 1$.

The Iwahori–Hecke algebra $H_m(q)$ is often defined as the algebra generated over $\mathbb{C}(q)$ by $1, g'_1, ..., g'_{m-1}$ subject to the relations

(B1) $g'_i g'_{i+1} g'_i = g'_{i+1} g'_i g'_{i+1}$,

(B2) $g'_i g'_j = g'_j g'_i \quad$ if $\quad |i-j| \geqslant 2$,

(IH) $(g'_i)^2 = (q-1)\, g'_i + q$.

One can pass from one presentation to the other by setting $g_i = g'_i / q$.

(6.1) COROLLARY. *Let $\mathfrak{U}$ be the Drinfel'd–Jimbo quantum group $\mathfrak{U}_h(\mathfrak{sl}(r+1))$ and let $V = \Lambda_{\omega_1}$ be the irreducible $\mathfrak{U}$-module indexed by the fundamental weight ω_1. The centralizer $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ is a quotient of the Iwahori–Hecke algebra of type A, $H_m(q^2)$.*

Proof. This follows immediately from Proposition (4.4) and Corollary (4.15). ∎

In fact, the classical Schur–Weyl duality gives that $\mathscr{Z}_m$ is isomorphic to $H_m(q^2)$ if $\mathfrak{U} = \mathfrak{U}_h(\mathfrak{sl}(r+1))$, $r \geqslant m$.

The Young lattice Y is the Bratteli diagram given in Fig. 1. The shapes of Y which are on level m are the partitions of m;

$$\hat{Y}_m = \{\lambda \vdash m\}.$$

A partition $\lambda \in \hat{Y}_m$ is connected by an edge to a partition $\mu \in \hat{Y}_{m+1}$ if μ can be obtained by adding a box to λ. Each tableau $T \in \mathscr{T}^\lambda$ in the Bratteli diagram Y can be identified in a natural way with a standard tableau of shape λ.

Let $S = (\sigma^{(0)}, ..., \sigma^{(m)})$ be a standard tableau, i.e. a tableau in the Bratteli diagram Y. Define

$$\diamondsuit_{m-1}(S, S) = \sigma_k^{(m)} - \sigma_l^{(m-1)} - k + l, \tag{6.2}$$

when $\sigma^{(m)}$ is obtained by adding a box to the kth row of $\sigma^{(m-1)}$ and $\sigma^{(m-1)}$ is obtained by adding a box to the lth row of $\sigma^{(m-2)}$.

The following result follows immediately from Theorem (4.8) and Corollary (6.1).

(6.3) THEOREM. *There is an identification of the Iwahori–Hecke algebras $H_m(q^2)$ with the path algebras corresponding to the Young lattice so that the generators g_i are given by the formula*

$$g_i = \sum_{(S,\,T) \in \Omega^{i+1}_{i-1}} (g_i)_{ST}\, E_{ST},$$

where for each $S \in \mathscr{T}^m$

$$(g_i)_{SS} = \frac{q^{\diamond_i(S,\,S)}}{[\diamond_i(S,\,S)]},$$

and for each pair $(S, T) \in \Omega^{i+1}_{i-1}$ such that $S \neq T$ we have

$$(g_i)_{ST} = \frac{\sqrt{[\diamond_i(S,\,S)-1][\diamond_i(S,\,S)+1]}}{[|\diamond_i(S,\,S)|]},$$

where $\diamond_i(S, S)$ is defined by (6.2).

The following corollaries are immediate consequences of the path algebra setup.

(6.4) COROLLARY. ([H], [W2]) *For each $\lambda \in \hat{Y}_m$ let $d_\lambda = Card(\mathscr{T}^\lambda)$ be the number of standard tableaux of shape λ. Define representations*

$$\begin{aligned} \pi^\lambda \colon H_m(q^2) &\to M_{d_\lambda}(\mathbb{C}(q)) \\ a &\mapsto (\pi^\lambda(a)_{ST})_{(S,\,T) \in \Omega^\lambda} \end{aligned}$$

of $H_m(q^2)$ by the following formulas:

For each $S \in \mathscr{T}^\lambda$,

$$\pi^\lambda(g_i)_{SS} = \frac{q^{\diamond_i(S,\,S)}}{[\diamond_i(S,\,S)]}$$

and for each pair $(S, T) \in \Omega^\lambda$ such that $S \neq T$,

$$\pi^\lambda(g_i)_{ST} = \begin{cases} \dfrac{\sqrt{[\diamond_i(S,\,S)-1][\diamond_i(S,\,S)+1]}}{[|\diamond_i(S,\,S)|]}, & \text{if } \sigma^{(j)} = \tau^{(j)} \text{ for all } j \neq i, \\ 0, & \text{otherwise,} \end{cases}$$

where $S = (\sigma^{(0)}, ..., \sigma^{(m)})$, $T = (\tau^{(0)}, ..., \tau^{(m)})$ and $\diamond_i(S, S)$ is defined by (6.2). *Then the representations π^λ, $\lambda \in \hat{Y}_m$, are nonisomorphic irreducible representations of $H_m(q^2)$.*

(6.5) COROLLARY. *For each* $\lambda \in \hat{Y}_m$ *let* $\mathscr{L}^\lambda$ *be a vector space with basis* v_S, $S \in \mathscr{T}^\lambda$. *If* $S = (\sigma^{(0)}, ..., \sigma^{(m)} = \lambda) \in \mathscr{T}^\lambda$ *then let* T *be a tableau of the form* $T = (\sigma^{(0)}, ..., \sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)}, ..., \sigma^{(m)})$ *such that* $\tau^{(i)} \neq \sigma^{(i)}$. *In view of* (4.1), *if* T *exists then it is unique. Let* $\pi^\lambda(g_i)_{SS}$ *and* $\pi^\lambda(g_i)_{ST}$ *be as given in the previous corollary. Define an action of* $H_m(q^2)$ *on* $\mathscr{L}^\lambda$ *by defining*

$$g_i v_S = \begin{cases} \pi^\lambda(g_i)_{SS} v_S + \pi^\lambda(g_i)_{ST} v_T, & \text{if } T \text{ exists}, \\ \pi^\lambda(g_i)_{SS} v_S, & \text{if } T \text{ does not exist}, \end{cases}$$

for each $S \in \mathscr{T}^\lambda$. *Then the* $\mathscr{L}^\lambda$, $\lambda \in \hat{Y}_m$, *are a complete set of nonisomorphic irreducible* $H_m(q^2)$*-modules.*

The Birman–Wenzl Algebras $\mathrm{BW}_m(z, q)$

Let z and q be indeterminates. We define the Birman–Wenzl algebra $\mathrm{BW}_m(z, q)$ (defined in [BW] and [M1]) as the algebra generated over $\mathbb{C}(z, q)$ by $1, g_1, g_2, ..., g_{m-1}$, which are assumed to be invertible, subject to the relations

$$\begin{aligned} &(\mathrm{B1}) \quad g_i g_j = g_j g_i \quad \text{if} \quad |i-j| \geqslant 2, \\ &(\mathrm{B2}) \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \\ &(\mathrm{BW1}) \quad (g_i - z^{-1})(g_i + q^{-1})(g_i - q) = 0, \\ &(\mathrm{BW2}) \quad e_i g_{i-1}^{\pm 1} e_i = z^{\pm 1} e_i \text{ and } e_i g_{i+1}^{\pm 1} e_i = z^{\pm 1} e_i, \end{aligned}$$

where e_i is defined by the equation

$$(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}. \tag{6.6}$$

Letting

$$x = \frac{z - z^{-1}}{q - q^{-1}} + 1, \tag{6.7}$$

one has the following relations

$$e_i^2 = x e_i, \tag{6.8}$$

$$e_i g_i^{\pm 1} = g_i^{\pm 1} e_i = z^{\mp 1} e_i, \tag{6.9}$$

(6.10) COROLLARY. *Let* $\mathfrak{U}$ *be the Drinfel'd–Jimbo quantum group* $\mathfrak{U}_h(\mathfrak{so}(2r+1))$ *and let* $V = \Lambda_{\omega_1}$ *be the irreducible* $\mathfrak{U}$ *module indexed by the fundamental weight* ω_1. *Then centralizer* $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ *is isomorphic to a quotient of the Birman–Wenzl algebra,* $BW_m(q^{2r}, q)$.

Proof. This follows immediately from Proposition (5.10), Corollary (5.22) and the definition of the Birman–Wenzl algebras. ∎

Recall the Bratteli diagram B given in Fig. 2. The shapes $\lambda \in \hat{B}_m$ of B which are on level m are the partitions in the set

$$\hat{B}_m = \{\lambda \vdash m-2k,\ 0 \leqslant k \leqslant \lfloor m/2 \rfloor\}.$$

A partition $\lambda \in \hat{B}_m$ is connected by an edge to a partition $\mu \in \hat{B}_{m+1}$ if μ can be obtained from λ by adding a box to λ or removing a box from λ. The tableaux $T \in \mathcal{T}^{\lambda}$ in the Bratteli diagram B are called *up-down tableaux* since they are sequences of partitions in which each partition differs from the previous one by either adding or removing a box.

For the remainder of this section, unless otherwise specified, all paths and tableaux shall be from the Bratteli diagram B.

Let y be a formal symbol and for each integer d make the following notations:

$$\begin{aligned} [d] &= \frac{q^d - q^{-d}}{q - q^{-1}}, & [y+d] &= \frac{zq^d - z^{-1}q^{-d}}{q-q^{-1}}, \\ [-y+d] &= \frac{z^{-1}q^d - zq^{-d}}{q-q^{-1}}, & \left[\frac{1}{d}\right] &= \frac{q^d}{[d]}, \\ \left[\frac{1}{y+d}\right] &= \frac{zq^d}{[y+d]}, & \left[\frac{1}{-y+d}\right] &= \frac{z^{-1}q^d}{[-y+d]}. \end{aligned} \tag{6.11}$$

Let λ be a partition. Let λ_i denote the length of the ith column and let λ'_j denote the length of the jth column. Define the hook length at a box $(i, j) \in \lambda$ to be

$$h(i,j) = \lambda_i - i + \lambda'_j - j + 1,$$

and, for each box $(i, j) \in \lambda$, define

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1, & \text{if } \ i \leqslant j, \\ -\lambda'_i - \lambda'_j + i + j - 1, & \text{if } \ i > j. \end{cases} \tag{6.12}$$

Following [W3] we define rational functions $Q_\lambda(z, q)$ as follows

$$Q_\lambda(z,q) = \prod_{(j,j)\in\lambda} \frac{[y + \lambda_j - \lambda'_j] + [h(j,j)]}{[h(j,j)]} \prod_{\substack{(i,j)\in\lambda \\ i \neq j}} \frac{[y + d(i,j)]}{[h(i,j)]}. \tag{6.13}$$

The important property of these functions ([W3], Theorem 5.5) is that, if $\mathfrak{U} = \mathfrak{U}_h\mathfrak{so}(2r+1)$, then for all $\lambda \in \hat{\mathfrak{U}}$, $Q_\lambda(q^{2r}, q) = \dim_q(\Lambda_\lambda)$, where Λ_λ is the irreducible $\mathfrak{U}$-module corresponding to the partition λ. Thus, $Q_\lambda(z, q)$ is a two parameter version of the quantum dimension.

Now let us define two parameter versions of the constants $\nabla_m(S)$ and $\diamond_{m-1}(S, T)$ which are given in Proposition (5.13).

Let $S = (\sigma^{(0)}, ..., \sigma^{(m)})$ be a tableau in the Bratteli diagram B. Define

$$\tilde{\nabla}_m(S) = \begin{cases} q^{2(\sigma_k^{(m-1)} - k + 1)}, & \text{when} \quad \sigma^{(m)} = \sigma^{(m-1)} + \varepsilon_k; \\ z^{-2} q^{2(-\sigma_k^{(m-1)} + k)}, & \text{when} \quad \sigma^{(m)} = \sigma^{(m-1)} - \varepsilon_k, \end{cases} \tag{6.14}$$

Let $S = (\sigma^{(m-2)}, \sigma^{(m-1)}, \sigma^{(m)})$ and $T = (\sigma^{(m-2)}, \tau^{(m-1)}, \sigma^{(m)})$ be such that $(S, T) \in \Omega^m_{m-2}$. Then define

$$\diamond_{m-1}(S, T) = \begin{cases} \pm(\sigma_k^{(m)} - k - \tau_l^{(m-1)} + l), \\ \qquad \text{if} \quad \tau^{(m-1)} = \tau^{(m-2)} \pm \varepsilon_l \text{ and } \sigma^{(m)} = \sigma^{(m-1)} \pm \varepsilon_k, \\ \pm(y + \tau_l^{(m-1)} - l + \sigma_k^{(m)} - k + 1), \\ \qquad \text{if} \quad \sigma^{(m)} = \sigma^{(m-1)} \pm \varepsilon_k \text{ and } \tau^{(m-1)} = \tau^{(m-2)} \mp \varepsilon_l. \end{cases}$$

(6.15) THEOREM. *There is an identification of the Birman–Wenzl algebras $BW_m(z, q)$ with the path algebras corresponding to the Bratteli diagram B. With this identification*:

(a) *The elements $D_m = g_{m-1} g_{m-2} \cdots g_1 g_1 \cdots g_{m-2} g_{m-1}$ are given by the formula,*

$$D_m = \sum_{S \in \mathcal{T}^m} (D_m)_{SS} E_{SS}, \qquad \textit{where} \quad (D_m)_{SS} = \tilde{\nabla}_m(S).$$

(b) *The elements e_i are given by the formula,*

$$e_i = \sum_{(S, T) \in \Omega^{i+1}_{i-1}} (e_i)_{ST} E_{ST}$$

where, if $S = (\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)})$ and $T = (\sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)})$, then

$$(e_i)_{ST} = \begin{cases} \dfrac{\sqrt{Q_{\sigma^{(i)}}(z, q)\, Q_{\tau^{(i)}}(z, q)}}{Q_{\sigma^{(i-1)}}(z, q)} & \textit{if} \quad \sigma^{(i-1)} = \sigma^{(i+1)}, \\ 0 & \textit{otherwise.} \end{cases}$$

(c) *The generators g_i are given by the formula*

$$g_i = \sum_{(S, T) \in \Omega^{i+1}_{i-1}} (g_i)_{ST} E_{ST},$$

where, for each $S=(\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)})$,

$$(g_i)_{SS}=\begin{cases} \left[\dfrac{1}{\diamondsuit_i(S,S)}\right], & \textit{if} \quad \sigma^{(i-1)} \neq \sigma^{(i+1)}, \\ \left[\dfrac{1}{\diamondsuit_i(S,S)}\right]\left(1-\dfrac{Q_{\sigma^{(i)}}(z,q)}{Q_{\sigma^{(i-1)}}(z,q)}\right), & \textit{if} \quad \sigma^{(i-1)} = \sigma^{(i+1)} \end{cases}$$

and for each pair $(S,T)=((\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)}), (\sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)})) \in \Omega_{i-1}^{i+1}$ *such that* $S \neq T$,

$$(g_i)_{ST}=\begin{cases} \dfrac{\sqrt{[\diamondsuit_i(S,S)-1][\diamondsuit_i(S,S)+1]}}{[|\diamondsuit_i(S,S)|]}, & \textit{if} \quad \sigma^{(i-1)} \neq \sigma^{(i+1)}, \\ -\left[\dfrac{1}{\diamondsuit_i(S,T)}\right]\dfrac{\sqrt{Q_{\sigma^{(i)}}(z,q)\, Q_{\tau^{(i)}}(z,q)}}{Q_{\sigma^{(i-1)}}(z,q)}, & \textit{if} \quad \sigma^{(i-1)} = \sigma^{(i+1)}, \end{cases}$$

where $\diamondsuit_i(S,T)$ *is given by* (6.14).

Proof. If z is specialized to q^{2r}, $r>m$, then the formulas given above coincide with the formulas given in (2.25), (5.9), and Theorem (5.14). In view of the results in Corollary (2.25), Theorem (3.12), and Theorem (5.14) it follows that this theorem holds whenever z is specialized to q^{2r}, $r>m$. Thus, for an infinite number of specializations of the parameter z, the theorem holds. This is sufficient to guarantee that the theorem holds over $\mathbb{C}(z,q)$. ∎

(6.16) COROLLARY. *For each* $\lambda \in \hat{B}_m$ *let* $d_\lambda = \mathrm{Card}(\mathcal{T}^\lambda)$ *be the number of up-down tableaux of shape* λ. *Define representations*

$$\pi^\lambda : BW_m(z,q) \to M_{d_\lambda}(\mathbb{C}(z,q))$$
$$a \mapsto (\pi^\lambda(a)_{ST})_{(S,T)\in\Omega^\lambda}$$

of $BW_m(z,q)$ *by defining*

For each $S \in \mathcal{T}^\lambda$,

$$\pi^\lambda(g_i)_{SS}=\begin{cases} \left[\dfrac{1}{\diamondsuit_i(S,S)}\right], & \textit{if} \quad \sigma^{(i-1)} \neq \sigma^{(i+1)}, \\ \left[\dfrac{1}{\diamondsuit_i(S,S)}\right]\left(1-\dfrac{Q_{\sigma^{(i)}}(z,q)}{Q_{\sigma^{(i-1)}}(z,q)}\right), & \textit{if} \quad \sigma^{(i-1)} = \sigma^{(i+1)}, \end{cases}$$

and for each pair $(S, T) \in \Omega^\lambda$ *such that* $S \neq T$,

$$\pi^\lambda(g_i)_{ST} = \begin{cases} \dfrac{\sqrt{[\diamondsuit_i(S,S)-1][\diamondsuit_i(S,S)+1]}}{[|\diamondsuit_i(S,S)|]} \\ \qquad \text{if} \quad \sigma^{(j)} = \tau^{(j)} \text{ for all } j \neq i \text{ and } \sigma^{(i-1)} \neq \sigma^{(i+1)}, \\ -\left[\dfrac{1}{\diamondsuit_i(S,T)}\right] \dfrac{\sqrt{Q_{\sigma^{(i)}}(z,q)\, Q_{\tau^{(i)}}(z,q)}}{Q_{\sigma^{(i-1)}}(z,q)}, \\ \qquad \text{if} \quad \sigma^{(j)} = \tau^{(j)} \text{ for all } j \neq i \text{ and } \sigma^{(i-1)} = \sigma^{(i+1)}, \\ 0, \qquad \text{otherwise}, \end{cases}$$

where $S = (\sigma^{(0)}, ..., \sigma^{(m)})$, $T = (\tau^{(0)}, ..., \tau^{(m)})$ *and* $\diamondsuit_i(S, S)$ *is given by* (6.14). *Then the representations* π^λ, $\lambda \in \hat{B}_m$, *are nonisomorphic irreducible representations of* $BW_m(z, q)$.

Let $\lambda \in \hat{B}_m$. If $S = (\sigma^{(0)}, ..., \sigma^{(m)}) \in \mathcal{T}^\lambda$ such that $\sigma^{(i-1)} \neq \sigma^{(i+1)}$ then let $s_i S$ be the tableau

$$s_i S = (\sigma^{(0)}, ..., \sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)}, ..., \sigma^{(m)})$$

such that $\tau^{(i)} \neq \sigma^{(i)}$. In view of Lemma (5.1), if $s_i S$ exists then it is unique. If $S = (\sigma^{(0)}, ..., \sigma^{(m)}) \in \mathcal{T}^\lambda$ such that $\sigma^{(i-1)} = \sigma^{(i+1)}$ then define

$$e_i S = \{T = (\tau^{(0)}, ..., \tau^{(m)}) \in \mathcal{T}^\lambda \mid S \neq T \text{ and } \tau^{(j)} = \sigma^{(j)} \text{ for all } j \neq i\}.$$

With this notation we have the following.

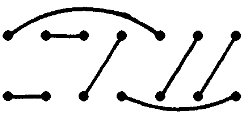
(6.17) Corollary. *For each* $\lambda \in \hat{B}_m(r)$ *let* $\mathcal{Z}^\lambda$ *be a vector space with basis* v_S, $S \in \mathcal{T}^\lambda$. *Let constants* $\pi^\lambda(g_i)_{ST}$, $(S, T) \in \Omega^\lambda$, *be as given in Corollary* (6.16). *Define an action of* $BW_m(z, q)$ *on* $\mathcal{Z}^\lambda$ *by defining*

$$g_i v_S = \begin{cases} \pi^\lambda(g_i)_{SS}\, v_S + \pi^\lambda(g_i)_{S, s_i S} v_{s_i S}, \\ \qquad \text{if} \quad \sigma^{(i-1)} \neq \sigma^{(i+1)} \text{ and } s_i S \text{ exists}, \\ \pi^\lambda(g_i)_{SS}\, v_S, \\ \qquad \text{if} \quad \sigma^{(i-1)} \neq \sigma^{(i+1)} \text{ and } s_i S \text{ does not exist}, \\ \pi^\lambda(g_i)_{SS}\, v_S + \sum\limits_{T \in e_i S} \pi^\lambda(g_i)_{ST}\, v_T, \\ \qquad \text{if} \quad \sigma^{(i-1)} = \sigma^{(i+1)}. \end{cases}$$

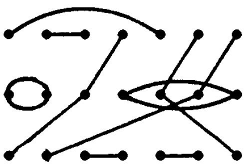
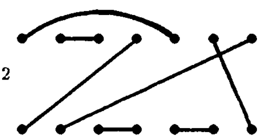
for each $S = (\sigma^{(0)}, ..., \sigma^{(m)}) \in \mathcal{T}^\lambda$. *Then the* $\mathcal{Z}^\lambda$, $\lambda \in \hat{B}_m(r)$, *are nonisomorphic irreducible* $BW_m(z, q)$*-modules.*

The Brauer Algebra

An m-diagram is a graph on two rows of m-vertices, one above the other, and m edges such that each vertex is incident to precisely one edge. The number of m-diagrams is $(2m)!! = (2m-1)(2m-3) \cdots 3 \cdot 1$. We multiply two m-diagrams d_1 and d_2 by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains m paths and some number γ of closed cycles. Let d be the m-diagram whose edges are the paths in this graph (with the cycles removed). Then the product $d_1 d_2$ is given by $d_1 d_2 = x^{\gamma} d$. For example, if

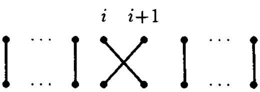
$d_1 =$  and $d_2 =$ 

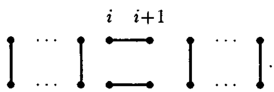
then

$d_1 d_2 =$  $= x^2$ 

Let x be an indeterminate. The *Brauer algebra* $B_m(x)$ (defined originally by R. Brauer [Br]) is the $\mathbb{C}(x)$-span of the m-diagrams. Diagram multiplication makes $B_m(x)$ an associative algebra whose identity id_m is given by the diagram having each vertex in the top row connected to the vertex just below it in the bottom row. By convention $B_0(x) = B_1(x) = \mathbb{C}(x)$.

The group algebra $\mathbb{C}(x)[\mathscr{S}_m]$ of the symmetric group $\mathscr{S}_m$ is embedded in $B_m(x)$ as the span of the diagrams with only vertical edges. For $1 \leqslant i \leqslant m-1$, let

$s_i =$ 

and $e_i =$ 

Then $e_i^2 = x e_i$, and the elements of the set $\{s_i, e_i \mid 1 \leqslant i \leqslant m-1\}$ generate $B_m(x)$. Note that the s_i correspond to the simple transpositions $(i, i+1)$ of $\mathscr{S}_m$ and that the s_i, $1 \leqslant i \leqslant m-1$, generate $\mathbb{C}(x)[\mathscr{S}_m]$.

For each complex number $\xi \in \mathbb{C}$ one defines a Brauer algebra $B_m(\xi)$ over $\mathbb{C}$ as the linear span of m-diagrams where the multiplication is given as above except with x replaced by ξ. R. Brauer [Br] originally introduced the

Brauer algebra $B_m(n)$ in his study of the centralizer of the tensor representation of the complex orthogonal group $O(n)=\{g\in M_n(\mathbb{C}) \mid gg^t=I\}$. Let $V=\mathbb{C}^n$ be the standard or fundamental representation for $O(n)$. The tensor space $V^{\otimes m}$ is a completely reducible $O(n)$-module with irreducible summands labeled by partitions in the set

$$\hat{B}_m(n)=\{\lambda \vdash (m-2k) \mid 0\leqslant k\leqslant \lfloor m/2 \rfloor, \lambda_1'+\lambda_2'\leqslant n\}.$$

Note that when n is sufficiently large $\hat{B}_m(n)=\hat{B}_m$ where $\hat{B}_m$ is as defined in Section 5. Brauer gives an action of $B_m(n)$ on $V^{\otimes m}$ which commutes with the action of $O(n)$. This action is such that s_i is the permutation which transposes the ith and the $(i+1)$st tensor factors of $V^{\otimes m}$ and e_1 is $(2r+1)$ times the projection onto the invariants in the first two tensor factors of $V^{\otimes m}$ (see [R1] for details). Brauer showed that the action of the Brauer algebra generates the full centralizer of the orthogonal group action on $V^{\otimes m}$. Provided we assume that $r\geqslant m$, all of these results hold if the group $O(n)$ is replaced by the group $SO(2r+1)$.

Let $r\geqslant 2$ and set $G=SO(2r+1)$. Let $\tilde{V}=\mathbb{C}^{2r+1}$ be the standard module for G and let $\tilde{\mathscr{Z}}_2=\mathrm{End}_G(\tilde{V}\otimes\tilde{V})$. As $SO(2r+1)$ modules,

$$\tilde{V}\otimes\tilde{V}\cong V^{(0)}\oplus V^{(1^2)}\oplus V^{(2)},$$

where V^λ denotes the irreducible G-module indexed by the partition λ. Let $\tilde{E}_{\varnothing\varnothing}, \tilde{E}_{(1^2)(1^2)}$, and $\tilde{E}_{(2),(2)}$ be the G-invariant projections onto the irreducible summands $V^{\varnothing}$, $V^{(1^2)}$, and $V^{(2)}$ respectively. We have chosen this notation so that it is suggestive of the identification of the centralizer algebra $\tilde{\mathscr{Z}}_2$ with a path algebra corresponding to the Bratteli diagram B. It can easily be shown that $B_2(2r+1)$ is isomorphic to $\tilde{\mathscr{Z}}_2$ and that under this isomorphism

$$\begin{aligned} e_1&=(2r+1)\,\tilde{E}_{\varnothing\varnothing},\\ s_1&=\tilde{E}_{\varnothing\varnothing}+\tilde{E}_{(2),(2)}-\tilde{E}_{(1^2)(1^2)}. \end{aligned} \tag{6.19}$$

Let $\mathfrak{U}=\mathfrak{U}_h(\mathfrak{so}(2r+1))$ and $V=\Lambda_{\omega_1}$ be the irreducible $\mathfrak{U}$-module indexed by the fundamental weight ω_1. As $\mathfrak{U}$ modules,

$$V\otimes V\cong\Lambda_{(0)}\oplus\Lambda_{(1^2)}\oplus\Lambda_{(2)},$$

where Λ_λ denotes the irreducible $\mathfrak{U}$-module indexed by the partition λ. Let $E_{\varnothing\varnothing}, E_{(1^2)(1^2)}$, and $E_{(2),(2)}$ be the $\mathfrak{U}$-invariant projections onto the irreducible summands $\Lambda_\varnothing$, $\Lambda_{(1^2)}$, and $\Lambda_{(2)}$ respectively. It follows from (5.9) and Theorem (5.14), or by direct calculation, that the elements $\check{R}_1, \check{E}_1 \in \mathscr{Z}_2 = \mathrm{End}_{\mathfrak{U}}(V \otimes V)$ are given by

$$\begin{aligned} \check{E}_1 &= ([2r]+1)\, E_{\varnothing\varnothing}, \\ \check{R}_1 &= q^{-2r} E_{\varnothing\varnothing} + qE_{(2),(2)} - q^{-1} E_{(1^2)(1^2)}. \end{aligned} \tag{6.20}$$

By comparing (6.19) and (6.20) we see that, at $q=1$, the transformations $\check{R}_1$ and $\check{E}_1$ are the transformations s_1 and e_1 respectively. The transformations $\check{R}_i$ and $\check{E}_i$ in $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ are the same transformations as $\check{R}_1$ and $\check{E}_1$ respectively, except that they act on the ith and the $(i+1)$st tensor factors of $V^{\otimes m}$ instead of the first and second tensor factors. Similarly, the transformations $\check{s}_i$ and $\check{e}_i$ in $\tilde{\mathscr{Z}}_m = \mathrm{End}_G(\tilde{V}^{\otimes m})$ are the same transformations as s_1 and e_1 respectively, except that they act on the ith and the $(i+1)$st tensor factors of $\tilde{V}^{\otimes m}$ instead of the first and second tensor factors. Since, at $q=1$, the transformations $\check{R}_1$ and $\check{E}_1$ are the same as s_1 and e_1 respectively, it follows that, at $q=1$, $\check{R}_i$ and $\check{E}_i$ are the same as s_i and e_i respectively. Hence, at $q=1$, the centralizer algebras $\mathscr{Z}_m = \mathrm{End}_{\mathfrak{U}}(V^{\otimes m})$ are the centralizer algebras $\tilde{\mathscr{Z}}_m = \mathrm{End}_G(\tilde{V}^{\otimes m})$.

Following [El-K], for each partition λ, define polynomials

$$P_\lambda(x) = \prod_{(i,j)\in\lambda} \frac{x-1+d(i,j)}{h(i,j)}, \tag{6.21}$$

where the constants $d(i,j)$ and $h(i,j)$ are as given in (6.12). These polynomials have the important property that $P_\lambda(2r+1) = \dim(V^\lambda)$, for each irreducible representation V^λ of the orthogonal group $SO(2r+1)$.

Let $S = (\sigma^{(m-2)}, \sigma^{(m-1)}, \sigma^{(m)})$ and $T = (\sigma^{(m-2)}, \tau^{(m-1)}, \sigma^{(m)})$ be such that $(S, T) \in \Omega^m_{m-2}$. Then define

$$\diamond_{m-1}(S,T) = \begin{cases} \pm(\sigma^{(m)}_k - k - \tau^{(m-1)}_l + l), \\ \qquad \text{if} \quad \tau^{(m-1)} = \tau^{(m-2)} \pm \varepsilon_l \text{ and } \sigma^{(m)} = \sigma^{(m-1)} \pm \varepsilon_k, \\ \pm(x + \tau^{(m-1)}_l - l + \sigma^{(m)}_k - k), \\ \qquad \text{if} \quad \sigma^{(m)} = \sigma^{(m-1)} \pm \varepsilon_k \text{ and } \tau^{(m-1)} = \tau^{(m-2)} \mp \varepsilon_l. \end{cases}$$

(6.22) THEOREM. *There is an identification of the Brauer algebras $B_m(x)$ with the path algebras corresponding to the Bratteli diagram B. With this identification*:

(a) *The elements e_i are given by the formula*

$$e_i = \sum_{(S,T)\in\Omega^{i+1}_{i-1}} (e_i)_{ST} E_{ST}$$

where, if $S=(\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)})$ *and* $T=(\sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)})$, *then*

$$(e_i)_{ST}=\begin{cases} \dfrac{\sqrt{P_{\sigma^{(i)}}(x)\,P_{\tau^{(i)}}(x)}}{P_{\sigma^{(i-1)}}(x)} & \text{if} \quad \sigma^{(i-1)}=\sigma^{(i+1)}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *The elements* s_i *are given by the formula*

$$s_i=\sum_{(S,\,T)\in\Omega_{i-1}^{i+1}} (s_i)_{ST}\,E_{ST},$$

where, for each $S=(\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)})$,

$$(s_i)_{SS}=\begin{cases} \dfrac{1}{\diamondsuit_i(S,S)}, & \text{if} \quad \sigma^{(i-1)}\neq\sigma^{(i+1)}, \\ \dfrac{1}{\diamondsuit_i(S,S)}\left(1-\dfrac{P_{\sigma^{(i)}}(x)}{P_{\sigma^{(i-1)}}(x)}\right), & \text{if} \quad \sigma^{(i-1)}=\sigma^{(i+1)}, \end{cases}$$

and for each pair $(S,T)=((\sigma^{(i-1)}, \sigma^{(i)}, \sigma^{(i+1)}),\ (\sigma^{(i-1)}, \tau^{(i)}, \sigma^{(i+1)}))\in\Omega_{i-1}^{i+1}$ *such that* $S\neq T$,

$$(s_i)_{ST}=\begin{cases} \dfrac{\sqrt{(\diamondsuit_i(S,S)-1)(\diamondsuit_i(S,S)+1)}}{|\diamondsuit_i(S,S)|}, & \text{if} \quad \sigma^{(i-1)}\neq\sigma^{(i+1)}, \\ -\dfrac{1}{\diamondsuit_i(S,T)}\,\dfrac{\sqrt{P_{\sigma^{(i)}}(x)\,P_{\tau^{(i)}}(x)}}{P_{\sigma^{(i-1)}}(x)}, & \text{if} \quad \sigma^{(i-1)}=\sigma^{(i+1)}, \end{cases}$$

where $\diamondsuit_i(S,T)$ *is as given just before Theorem* (6.22).

Proof. It follows from the discussion above that, at $q=1$, the centralizer algebras $\mathscr{Z}_m$ are the same as the centralizer algebras $\widetilde{\mathscr{Z}}_m$. Note that the formulas in Theorem (5.14) all specialize to well defined rational numbers at $q=1$. Thus, there is an identification of the centralizer algebras $\widetilde{\mathscr{Z}}_m$ with the path algebras corresponding to the centralizer algebras so that the elements e_i and s_i are given by the formulas in Theorem (5.14) evaluated at $q=1$. These specializations are well defined and are equal to the formulas in the statement of Theorem (6.15) except with x replaced by $2r+1$.

The centralizer algebras $\widetilde{\mathscr{Z}}_m$ are quotients of the Brauer algebras $B_m(2r+1)$. If $r>m$ these algebras are isomorphic [Br]. Thus, it follows from the previous paragraph that there is an identification of the Brauer algebras $B_m(2r+1)$, $r>m$, with the path algebras corresponding to the centralizer algebras so that the elements e_i and s_i are given as in the above statement except with x replaced by $2r+1$. So Theorem (6.15) is true for an infinite number of specializations of the parameter x. The result follows. ∎

APPENDIX

Weight Polynomials

$$Q_\lambda(z,q)=\prod_{(i,i)\in\lambda}\frac{[y+\lambda_i-\lambda_i']+[h_\lambda(i,i)]}{[h_\lambda(i,i)]}\prod_{\substack{(i,j)\in\lambda\\ i\neq j}}\frac{[y+d_\lambda(i,j)]}{[h_\lambda(i,j)]}$$

$$P_\lambda(x)=\prod_{(i,j)\in\lambda}\frac{x-1+d_\lambda(i,j)}{h_\lambda(i,j)}$$

λ	$Q_\lambda(z,q)$	$P_\lambda(x)$
$\emptyset$	1	1
(1)	$\frac{[y+0]+[1]}{[1]}$	x
(1^2)	$\frac{[y-1]+[2]}{[2]}\,\frac{[y+0]}{[1]}$	$\frac{x(x-1)}{2}$
(2)	$\frac{[y+1]+[2]}{[2]}\,\frac{[y+0]}{[1]}$	$\frac{(x+2)(x-1)}{2}$
(1^3)	$\frac{[y-2]+[3]}{[3]}\,\frac{[y-1]}{[2]}\,\frac{[y+0]}{[1]}$	$\frac{x(x-2)(x-1)}{3!}$
$(2,1)$	$\frac{[y+0]+[3]}{[3]}\,\frac{[y+1]}{[1]}\,\frac{[y-1]}{[1]}$	$\frac{(x+2)\,x(x-2)}{3}$

(3)	$\dfrac{[y+2]+[3]}{[3]}\,\dfrac{[y+1]}{[2]}\,\dfrac{[y+0]}{[1]}$	$\dfrac{(x+4)\,x(x-1)}{3!}$
(1^4)	$\dfrac{[y-3]+[4]}{[4]}\,\dfrac{[y-2]}{[3]}\,\dfrac{[y-1]}{[2]}\,\dfrac{[y+0]}{[1]}$	$\dfrac{x(x-3)(x-2)(x-1)}{4!}$
$(2, 1^2)$	$\dfrac{[y-1]+[4]}{[4]}\,\dfrac{[y+1]}{[1]}\,\dfrac{[y-2]}{[2]}\,\dfrac{[y+0]}{[0]}$	$\dfrac{(x+1)\,x(x-3)(x-1)}{4\cdot 2}$
(2^2)	$\dfrac{[y+0]+[3]}{[3]}\,\dfrac{[y+0]+[1]}{[1]}\,\dfrac{[y+0]}{[2]}\,\dfrac{[y+0]}{[2]}$	$\dfrac{x(x+2)(x+1)(x-3)}{3\cdot 2\cdot 2}$
$(3, 1)$	$\dfrac{[y+1]+[4]}{[4]}\,\dfrac{[y+2]}{[2]}\,\dfrac{[y+0]}{[1]}\,\dfrac{[y-1]}{[1]}$	$\dfrac{(x+4)(x+1)(x-1)(x-2)}{4\cdot 2}$
(4)	$\dfrac{[y+3]+[4]}{[4]}\,\dfrac{[y+2]}{[3]}\,\dfrac{[y+1]}{[2]}\,\dfrac{[y+0]}{[1]}$	$\dfrac{(x+6)(x+1)\,x(x-1)}{4!}$

$$[n]=\frac{q^n-q^{-n}}{q-q^{-1}} \qquad h_\lambda(i,j)=\lambda_i-i+\lambda'_j-j+1$$

$$[y+n]=\frac{zq^n-z^{-1}q^{-1}}{q-q^{-1}} \qquad d_\lambda(i,j)=\begin{cases}\lambda_i+\lambda_j-i-j+1 & i\leqslant j\\ -\lambda'_i-\lambda'_j+i+j-1 & i>j\end{cases}$$

$$\lim_{q\to 1} Q_\lambda(q^{2r},q)=P_\lambda(2r+1)$$

Representations of $BW(z, q)$

We give the representing matrices $\pi^\lambda(g_i)$ for the irreducible representations π^λ of $BW_m(z, q)$, $m = 2, 3, 4$ with respect to a given ordered basis $v_1, v_2, ..., v_{d_\lambda}$ of $\mathscr{Z}^\lambda$ labeled by paths in the Bratelli diagram B. The representations π^λ of $BW_m(z, q)$ such that $|\lambda| = m$ are the irreducible representations of the Iwahori–Hecke algebra $H_m(q^2)$ of type A.

$BW_2(z, q)$

$$\pi^{\emptyset}(g_1) = \left(\frac{z^{-1}}{[-y]}\left(1 - \frac{Q_\Box}{Q_\emptyset}\right)\right) = (z^{-1}) \qquad \pi^{\ydiagram{1,1}}(g_1) = \left(\frac{q^{-1}}{[-1]}\right) \qquad \pi^{\ydiagram{2}}(g_1) = \left(\frac{q}{[1]}\right)$$

$$v_1 = (\emptyset, \Box, \emptyset) \qquad v_1 = \left(\emptyset, \Box, \ydiagram{1,1}\right) \qquad v_1 = (\emptyset, \Box, \ydiagram{2})$$

$BW_3(z, q)$

$$\pi^{\ydiagram{1,1,1}}(g_2) = \pi^{\ydiagram{1,1,1}}(g_1) = \left(\frac{q^{-1}}{[-1]}\right) \qquad v_1 = \left(\emptyset, \Box, \ydiagram{1,1}, \ydiagram{1,1,1}\right)$$

$$\pi^{\ydiagram{3}}(g_2) = \pi^{\ydiagram{3}}(g_1) = \left(\frac{q}{[1]}\right) \qquad v_1 = (\emptyset, \Box, \ydiagram{2}, \ydiagram{3})$$

$$\pi^{\ydiagram{2,1}}(g_1) = \begin{pmatrix} \frac{q^{-1}}{[-1]} & 0 \\ 0 & \frac{q}{[1]} \end{pmatrix} \qquad \pi^{\ydiagram{2,1}}(g_2) = \begin{pmatrix} \frac{q^2}{[2]} & \frac{\sqrt{[1]\cdot[3]}}{[2]} \\ \frac{\sqrt{[1]\cdot[3]}}{[2]} & \frac{q^{-2}}{[-2]} \end{pmatrix}$$

$$v_1 = \left(\emptyset, \Box, \ydiagram{1,1}, \ydiagram{2,1}\right)$$
$$v_2 = \left(\emptyset, \Box, \ydiagram{2}, \ydiagram{2,1}\right)$$

$$\pi^{\square}(g_1)=\begin{pmatrix} \frac{z^{-1}}{[-y]}\left(1-\frac{Q_{\square}}{Q_{\emptyset}}\right) & 0 & 0 \\ 0 & \frac{q^{-1}}{[-1]} & 0 \\ 0 & 0 & \frac{q}{[1]} \end{pmatrix}$$

$$\pi^{\square}(g_2)=\begin{pmatrix} \frac{z}{[y]}\left(1-\frac{Q_{\emptyset}}{Q_{\square}}\right) & -\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\substack{\square\\\square}}}}{Q_{\square}} & -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\square\square}}}{Q_{\square}} \\ -\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\substack{\square\\\square}}}}{Q_{\square}} & \frac{z^{-1}q^{2}}{[-y+2]}\left(1-\frac{Q_{\substack{\square\\\square}}}{Q_{\square}}\right) & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\substack{\square\\\square}}Q_{\square\square}}}{Q_{\square}} \\ -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\square\square}}}{Q_{\square}} & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\substack{\square\\\square}}Q_{\square\square}}}{Q_{\square}} & \frac{z^{-1}q^{-2}}{[-y-2]}\left(1-\frac{Q_{\square\square}}{Q_{\square}}\right) \end{pmatrix}$$

$$v_1=(\emptyset,\ \square,\ \emptyset,\ \square)$$

$$v_2=\left(\emptyset,\ \square,\ \begin{smallmatrix}\square\\\square\end{smallmatrix},\ \square\right)$$

$$v_3=(\emptyset,\ \square,\ \square\square,\ \square)$$

$BW_4(z,q)$

$$\pi^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(g_3)=\pi^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(g_2)=\pi^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(g_1)=\left(\frac{q^{-1}}{[-1]}\right)$$

$$v_1=\left(\emptyset,\ \square,\ \begin{smallmatrix}\square\\\square\end{smallmatrix},\ \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\ \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)$$

$$\pi^{\square\square\square}(g_3)=\pi^{\square\square\square}(g_2)=\pi^{\square\square\square}(g_1)=\left(\frac{q}{[1]}\right)$$

$$v_1=(\emptyset,\ \square,\ \square\square,\ \square\square\square,\ \square\square\square\square)$$

$$\pi^{\yng(2,1,1)}(g_1)=\begin{pmatrix} \frac{q^{-1}}{[-1]} & 0 & 0 \\ 0 & \frac{q^{-1}}{[-1]} & 0 \\ 0 & 0 & \frac{q}{[1]} \end{pmatrix} \qquad \pi^{\yng(2,1,1)}(g_2)=\begin{pmatrix} \frac{q^{-1}}{[-1]} & 0 & 0 \\ 0 & \frac{q^{2}}{[2]} & \frac{\sqrt{[1]\cdot[3]}}{[2]} \\ 0 & \frac{\sqrt{[1]\cdot[3]}}{[2]} & \frac{q^{-2}}{[-2]} \end{pmatrix}$$

$$v_1=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(1,1,1),\ \yng(2,1,1)\right)$$
$$v_2=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(2,1),\ \yng(2,1,1)\right)$$
$$v_3=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(2,1),\ \yng(2,1,1)\right)$$

$$\pi^{\yng(2,1,1)}(g_3)=\begin{pmatrix} \frac{q^{3}}{[3]} & \frac{\sqrt{[2]\cdot[4]}}{[3]} & 0 \\ \frac{\sqrt{[2]\cdot[4]}}{[3]} & \frac{q^{-3}}{[-3]} & 0 \\ 0 & 0 & \frac{q^{-1}}{[-1]} \end{pmatrix}$$

$$\pi^{\yng(2,2)}(g_3)=\pi^{\yng(2,2)}(g_1)=\begin{pmatrix} \frac{q^{-1}}{[-1]} & 0 \\ 0 & \frac{q}{[1]} \end{pmatrix} \qquad \pi^{\yng(2,2)}(g_2)=\begin{pmatrix} \frac{q^{2}}{[2]} & \frac{\sqrt{[1]\cdot[3]}}{[2]} \\ \frac{\sqrt{[1]\cdot[3]}}{[2]} & \frac{q^{-2}}{[-2]} \end{pmatrix}$$

$$v_1=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(2,1),\ \yng(2,2)\right)$$
$$v_2=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(2,1),\ \yng(2,2)\right)$$

$$\pi^{\ydiagram{3,1}}(g_1)=\begin{pmatrix}\frac{q^{-1}}{[-1]} & 0 & 0\\ 0 & \frac{q}{[1]} & 0\\ 0 & 0 & \frac{q}{[1]}\end{pmatrix}\qquad \pi^{\ydiagram{3,1}}(g_2)=\begin{pmatrix}\frac{q^{2}}{[2]} & \frac{\sqrt{[1]\cdot[3]}}{[2]} & 0\\ \frac{\sqrt{[1]\cdot[3]}}{[2]} & \frac{q^{-2}}{[-2]} & 0\\ 0 & 0 & \frac{q}{[1]}\end{pmatrix}$$

$$v_1=\left(\emptyset,\ \square,\ \ydiagram{1,1},\ \ydiagram{2,1},\ \ydiagram{3,1}\right)$$
$$v_2=\left(\emptyset,\ \square,\ \ydiagram{2},\ \ydiagram{2,1},\ \ydiagram{3,1}\right)$$
$$v_3=\left(\emptyset,\ \square,\ \ydiagram{2},\ \ydiagram{3},\ \ydiagram{3,1}\right)$$

$$\pi^{\ydiagram{3,1}}(g_3)=\begin{pmatrix}\frac{q}{[1]} & 0 & 0\\ 0 & \frac{q^{3}}{[3]} & \frac{\sqrt{[2]\cdot[4]}}{[3]}\\ 0 & \frac{\sqrt{[2]\cdot[4]}}{[3]} & \frac{q^{-3}}{[-3]}\end{pmatrix}$$

$$\pi^{\emptyset}(g_3)=\pi^{\emptyset}(g_1)=\begin{pmatrix}\frac{z^{-1}}{[-y]}\left(1-\frac{Q_{\square}}{Q_{\emptyset}}\right) & 0 & 0\\ 0 & \frac{q^{-1}}{[-1]} & 0\\ 0 & 0 & \frac{q}{[1]}\end{pmatrix}$$

$$v_1=(\emptyset,\ \square,\ \emptyset,\ \square,\ \emptyset)$$
$$v_2=\left(\emptyset,\ \square,\ \ydiagram{1,1},\ \square,\ \emptyset\right)$$
$$v_3=(\emptyset,\ \square,\ \ydiagram{2},\ \square,\ \emptyset)$$

$$\pi^{\emptyset}(g_2)=\begin{pmatrix} \frac{z}{[y]}\left(1-\frac{Q_{\emptyset}}{Q_{\square}}\right) & -\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\boxminus}}}{Q_{\square}} & -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\square\square}}}{Q_{\square}} \\ -\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\boxminus}}}{Q_{\square}} & \frac{z^{-1}q^{2}}{[-y+2]}\left(1-\frac{Q_{\boxminus}}{Q_{\square}}\right) & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\boxminus}Q_{\square\square}}}{Q_{\square}} \\ -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\square\square}}}{Q_{\square}} & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\boxminus}Q_{\square\square}}}{Q_{\square}} & \frac{z^{-1}q^{-2}}{[-y-2]}\left(1-\frac{Q_{\square\square}}{Q_{\square}}\right) \end{pmatrix}$$

$$\pi^{\boxminus}(g_1)=\begin{pmatrix} \frac{z^{-1}}{[-y]}\left(1-\frac{Q_{\square}}{Q_{\emptyset}}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q^{-1}}{[-1]} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q}{[1]} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q^{-1}}{[-1]} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q^{-1}}{[-1]} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q}{[1]} \end{pmatrix}$$

$$\pi^{\substack{\square\\\square}}(g_2)=\begin{pmatrix}
\frac{z}{[y]}\left(1-\frac{Q_{\emptyset}}{Q_{\square}}\right) & -\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\substack{\square\\\square}}}}{Q_{\square}} & -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\square\square}}}{Q_{\square}} & 0 & 0 & 0\\
-\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\substack{\square\\\square}}}}{Q_{\square}} & \frac{z^{-1}q^{2}}{[-y+2]}\left(1-\frac{Q_{\substack{\square\\\square}}}{Q_{\square}}\right) & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\substack{\square\\\square}}Q_{\square\square}}}{Q_{\square}} & 0 & 0 & 0\\
-\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\square\square}}}{Q_{\square}} & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\substack{\square\\\square}}Q_{\square\square}}}{Q_{\square}} & \frac{z^{-1}q^{-2}}{[-y-2]}\left(1-\frac{Q_{\square\square}}{Q_{\square}}\right) & 0 & 0 & 0\\
0 & 0 & 0 & \frac{q^{-1}}{[-1]} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{q^{2}}{[2]} & \frac{\sqrt{[1]\cdot[3]}}{[2]}\\
0 & 0 & 0 & 0 & \frac{\sqrt{[1]\cdot[3]}}{[2]} & \frac{q^{-2}}{[-2]}
\end{pmatrix}$$

$$\pi^{(1,1)}(g_3)=\begin{pmatrix} \frac{q^{-1}}{[-1]} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{zq^{-2}}{[y-2]}\left(1-\frac{Q_{(1)}}{Q_{(1,1)}}\right) & 0 & -\frac{q}{[1]}\frac{\sqrt{Q_{(1)}Q_{(1,1,1)}}}{Q_{(1,1)}} & -\frac{q^{-2}}{[-2]}\frac{\sqrt{Q_{(2,1)}Q_{(1)}}}{Q_{(1,1)}} & 0 \\ 0 & 0 & \frac{z}{[y]} & 0 & 0 & \frac{\sqrt{[y+1][y-1]}}{[y]} \\ 0 & -\frac{q}{[1]}\frac{\sqrt{Q_{(1)}Q_{(1,1,1)}}}{Q_{(1,1)}} & 0 & \frac{z^{-1}q^{4}}{[-y+4]}\left(1-\frac{Q_{(1,1,1)}}{Q_{(1,1)}}\right) & -\frac{z^{-1}q}{[-y+1]}\frac{\sqrt{Q_{(2,1)}Q_{(1,1,1)}}}{Q_{(1,1)}} & 0 \\ 0 & -\frac{q^{-2}}{[-2]}\frac{\sqrt{Q_{(2,1)}Q_{(1)}}}{Q_{(1,1)}} & 0 & -\frac{z^{-1}q}{[-y+1]}\frac{\sqrt{Q_{(2,1)}Q_{(1,1,1)}}}{Q_{(1,1)}} & \frac{z^{-1}q^{-2}}{[-y-2]}\left(1-\frac{Q_{(2,1)}}{Q_{(1,1)}}\right) & 0 \\ 0 & 0 & \frac{\sqrt{[y+1][y-1]}}{[y]} & 0 & 0 & \frac{z^{-1}}{[-y]} \end{pmatrix}$$

$$v_1=\left(\emptyset, (1), \emptyset, (1), (1,1)\right) \qquad v_4=\left(\emptyset, (1), (1,1), (1,1,1), (1,1)\right)$$

$$v_2=\left(\emptyset, (1), (1,1), (1), (1,1)\right) \qquad v_5=\left(\emptyset, (1), (1,1), (2,1), (1,1)\right)$$

$$v_3=\left(\emptyset, (1), (2), (1), (1,1)\right) \qquad v_6=\left(\emptyset, (1), (2), (2,1), (1,1)\right)$$

$$\pi^{\square\square}(g_1) = \begin{pmatrix} \frac{z^{-1}}{[-y]}\left(1 - \frac{Q_{\square}}{Q_{\emptyset}}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q^{-1}}{[-1]} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q}{[1]} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q^{-1}}{[-1]} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q^{-1}}{[1]} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q}{[1]} \end{pmatrix}$$

$$
\pi^{\Box\Box}(g_2)=\begin{pmatrix}
\frac{z}{[y]}\left(1-\frac{Q_{\emptyset}}{Q_{\Box}}\right) & -\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\substack{\Box\\\Box}}}}{Q_{\Box}} & -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\Box\Box}}}{Q_{\Box}} & 0 & 0 & 0 \\
-\frac{q}{[1]}\frac{\sqrt{Q_{\emptyset}Q_{\substack{\Box\\\Box}}}}{Q_{\Box}} & \frac{z^{-1}q^{2}}{[-y+2]}\left(1-\frac{Q_{\substack{\Box\\\Box}}}{Q_{\Box}}\right) & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\substack{\Box\\\Box}}Q_{\Box\Box}}}{Q_{\Box}} & 0 & 0 & 0 \\
-\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\emptyset}Q_{\Box\Box}}}{Q_{\Box}} & -\frac{z^{-1}}{[-y]}\frac{\sqrt{Q_{\substack{\Box\\\Box}}Q_{\Box\Box}}}{Q_{\Box}} & \frac{z^{-1}q^{-2}}{[-y-2]}\left(1-\frac{Q_{\Box\Box}}{Q_{\Box}}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{q^{2}}{[2]} & \frac{\sqrt{[1]\cdot[3]}}{2]} & 0 \\
0 & 0 & 0 & \frac{\sqrt{[1]\cdot[3]}}{[2]} & \frac{q^{-2}}{[-2]} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{q}{[1]}
\end{pmatrix}
$$

$$\pi^{\ydiagram{2}}(g_3)=\begin{pmatrix} \frac{q}{[1]} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{z}{[y]} & 0 & \frac{\sqrt{[y+1][y-1]}}{[y]} & 0 & 0 \\ 0 & 0 & \frac{zq^2}{[y+2]}\left(1-\frac{Q_{\ydiagram{1}}}{Q_{\ydiagram{2}}}\right) & 0 & -\frac{g^2}{[2]}\frac{\sqrt{Q_{\ydiagram{1}}Q_{\ydiagram{2,1}}}}{Q_{\ydiagram{2}}} & -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\ydiagram{1}}Q_{\ydiagram{3}}}}{Q_{\ydiagram{2}}} \\ 0 & \frac{\sqrt{[y+1][y-1]}}{[y]} & 0 & \frac{z^{-1}}{[-y]} & 0 & 0 \\ 0 & 0 & -\frac{q^2}{[2]}\frac{\sqrt{Q_{\ydiagram{1}}Q_{\ydiagram{2,1}}}}{Q_{\ydiagram{2}}} & 0 & \frac{z^{-1}q^2}{[-y+2]}\left(1-\frac{Q_{\ydiagram{2,1}}}{Q_{\ydiagram{2}}}\right) & -\frac{z^{-1}q^{-1}}{[-y-1]}\frac{\sqrt{Q_{\ydiagram{2,1}}Q_{\ydiagram{4}}}}{QA_{\ydiagram{2}}} \\ 0 & 0 & -\frac{q^{-1}}{[-1]}\frac{\sqrt{Q_{\ydiagram{1}}Q_{\ydiagram{3}}}}{Q_{\ydiagram{2}}} & 0 & -\frac{z^{-1}q^{-1}}{[-y-1]}\frac{\sqrt{Q_{\ydiagram{2,1}}Q_{\ydiagram{4}}}}{Q_{\ydiagram{2}}} & \frac{z^{-1}q^{-4}}{[-y-4]}\left(1-\frac{Q_{\ydiagram{3}}}{Q_{\ydiagram{2}}}\right) \end{pmatrix}$$

$$v_1=(\emptyset,\ \ydiagram{1},\ \emptyset,\ \ydiagram{1},\ \ydiagram{2}) \qquad v_4=\left(\emptyset,\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{2,1},\ \ydiagram{2}\right)$$

$$v_2=\left(\emptyset,\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{1},\ \ydiagram{2}\right) \qquad v_5=\left(\emptyset,\ \ydiagram{1},\ \ydiagram{2},\ \ydiagram{2,1},\ \ydiagram{2}\right)$$

$$v_3=(\emptyset,\ \ydiagram{1},\ \ydiagram{2},\ \ydiagram{1},\ \ydiagram{2}) \qquad v_6=(\emptyset,\ \ydiagram{1},\ \ydiagram{2},\ \ydiagram{3},\ \ydiagram{2})$$

Representations of $B(x)$

We give the representing matrices $\pi^{\lambda}(s_i)$ and $\pi^{\lambda}(e_i)$ for the irreducible representations of π^{λ} of $B_m(x)$, $m=2, 3, 4$ with respect to a given ordered basis $v_1, v_2, ..., v_{d_\lambda}$ of Z^{λ} labeled by paths in the Bratelli diagram B. The representations π^{λ} of $B_m(x)$ such that $|\lambda|=m$ are the irreducible representations of the group algebra of the symmetric group S_m. In these cases $\pi^{\lambda}(e_i)=0$ for all $1 \leqslant i \leqslant m-1$.

$B_2(x)$

$$\pi^{\emptyset}(s_1)=\left(\frac{1}{1-x}\left(1-\frac{P_{\square}}{P_{\emptyset}}\right)\right)=(1) \qquad \pi^{(1,1)}(s_1)=(-1) \qquad \pi^{(2)}(s_1)=(1)$$

$$\pi^{\emptyset}(e_1)=\left(\frac{P_{\square}}{P_{\emptyset}}\right)=(x) \qquad v_1=(\emptyset, \square, (1,1)) \qquad v_1=(\emptyset, \square, (2))$$

$$v_1=(\emptyset, \square, \emptyset)$$

$B_3(x)$

$$\pi^{(1,1,1)}(s_2)=\pi^{(1,1,1)}(s_1)=(-1) \qquad v_1=(\emptyset, \square, (1,1), (1,1,1))$$

$$\pi^{(3)}(s_2)=\pi^{(3)}(s_1)=(1) \qquad v_1=(\emptyset, \square, (2), (3))$$

$$\pi^{(2,1)}(s_1)=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \pi^{(2,1)}(s_2)=\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{1\cdot 3}}{2} \\ \frac{\sqrt{1\cdot 3}}{2} & \frac{1}{-2} \end{pmatrix} \qquad \begin{aligned} v_1&=(\emptyset, \square, (1,1), (2,1)) \\ v_2&=(\emptyset, \square, (3), (2,1)) \end{aligned}$$

$$v_1 = (\emptyset,\ \square,\ \emptyset,\ \square)$$

$$v_2 = \left(\emptyset,\ \square,\ \substack{\square\\\square},\ \square\right)$$

$$v_3 = (\emptyset,\ \square,\ \square\square,\ \square)$$

$$\pi^{\square}(s_1) = \begin{pmatrix} \frac{1}{1-x}\left(1-\frac{P_{\square}}{P_{\emptyset}}\right) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \pi^{\square}(e_1) = \begin{pmatrix} \frac{P_{\square}}{P_{\emptyset}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\pi^{\square}(s_2) = \begin{pmatrix} \frac{1}{x-1}\left(1-\frac{P_{\emptyset}}{P_{\square}}\right) & -\frac{\sqrt{P_{\emptyset}P_{\substack{\square\\\square}}}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} \\ -\frac{\sqrt{P_{\emptyset}P_{\substack{\square\\\square}}}}{P_{\square}} & \frac{1}{3-x}\left(1-\frac{P_{\substack{\square\\\square}}}{P_{\square}}\right) & -\frac{1}{1-x}\frac{\sqrt{P_{\substack{\square\\\square}}P_{\square\square}}}{P_{\square}} \\ \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} & -\frac{1}{1-x}\frac{\sqrt{P_{\substack{\square\\\square}}P_{\square\square}}}{P_{\square}} & \frac{1}{-(x+1)}\left(1-\frac{P_{\square\square}}{P_{\square}}\right) \end{pmatrix}$$

$$\pi^{\square}(e_2) = \begin{pmatrix} \frac{P_{\emptyset}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\substack{\square\\\square}}}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} \\ \frac{\sqrt{P_{\substack{\square\\\square}}P_{\emptyset}}}{P_{\square}} & \frac{P_{\substack{\square\\\square}}}{P_{\square}} & \frac{\sqrt{P_{\substack{\square\\\square}}P_{\square\square}}}{P_{\square}} \\ \frac{\sqrt{P_{\square\square}P_{\emptyset}}}{P_{\square}} & \frac{\sqrt{P_{\square\square}P_{\substack{\square\\\square}}}}{P_{\square}} & \frac{P_{\square\square}}{P_{\square}} \end{pmatrix}$$

$B_4(x)$

$$\pi^{\yng(1,1,1,1)}(s_3)=\pi^{\yng(1,1,1,1)}(s_2)=\pi^{\yng(1,1,1,1)}(s_1)=(-1) \qquad v_1=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(1,1,1),\ \yng(1,1,1,1)\right)$$

$$\pi^{\yng(4)}(s_3)=\pi^{\yng(4)}(s_2)=\pi^{\yng(4)}(s_1)=(1) \qquad v_1=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(3),\ \yng(4)\right)$$

$$\pi^{\yng(2,1,1)}(s_1)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \pi^{\yng(2,1,1)}(s_2)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{1\cdot 3}}{2} \\ 0 & \frac{\sqrt{1\cdot 3}}{2} & \frac{1}{-2} \end{pmatrix}$$

$$v_1=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(1,1,1),\ \yng(2,1,1)\right)$$

$$v_2=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(2,1),\ \yng(2,1,1)\right)$$

$$v_3=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(2,1),\ \yng(2,1,1)\right)$$

$$\pi^{\yng(2,1,1)}(s_3)=\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2\cdot 4}}{3} & 0 \\ \frac{\sqrt{2\cdot 4}}{3} & \frac{1}{-3} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\pi^{\yng(2,2)}(s_3)=\pi^{\yng(2,2)}(s_1)=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \pi^{\yng(2,2)}(s_2)=\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{1\cdot 3}}{2} \\ \frac{\sqrt{1\cdot 3}}{2} & \frac{1}{-2} \end{pmatrix}$$

$$v_1=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(2,1),\ \yng(2,2)\right)$$

$$v_2=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(2,1),\ \yng(2,2)\right)$$

$$v_1 = \left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(2,1),\ \yng(3,1)\right)$$

$$v_2 = \left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(2,1),\ \yng(3,1)\right)$$

$$v_3 = \left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(3),\ \yng(3,1)\right)$$

$$\pi^{\yng(3,1)}(s_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \pi^{\yng(3,1)}(s_2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{1\cdot 3}}{2} & 0 \\ \frac{\sqrt{1\cdot 3}}{2} & \frac{1}{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\pi^{\yng(3,1)}(s_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{2\cdot 4}}{3} \\ 0 & \frac{\sqrt{2\cdot 4}}{3} & \frac{1}{-3} \end{pmatrix}$$

$$v_1 = (\emptyset,\ \square,\ \emptyset,\ \square,\ \emptyset)$$

$$v_2 = \left(\emptyset,\ \square,\ \substack{\square\\\square},\ \square,\ \emptyset\right)$$

$$v_3 = (\emptyset,\ \square,\ \square\square,\ \square,\ \emptyset)$$

$$\pi^{\emptyset}(s_3) = \pi^{\emptyset}(s_1) = \begin{pmatrix} \dfrac{1}{1-x}\left(1 - \dfrac{P_{\square}}{P_{\emptyset}}\right) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \pi^{\emptyset}(e_3) = \pi^{\emptyset}(e_1) = \begin{pmatrix} \dfrac{P_{\square}}{P_{\emptyset}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\pi^{\emptyset}(s_2) = \begin{pmatrix} \dfrac{1}{x-1}\left(1 - \dfrac{P_{\emptyset}}{P_{\square}}\right) & -\dfrac{\sqrt{P_{\emptyset} P_{\substack{\square\\\square}}}}{P_{\square}} & \dfrac{\sqrt{P_{\emptyset} P_{\square\square}}}{P_{\square}} \\ -\dfrac{\sqrt{P_{\emptyset} P_{\substack{\square\\\square}}}}{P_{\square}} & \dfrac{1}{3-x}\left(1 - \dfrac{P_{\substack{\square\\\square}}}{P_{\square}}\right) & -\dfrac{1}{1-x}\dfrac{\sqrt{P_{\substack{\square\\\square}} P_{\square\square}}}{P_{\square}} \\ \dfrac{\sqrt{P_{\emptyset} P_{\square\square}}}{P_{\square}} & -\dfrac{1}{1-x}\dfrac{\sqrt{P_{\substack{\square\\\square}} P_{\square\square}}}{P_{\square}} & \dfrac{1}{-(x+1)}\left(1 - \dfrac{P_{\square\square}}{P_{\square}}\right) \end{pmatrix}$$

$$\pi^{\emptyset}(e_2) = \begin{pmatrix} \dfrac{P_{\emptyset}}{P_{\square}} & \dfrac{\sqrt{P_{\emptyset} P_{\substack{\square\\\square}}}}{P_{\square}} & \dfrac{\sqrt{P_{\emptyset} P_{\square\square}}}{P_{\square}} \\ \dfrac{\sqrt{P_{\substack{\square\\\square}} P_{\emptyset}}}{P_{\square}} & \dfrac{P_{\substack{\square\\\square}}}{P_{\square}} & \dfrac{\sqrt{P_{\substack{\square\\\square}} P_{\square\square}}}{P_{\square}} \\ \dfrac{\sqrt{P_{\square\square} P_{\emptyset}}}{P_{\square}} & \dfrac{\sqrt{P_{\square\square} P_{\substack{\square\\\square}}}}{P_{\square}} & \dfrac{P_{\square\square}}{P_{\square}} \end{pmatrix}$$

$$\pi^{\boxminus}(s_1) = \begin{pmatrix} \frac{1}{1-x}\left(1-\frac{P_{\square}}{P_{\emptyset}}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi^{\boxminus}(s_2) = \begin{pmatrix} \frac{1}{x-1}\left(1-\frac{P_{\emptyset}}{P_{\square}}\right) & -\frac{\sqrt{P_{\emptyset}P_{\boxminus}}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} & 0 & 0 & 0 \\ -\frac{\sqrt{P_{\emptyset}P_{\boxminus}}}{P_{\square}} & \frac{1}{3-x}\left(1-\frac{P_{\boxminus}}{P_{\square}}\right) & -\frac{1}{1-x}\frac{\sqrt{P_{\boxminus}P_{\square\square}}}{P_{\square}} & 0 & 0 & 0 \\ \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} & -\frac{1}{1-x}\frac{\sqrt{P_{\boxminus}P_{\square\square}}}{P_{\square}} & \frac{1}{-(x+1)}\left(1-\frac{P_{\square\square}}{P_{\square}}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{1\cdot 3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{1\cdot 3}}{2} & \frac{1}{-2} \end{pmatrix}$$

$$\pi^{(1,1)}(s_3)=\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{x-3}\left(1-\frac{P_{\square}}{P_{(1,1)}}\right) & 0 & -\frac{\sqrt{P_{\square}P_{(1,1,1)}}}{P_{(1,1)}} & -\frac{1}{-2}\frac{\sqrt{P_{(2,1)}P_{\square}}}{P_{(1,1)}} & 0\\
0 & 0 & \frac{1}{x-1} & 0 & 0 & \frac{\sqrt{x(x-2)}}{x-1}\\
0 & -\frac{\sqrt{P_{\square}P_{(1,1,1)}}}{P_{(1,1)}} & 0 & \frac{1}{5-x}\left(1-\frac{P_{(1,1,1)}}{P_{(1,1)}}\right) & -\frac{1}{2-x}\frac{\sqrt{P_{(2,1)}P_{(1,1,1)}}}{P_{(1,1)}} & 0\\
0 & -\frac{1}{-2}\frac{\sqrt{P_{(2,1)}P_{\square}}}{P_{(1,1)}} & 0 & -\frac{1}{2-x}\frac{\sqrt{P_{(2,1)}P_{(1,1,1)}}}{P_{(1,1)}} & \frac{1}{-(x+1)}\left(1-\frac{P_{(2,1)}}{P_{(1,1)}}\right) & 0\\
0 & 0 & \frac{\sqrt{x(x-2)}}{x-1} & 0 & 0 & \frac{1}{1-x}
\end{pmatrix}$$

$$v_1=\left(\emptyset,\ \square,\ \emptyset,\ \square,\ (1,1)\right)\qquad v_4=\left(\emptyset,\ \square,\ (1,1),\ (1,1,1),\ (1,1)\right)$$

$$v_2=\left(\emptyset,\ \square,\ (1,1),\ \square,\ (1,1)\right)\qquad v_5=\left(\emptyset,\ \square,\ (1,1),\ (2,1),\ (1,1)\right)$$

$$v_3=\left(\emptyset,\ \square,\ (2),\ \square,\ (1,1)\right)\qquad v_6=\left(\emptyset,\ \square,\ (2),\ (2,1),\ (1,1)\right)$$

$$\pi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(e_1)=\begin{pmatrix} \frac{P_{\square}}{P_{\emptyset}} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\pi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(e_2)=\begin{pmatrix} \frac{P_{\emptyset}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} & 0 & 0 & 0\\ \frac{\sqrt{P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}P_{\emptyset}}}{P_{\square}} & \frac{P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}{P_{\square}} & \frac{\sqrt{P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}P_{\square\square}}}{P_{\square}} & 0 & 0 & 0\\ \frac{\sqrt{P_{\square\square}P_{\emptyset}}}{P_{\square}} & \frac{\sqrt{P_{\square\square}P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}}{P_{\square}} & \frac{P_{\square\square}}{P_{\square}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\pi^{\yng(1,1)}(e_3)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P_{\yng(1)}}{P_{\yng(1,1)}} & 0 & \frac{\sqrt{P_{\yng(1)}P_{\yng(1,1,1)}}}{P_{\yng(1,1)}} & \frac{\sqrt{P_{\yng(1)}P_{\yng(2,1)}}}{P_{\yng(1,1)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{P_{\yng(1,1,1)}P_{\yng(1)}}}{P_{\yng(1,1)}} & 0 & \frac{P_{\yng(1,1,1)}}{P_{\yng(1,1)}} & \frac{\sqrt{P_{\yng(1,1,1)}P_{\yng(2,1)}}}{P_{\yng(1,1)}} & 0 \\ 0 & \frac{\sqrt{P_{\yng(2,1)}P_{\yng(1)}}}{P_{\yng(1,1)}} & 0 & \frac{\sqrt{P_{\yng(2,1)}P_{\yng(1,1,1)}}}{P_{\yng(1,1)}} & \frac{P_{\yng(2,1)}}{P_{\yng(1,1)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$v_1=\left(\emptyset,\ \yng(1),\ \emptyset,\ \yng(1),\ \yng(1,1)\right) \qquad v_4=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(1,1,1),\ \yng(1,1)\right)$$

$$v_2=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(1),\ \yng(1,1)\right) \qquad v_5=\left(\emptyset,\ \yng(1),\ \yng(1,1),\ \yng(2,1),\ \yng(1,1)\right)$$

$$v_3=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(1),\ \yng(1,1)\right) \qquad v_6=\left(\emptyset,\ \yng(1),\ \yng(2),\ \yng(2,1),\ \yng(1,1)\right)$$

$$\pi^{\square\square}(s_1)=\begin{pmatrix} \frac{1}{1-x}\left(1-\frac{P_{\square}}{P_{\emptyset}}\right) & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi^{\square\square}(s_2)=\begin{pmatrix} \frac{1}{x-1}\left(1-\frac{P_{\emptyset}}{P_{\square}}\right) & -\frac{\sqrt{P_{\emptyset}P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}}{P_{\square}} & \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} & 0 & 0 & 0\\ -\frac{\sqrt{P_{\emptyset}P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}}{P_{\square}} & \frac{1}{3-x}\left(1-\frac{P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}{P_{\square}}\right) & -\frac{1}{1-x}\frac{\sqrt{P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}P_{\square\square}}}{P_{\square}} & 0 & 0 & 0\\ \frac{\sqrt{P_{\emptyset}P_{\square\square}}}{P_{\square}} & -\frac{1}{1-x}\frac{\sqrt{P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}P_{\square\square}}}{P_{\square}} & \frac{1}{-(x+1)}\left(1-\frac{P_{\square\square}}{P_{\square}}\right) & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{1\cdot 3}}{2} & 0\\ 0 & 0 & 0 & \frac{\sqrt{1\cdot 3}}{2} & \frac{1}{-2} & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi^{\Box\!\Box}(s_3)=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{x-1} & 0 & \frac{\sqrt{x(x-2)}}{x-1} & 0 & 0\\
0 & 0 & \frac{1}{x+1}\left(1-\frac{P_{\Box}}{P_{\Box\!\Box}}\right) & 0 & -\frac{1}{2}\frac{\sqrt{P_{\Box}P_{\begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix}}}}{P_{\Box\!\Box}} & \frac{\sqrt{P_{\Box}P_{\Box\!\Box\!\Box}}}{P_{\Box\!\Box}}\\
0 & \frac{\sqrt{x(x-2)}}{x-1} & 0 & \frac{1}{1-x} & 0 & 0\\
0 & 0 & -\frac{1}{2}\frac{\sqrt{P_{\Box}P_{\begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix}}}}{P_{\Box\!\Box}} & 0 & \frac{1}{3-x}\left(1-\frac{P_{\begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix}}}{P_{\Box\!\Box}}\right) & -\frac{1}{-x}\frac{\sqrt{P_{\begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix}}P_{\Box\!\Box\!\Box}}}{P_{\Box\!\Box}}\\
0 & 0 & \frac{\sqrt{P_{\Box}P_{\Box\!\Box\!\Box}}}{P_{\Box\!\Box}} & 0 & -\frac{1}{-x}\frac{\sqrt{P_{\begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix}}P_{\Box\!\Box\!\Box}}}{P_{\Box\!\Box}} & \frac{1}{-(x+3)}\left(1-\frac{P_{\Box\!\Box\!\Box\!\Box}}{P_{\Box\!\Box}}\right)
\end{pmatrix}$$

$$v_1=(\emptyset,\ \Box,\ \emptyset,\ \Box,\ \Box\!\Box) \qquad v_4=\left(\emptyset,\ \Box,\ \begin{smallmatrix}\Box\\ \Box\end{smallmatrix},\ \begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix},\ \Box\!\Box\right)$$

$$v_2=\left(\emptyset,\ \Box,\ \begin{smallmatrix}\Box\\ \Box\end{smallmatrix},\ \Box,\ \Box\!\Box\right) \qquad v_5=\left(\emptyset,\ \Box,\ \Box\!\Box,\ \begin{smallmatrix}\Box\!\Box\\ \Box\,\end{smallmatrix},\ \Box\!\Box\right)$$

$$v_3=(\emptyset,\ \Box,\ \Box\!\Box,\ \Box,\ \Box\!\Box) \qquad v_6=(\emptyset,\ \Box,\ \Box\!\Box,\ \Box\!\Box\!\Box,\ \Box\!\Box)$$

$$
\pi^{\square\square}(e_1) = \begin{pmatrix}
\frac{P_{\square}}{P_{\emptyset}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\pi^{\square\square}(e_2) = \begin{pmatrix}
\frac{P_{\emptyset}}{P_{\square}} & \frac{\sqrt{P_{\emptyset} P_{\substack{\square\\\square}}}}{P_{\square}} & \frac{\sqrt{P_{\emptyset} P_{\square\square}}}{P_{\square}} & 0 & 0 & 0 \\
\frac{\sqrt{P_{\substack{\square\\\square}} P_{\emptyset}}}{P_{\square}} & \frac{P_{\substack{\square\\\square}}}{P_{\square}} & \frac{\sqrt{P_{\substack{\square\\\square}} P_{\square\square}}}{P_{\square}} & 0 & 0 & 0 \\
\frac{\sqrt{P_{\square\square} P_{\emptyset}}}{P_{\square}} & \frac{\sqrt{P_{\square\square} P_{\substack{\square\\\square}}}}{P_{\square}} & \frac{P_{\square\square}}{P_{\square}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\pi^{\ydiagram{2}}(e_3)=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{P_{\ydiagram{1}}}{P_{\ydiagram{2}}} & 0 & \frac{\sqrt{P_{\ydiagram{1}}P_{\ydiagram{2,1}}}}{P_{\ydiagram{2}}} & \frac{\sqrt{P_{\ydiagram{1}}P_{\ydiagram{3}}}}{P_{\ydiagram{2}}}\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{P_{\ydiagram{2,1}}P_{\ydiagram{1}}}}{P_{\ydiagram{2}}} & 0 & \frac{P_{\ydiagram{2,1}}}{P_{\ydiagram{2}}} & \frac{\sqrt{P_{\ydiagram{2,1}}P_{\ydiagram{3}}}}{P_{\ydiagram{2}}}\\
0 & 0 & \frac{\sqrt{P_{\ydiagram{3}}P_{\ydiagram{1}}}}{P_{\ydiagram{2}}} & 0 & \frac{\sqrt{P_{\ydiagram{3}}P_{\ydiagram{2,1}}}}{P_{\ydiagram{2}}} & \frac{P_{\ydiagram{3}}}{P_{\ydiagram{2}}}
\end{pmatrix}$$

$$v_1=(\emptyset,\ \ydiagram{1},\ \emptyset,\ \ydiagram{1},\ \ydiagram{2}) \qquad v_4=\left(\emptyset,\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{2,1},\ \ydiagram{2}\right)$$

$$v_2=\left(\emptyset,\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{1},\ \ydiagram{2}\right) \qquad v_5=\left(\emptyset,\ \ydiagram{1},\ \ydiagram{2},\ \ydiagram{2,1},\ \ydiagram{2}\right)$$

$$v_3=(\emptyset,\ \ydiagram{1},\ \ydiagram{2},\ \ydiagram{1},\ \ydiagram{2}) \qquad v_6=(\emptyset,\ \ydiagram{1},\ \ydiagram{2},\ \ydiagram{3},\ \ydiagram{2})$$

ACKNOWLEDGMENTS

We thank Georgia Benkart and Louis Solomon for numerous interesting discussions and their avid interest in our work. R. Leduc expresses his gratitude to Georgia Benkart for graudate student support from her National Science Foundation grant. A. Ram thanks all of the members of the audience of the quantum groups course at the University of Wisconsin during the spring of 1994 for their interest and helpful suggestions, and for coming three days a week so early in the morning.

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