Quantum groups:
A survey of definitions, motivations, and results

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Introduction

I. Hopf algebras and quasitriangular Hopf algebras

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Introduction

(1.1) Goal of this survey

The theory of quantum groups began its development in about 1982-1985. It is now 10 years since the 1986 ICM address of V.G. Drinfel'd ignited a wild frenzy of research activity in this area and things related to it. During this time quantum groups have become a "household" term in Lie theory in much the same way that Kac-Moody Lie algebras did in the 1970's. Given that quantum groups are now a part of every day Lie theory it seems desirable that there are treatments of the subject which are accessible to graduate students.

It has been my goal to produce a survey which is accessible to graduate students, and which contains the necessary background and the main results in the theory. I have chosen to make this a compendium of motivation, definitions and results. A secondary goal has been to write this in a relatively small space (long works are usually too daunting) and with this in mind I have chosen not to include any proofs. In many cases, providing a full proof would require introducing and developing some fairly sophisticated tools.

My main focus in these notes is to give a description of what the Drinfel'd-Jimbo quantum groups are, how one arrives at them and why they are natural. In the last chapter I shall explain how the Drinfel'd-Jimbo quantum groups are applied to get link invariants such as the Jones polynomial.

(1.2) References for quantum groups

Drinfel'd's paper in the proceedings of the ICM 1986 is a dense summary of many of the amazing results that he had obtained. This paper still remains a basic reference.

[Dr] V.G. Drinfeld, *Quantum Groups*, in Proceedings of the International Congress of Mathematicians, A.M. Gleason ed., pp. 798-820, American Mathematical Society, Providence 1987.

Between 1987 and 1995 literally thousands of papers on quantum groups have been published. The book by V. Chari and A. Pressley which appeared in 1994 has 70 pages of references in minuscule type! Instead of wading through this mass of literature I have decided to only refer you to the books on quantum groups which have begun to appear recently, as follows:

[CP] V. Chari and A. Pressley, "A Guide to Quantum Groups", Cambridge University Press, Cambridge, 1994.

[Ja] J. Jantzen, "Lectures on Quantum Groups", Graduate Studies in Mathematics Vol. **6**, American Mathematical Society, 1995.

[Jo] A. Joseph, "Quantum groups and their Primitive Ideals", Ergebnisse der Mathematik und ihrer Grenzgebiete; **3** Folge, Bd. **29**, Springer-Verlag, New York-Berlin, 1995.

[Ka] C. Kassel, “Quantum groups”, Graduate Texts in Mathematics **155**, Springer-Verlag, New York, 1995.

[Lu] G. Lusztig, “Introduction to Quantum Groups”, Progress in Mathematics **110**, Birkhauser, Boston, 1993.

[Ma] S. Majid, “Foundations of quantum group theory”, Cambridge University Press, 1995.

[SS] S. Shnider and S. Sternberg, “Quantum groups: From Coalgebras to Drinfel’d Algebras”, Graduate Texts in Mathematical Physics Vol. **2**, International Press, Cambridge, MA 1995.

I recommend [CP] for obtaining a basic understanding of what quantum groups are, where they came from, what the main results are, and what was known as of about the end of 1993. It contains only easy proofs and sketches of more involved proofs, very often referring the reader to the original papers for the full details of proofs. This book, however, is very useful for understanding what is going on. The recent book [Ja] is written specifically for graduate students. It has an excellent choice of topics, thorough descriptions of the motivations at each stage and detailed proofs. The book [SS] treats the deformation theory aspect of quantum groups in detail and the book [Lu] is the only one that covers the connection between the quantum group and perverse sheaves.

(1.3) Some missing topics and where to find them

There are many beautiful things in the theory of quantum groups that we won’t even have time to mention. A *few* of these are:

(a) Canonical and crystal bases and the Littelmann path model for representations, see [Jo] Chapt. 5-6 and [Ja] Chapt. 9-11.

(b) Yangians, see [CP] Chapt. 12.

(c) Quasi-Hopf algebras and twisting, see [CP] Chapt. 16 and [SS] Chapt. 8.

(d) The Knizhnik-Zamalodchikov equation and hypergeometric functions, see [CP] Chapt. 16, [Ka] Chapt. 19 and [SS] Chapt. 12.

(e) Lie bialgebras, Poisson Lie groups, and symplectic leaves, see [CP] Chapt. 1.

(f) Representations at roots of unity and the connection to representations of algebraic groups over a finite field, see [CP] Chapt. 11 and [AJS].

(g) The connection between representations of quantum groups at roots of unity and representations of affine Lie algebras at negative level, see [CP] Chapt. 11 and Chapt. 16 and [KL].

(1.4) Further references for the background topics

Chapters I-IV consist of background material needed for the material on quantum groups. These chapters are:

I. Hopf algebras and braided tensor categories
II. Lie algebras and enveloping algebras
III. Deformations of Hopf algebras
IV. Perverse Sheaves

The following book contains a very nice up-to-date account of the theory of Hopf algebras, and it also includes some useful things on quantum groups.

[Mo] S. Montgomery, "Hopf Algebras and their Actions on Rings", Regional Conference Series in Mathematics **82**, American Mathematical Society, 1992.

The book by Chari and Pressley [CP] contains a nice introduction to monoidal categories and braided monoidal categories.

The following little book is a beautiful summary of the main results in semisimple Lie theory.

[Se] J.-P. Serre, "Complex Semisimple Lie algebras", Springer-Verlag, New York, 1987.

Comprehensive accounts of the theory of Lie algebras and enveloping algebras can be found in Bourbaki and in the book by Dixmier.

[Bou] N. Bourbaki, "Groupes et Algèbres de Lie, Chapitres I-VIII", Masson, Paris, 1972.

[Dix] J. Dixmier, "Enveloping algebras", Amer. Math. Soc. (1994); originally published in French by Gauthier-Villars, Paris 1974 and in English by North Holland, Amsterdam 1977.

The following are standard (and very useful) texts in Lie theory.

[Hu] J. Humphreys, "Introduction to Lie algebras and representation theory", Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Berlin, (3rd printing) 1980.

[K] V. Kac, "Infinite dimensional Lie algebras", Birkhauser, Boston, 1983.

The most comprehensive reference for modern deformation theory, especially in regard to deformations of Hopf algebras, is the book by Shnider and Sternberg [SS] listed above. The book [CP] also contains a very informative chapter on deformation theory.

Unfortunately, to my knowledge, there is no good introductory text on the theory of perverse sheaves. The classical reference is the following monograph.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982), Soc. Math. France.

On the other hand, much of the background material to perverse sheaves, such as homological algebra and sheaf theory is classical and appears in many books. The first few chapters of the following book contain an introduction to these topics.

[KS] M. Kashiwara and P. Schapira, “Sheaves on Manifolds”, Grundlehren der mathematischen Wissenschaften **292**, Springer-Verlag, New York-Berlin, 1980.

(1.5) On reading these notes

I advise the reader to begin immediately with Chapter V and find out what a quantum group is. One can always peek back at the earlier chapters and find out the definitions later. This makes it more fun and provides good motivation for learning the earlier background material. It also avoids getting bogged down before one even gets to the quantum group.

In a number of places I have chosen to make these notes “nonlinear”. There have been some occasions when I have decided to repeat some definition or some statement. Also in a few places, I have used some terms and notations that have not been defined yet, with an appropriate reference to the place later in the text where the definitions and notations can be found. I have done this with the intention of making each section a somewhat complete set of ideas without disrupting any particular section with a myriad of lengthy definitions. Even though we may wish it so, ideas in mathematics are not really linear and this has been reflected in these notes. *The reader should feel free to skip around in the notes whenever the inclination arises.*

I have included a complete table of contents in the hope that it will be helpful to the reader as a tool for finding definitions and for organizing and motivating the structures. For the same reason I have given every small section a title. This way the reader can follow the process of the development, as well as the details. Think of the table of contents as a flow chart for the mathematics.

(1.6) Disclaimer

Even though the theory of quantum groups is less than 15 years old I shall not undertake the complicated task of giving appropriate references and credits concerning the sources of the theorems and their first proofs. I refer the reader to the above books on quantum groups for this information.

Let me stress that none of the theorems stated in this manuscript are due to me with two possible exceptions. Chapt. I Proposition (5.5) and Chapt. VII Theorem (5.2) are more general than I know of in the existing literature. Chapt. I Proposition (5.5) is well known in the context of the quantum group and I am only pointing out here that the well known proof, see [Ta] Prop. 2.2.1, works for any quantum double. Chapt. VII Theorem (5.2) is a nontrivial, but very natural, extension of well known results which appear, for example, in [Ja] Chapt. 8. The crucial part of the proof is similar to the proof of [Ja] Lemma 8.3.

I have tried to indicate, at the beginning of each chapter, where one can find proofs of the theorems stated in that chapter. In many instances I have had to make minor changes in notations and statements in order to be consistent with the definitions that I have given. Especially since I have not included proofs the reader should be watchful and open to the possibility that there may be some minor errors.

(1.7) Acknowledgements

First and foremost I thank Hans Wenzl for introducing me to the world of quantum groups and encouraging me to pursue research in related topics. He taught me the basics of quantum group theory and notes from a course he gave at University of California, San Diego have been tremendously useful over the years. I thank all of the audience members in my course in quantum groups at University of Wisconsin during Spring of 1994 for their interest, their suggestions and for coming so early in the morning.

I thank Gus Lehrer for inviting me to Australia, for making my year there a wonderful one and for suggesting my name for various invitations via which these notes have come into being. I thank Chuck Miller and John Cossey for the invitation to give a series of lectures on quantum groups to graduate students at the Workshop on Algebra, Geometry and Topology at Australian National University in Canberra during January 1996. These notes are an expanded version of the notes I distributed there. I thank Michael Murray and Alan Carey for inviting me to speak at the Australian Lie Groups Conference 1996 and for inviting me to contribute to these proceedings. Finally, I thank Dave Benson for some very helpful proofreading.

I. Hopf algebras and quasitriangular Hopf algebras

Let k be a field. Unless otherwise specified all maps between vector spaces over k are assumed to be k-linear and, if V is a vector space over k, then $\mathrm{id}_V: V \to V$ denotes the identity map from V to V.

The proofs of most of the statements in this chapter can be found in [Mo]. The proof that the antipode is an antihomomorphism (2.1) is given in [Sw] 4.0.1. The statement of Theorem (5.3), giving the construction of the quantum double, is given explicitly in [D1] §13, and the proof can be found in [Ma] p. 287-289. A statement similar to Proposition (5.5) is in [Ta] Prop. 2.2.1 and the proof is similar to the proof given there.

1. SRMCwMFFs

(1.1) Definition of an algebra

An *algebra over k* is a vector space A over k with a *multiplication*

$$\begin{array}{rccc} m: & A\otimes A & \longrightarrow & A \\ & a\otimes b & \longmapsto & a\cdot b = ab \end{array}$$

and an *identity element* $1_A \in A$ such that

(a) m is *associative*, i.e. $(ab)c = a(bc)$, for all $a, b, c \in A$, and

(b) $1_A \cdot a = a \cdot 1_A = a$, for all $a \in A$.

Equivalently, an *algebra over k* is a vector space A over k with a *multiplication* $m: A\otimes A \to A$ and a *unit* $\iota: k \to A$ such that

(a) m is *associative*, i.e. $m\circ(m\otimes \mathrm{id}_A) = m\circ(\mathrm{id}_A \otimes m)$, and

(b) (*unit condition*) $m\circ(\iota\otimes\mathrm{id}_A) = m\circ(\mathrm{id}_A\otimes\iota) = \mathrm{id}_A$.

The relationship between the identity $1_A \in A$ and the unit $\iota: k \to A$ is $\iota(1) = 1_A$. If we are being precise we should denote an algebra over k by a triple (A, m, ι) or $(A, m, 1_A)$ but we shall usually be lazy and simply write A.

(1.2) Definition of a module

Let A be an algebra over k. An *A-module* is a vector space M over k with an A-action

$$\begin{array}{ccc} A\otimes M & \longrightarrow & M \\ a\otimes m & \longmapsto & a\cdot m = am \end{array}$$

such that

(a) $(ab)m = a(bm)$, for all $a, b \in A$ and $m \in M$, and

(b) $1_A m = m$, for all $m \in M$.

Let M and N be A-modules. An *A-module morphism from M to N* is a map $\varphi: M \to N$ such that
$$\varphi(am) = a\varphi(m), \quad \text{for all } a \in A \text{ and } m \in M.$$
The set of A-module morphisms from M to N is denoted $\mathrm{Hom}_A(M, N)$. An A-module is *finite dimensional* if it is finite dimensional as a vector space over k.

(1.3) Motivation for SRMCwMFFs

Our interest will be in special algebras for which the category of finite dimensional A-modules has a lot of nice structure. We want to be able to take the tensor product of two A-modules and get a new A-module, we want to be able to take the dual of an A-module and get a new A-module and we want to have a 1-dimensional "trivial" A-module.

(1.4) Definition of SRMCwMFFs

Let A be an algebra over k. The category of finite dimensional A-modules is a *strict rigid monoidal category such that the forgetful functor is monoidal* (a SRMCwMFF for short) if

(a) For every pair M, N of finite dimensional A-modules there is a given A-module structure on $M \otimes N$,

(b) For every finite dimensional A-module M there is a given A-module structure on $M^* = \mathrm{Hom}_k(M, k)$,

(c) There is a distinguished one-dimensional A-module $\mathbf{1}$ with a distinguished basis element $1 \in \mathbf{1}$,

and the following conditions are satisfied:

(1) For all finite dimensional A-modules M, N, and P,
$$(M \otimes N) \otimes P \; = \; M \otimes (N \otimes P)$$
as A-modules*.

(2) The maps
$$\begin{array}{ccc} \mathbf{1} \otimes M & \xrightarrow{\sim} & M \\ 1 \otimes m & \longmapsto & m \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M \otimes \mathbf{1} & \xrightarrow{\sim} & M \\ m \otimes 1 & \longmapsto & m \end{array}$$
are A-module isomorphisms.

(3) For each finite dimensional A-module M, the maps
$$\begin{array}{ccc} M^* \otimes M & \xrightarrow{\sim} & \mathbf{1} \\ \varphi \otimes m & \longmapsto & \varphi(m) \cdot 1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{1} & \xrightarrow{\sim} & M \otimes M^* \\ 1 & \longmapsto & \sum_i \, m_i \otimes \varphi_i \end{array}$$
are A-module morphisms.

* Strictly speaking we can only identify $(M \otimes N) \otimes P$ and $M \otimes (N \otimes P)$ up to coherent natural isomorphisms. If we are being precise this is crucial, but conceptually these two spaces are "equal".

In condition (3) the set $\{m_i\}$ is a basis of M and the set $\{\varphi_i\}$ is the dual basis in M^*, i.e. $\varphi_i \in M^*$ is such that $\varphi_i(m_j) = \delta_{ij}$ for all i, j.

The distinguished one-dimensional A-module $\mathbf{1}$ is called the *trivial* A module.

2. Hopf algebras

(2.1) Definition of Hopf algebras

A *Hopf algebra* is a vector space A over k with

a multiplication,	$m\colon A \otimes A \longrightarrow A$,
a comultiplication,	$\Delta\colon A \longrightarrow A \otimes A$,
a unit,	$\iota\colon k \longrightarrow A$,
a counit,	$\epsilon\colon A \longrightarrow k$, and
an antipode,	$S\colon A \longrightarrow A$,

such that

(1) m is *associative*,

$$m \circ (\mathrm{id}_A \otimes m) = m \circ (m \otimes \mathrm{id}_A),$$

(2) Δ is *coassociative*,

$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta,$$

(3) (unit condition),

$$m \circ (\mathrm{id}_A \otimes \iota) = m \circ (\iota \otimes \mathrm{id}_A) = \mathrm{id}_A,$$

(4) (counit condition),

$$(\mathrm{id}_A \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A,$$

(5) Δ is an algebra homomorphism,

$$\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta),$$

(6) ϵ is an algebra homomorphism,

$$\epsilon \circ m = \epsilon \otimes \epsilon,$$

(7) (antipode condition),

$$m \circ (\mathrm{id}_A \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}_A) \circ \Delta = \iota \circ \epsilon.$$

In condition (5) the algebra structure on $A \otimes A$ is given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd, \quad \text{for all } a, b, c, d \in A,$$

and the map τ is given by

$$\begin{array}{rccc} \tau\colon & A \otimes A & \longrightarrow & A \otimes A \\ & a \otimes b & \longmapsto & b \otimes a. \end{array}$$

In condition (6) we have identified the vector space $k \otimes k$ with k. One can show that the antipode $S\colon A \to A$ is always an anti-homomorphism,

$$S(ab) = S(b)S(a), \quad \text{for all } a, b \in A.$$

(2.2) Sweedler notation for the comultiplication

Let A be a Hopf algebra over k. If $a \in A$ we write

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$$

to express $\Delta(a)$ as an element of $A \otimes A$. This unusual notation is called Sweedler notation and is a standard notation for working with Hopf algebras. Don't let it bother you, we are simply trying to write $\Delta(a)$ so that it looks like an element of $A \otimes A$, without having to go through the rigmarole of actually choosing a basis in A.

(2.3) Hopf algebras give us SRMCwMFFs!

Let $(A, m, \Delta, \iota, \epsilon, S)$ be a Hopf algebra over k.

(a) If M_1 and M_2 are A-modules define an A-module structure on $M_1 \otimes M_2$ by

$$a(m_1 \otimes m_2) = \Delta(a)(m_1 \otimes m_2) = \sum_a a_{(1)}m_1 \otimes a_{(2)}m_2\ ,$$

for each $a \in A$, $m_1 \in M_1$, and $m_2 \in M_2$.

(b) Define $\mathbf{1}$ to be the vector space $\mathbf{1} = k \cdot 1$ and define an action of A on $\mathbf{1}$ by

$$a \cdot 1 \ = \ \epsilon(a) \cdot 1, \quad \text{for each } a \in A.$$

(c) If M is a finite dimensional A-module define an A-module structure on $M^* = \mathrm{Hom}_k(M, k)$ by

$$(a\varphi)(m) = \varphi(S(a)m), \quad \text{for each } a \in A,\ \varphi \in M^*, \text{ and } m \in M.$$

The point is that if A is a Hopf algebra then, with the definitions in (a)-(c) above, the category of finite dimensional A-modules is very nice; it is a strict rigid monoidal category such that the forgetful functor is monoidal.

(2.4) Group algebras are Hopf algebras

Let G be a group. The *group algebra of G over k* is the vector space kG of finite k-linear combinations of elements of G,

$$kG = \left\{ \sum_g c_g g \mid c_g \in k \text{ and all but a finite number of } c_g = 0 \right\} ,$$

with multiplication given by the k-linear extension of the multiplication in G. A *G-module* is a kG-module.

(a) If M_1 and M_2 are G-modules define a G-module structure on $M_1 \otimes M_2$ by

$$g(m_1 \otimes m_2) = gm_1 \otimes gm_2, \quad \text{for all } g \in G,\ m_1 \in M_1, \text{ and } m_2 \in M_2.$$

(b) The *trivial* G-module is the 1-dimensional vector space $\mathbf{1}$ with G-action given by

$$g \cdot v = v, \quad \text{for all } g \in G,\ v \in \mathbf{1}.$$

(c) If M is a finite dimensional G-module define a G-module structure on $M^* = \mathrm{Hom}_k(M, k)$ by

$$(g\varphi)(m) \ = \ \varphi(g^{-1}m) \ , \quad \text{for all } g \in G,\ m \in M, \text{ and } \varphi \in M^*.$$

With these definitions the category of finite dimensional G-modules is a strict monoidal category such that the forgetful functor is monoidal.

The group algebra kG is a Hopf algebra if we define

(a) a comultiplication, $\Delta: kG \to kG \otimes kG$, by

$$\Delta(g) = g \otimes g, \quad \text{for all } g \in G,$$

(b) a counit, $\epsilon: kG \to k$, by

$$\epsilon(g) = 1, \quad \text{for all } g \in G,$$

(c) and an antipode, $S: kG \to kG$, by

$$S(g) = g^{-1}, \quad \text{for all } g \in G.$$

(2.5) Enveloping algebras of Lie algebras are Hopf algebras

Let $\mathfrak{g}$ be a Lie algebra over k and let $\mathfrak{U}\mathfrak{g}$ be its enveloping algebra. (See II (1.1) and II (4.2) for definitions of Lie algebras and enveloping algebras.)

(a) If M_1 and M_2 are $\mathfrak{g}$-modules we define a $\mathfrak{g}$-module structure on $M_1 \otimes M_2$ by

$$x(m_1 \otimes m_2) = xm_1 \otimes m_2 + m_1 \otimes xm_2, \quad \text{for all } x \in \mathfrak{g},\ m_1 \in M_1, \text{ and } m_2 \in M_2.$$

(b) The *trivial* $\mathfrak{g}$-module is the 1-dimensional vector space $\mathbf{1}$ with $\mathfrak{g}$-action given by

$$xv = 0\ , \quad \text{for all } x \in \mathfrak{g},\ v \in \mathbf{1}.$$

(c) If M is a finite dimensional $\mathfrak{g}$-module we define a $\mathfrak{g}$-module structure on $M^* = \mathrm{Hom}_k(M, k)$ by

$$(x\varphi)(m) = \varphi(-xm), \quad \text{for all } x \in \mathfrak{g},\ \varphi \in M^*, \text{ and } m \in M.$$

With these definitions the category of finite dimensional $\mathfrak{g}$-modules is a strict rigid monoidal category such that the forgetful functor is monoidal.

The enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ is a Hopf algebra if we define

(a) a comultiplication, $\Delta\colon \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$, by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \text{for all } x \in \mathfrak{g},$$

(b) a counit, $\epsilon\colon \mathfrak{U}\mathfrak{g} \to k$, by

$$\epsilon(x) = 0, \quad \text{for all } x \in \mathfrak{g},$$

(c) and an antipode, $S\colon \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g}$, by

$$S(x) = -x, \quad \text{for all } x \in \mathfrak{g}.$$

(2.6) Definition of the adjoint action of a Hopf algebra on itself

Let $(A, m, \Delta, \iota, \epsilon, S)$ be a Hopf algebra. The vector space A is an A-module where the action of A on A is given by

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ a \otimes b & \longmapsto & \sum_a a_{(1)} b S(a_{(2)}) \end{array}, \quad \text{where} \quad \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}.$$

The linear transformation of A determined by the action of an element $a \in A$ is denoted ad_a. Thus,

$$\mathrm{ad}_a(b) = \sum_a a_{(1)} b S(a_{(2)}), \quad \text{for all } b \in A.$$

(2.7) Motivation for the definition of the adjoint action

Let M be an A-module and let $\rho: A \to \mathrm{End}(M)$ be the corresponding representation of A, i.e. the map

$$\begin{array}{cccc} \rho: & A & \longrightarrow & \mathrm{End}(M) \\ & a & \longmapsto & \rho(a) \end{array}$$

where $\rho(a)$ is the linear transformation of M determined by the action of a. Note that $\mathrm{End}(M) \cong M \otimes M^*$ as a vector space. On the other hand $M \otimes M^*$ is an A-module. If we view A as an A-module under the adjoint action then the composite map

$$\rho: A \to \mathrm{End}(M) \cong M \otimes M^*$$

is a homomorphism of A-modules.

(2.8) Definition of an ad-invariant bilinear form on a Hopf algebra

Let A be a Hopf algebra with antipode S and let M be an A-module. A bilinear form

$$\begin{array}{cccc} (,): & M \otimes M & \to & k \\ & m \otimes n & \mapsto & (m,n) \end{array} \quad \text{is } \textit{invariant} \text{ if} \quad (am_1, m_2) = (m_1, S(a)m_2),$$

for all $a \in A$, $m_1, m_2 \in M$. This is equivalent to the condition that the map $(,)$ is a homomorphism of A-modules when we identify k with the trivial A-module $\mathbf{1}$..

A bilinear form

$$(,): A \otimes A \to k \quad \text{is } \textit{ad-invariant} \text{ if} \quad (\mathrm{ad}_a(b_1), b_2) = (b_1, \mathrm{ad}_{S(a)}(b_2)),$$

for all $a, b_1, b_2 \in A$. In other words, the bilinear form is invariant if we view A as an A-module via the adjoint action.

3. Braided SRMCwMFFs

(3.1) Motivation for braided SRMCwMFFs

Our interest here will be in even more special algebras for which the category of finite dimensional A-modules is "braided". Specifically, we want the two tensor product modules $M \otimes N$ and $N \otimes M$ to be isomorphic.

(3.2) Definition of braided SRMCwMFFs

Let A be an algebra over k. The category of finite dimensional A-modules is a *braided strict rigid monoidal category such that the forgetful functor is monoidal* (a braided SRMCwMFF for short) if it is a strict rigid monoidal category such that the forgetful functor is monoidal and

(a) There is a family of *braiding isomorphisms*

$$\check{R}_{M,N} : M \otimes N \longrightarrow N \otimes M,$$

which are natural isomorphisms (in the sense of the theory of categories).

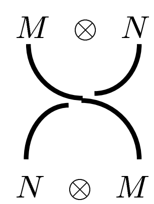
(b) For all finite dimensional A-modules M, N, P

$$\begin{aligned}
\check{R}_{M\otimes N,P} &= (\check{R}_{M,P} \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \check{R}_{N,P}), \\
\check{R}_{M,N\otimes P} &= (\mathrm{id}_N \otimes \check{R}_{M,P}) \circ (\check{R}_{M,N} \otimes \mathrm{id}_P), \quad \text{and} \\
\check{R}_{\mathbf{1},M} &= \mathrm{id}_M = \check{R}_{M,\mathbf{1}},
\end{aligned}$$

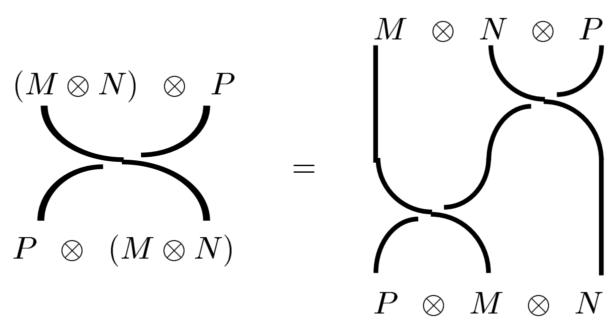
where $\mathbf{1}$ denotes the trivial module and we identify M, $\mathbf{1} \otimes M$, and $M \otimes \mathbf{1}$.

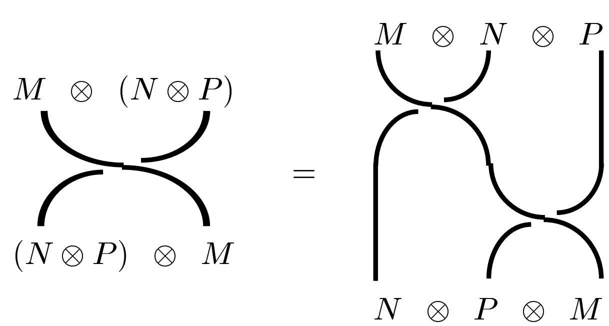
(3.3) Pictorial representation of braiding isomorphisms

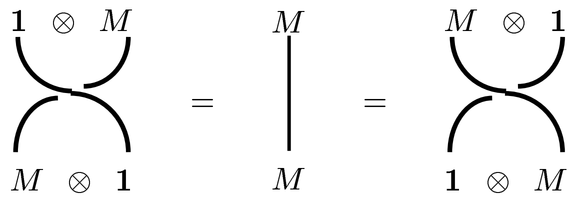
Sometimes it is convenient to denote the isomorphism $\check{R}_{M,N} : M \otimes N \longrightarrow N \otimes M$ by the picture



With this notation the relations defining a braided SRMCwMFF can be written in the form





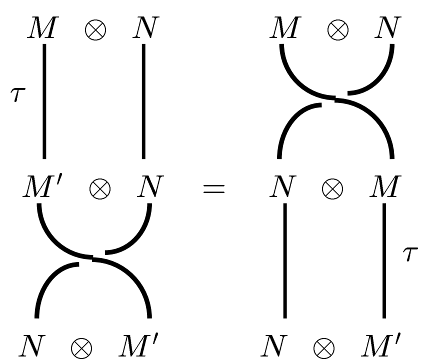


(3.4) What “natural isomorphism” means

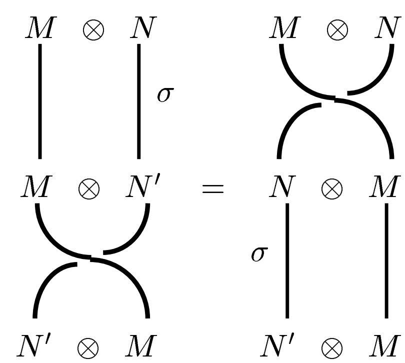
Let M, M', N, N' be A-modules and let $\tau: M \to M'$ and $\sigma: N \to N'$ be A-module isomorphisms. Then the naturality condition on the isomorphisms $\check{R}_{M,N}$ means that the following diagrams commute.

$$\begin{tikzcd} M \otimes N \arrow[r, "\tau\otimes \mathrm{id}_N"] \arrow[d, "\check{R}_{M,N}"'] & M' \otimes N \arrow[d, "\check{R}_{M',N}"] \\ N \otimes M \arrow[r, "\mathrm{id}_N \otimes \tau"] & N \otimes M' \end{tikzcd} \qquad \begin{tikzcd} M \otimes N \arrow[r, "\mathrm{id}_M\otimes \sigma"] \arrow[d, "\check{R}_{M,N}"'] & M \otimes N' \arrow[d, "\check{R}_{M,N'}"] \\ N \otimes M \arrow[r, "\sigma \otimes \mathrm{id}_M"] & N' \otimes M \end{tikzcd}$$

Pictorially we have

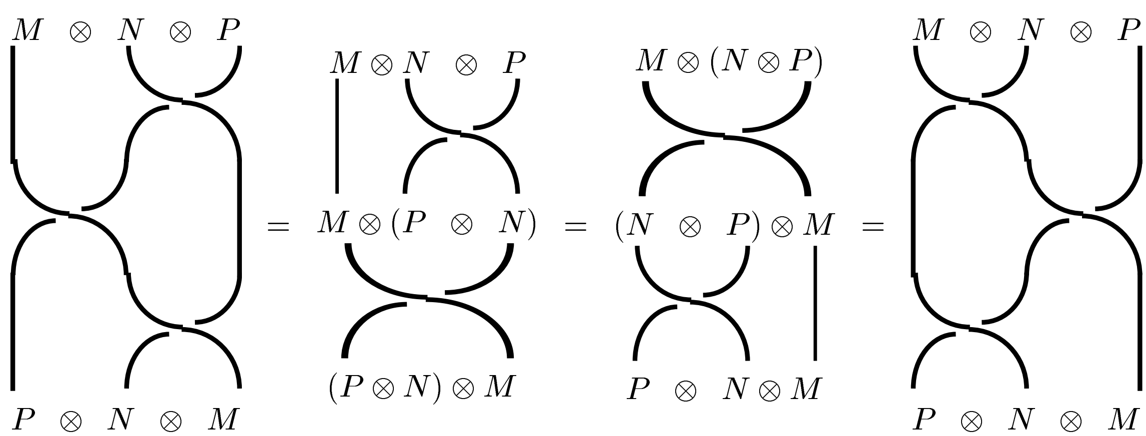


and



(3.5) The braid relation

The relations in (3.3) imply the following relation which is usually called the *braid relation*.



where the middle equality is a consequence of the naturality property and the fact that the map $\check{R}_{N,P}$ is an isomorphism.

4. Quasitriangular Hopf algebras

(4.1) Motivation for quasitriangular Hopf algebras

In addition to the definition of a braided SRMCwMFF the following observations help to motivate the definition of a quasitriangular Hopf algebra.

Let $(A, m, \Delta, \epsilon, \iota, S)$ be a Hopf algebra and let τ be the k-linear map

$$\begin{array}{rccc} \tau\colon & A\otimes A & \longrightarrow & A\otimes A \\ & a\otimes b & \longmapsto & b\otimes a. \end{array}$$

Let $\Delta^{\mathrm{op}} = \tau\circ\Delta$ so that, if $a\in A$ and

$$\Delta(a) = \sum_a a_{(1)}\otimes a_{(2)}, \quad \text{then} \quad \Delta^{\mathrm{op}}(a) = \sum_a a_{(2)}\otimes a_{(1)}.$$

Then $(A, m, \Delta^{\mathrm{op}}, \iota, \epsilon, S^{-1})$ is a Hopf algebra.

The map $\tau : A\otimes A\to A\otimes A$ is an algebra automorphism of $A\otimes A$ (the algebra structure on $A\otimes A$ is as given in (2.1)) and the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}_A"'] & A\otimes A \arrow[d, "\tau"] \\ A \arrow[r, "\Delta^{\mathrm{op}}"] & A\otimes A \end{tikzcd}$$

Sometimes we are lucky and we can replace τ by an *inner* automorphism.

(4.2) Definition of quasitriangular Hopf algebras

A *quasitriangular Hopf algebra* is a pair $(A, \mathcal{R})$ where A is a Hopf algebra and $\mathcal{R}$ is an invertible element of $A\otimes A$ such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad \text{for all } a\in A, \text{ and}$$

$$(\Delta\otimes\mathrm{id}_A)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad \text{and} \quad (\mathrm{id}_A\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

where, if $\mathcal{R} = \sum a_i\otimes b_i$, then

$$\mathcal{R}_{12} = \sum a_i\otimes b_i\otimes 1, \;\; \mathcal{R}_{13} = \sum a_i\otimes 1\otimes b_i, \quad \text{and} \quad \mathcal{R}_{23} = \sum 1\otimes a_i\otimes b_i.$$

(4.3) Quasitriangular Hopf algebras give braided SRMCwMFFs

Let $(A, \mathcal{R})$ be a quasitriangular Hopf algebra. For each pair of finite dimensional A-modules M, N define

$$\begin{array}{rccc} \check{R}_{M,N}\colon & M \otimes N & \longrightarrow & N \otimes M \\ & m \otimes n & \longmapsto & \sum b_i n \otimes a_i m, \end{array}$$

where $\mathcal{R} = \sum a_i \otimes b_i \in A \otimes A$. Then the category of finite dimensional A-modules is a braided strict rigid monoidal category such that the forgetful functor is monoidal.

5. The quantum double

(5.1) Motivation for the quantum double

In general it can be very difficult to find quasitriangular Hopf algebras, especially ones where the element $\mathcal{R}$ is different from $1 \otimes 1$. The construction in (5.3) says that, given a Hopf algebra A, we can sort of paste it and its dual A^* together to get a quasitriangular Hopf algebra $D(A)$ and that the $\mathcal{R}$ for this new quasitriangular Hopf algebra is both a natural one and is nontrivial.

(5.2) Construction of the Hopf algebra $A^{*\mathrm{coop}}$

Let $(A, m, \Delta, \iota, \epsilon, S)$ be a finite dimensional Hopf algebra over k. Let $A^* = \mathrm{Hom}_k(A, k)$ be the dual of A. There is a natural bilinear pairing $\langle , \rangle\colon A^* \otimes A \longrightarrow k$ between A and A^* given by

$$\langle \alpha, a \rangle = \alpha(a), \quad \text{for all } \alpha \in A^* \text{ and } a \in A.$$

Extend this notation so that if $\alpha_1, \alpha_2 \in A^*$ and $a_1, a_2 \in A$ then

$$\langle \alpha_1 \otimes \alpha_2, a_1 \otimes a_2 \rangle = \langle \alpha_1, a_1 \rangle \langle \alpha_2, a_2 \rangle.$$

We make A^* into a Hopf algebra, which is denoted $A^{*\mathrm{coop}}$, by defining a multiplication and a comultiplication Δ on A^* via the equations

$$\langle \alpha_1 \alpha_2, a \rangle = \langle \alpha_1 \otimes \alpha_2, \Delta(a) \rangle \quad \text{and} \quad \langle \Delta^{\mathrm{op}}(\alpha), a_1 \otimes a_2 \rangle = \langle \alpha, a_1 a_2 \rangle,$$

for all $\alpha, \alpha_1, \alpha_2 \in A^*$ and $a, a_1, a_2 \in A$. The definition of Δ^{op} is in (4.1).

(a) The identity in $A^{*\mathrm{coop}}$ is the counit $\epsilon\colon A \to k$ of A.

(b) The counit of $A^{*\mathrm{coop}}$ is the map

$$\begin{array}{rccc} \epsilon\colon & A^* & \to & k \\ & \alpha & \mapsto & \alpha(1), \end{array}$$

where 1 is the identity in A.

(c) The antipode of $A^{*\mathrm{coop}}$ is given by the identity $\langle S(\alpha), a \rangle = \langle \alpha, S^{-1}(a) \rangle$, for all $\alpha \in A^*$ and all $a \in A$.

(5.3) Construction of the quantum double

We want to paste the algebras A and $A^{*\mathrm{coop}}$ together in order to make a quasitriangular Hopf algebra $D(A)$. There are three main steps.

(1) We paste A and $A^{*\mathrm{coop}}$ together by letting

$$D(A) = A \otimes A^{*\mathrm{coop}}.$$

Write elements of $D(A)$ as $a\alpha$ instead of as $a \otimes \alpha$.

(2) We want the multiplication in $D(A)$ to reflect the multiplication in A and the multiplication in $A^{*\mathrm{coop}}$. Similarly for the comultiplication.

(3) We want the $\mathcal{R}$-matrix to be

$$\mathcal{R} = \sum_i b_i \otimes b^i,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is the dual basis in A^*.

The condition in (2) determines the comultiplication in $D(A)$,

$$\Delta(\alpha a) = \Delta(\alpha)\Delta(a) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)},$$

where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)}$. The condition in (2) doesn't quite determine the multiplication in $D(A)$. We need to be able to expand products like $(a_1\alpha_1)(a_2\alpha_2)$. If we knew

$$\alpha_1 a_2 = \sum_j b_j\beta_j, \quad \text{for some elements } \beta_j \in A^{*\mathrm{coop}} \text{ and } b_j \in A,$$

then we would have

$$(a_1\alpha_1)(a_2\alpha_2) = \sum_j (a_1 b_j)(\beta_j\alpha_2)$$

which is a well defined element of $D(A)$. Miraculously, the condition in (3) and the equation

$$\mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a), \quad \text{for all } a \in A,$$

force that if $\alpha \in A^{*\mathrm{coop}}$ and $a \in A$ then, in $D(A)$,

$$\alpha a = \sum_{\alpha,a} \langle \alpha_{(1)}, S^{-1}(a_{(1)})\rangle \langle \alpha_{(3)}, a_{(3)}\rangle a_{(2)}\alpha_{(2)}, \quad \text{and}$$

$$a\alpha = \sum_{\alpha,a} \langle \alpha_{(1)}, a_{(1)}\rangle \langle \alpha_{(3)}, S^{-1}(a_{(3)})\rangle \alpha_{(2)}a_{(2)},$$

where, if Δ is the comultiplication in $D(A)$,

$$(\Delta\otimes \mathrm{id})\circ\Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}\otimes a_{(3)},\quad\text{and}\quad (\Delta\otimes \mathrm{id})\circ\Delta(\alpha)=\sum_\alpha \alpha_{(1)}\otimes \alpha_{(2)}\otimes \alpha_{(3)}.$$

These relations completely determine the multiplication in $D(A)$. This construction is summarized in the following theorem.

Theorem. *Let A be a finite dimensional Hopf algebra over k and let $A^{*\mathrm{coop}}$ be the Hopf algebra $A^*=\mathrm{Hom}_k(A,k)$ except with opposite comultiplication. Then there exists a unique quasitriangular Hopf algebra $(D(A),\mathcal{R})$ given by*

(1) The k-linear map

$$\begin{array}{ccc} A\otimes A^{*\mathrm{coop}} & \longrightarrow & D(A) \\ a\otimes\alpha & \longmapsto & a\alpha \end{array}$$

is bijective.

(2) $D(A)$ contains A and $A^{*\mathrm{coop}}$ as Hopf subalgebras.

(3) The element $\mathcal{R}\in D(A)\otimes D(A)$ is given by

$$\mathcal{R}=\sum_i b_i\otimes b^i\ ,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is dual basis in $A^{*\mathrm{coop}}$.

In condition (2), A is identified with the image of $A\otimes 1$ under the map in (1) and $A^{*\mathrm{coop}}$ is identified with the image of $1\otimes A^{*\mathrm{coop}}$ under the map in (1).

(5.4) If A is an infinite dimensional Hopf algebra

It is sometimes possible to do an analogous construction when A is infinite dimensional if one is careful about what the dual of A is and how to express the (now infinite) sum $\mathcal{R}=\sum_i b_i\otimes b^i$. To get an idea of how this is done see VII (7.1) and [Lu] Chapt. 4.

(5.5) An ad-invariant pairing on the quantum double

Proposition. *Let $(A,m,\Delta,\iota,\epsilon,S)$ be a Hopf algebra. The bilinear form on the quantum double $D(A)$ of A which is defined by*

$$\langle a\alpha,b\beta\rangle=\langle\beta,S(a)\rangle\langle\alpha,S^{-1}(b)\rangle,\quad \text{for all } a,b\in A \text{ and all } \alpha,\beta\in A^{*\mathrm{coop}},$$

satisfies

$$\langle ad_u(x),y\rangle=\langle x,\mathrm{ad}_{S(u)}(y)\rangle,\qquad \text{for all } u,x,y\in D(A).$$

The proposition says that the bilinear form is ad-invariant, as defined in (2.8). This bilinear form is *not* necessarily symmetric,

$$\langle y,x\rangle=\langle x,S^2(y)\rangle,\quad \text{for all } x,y\in D(A).$$

II. Lie algebras and enveloping algebras

All of the statements in §1 are proved in [Se] Chapts. I-III. The statements in §3, except possibly (3.6), are proved in [Dix] Chapt. 2. The proof that the Lie algebra can be recovered from its enveloping algebra (3.6) can be found in [Bou] II §1.4. The classification theorem for semisimple Lie algebras, Theorem (2.2), is proved in [Se] VI §4 Theorem 7. The results in (2.4) and (2.5) on the classification of finite dimensional modules for simple Lie algebras are proved in [Se] VII §1-4. Theorem (2.8) is proved in [Bou] Chapt 6 §1.3 and Proposition (2.8) is proved in [Bou] Chapt 6 §1.6 Cor. 2 and Cor. 3.

1. Semisimple Lie algebras

(1.1) Definition of a Lie algebra

Let k be a field. A *Lie algebra over* k is a vector space $\mathfrak{g}$ over k with a bracket $[,]: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ which satisfies

$$[x,x]=0, \quad \text{for all } x\in\mathfrak{g},$$
$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0, \quad \text{for all } x,y,z\in\mathfrak{g}.$$

The first relation is the skew-symmetric relation and is equivalent to $[x,y]=-[y,x]$, for all $x,y\in\mathfrak{g}$, provided that char $k\neq 2$. The second relation is the *Jacobi identity.* A Lie algebra $\mathfrak{g}$ over k is *finite dimensional* if it is finite dimensional as a vector space over k, and it is *complex* if $k=\mathbb{C}$.

(1.2) Definition of a simple Lie algebra

An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{a}\subseteq\mathfrak{g}$ such that

$$[x,a]\in\mathfrak{a}, \quad \text{for all } x\in\mathfrak{g}, \text{ and } a\in\mathfrak{a}\ .$$

A Lie algebra $\mathfrak{g}$ is *abelian* if $[x,y]=0$ for all $x,y\in\mathfrak{g}$. A finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic 0 is *simple* if

(1) $\mathfrak{g}$ is not the one dimensional abelian Lie algebra,

(2) The only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$.

(1.3) Definition of the radical of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic 0. If $\mathfrak{a}\subseteq\mathfrak{g}$ is an ideal of $\mathfrak{g}$ define

$$D^1\mathfrak{a}=\mathfrak{a}, \quad \text{and} \quad D^n\mathfrak{a}=[D^{n-1}\mathfrak{a},D^{n-1}\mathfrak{a}], \quad \text{for } n\geq 2.$$

An ideal $\mathfrak{a}$ of $\mathfrak{g}$ is *solvable* if there exists a positive integer n such that $D^n\mathfrak{a} = 0$. The *radical* of $\mathfrak{g}$ is the largest solvable ideal of $\mathfrak{g}$. A finite dimensional Lie algebra is *semisimple* if its radical is 0.

(1.4) Definition of simple modules for a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over a field k. A $\mathfrak{g}$-*module* is a vector space V over k with a $\mathfrak{g}$-action

$$\begin{array}{ccc} \mathfrak{g}\otimes V & \longrightarrow & V \\ x\otimes v & \longmapsto & x\cdot v = xv \end{array}$$

such that

$$[x,y]\cdot v = x(yv) - y(xv), \quad \text{for all } x,y\in\mathfrak{g}, \text{ and } v\in V.$$

A *representation* of $\mathfrak{g}$ on a vector space V is a map

$$\begin{array}{cccc} \rho: & \mathfrak{g} & \longrightarrow & \mathrm{End}(V) \\ & x & \longmapsto & \rho(x) \end{array} \quad \text{such that} \quad \rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x),$$

for all $x,y\in\mathfrak{g}$. Every $\mathfrak{g}$-module V determines a representation of $\mathfrak{g}$ on V (and vice versa) by the formula

$$\rho(x)v = xv, \quad \text{for all } x\in\mathfrak{g}, \text{ and } v\in V.$$

A *submodule* of a $\mathfrak{g}$-module V is subspace $W\subseteq V$ such that $xw\in W$ for all $x\in\mathfrak{g}$ and $w\in W$. A *simple* or *irreducible* $\mathfrak{g}$-*module* is a $\mathfrak{g}$-module V such that the only submodules of V are 0 and V. A $\mathfrak{g}$-module V is *completely decomposable* if V is a direct sum of simple submodules.

(1.5) Definition of the adjoint representation of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k. The vector space $\mathfrak{g}$ is a $\mathfrak{g}$-module where the action of $\mathfrak{g}$ on $\mathfrak{g}$ is given by

$$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ x\otimes y & \longmapsto & [x,y]. \end{array}$$

The linear transformation of $\mathfrak{g}$ determined by the action of an element $x\in\mathfrak{g}$ is denoted ad_x. Thus,

$$\mathrm{ad}_x(y) = [x,y], \quad \text{for all } y\in\mathfrak{g}.$$

The representation

$$\begin{array}{cccc} \mathrm{ad}: & \mathfrak{g} & \longrightarrow & \mathrm{End}(\mathfrak{g}) \\ & x & \longmapsto & \mathrm{ad}_x \end{array}$$

is the *adjoint representation* of $\mathfrak{g}$.

(1.6) Definition of the Killing form

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k. The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to k$ given by

$$\langle x, y\rangle = Tr(\mathrm{ad}_x \mathrm{ad}_y), \quad \text{for all } x, y \in \mathfrak{g}.$$

The Killing form $\langle , \rangle$ is *invariant*, i.e.

$$\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0, \quad \text{for all } x, y, z \in \mathfrak{g}.$$

(1.7) Characterizations of semisimple Lie algebras

Theorem. *A finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic 0 is semisimple if any of the following equivalent conditions holds:*

(1) *$\mathfrak{g}$ is a direct sum of simple Lie subalgebras.*

(2) *The radical of $\mathfrak{g}$ is 0.*

(3) *Every finite dimensional $\mathfrak{g}$ module is completely decomposable and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

(4) *The Killing form on $\mathfrak{g}$ is non-degenerate.*

2. Finite dimensional complex simple Lie algebras

(2.1) Dynkin diagrams and Cartan matrices

A *Dynkin diagram* is one of the graphs in Table 1. A *Cartan matrix* is one of the matrices in Table 2. The (i, j) entry of a Cartan matrix is denoted $\alpha_j(H_i)$. Notice that every Cartan matrix satisfies the conditions,

(1) $\alpha_i(H_i) = 2$, for all $1 \le i \le r$,

(2) $\alpha_j(H_i)$ is a non positive integer, for all $i \ne j$,

(3) $\alpha_i(H_j) = 0$ if and only if $\alpha_j(H_i) = 0$.

If C is a Cartan matrix the vertices of the corresponding Dynkin diagram are labeled by $\alpha_i, 1 \le i \le r$, such that $\alpha_i(H_j)\alpha_j(H_i)$ is the number of lines connecting vertex α_i to vertex α_j. If $\alpha_j(H_i) > \alpha_i(H_j)$ then there is a $>$ sign on the edge connecting vertex α_j to vertex α_i, with the point towards α_i. With these conventions it is clear that the Cartan matrix contains exactly the same information as the Dynkin diagram; each can be constructed from the other.

(2.2) Classification of finite dimensional complex simple Lie algebras

Fix a Cartan matrix $C = (\alpha_j(H_i))_{1 \le i,j \le r}$. Let $\mathfrak{g}_C$ be the Lie algebra over $\mathbb{C}$ given by generators

$$X_1^-, X_2^-, \dots, X_r^-, \quad H_1, H_2, \dots, H_r, \quad X_1^+, X_2^+, \dots, X_r^+,$$

and relations

$$\begin{aligned}
&[H_i, H_j] = 0, && \text{for all } 1 \le i, j \le r, \\
&[H_i, X_j^+] = \alpha_j(H_i) X_j^+, && \\
& && \text{for all } 1 \le i, j \le r, \\
&[H_i, X_j^-] = -\alpha_j(H_i) X_j^-, && \\
&[X_i^+, X_j^-] = \delta_{ij} H_i, && \text{for } 1 \le i, j \le r, \\
&\underbrace{[X_i^+, [X_i^+, \cdots [X_i^+,}_{-\alpha_j(H_i)+1 \text{ brackets}} X_j^+]] \cdots] = 0, && \\
& && \text{for } i \neq j. \\
&\underbrace{[X_i^-, [X_i^-, \cdots [X_i^-,}_{-\alpha_j(H_i)+1 \text{ brackets}} X_j^-]] \cdots] = 0, &&
\end{aligned}$$

Theorem. *Let C be a Cartan matrix and let $\mathfrak{g}_C$ be the Lie algebra defined above.*

(1) The Lie algebra $\mathfrak{g}_C$ is a finite dimensional complex simple Lie algebra.

(2) Every finite dimensional complex simple Lie algebra is isomorphic to $\mathfrak{g}_C$ for some Cartan matrix C.

(3) If C, C' are Cartan matrices then

$$\mathfrak{g}_C \simeq \mathfrak{g}_{C'} \quad \textit{if and only if} \quad C = C'.$$

(2.3) Triangular decomposition

Fix a Cartan matrix $C = (\alpha_i(H_j))_{1 \le i,j \le r}$ and let $\mathfrak{g} = \mathfrak{g}_C$. Define

$$\begin{aligned}
\mathfrak{n}^- &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^-. \\
\mathfrak{h} &= \mathbb{C}\text{-span } \{H_1, H_2, \ldots, H_r\}, \\
\mathfrak{n}^+ &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+,
\end{aligned}$$

The elements $X_1^-, X_2^- \ldots, X_r^-, H_1, \ldots, H_r, X_1^+, X_2^+, \ldots, X_r^+$ are linearly independent in $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

The Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* of $\mathfrak{g}$ and the Lie subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a *Borel subalgebra* of $\mathfrak{g}$. The *rank* of $\mathfrak{g}$ is $r = \dim \mathfrak{h}$.

(2.4) Weights and weight spaces

Fix a Cartan matrix $C = (\alpha_j(H_i))_{1\le i,j\le r}$ and let $\mathfrak{g} = \mathfrak{g}_C$. Let $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ and define the *fundamental weights* $\omega_1, \ldots, \omega_r \in \mathfrak{h}^*$ by

$$\omega_i(H_j) = \delta_{ij}, \quad \text{for } 1 \le i, j \le r.$$

Let V be a $\mathfrak{g}$-module and let $\mu = \sum_{i=1}^r \mu_i\omega_i \in \mathfrak{h}^*$. The subspace

$$\begin{aligned} V_\mu &= \{v \in V \mid hv = \mu(h)v, \quad \text{for} \quad h \in \mathfrak{h}\} \\ &= \{v \in V \mid H_i v = \mu_i v, \quad \text{for} \quad 1 \le i \le r\} \end{aligned}$$

is the *μ-weight space* of V. Vectors $v \in V_\mu$ are *weight vectors* of V of *weight* μ, $\mathrm{wt}(v) = \mu$. The *weights* of the $\mathfrak{g}$-module V are the elements $\mu \in \mathfrak{h}^*$ such that $V_\mu \ne 0$. If μ is a weight of V, the *multiplicity* of μ in V is $\dim(V_\mu)$. A *highest weight vector* in a $\mathfrak{g}$-module V is a weight vector $v \in V$ such that $\mathfrak{n}^+ v = 0$ or, equivalently, a weight vector $v \in V$ such that $X_i^+ v = 0$, for $1 \le i \le r$.

The set of *dominant integral weights* P^+ and the *weight lattice* P are the subsets of $\mathfrak{h}^*$ given by

$$P^+ = \sum_{i=1}^r \mathbb{N}\omega_i \quad \text{and} \quad P = \sum_{i=1}^r \mathbb{Z}\omega_i, \quad \text{respectively,}$$

where $\mathbb{N} = \mathbb{Z}_{\ge 0}$.

(2.5) Classification of simple $\mathfrak{g}$-modules

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. Every finite dimensional $\mathfrak{g}$-module V is a direct sum of its weight spaces and all weights of V are elements of P,*

$$V = \bigoplus_{\mu\in P} V_\mu.$$

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra.*

(1) Every finite dimensional irreducible $\mathfrak{g}$-module V contains a unique, up to constant multiples, highest weight vector $v^+ \in V$ and $wt(v^+) \in P^+$.

(2) Conversely, if $\lambda \in P^+$, then there is a unique (up to isomorphism) finite dimensional irreducible $\mathfrak{g}$-module, V^λ, with highest weight vector of weight λ.

(2.6) Roots and the root lattice

Fix a Cartan matrix $C = (\alpha_j(H_i))_{1\le i,j\le r}$ and let $\mathfrak{g} = \mathfrak{g}_C$. The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ (see (1.5)) makes $\mathfrak{g}$ into a finite dimensional $\mathfrak{g}$-module. An element $\alpha \in P$, $\alpha \ne 0$ is a *root* if the weight space $\mathfrak{g}_\alpha \ne 0$. A root is *positive*, $\alpha > 0$, if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}^+$ and *negative*, $\alpha < 0$, if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}^-$. We have

$$\dim \mathfrak{g}_\alpha \; = \; 1 \quad \text{for all roots } \alpha,$$

$$\mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha, \quad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}^+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{g} \; = \; \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The roots α_i, $1 \le i \le r$, given by $\mathfrak{g}_{\alpha_i} = \mathbb{C}X_i^+$ are the *simple roots*. The Cartan matrix is the transition matrix between the simple roots and the fundamental weights,

$$\alpha_i \; = \; \sum_{j=1}^{r} \alpha_i(H_j)\omega_j, \quad \text{for } 1 \le i \le r.$$

The *root lattice* is the lattice $Q \subseteq P \subseteq \mathfrak{h}^*$ given by $Q = \sum_{i=1}^{r} \mathbb{Z}\alpha_i$.

(2.7) The inner product on $\mathfrak{h}^*_{\mathbb{R}}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. There exist unique positive integers $d_1, d_2, \cdots, d_r$ such that $gcd(d_1, \cdots, d_r) = 1$ and the matrix $(d_i\alpha_j(H_i))_{1\le i,j\le r}$ is symmetric. The integers $d_1, d_2, \cdots, d_r$ are given explicitly by

$A_r,\ D_r,\ E_6,\ E_7,\ E_8$:	$d_i = 1$ for all $1 \le i \le r$,
B_r :	$d_i = 1$ for $1 \le i \le r-1$, and $d_r = 2$,
C_r :	$d_i = 2$, for $1 \le i \le r-1$, and $d_r = 1$,
F_4 :	$d_1 = d_2 = 1$, and $d_3 = d_4 = 2$,
G_2:	$d_1 = 3$, and $d_2 = 1$.

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$. Define

$$\mathfrak{h}^*_{\mathbb{R}} = \sum_{i=1}^{r} \mathbb{R}\alpha_i,$$

so that $\mathfrak{h}^*_{\mathbb{R}}$ is a real vector space of dimension r. Define an symmetric inner product on $\mathfrak{h}^*_{\mathbb{R}}$ by

$$(\alpha_i, \alpha_j) = d_i\alpha_i(H_j), \qquad \text{for } \le i, j \le r,$$

where the values $\alpha_j(H_i)$ are the entries of the Cartan matrix corresponding to $\mathfrak{g}$.

(2.8) The Weyl group corresponding to $\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let R be the set of roots of $\mathfrak{g}$ and let $\alpha_1, \ldots, \alpha_r$ be the simple roots. For each root $\alpha \in R$ define a linear transformation of $\mathfrak{h}_{\mathbb{R}}^*$ by

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \text{where} \quad \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}.$$

The Weyl group corresponding to $\mathfrak{g}$ is the group of linear transformations of $\mathfrak{h}_{\mathbb{R}}^*$ generated by the reflections s_α, $\alpha \in R$,

$$W \ = < s_\alpha \mid \alpha \in R > .$$

The *simple reflections* in W are the elements $s_i = s_{\alpha_i}$, $1 \le i \le r$.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let W be the Weyl group corresponding to $\mathfrak{g}$.*

(a) The Weyl group W is a finite group.

(b) The Weyl group W can be presented by generators $s_1, \ldots, s_r$ and relations

$$\begin{aligned} s_i^2 &= 1, && 1 \le i \le r, \\ \underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}\ \text{factors}} &= \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij}\ \text{factors}} && \text{for } i \ne j, \end{aligned}$$

where

$$m_{ij} = \begin{cases} 2, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 0, \\ 3, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 1, \\ 4, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 2, \\ 6, & \text{if } \alpha_i(H_j)\alpha_j(H_i) = 3. \end{cases}$$

Let $w \in W$. A *reduced decomposition for* w is an expression

$$w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$$

of w as a product of generators which is as short as possible. The length $\ell(w)$ of this expression is the *length of* w.

Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let W be the Weyl group corresponding to $\mathfrak{g}$.*

(a) There is a unique longest element w_0 in W.

(b) Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced decomposition for the longest element of W. Then the elements

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N}),$$

are the positive roots of $\mathfrak{g}$.

3. Enveloping algebras

(3.1) Motivation for the enveloping algebra

A Lie algebra $\mathfrak{g}$ is not an algebra, at least as defined in I (1.1), because the bracket is not associative. We would like to find an algebra, or even better a Hopf algebra, $\mathfrak{U}\mathfrak{g}$, for which the category of modules for $\mathfrak{U}\mathfrak{g}$ is the same as the category of modules for $\mathfrak{g}$. In other words we want $\mathfrak{U}\mathfrak{g}$ to carry all the information that $\mathfrak{g}$ does and to be a Hopf algebra.

(3.2) Definition of the enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra over k. Let $T(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k}$ be the tensor algebra of $\mathfrak{g}$ and let J be the ideal of $T(\mathfrak{g})$ generated by the tensors

$$x \otimes y - y \otimes x - [x, y], \quad \text{where} \quad x, y \in \mathfrak{g}.$$

The *enveloping algebra of* $\mathfrak{g}$, $\mathfrak{U}\mathfrak{g}$, is the associative algebra

$$\mathfrak{U}\mathfrak{g} = \frac{T(\mathfrak{g})}{J}.$$

There is a canonical map

$$\begin{array}{rccc} \alpha_0 : & \mathfrak{g} & \longrightarrow & \mathfrak{U}\mathfrak{g} \\ & x & \longmapsto & x + J . \end{array}$$

The algebra $\mathfrak{U}\mathfrak{g}$ can be given by the following universal property:

> Let $\alpha : \mathfrak{g} \to A$ be a mapping of $\mathfrak{g}$ into an associative algebra A over k such that
>
> $$\alpha([x, y]) = \alpha(x)\alpha(y) - \alpha(y)\alpha(x),$$
>
> for all $x, y \in \mathfrak{g}$, and let 1 and 1_A denote the identities in $\mathfrak{U}\mathfrak{g}$ and A respectively. Then there exists a unique algebra homomorphism $\tau : \mathfrak{U}\mathfrak{g} \to A$ such that $\tau(1) = 1_A$ and $\alpha = \tau \circ \alpha_0$, i.e. the following diagram commutes.
>
> $$\begin{tikzcd} \mathfrak{g} \arrow[r, "\alpha_0"] \arrow[dr, "\alpha"'] & \mathfrak{U}\mathfrak{g} \arrow[d, "\tau"] \\ & A \end{tikzcd}$$

(3.3) A functorial way of realising the enveloping algebra

If A is an algebra over k, as defined in I (1.1), then define a bracket on A by

$$[x, y] = xy - yx, \quad \text{for all } x, y \in A.$$

This defines a Lie algebra structure on A and we denote the resulting Lie algebra by $L(A)$ to distinguish it from A. L is a functor from the category of algebras to the category of

Lie algebras. $\mathfrak{U}$ is a functor from the category of Lie algebras to the category of algebras. In fact $\mathfrak{U}$ is the left adjoint of the functor L since

$$\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{U}\mathfrak{g}, A) = \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, L(A))$$

for all Lie algebras $\mathfrak{g}$ and all algebras A.

(3.4) The enveloping algebra is a Hopf algebra

The enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ is a Hopf algebra if we define

(a) a comultiplication, $\Delta\colon \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$, by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \text{for all } x \in \mathfrak{g},$$

(b) a counit, $\epsilon\colon \mathfrak{U}\mathfrak{g} \to k$, by

$$\epsilon(x) = 0, \quad \text{for all } x \in \mathfrak{g},$$

(c) and an antipode, $S\colon \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g}$, by

$$S(x) = -x, \quad \text{for all } x \in \mathfrak{g}.$$

(3.5) Modules for the enveloping algebra and the Lie algebra are the same!

Every $\mathfrak{g}$-module M is a $\mathfrak{U}\mathfrak{g}$-module and vice versa, since there is a unique extension of the action of $\mathfrak{g}$ on M to a $\mathfrak{U}\mathfrak{g}$-action on M.

(3.6) The Lie algebra can be recovered from its enveloping algebra!

An element x of a Hopf algebra A is *primitive* if

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

It can be shown that if $\mathrm{char}\, k = 0$ then the subspace $\mathfrak{g}$ of $\mathfrak{U}\mathfrak{g}$ is the set of primitive elements of $\mathfrak{U}\mathfrak{g}$. Thus, if $\mathrm{char}\, k = 0$, we can "determine" the Lie algebra $\mathfrak{g}$ from the algebra $\mathfrak{U}\mathfrak{g}$ and the Hopf algebra structure on it.

(3.7) A basis for the enveloping algebra

The following statement is the *Poincaré-Birkhoff-Witt* theorem.

> Suppose that $\mathfrak{g}$ has a totally ordered basis $(x_i)_{i\in\Lambda}$. Then the elements
>
> $$x_{i_1} x_{i_2} \cdots x_{i_n}$$
>
> in the enveloping algebra $\mathfrak{U}\mathfrak{g}$, where $i_1 \le i_2 \le \cdots \le i_n$ is an arbitrary increasing finite sequence of elements of Λ, form a basis a $\mathfrak{U}\mathfrak{g}$.

5. The enveloping algebra of a complex simple Lie algebra

(5.1) A presentation by generators and relations

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. Then the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ can be presented as the algebra over $\mathbb{C}$ generated by

$$X_1^-, X_2^-, \ldots, X_r^-, \quad H_1, H_2, \ldots, H_r, \quad X_1^+, X_2^+, \ldots, X_r^+,$$

with relations

$$\begin{aligned}
&[H_i, H_j] = 0, && \text{for all } 1 \le i,j \le r,\\
&[H_i, X_j^+] = \alpha_j(H_i) X_j^+, && \\
& && \text{for all } 1 \le i,j \le r,\\
&[H_i, X_j^-] = -\alpha_j(H_i) X_j^-, && \\
&[X_i^+, X_j^-] = \delta_{ij} H_i, && \text{for } 1 \le i,j \le r,
\end{aligned}$$

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \binom{1-\alpha_j(H_i)}{s} (X_i^\pm)^s X_j^\pm (X_i^\pm)^t = 0, \qquad \text{for } i \ne j,$$

where, if $a, b \in \mathfrak{U}\mathfrak{g}$, we use the notation $[a,b] = ab - ba$. Note that since

$$\underbrace{[a,[a,\cdots[a,}_{\ell \text{ brackets}} b]]\cdots] = \sum_{s+t=\ell} (-1)^s \binom{\ell}{s} a^s b a^t,$$

for any two elements $a, b \in \mathfrak{U}\mathfrak{g}$ and any positive integer ℓ, the relations for $\mathfrak{U}\mathfrak{g}$ are exactly the same as the relations for $\mathfrak{g}$ given in (2.2).

(5.2) Triangular decomposition

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra as presented in (2.2). Recall from (2.3) that $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where

$$\begin{aligned}
\mathfrak{n}^- &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^- .\\
\mathfrak{h} \ &= \mathbb{C}\text{-span } \{H_1, H_2, \ldots, H_r\},\\
\mathfrak{n}^+ &= \text{Lie subalgebra of } \mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+
\end{aligned}$$

It follows from this and the Poincaré-Birkhoff-Witt theorem that

$$\mathfrak{U}\mathfrak{g} \cong \mathfrak{U}\mathfrak{n}^- \otimes \mathfrak{U}\mathfrak{h} \otimes \mathfrak{U}\mathfrak{n}^+, \qquad \text{as vector spaces.}$$

(5.3) Grading on $\mathfrak{U}\mathfrak{n}^+$ and $\mathfrak{U}\mathfrak{n}^-$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra as presented in (2.2). Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$ and let

$$Q^+ \;=\; \sum_i \mathbb{N}\alpha_i, \qquad \text{where } \mathbb{N} = \mathbb{Z}_{\geq 0}.$$

For each element $\nu = \sum_{i=1}^r \nu_i\alpha_i \in Q^+$ define

$$\begin{aligned} (\mathfrak{U}\mathfrak{n}^+)_\nu &= \text{span-}\{X_{i_1}^+ \cdots X_{i_p}^+ \mid X_{i_1}^+ \cdots X_{i_p}^+ \;\text{ has } \nu_j\text{-factors of type } X_j^+\} \\ (\mathfrak{U}\mathfrak{n}^-)_\nu &= \text{span-}\{X_{i_1}^- \cdots X_{i_p}^- \mid X_{i_1}^- \cdots X_{i_p}^- \;\text{ has } \nu_j\text{-factors of type } X_j^-\}. \end{aligned}$$

Then

$$\mathfrak{U}\mathfrak{n}^- = \bigoplus_{\nu\in Q^+} (\mathfrak{U}\mathfrak{n}^-)_\nu, \quad \text{and} \quad \mathfrak{U}\mathfrak{n}^+ = \bigoplus_{\nu\in Q^+} (\mathfrak{U}\mathfrak{n}^+)_\nu,$$

as vector spaces.

(5.4) Poincaré-Birkhoff-Witt bases of $\mathfrak{U}\mathfrak{n}^-$, $\mathfrak{U}\mathfrak{h}$, and $\mathfrak{U}\mathfrak{n}^+$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra as presented in (2.2), let $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{h}$ be as in (2.3) and recall the root spaces $\mathfrak{g}_\alpha$ from (2.6). Let W be the Weyl group corresponding to $\mathfrak{g}$. Fix a reduced decomposition of the longest element $w_0 \in W$, $w_0 = s_{i_1} \cdots s_{i_N}$, and define

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = s_{i_1}(\alpha_{i_2}), \qquad \ldots, \qquad \beta_N = s_{i_1}s_{i_2}\cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The elements $\beta_1, \ldots, \beta_N$ are the positive roots $\mathfrak{g}$ and the elements $-\beta_1, \ldots, -\beta_N$ are the negative roots of $\mathfrak{g}$.

$$\text{For each root } \alpha, \text{ fix an element } X_\alpha \in \mathfrak{g}_\alpha.$$

Since $\mathfrak{g}_\alpha$ is 1-dimensional X_α is uniquely defined, up to multiplication by a constant. Since

$$\mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha, \quad \mathfrak{n}^+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \quad \mathfrak{h} = \text{span-}\{H_1, H_2, \ldots H_r\} \quad \text{and} \quad \mathfrak{g} \;=\; \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

it follows that

$$\begin{aligned} &\{X_{\beta_1}, \ldots, X_{\beta_N}\} && \text{is a basis of } \mathfrak{n}^+, \\ &\{X_{-\beta_1}, \ldots, X_{-\beta_N}\} && \text{is a basis of } \mathfrak{n}^-, \text{ and} \\ &\{H_1, H_2, \ldots, H_r\} && \text{is a basis of } \mathfrak{h}. \end{aligned}$$

Then, by the Poincaré-Birkhoff-Witt theorem,

$$\{X_{\beta_1}^{p_1}X_{\beta_2}^{p_2}\cdots X_{\beta_N}^{p_N} \mid p_1,\ldots,p_N\in\mathbb{Z}_{\geq 0}\} \qquad \text{is a basis of } \mathfrak{U}\mathfrak{n}^+,$$

$$\{X_{-\beta_1}^{n_N}X_{-\beta_2}^{n_2}\cdots X_{-\beta_1}^{n_1} \mid n_1,\ldots,n_N\in\mathbb{Z}_{\geq 0}\} \qquad \text{is a basis of } \mathfrak{U}\mathfrak{n}^-, \text{ and}$$

$$\{H_1^{s_1}H_2^{s_2}\cdots H_1^{s_r} \mid s_1,\ldots,s_N\in\mathbb{Z}_{\geq 0}\} \qquad \text{is a basis of } \mathfrak{U}\mathfrak{h}.$$

(5.5) The Casimir element in $\mathfrak{U}\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra and let $\langle,\rangle$ be the Killing form on $\mathfrak{g}$ (see (1.6)). Let $\{b_i\}$ be a basis of $\mathfrak{g}$ and let $\{b^i\}$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form. Let c be the element of the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$ given by

$$c=\sum_i b_i b^i.$$

Then

$$c \text{ is in the center of } \mathfrak{U}\mathfrak{g}.$$

Any central element of $\mathfrak{U}\mathfrak{g}$ must act on each finite dimensional simple module by a constant. For each dominant integral weight λ let V^λ be the finite dimensional simple $\mathfrak{U}\mathfrak{g}$-module indexed by λ (see (2.5)). Let ρ be the element of $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\rho=\tfrac{1}{2}\sum_{\alpha>0}\alpha,$$

where the sum is over all positive roots for $\mathfrak{g}$. Then the element

$$c \text{ acts on } V^\lambda \text{ by the constant } (\lambda+\rho,\lambda+\rho)-(\rho,\rho),$$

where inner product on $\mathfrak{h}_{\mathbb{R}}^*$ is as given in (2.7).

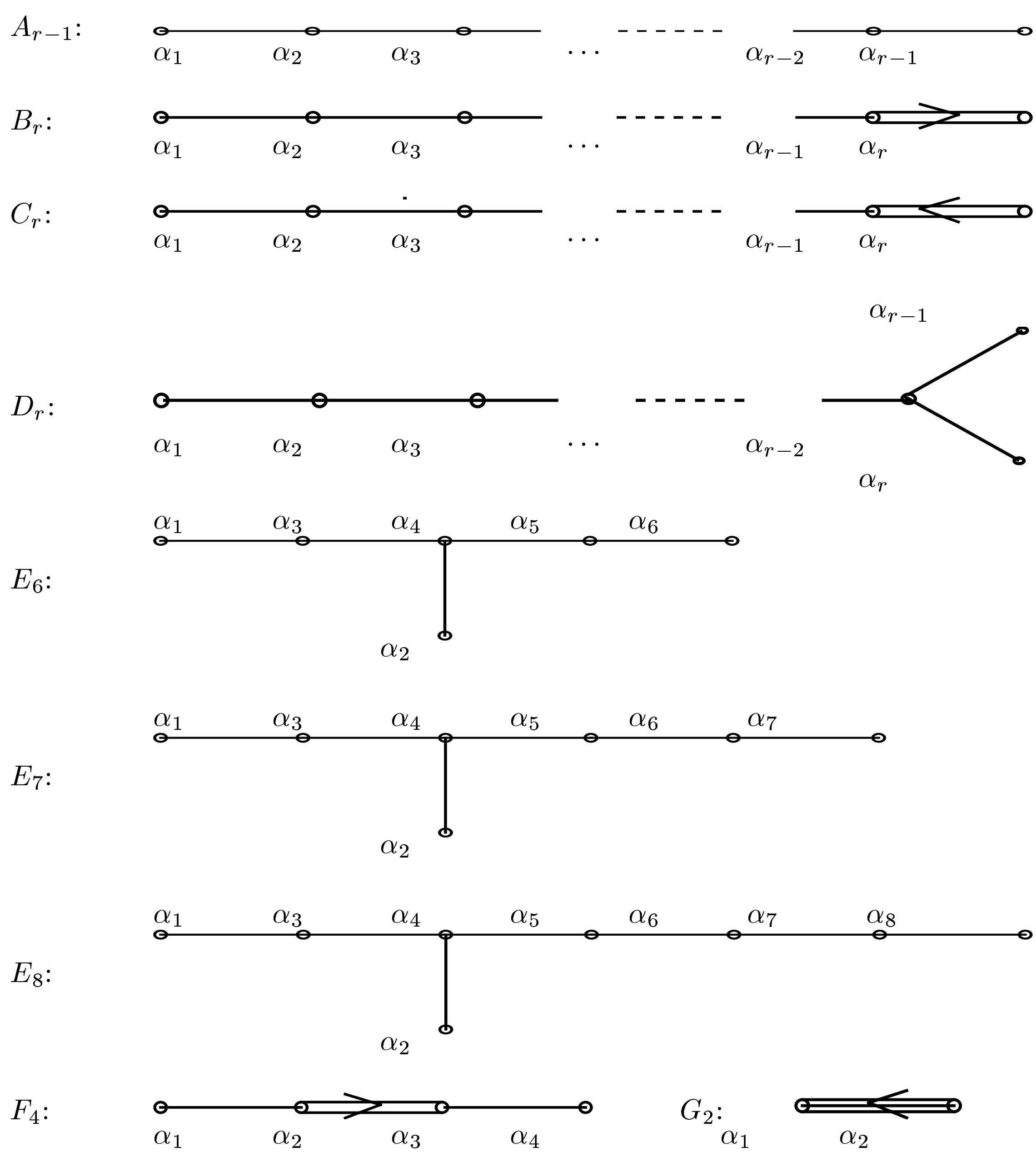


Table 1. Dynkin diagrams corresponding to finite dimensional complex simple Lie algebras

$$A_{r-1}: \quad \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix} \qquad B_r: \quad \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -2 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix}$$

$$C_r: \quad \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ 0 & -1 & 2 & \cdots & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -2 & 2 \end{pmatrix} \qquad D_r: \quad \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6: \quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \qquad E_7: \quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8: \quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4: \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad G_2: \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Table 2. Cartan matrices corresponding to finite dimensional complex simple Lie algebras

III. Deformations of Hopf algebras

The basic material on completions given in §1 can be found in many books, in particular, [AM] Chapt 10. The book [SS] has a comprehensive treatment of deformation theory. Theorem (2.6) is stated and proved in [SS] Prop. 11.3.1.

1. h-adic completions

(1.1) Motivation for h-adic completions

We will be working with algebras over $\mathbb{C}[[h]]$, the ring of formal power series in a variable h with coefficients in $\mathbb{C}$. A typical element of $\mathbb{C}[[h]]$ which is not in $\mathbb{C}[h]$ is the element

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots.$$

The ring $\mathbb{C}[[h]]$ is just $\mathbb{C}[h]$ extended a little bit so that some nice elements that we want to write down, like e^h, are in $\mathbb{C}[[h]]$.

An algebra over $\mathbb{C}[[h]]$ is a vector space over $\mathbb{C}[[h]]$, i.e. a free $\mathbb{C}[[h]]$-module, which has a multiplication and an identity which satisfy the conditions in I (1.1). If A is an algebra over $\mathbb{C}$ then we can extend coefficients and get a new algebra $A \otimes_{\mathbb{C}} \mathbb{C}[[h]]$ which is over $\mathbb{C}[[h]]$. But sometimes this new algebra is not quite big enough so we need to extend it a little bit and work with the h-adic completion $A[[h]]$ which contains all the nice elements that we want to write down.

Continuing in this vein we will want to consider the tensor product $A[[h]] \otimes A[[h]]$. Again, this algebra is not quite big enough and we extend it to get a slightly bigger object $A[[h]]\hat{\otimes}A[[h]]$ so that all the elements we want are available.

(1.2) The algebra $A[[h]]$, an example of an h-adic completion

If A is an algebra over k then the set

$$A[[h]] = \{a_0 + a_1 h + a_2 h^2 + \cdots \mid a_i \in A\}$$

of formal power series with coefficients in A is the completion of the $k[[h]]$-module $k[[h]] \otimes_k A$ in the h-adic topology. The $k[[h]]$-linear extension of the multiplication in A gives $A[[h]]$ the structure of a $k[[h]]$-algebra. The ring $A[[h]]$ is, in general, larger than $k[[h]] \otimes_k A$. For each element $a = \sum_{j\geq 0} a_j h^j \in A[[h]]$ the element

$$e^{ha} = \sum_{\ell \geq 0} \frac{(ha)^\ell}{\ell!} = 1 + a_0 h + (a_0^2 + 2a_1)\left(\frac{h^2}{2}\right) + (a_0^3 + 3(a_0 a_1 + a_1 a_0) + 6a_2)\left(\frac{h^3}{3!}\right) + \cdots$$

is a well defined element of $A[[h]]$.

(1.3) Definition of the h-adic topology

Let k be a field and let h be an indeterminate. The ring $k[[h]]$ is a local ring with unique maximal ideal (h). Let M be a $k[[h]]$-module. The sets

$$m + h^n M, \quad m \in M,\, n \in \mathbb{N},$$

form a basis for a topology on M called the *h-adic topology*. Define a map $d: M \times M \to \mathbb{R}$ by

$$d(x,y) = e^{-v(x-y)}, \quad \text{for all } x, y \in M,$$

where e is a real number $e > 1$ and $v(x)$ is the largest nonnegative integer n such that $x \in h^n M$. Then d is a metric on M which generates the h-adic topology.

(1.4) Definition of an h-adic completion

Let M be a $k[[h]]$-module. The completion of the metric space M is a metric space $\hat{M}$ which contains M in a natural way and which has a natural $k[[h]]$-module structure. The completion $\hat{M}$ of M is defined in the usual way, as a set of equivalence classes of Cauchy sequences of elements of M. Let us review this construction.

A sequence of elements $\{p_n\}$ in M is a *Cauchy sequence* in the h-adic topology if for every positive integer $\ell > 0$ there exists a positive integer N such that

$$p_n - p_m \in h^\ell M, \quad \text{for all } m, n > N,$$

i.e. $p_n - p_m$ is "divisible" by h^ℓ for all $n, m > N$. Two Cauchy sequences $P = \{p_n\}$ and $Q = \{q_n\}$ are *equivalent* if the sequence $\{p_n - q_n\}$ converges to 0, i.e.

$$P \sim Q \text{ if for every } \ell \text{ there exists an } N \text{ such that } p_n - q_n \in h^\ell M \text{ for all } n > N.$$

The set of all equivalence classes of Cauchy sequences in M is the *completion* $\hat{M}$ of M.

The completion $\hat{M}$ is a $k[[h]]$-module where the operations are determined by

$$P + Q = \{p_n + q_n\}, \quad \text{and} \quad aP = \{ap_n\},$$

where $P = \{p_n\}$ and $Q = \{q_n\}$ are Cauchy sequences with elements in M and $a \in k[[h]]$. Define a map

$$\begin{array}{cccc} \phi: & M & \longrightarrow & \hat{M} \\ & m & \longmapsto & [(m, m, m, \ldots)], \end{array}$$

i.e. $\phi(m)$ is the equivalence class of the sequence $\{p_n\}$ such that $p_n = m$ for all n. This map is injective and thus we can view M as a submodule of $\hat{M}$.

2. Deformations

(2.1) Motivation for deformations

We are going to make the quantum group by deforming the enveloping algebra $\mathfrak{U}\mathfrak{g}$ of a complex simple Lie algebra $\mathfrak{g}$ *as a Hopf algebra.* This last condition is important because the enveloping algebra $\mathfrak{U}\mathfrak{g}$ does not have any deformations as an algebra.

(2.2) Deformation as a Hopf algebra

Assume that $(A, m, \iota, \Delta, \epsilon, S)$ is a Hopf algebra over k. Let $A[[h]]\hat{\otimes}A[[h]]$ denote the completion of $A[[h]] \otimes_{k[[h]]} A[[h]]$ in the h-adic topology. A *deformation of* A *as a Hopf algebra* is a tuple $(A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h)$ where

$$\begin{gathered} m_h\colon A[[h]]\hat{\otimes}A[[h]] \longrightarrow A[[h]], \qquad \Delta_h\colon A[[h]] \longrightarrow A[[h]]\hat{\otimes}A[[h]], \\ \iota_h\colon k[[h]] \longrightarrow A[[h]], \quad \epsilon_h\colon A[[h]] \longrightarrow k[[h]], \quad \text{and} \quad S_h\colon A[[h]] \longrightarrow A[[h]], \end{gathered}$$

are $k[[h]]$-linear maps which are continuous in the h-adic topology, satisfy axioms (1) - (7) in the definition of a Hopf algebra, and can be written in the form

$$\begin{aligned} m_h &= m + m_1 h + m_2 h^2 + \cdots \\ \Delta_h &= \Delta + \Delta_1 h + \Delta_2 h^2 + \cdots \\ \iota_h &= \iota + \iota_1 h + \iota_2 h^2 + \cdots \\ \epsilon_h &= \epsilon + \epsilon_1 h + \epsilon_2 h^2 + \cdots \\ S_h &= S + S_1 h + S_2 h^2 + \cdots \end{aligned}$$

where, for each positive integer i,

$$\begin{gathered} m_i\colon A \otimes A \longrightarrow A, \qquad \Delta_i\colon A \longrightarrow A \otimes A, \\ \iota_i\colon k \longrightarrow A, \qquad \epsilon_i\colon A \longrightarrow k, \quad \text{and} \quad S_i\colon A \longrightarrow A, \end{gathered}$$

are k-linear maps which are extended first $k[[h]]$-linearly and then to the h-adic completion. We shall abuse language (only slightly) and call $(A[[h]], m_h, \iota_h, \epsilon_h, \Delta_h, S_h)$ a Hopf algebra over $k[[h]]$.

(2.3) Definition of equivalent deformations

Two Hopf algebra deformations $(A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h)$ and $(A[[h]], m'_h, \iota'_h, \Delta'_h, S'_h)$ of a Hopf algebra $(A, m, \iota, \Delta, \epsilon, S)$ are *equivalent* if there is an isomorphism

$$f_h : (A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h) \longrightarrow (A[[h]], m'_h, \iota'_h, \Delta'_h, \epsilon'_h, S'_h)$$

of h-adically complete Hopf algebras over $k[[h]]$ which can be written in the form

$$f_h \ = \ \mathrm{id}_A + f_1 h + f_2 h^2 + \cdots$$

such that, for each positive integer i, $f_i : A \to A$ is a k-linear map which is extended $k[[h]]$-linearly to $k[[h]] \otimes_k A$ and then to the h-adic completion $A[[h]]$.

(2.4) Definition of the trivial deformation as a Hopf algebra

Let $(A, m, \iota, \Delta, \epsilon, S)$ be a Hopf algebra. The *trivial deformation of A as a Hopf algebra* is the Hopf algebra $(A[[h]], m_h, \iota_h, \Delta_h, \epsilon_h, S_h)$ over $k[[h]]$ such that $m_h = m$, $\iota_h = \iota$, $\Delta_h = \Delta$, $\epsilon_h = \epsilon$ and $S_h = S$ (extended to $A[[h]]$).

(2.5) Deformation as an algebra

Assume that (A, m, ι) is an algebra over k. Let $A[[h]]\hat{\otimes}A[[h]]$ denote the completion of $A[[h]] \otimes_{k[[h]]} A[[h]]$ in the h-adic topology. A *deformation of A as an algebra* is a tuple $(A[[h]], m_h, \iota_h)$ where

$$m_h\colon A[[h]]\hat{\otimes}A[[h]] \longrightarrow A[[h]], \qquad \iota_h\colon k[[h]] \longrightarrow A[[h]],$$

are $k[[h]]$-linear maps which are continuous in the h-adic topology, satisfy the axioms the definition of an algebra (see I (1.1)) and can be written in the form

$$\begin{aligned} m_h &= m + m_1 h + m_2 h^2 + \cdots \\ \iota_h &= \iota + \iota_1 h + \iota_2 h^2 + \cdots \end{aligned}$$

where, for each positive integer i,

$$m_i\colon A \otimes A \longrightarrow A, \qquad \iota_i\colon k \longrightarrow A,$$

are k-linear maps which are extended first $k[[h]]$-linearly and then to the h-adic completion. We shall abuse language (only slightly) and call $(A[[h]], m_h, \iota_h)$ an algebra over $k[[h]]$.

This definition is exactly like the definition of a deformation as a Hopf algebra in (2.2) above except that we only need to start with an algebra and we only require the result to be an algebra. We can define *equivalence of deformations as algebras* in exactly the same way that we defined them for deformations as Hopf algebras except that we only require the isomorphism f_h to be an algebra isomorphism instead of a Hopf algebra isomorphism.

(2.6) The trivial deformation as an algebra

Let (A, m, ι) be an algebra. The *trivial deformation* of A as an algebra is the algebra $(A[[h]], m_h, \iota_h)$ over $k[[h]]$ such that $m_h = m$ and $\iota_h = \iota$ (extended to $A[[h]]$). The deformation of the quantum group given in V (1.3) is even more incredible if one keeps the following theorem in mind.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$. Then $\mathfrak{U}\mathfrak{g}$ has no deformations as an algebra (up to equivalence of deformations).*

In other words, all deformations of $\mathfrak{U}\mathfrak{g}$ as an algebra are equivalent to the trivial deformation of $\mathfrak{U}\mathfrak{g}$.

IV. Perverse sheaves

To any reader that has not met sheaves before: I suggest that you don't read this section, only refer to it a few times while you are reading Chapter VIII of these notes. The most important thing, from the point of view of these notes, is to understand the basic structures given in Chapter VIII; anyone who is going to study these topics in more depth can always come back and learn these definitions later.

A large part of the material in this section is basic material about derived categories. This material can usually be found in texts which treat homological algebra. Everything in this section, except the definition and properties of perverse sheaves given in §3 can be found in [KS] Chapt. I-III. The definition of a perverse sheaf is in [BBD] 4.0 and the proof of Theorem (3.1) is in [BBD] Theorem 1.3.6. The Theorems in (3.2) are proved in [BBD] 2.1.9-2.1.11 and Theorem 4.3.1, respectively. We shall not review the definition of sheaves, it can be found in many textbooks, see [KS] Chapt. II.

1. The category $D^b_c(X)$

(1.1) Complexes of sheaves

Let X be an algebraic variety. A *complex of sheaves on X* is a sequence of sheaves A^i on X and morphisms of sheaves $d_i\colon A^i \to A^{i+1}$,

$$A \;=\; \left(\cdots \xrightarrow{d_{-2}} A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} \cdots\right) \qquad \text{such that} \qquad d_{i+1}d_i = 0.$$

The morphisms $d_i\colon A^i \to A^{i+1}$ are called the *differentials* of the complex A. Let A and B be complexes of sheaves. A *morphism* $f\colon A \to B$ is a set of maps $f_n\colon A^n \to B^n$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{-2}} & A^{-1} & \xrightarrow{d_{-1}} & A^0 & \xrightarrow{d_0} & A^1 & \xrightarrow{d_1} & \cdots \\
 & & \big\downarrow{\scriptstyle f_{-1}} & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_1} & & \\
\cdots & \xrightarrow{d_{-2}} & B^{-1} & \xrightarrow{d_{-1}} & B^0 & \xrightarrow{d_0} & B^1 & \xrightarrow{d_1} & \cdots
\end{array}$$

commutes.

The ith cohomology sheaf of a complex A is the sheaf

$$\mathcal{H}^i(A) = \frac{\ker(A^i \to A^{i+1})}{\mathrm{im}(A^{i-1} \to A_i)}$$

We have a well defined complex of sheaves $\mathcal{H}(A)$ given by

$$\cdots \xrightarrow{d_{-2}} \mathcal{H}^{-1}(A) \xrightarrow{d_{-1}} \mathcal{H}^0(A) \xrightarrow{d_0} \mathcal{H}^1(A) \xrightarrow{d_1} \cdots$$

A *quasi-isomorphism* $f: A \xrightarrow{\sim} B$ is a morphism $f: A \to B$ such that the induced morphism $\mathcal{H}(f): \mathcal{H}(A) \to \mathcal{H}(B)$ is an isomorphism. Note that every isomorphism is a quasi-isomorphism but *not* the other way around (even though the notation may be confusing).

(1.2) The category $K(X)$ and derived functors

Let X be an algebraic variety. Let A and B be complexes of sheaves on X. Two morphisms $f: A \to B$ and $g: A \to B$ are *homotopic* if there is a collection of morphisms $k_i: A^i \to B^{i-1}$ such that

$$f_n - g_n = k_{n+1}d_n + d_{n-1}k_n.$$

The motivation for this definition is that if f and g are homotopic then $\mathcal{H}(f) = \mathcal{H}(g)$.

Define $K(X)$ to be the category given by

Objects: Complexes of sheaves on X.

Morphisms: A $K(X)$-morphism from a complex A to a complex B is an homotopy equivalence class of morphisms from A to B.

This just means that, in the category $K(X)$, we identify homotopic morphisms.

Let A be a complex of sheaves on X. An *injective resolution* of A is a quasi-isomorphism $A \xrightarrow{\sim} J$ such that J^i is injective (an injective object in the category of sheaves on X) for all i. Let $Sh(X)$ denote the category of sheaves on X and let $F: Sh(X) \to Sh(X)$ be a functor. The *right derived functor* of F is the functor $RF: K(X) \to K(X)$ given by

$$RF(A) = F(J) \;=\; \left(\cdots \xrightarrow{F(d_{-2})} F(J^{-1}) \xrightarrow{F(d_{-1})} F(J^0) \xrightarrow{F(d_0)} F(J^1) \xrightarrow{F(d_1)} \cdots\right)$$

where J is an injective resolution of A. The *ith derived functor of* F is the functor $R^iF: K(X) \to Sh(X)$ given by

$$R^iF(A) \;=\; \mathcal{H}^i(F(J)),$$

where J is an injective resolution of A. In other words $R^iF(A)$ is the ith cohomology sheaf of the complex $RF(A)$.

(1.3) Bounded complexes and constructible complexes

A complex of sheaves A is *bounded* if there exists a positive integer n such that $A^m = 0$ and $A^{-m} = 0$ for all $m > n$.

An *algebraic stratification* of an algebraic variety X is a finite partition $X = \bigsqcup_\alpha X_\alpha$ of X into *strata* such that

(a) For each α, the stratum X_α is a smooth locally closed algebraic subvariety in X,

(b) The closure of each stratum is a union of strata, and

(c) The Whitney condition holds (see Verdier [Ver]).

Let l be a prime number and let $\overline{\mathbb{Q}_l}$ be the algebraic closure of the field $\mathbb{Q}_l$ of l-adic numbers. A sheaf F on X is $\overline{\mathbb{Q}_l}$*-constructible* if there is an algebraic stratification $X = \bigsqcup_\alpha X_\alpha$ such that, for each α, the restriction of F to X_α is a locally constant sheaf of finite dimensional vector spaces over $\overline{\mathbb{Q}_l}$. A complex $A \in K(X)$ is $\overline{\mathbb{Q}_l}$*-constructible* if $\mathcal{H}^i(A)$ is $\overline{\mathbb{Q}_l}$-constructible for all i.

(1.4) Definition of the category $D_c^b(X)$

Let X be a variety. Let A and B be complexes of sheaves on X. Define an equivalence relation on diagrams

$$A \overset{\sim}{\longleftarrow} C \longrightarrow B$$

in $K(X)$ which have A and B as end points by saying that the diagram $A \overset{\sim}{\longleftarrow} C \longrightarrow B$, is equivalent to the diagram $A \overset{\sim}{\longleftarrow} C' \longrightarrow B$, if there exists a commutative diagram

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[dr] & \\
A \arrow[r, "\sim"] & D \arrow[u] \arrow[d] \arrow[r] & B \\
 & C' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The notation $C \overset{\sim}{\longleftarrow} A$ denotes that the map is a quasi-isomorphism. The *bounded derived category of* $\overline{\mathbb{Q}}_l$*-constructible sheaves* on X is the category $D_c^b(X)$ given by

> *Objects:* Bounded, $\overline{\mathbb{Q}_l}$-constructible complexes of sheaves on X.
>
> *Morphisms:* A morphism from A to B is an equivalence class of diagrams $A \overset{\sim}{\longleftarrow} C \longrightarrow B$.

This definition of morphisms is a formal mechanism that inverts all quasi-isomorphisms. It ensures (in a coherent way) that "inverses" of quasi-isomorphisms are morphisms, i.e. that $A \overset{\sim}{\longleftarrow} B$ is a morphism from A to B.

Given two morphisms $A \overset{\sim}{\longleftarrow} D \longrightarrow B$ and $B \overset{\sim}{\longleftarrow} E \longrightarrow C$ in $D_c^b(X)$ one can show that there always exists a commutative diagram

$$\begin{tikzcd}
 & & F \arrow[dl] \arrow[dr] & & \\
 & D \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

and one defines the composition of the two morphisms $A \overset{\sim}{\longleftarrow} D \longrightarrow B$ and $B \overset{\sim}{\longleftarrow} E \longrightarrow C$ to be the morphism defined by the diagram $A \overset{\sim}{\longleftarrow} F \longrightarrow C$.

2. Functors

(2.1) The direct image with compact support functor $f_!$

A map $g: X \to Y$ between locally compact algebraic varieties is *compact* if the inverse image of every compact subset of Y is a compact subset of X.

Let $f: X \to Y$ be a morphism of locally compact algebraic varieties. Let F be a sheaf on X. The *support*, supp s, of a section s of F on an open set V is the complement in V of the union of open sets $U \subseteq V$ such that $s|_U = 0$.

The *direct image with compact support sheaf* $f_!F$, is the sheaf on Y defined by setting

$$\Gamma(U; f_!F) = \{s \in \Gamma(f^{-1}(U); F) \mid f : \text{supp } s \to U \text{ is compact}\},$$

for every open set U in Y. (For a sheaf F on X and an open set U in X, $\Gamma(U; F) = F(U)$.) This defines a functor $f_! : Sh(X) \to Sh(Y)$, where $Sh(X)$ denotes the category of sheaves on X.

Let $f: X \to Y$ be a morphism of locally compact algebraic varieties. The *direct image with compact support functor* $f_!: D_c^b(X) \to D_c^b(Y)$ is given by

$$f_! = Rf_!,$$

so that $f_!$ is the right derived functor of the functor $f_!: Sh(X) \to Sh(Y)$.

(2.2) The inverse image functor f^*

Let $f: X \to Y$ be a morphism of algebraic varieties. Let F be a sheaf on Y. The *inverse image* sheaf f^*F is the sheaf on X associated to the presheaf

$$V \longmapsto \lim_{U \supseteq f(V)} F(U), \qquad \text{for all } V \text{ open in } X,$$

where the limit is over all open sets U in Y which contain $f(V)$. This defines a functor $f^*: Sh(Y) \to Sh(X)$, where $Sh(X)$ denotes the category of sheaves on X. It is very common to denote this functor by f^{-1} but we shall follow [BBD] and [Lu] and use the notation f^*.

The *inverse image functor* $f^*: D_c^b(Y) \to D_c^b(X)$ is given by

$$f^* = Rf^*,$$

so that f^* is the right derived functor of the functor $f^*: Sh(Y) \to Sh(X)$.

(2.3) The functor $f_\flat$

Let $f: X \to Y$ be a morphism of algebraic varieties. Let $A \in D_c^b(X)$. Then $f_\flat A$ is the unique (up to isomorphism) complex on Y such that

$$A \cong f^*(f_\flat A).$$

Actually, I have cheated here: We can only be sure that the complex $f_\flat A$ is well defined if f is a locally trivial principal G-bundle, A is a semisimple G-equivariant complex on X and we require $f_\flat A$ to be a semisimple complex on Y, see [Lu] 8.1.7 and 8.1.8 for definitions and details.

(2.4) The shift functor $[n]$

Let A be a complex of sheaves on X. For each integer n define a new complex $A[n]$, with differentials $d[n]_i$, by

$$(A[n])^i = A^{n+i}, \qquad \text{and} \qquad (d[n])_i = (-1)^n d_{n+i}.$$

The *shift functor* is the functor

$$\begin{matrix} D_c^b(X) & \xrightarrow{[n]} & D_c^b(X) \\ A & \longrightarrow & A[n]. \end{matrix}$$

(2.5) The Verdier duality functor D

This definition is too involved for us to take the energy to repeat it here, we shall refer the reader to [KS] §3.1. The main thing that we will need to know is that this functor exists.

3. Perverse sheaves

(3.1) Definition of perverse sheaves

Let X be an algebraic variety. The *support*, supp F, of a sheaf F on X is the complement of the union of open sets $U \subseteq X$ such that $F\big|_U = 0$.

A complex $A \in D_c^b(X)$ is a *perverse sheaf* if

(a) dim supp $\mathcal{H}^i(A) = 0$ for $i \geq 0$ and dim supp $\mathcal{H}^i(A) \leq -i$ for $i < 0$, and

(b) dim supp $\mathcal{H}^i(D(A)) = 0$ for $i \geq 0$ and dim supp $\mathcal{H}^i(D(A)) \leq -i$ for $i < 0$,

where $D(A)$ is the Verdier dual of A.

An abelian category is a category which has a direct sum operation and for which every morphism has a kernel and a cokernel. See [KS] I §1.2 for a precise definition.

Theorem. *The full subcategory of $D_c^b(X)$ whose objects are perverse sheaves on X is an abelian category.*

(3.2) Intersection cohomology complexes

Theorem. *Let $Y \subseteq X$ be a smooth locally closed subvariety of complex dimension $d > 0$ and let $\mathcal{L}$ be a locally constant sheaf on Y. There is a unique complex $IC(Y, \mathcal{L})$ in $D_c^b(X)$ such that*

(1) $\mathcal{H}^i(IC(Y, \mathcal{L})) = 0$, if $i < -d$,

(2) $\mathcal{H}^{-d}(IC(Y,\mathcal{L}))\mid_Y = \mathcal{L}$,
(3) $\dim \operatorname{supp} \mathcal{H}^i(IC(Y,\mathcal{L})) \le -i$, *if* $i > -d$,
(4) $\dim \operatorname{supp} \mathcal{H}^i(D(IC(Y,\mathcal{L}))) \le -i$, *if* $i > -d$,

The complexes $IC(Y,\mathcal{L})$ are the *intersection cohomology* complexes and an explicit construction of these complexes is given in [BBD] Prop. 2.1.11.

Theorem. *The simple objects of the category of perverse sheaves are the intersection complexes* $IC(Y,\mathcal{L})$ *as* $\mathcal{L}$ *runs through the irreducible locally constant sheaves on various smooth locally closed subvarieties* $Y \subseteq X$.

V. Quantum groups

The definition of the quantum group and the uniqueness theorem, Theorem (1.4), are stated in [D1] §6 Example 6.2. Theorem (1.4) appears with proof in [SS] Theorem 11.4.1. The statements in (3.3) and (3.4) can be found in [CP] 9.2.1 and 9.3.1 and the treatment there gives references for where to find the proofs.

1. Definition, uniqueness, and existence

(1.1) Making the Cartan matrix symmetric

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. There exist unique positive integers $d_1, d_2, \cdots, d_r$ such that $gcd(d_1, \cdots, d_r) = 1$ and the matrix $(d_i\alpha_j(H_i))_{1\le i,j\le r}$ is symmetric. The integers $d_1, d_2, \cdots, d_r$ are given explicitly by

$A_r,\ D_r,$ $E_6,\ E_7,\ E_8$:	$d_i = 1$ for all $1 \le i \le r$,
B_r :	$d_i = 1$ for $1 \le i \le r-1$, and $d_r = 2$,
C_r :	$d_i = 2$, for $1 \le i \le r-1$, and $d_r = 1$,
F_4 :	$d_1 = d_2 = 1$, and $d_3 = d_4 = 2$,
G_2:	$d_1 = 3$, and $d_2 = 1$.

(1.2) The Poisson homomorphism δ

Let $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be the $\mathbb{C}$-linear map given by

$$\delta(H_i) = 0, \quad \delta(X_i^\pm) = d_i(X_i^\pm \otimes H_i - H_i \otimes X_i^\pm), \quad 1 \le i \le r.$$

There is a unique extension of the map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ to a $\mathbb{C}$-linear map $\delta : \mathfrak{U}\mathfrak{g} \to \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$ such that

$$\delta(xy) = \Delta(x)\delta(y) + \delta(x)\Delta(y), \quad \text{for all } x, y \in \mathfrak{U}\mathfrak{g}.$$

(1.3) The definition of the quantum group

A *Drinfel'd-Jimbo quantum group* $\mathfrak{U}_h\mathfrak{g}$ *corresponding* to $\mathfrak{g}$ is a deformation of $\mathfrak{U}\mathfrak{g}$ as a Hopf algebra over $\mathbb{C}$ such that

(1) Poisson condition :

$$\frac{\Delta_h(a) - \Delta_h^{\rm op}(a)}{h} \pmod h = \delta(a \bmod h), \quad \text{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

(If $\Delta_h(a) = \sum_a a_{(1)} \otimes a_{(2)}$ then $\Delta_h^{\rm op}(a) = \sum_a a_{(2)} \otimes a_{(1)}$.)

(2) Cartan subalgebra condition:

There is a subalgebra $\mathfrak{U}_h\mathfrak{h} \subseteq \mathfrak{U}_h\mathfrak{g}$ such that

(a) $\mathfrak{U}_h\mathfrak{h}$ is cocommutative, i.e. $\Delta_h(a) = \Delta_h^{\mathrm{op}}(a)$, for all $a \in \mathfrak{U}_h\mathfrak{h}$,

(b) The mapping $\mathfrak{U}_h\mathfrak{h}/h\mathfrak{U}_h\mathfrak{h} \to \mathfrak{U}\mathfrak{g}$ is injective with image $\mathfrak{U}\mathfrak{h}$.

(3) Cartan involution condition:

There is a mapping $\theta : \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ such that

(a) $\theta^2 = \mathrm{id}_{\mathfrak{U}_h\mathfrak{g}}$,

(b) $\theta(\mathfrak{U}_h\mathfrak{h}) = \mathfrak{U}_h\mathfrak{h}$,

(c) θ is an algebra homomorphism and a coalgebra antihomomorphism, i.e.

$$\begin{aligned} \theta(ab) &= \theta(a)\theta(b), \quad \text{for all } a, b \in \mathfrak{U}_h\mathfrak{g}, \text{ and} \\ \Delta_h(\theta(a)) &= (\theta \otimes \theta)\Delta_h^{\mathrm{op}}(a), \quad \text{for all } a \in \mathfrak{U}_h\mathfrak{g}, \end{aligned}$$

(d) θ mod h is the Cartan involution.

(1.4) Uniqueness of the quantum group

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. The Drinfel'd-Jimbo quantum group $\mathfrak{U}_h\mathfrak{g}$ corresponding to $\mathfrak{g}$ is unique (up to equivalence of deformations).*

(1.5) Definition of q-integers and q-factorials

For any symbol q define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \text{and}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \quad \text{for all positive integers } m \geq n,$$

(1.6) Presentation of the quantum group by generators and relations

Note the similarities (and the differences) between the following presentation of the quantum group by generators and relations and the presentation of the enveloping algebra of $\mathfrak{g}$ given in II (2.2).

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\leq i,j\leq r}$ be the corresponding Cartan matrix. The Drinfel'd-Jimbo quantum group $\mathfrak{U}_h\mathfrak{g}$ corresponding to $\mathfrak{g}$ can be presented as the algebra over $\mathbb{C}[[h]]$ generated (as a complete $\mathbb{C}[[h]]$-algebra in the h-adic topology) by*

$$X_1^-, X_2^-, \ldots, X_r^-, \quad H_1, H_2, \ldots, H_r, \quad X_1^+, X_2^+, \ldots, X_r^+,$$

with relations

$$
\begin{aligned}
&[H_i, H_j] = 0, && \text{for all } 1 \le i, j \le r, \\
&[H_i, X_j^+] = \alpha_j(H_i) X_j^+, && \\
& && \text{for all } 1 \le i, j \le r, \\
&[H_i, X_j^-] = -\alpha_j(H_i) X_j^-, && \\
&[X_i^+, X_j^-] = \delta_{ij} \frac{e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}}, && \text{for } 1 \le i, j \le r,
\end{aligned}
$$

$$
\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{e^{d_i h}} (X_i^\pm)^s X_j^\pm (X_i^\pm)^t = 0, \qquad \text{for } i \ne j,
$$

and with Hopf algebra structure given by

$$
\begin{gathered}
\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \\
\Delta_h(X_i^+) = X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+, \qquad \Delta_h(X_i^-) = X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-, \\
S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ e^{-d_i h H_i}, \quad S_h(X_i^-) = -e^{d_i h H_i} X_i^-, \\
\epsilon_h(H_i) = \epsilon_h(X_i^+) = \epsilon_h(X_i^-) = 0,
\end{gathered}
$$

Cartan subalgebra $\mathfrak{U}\mathfrak{h}[[h]] \subseteq \mathfrak{U}_h\mathfrak{g}$, *and Cartan involution* $\theta\colon \mathfrak{U}_h\mathfrak{g} \longrightarrow \mathfrak{U}_h\mathfrak{g}$ *determined by*

$$
\theta(X_i^+) = -X_i^-, \qquad \theta(X_i^-) = -X_i^+, \qquad \theta(H_i) = -H_i.
$$

2. The rational form of the quantum group

The rational form of the quantum group is an algebra which is similar to the algebra $\mathfrak{U}_h\mathfrak{g}$ except that it is over an arbitrary field k. There are two reasons for introducing this algebra.

(1) In the case when $k = \mathbb{C}(q)$ is the field this new algebra $U_q\mathfrak{g}$ has "integral forms" which can be used to specialize q to special values.

(2) In the case when $k = \mathbb{C}$ and q is a power of a prime then part of this algebra appears naturally as a Hall algebra of representations of quivers or, equivalently, as a Grothendieck ring of G-equivariant perverse sheaves on certain varieties E_V.

(2.1) Definition of the rational form of the quantum group

Many authors use the following form $U_q\mathfrak{g}$ of the quantum group as *the definition* of the quantum group.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix. Let k be a field and let $q \in k$ be an nonzero element of k. The *rational form of the Drinfel'd-Jimbo quantum group* $U_q\mathfrak{g}$ corresponding to $\mathfrak{g}$ is the algebra $U_q\mathfrak{g}$ over k generated by

$$F_1, F_2, \ldots, F_r, \quad K_1, K_2, \ldots, K_r, \quad K_1^{-1}, K_2^{-1}, \ldots, K_r^{-1}, \quad E_1, E_2, \ldots, E_r,$$

with relations

$$\begin{aligned}
&K_iK_j = K_jK_i, && \text{for all } 1 \le i,j \le r,\\
&K_iK_i^{-1} = K_i^{-1}K_i = 1, && \text{for all } 1 \le i \le r,\\
&K_iE_jK_i^{-1} = q^{d_i\alpha_j(H_i)}E_j, && \\
& && \text{for all } 1 \le i,j \le r,\\
&K_iF_jK_i^{-1} = q^{-d_i\alpha_j(H_i)}F_j, && \\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, && \text{for } 1 \le i,j \le r,
\end{aligned}$$

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{q^{d_i}} E_i^sE_jE_i^t = 0, \qquad \text{for } i \ne j,$$

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{q^{d_i}} F_i^sF_jF_i^t = 0, \qquad \text{for } i \ne j,$$

and with Hopf algebra structure given by

$$\begin{gathered}
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,\\
S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i,\\
\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0, \quad \epsilon(F_i) = 0.
\end{gathered}$$

It is very common to take q to be an indeterminate and to let $k = \mathbb{C}(q)$ be the field of rational functions in q.

(2.2) Relating the rational form and the original form of the quantum group

The relations in the rational form of the quantum group are obtained from the relations in the presentation of $\mathfrak{U}_h\mathfrak{g}$ by making the following replacements:

$$e^h \longrightarrow q, \quad e^{hd_iH_i} \longrightarrow K_i, \quad X_i^- \longrightarrow F_i, \quad X_i^+ \longrightarrow E_i.$$

The ring $U_q\mathfrak{g}$ is an algebra over k and $q \in k$ while the ring $\mathfrak{U}_h\mathfrak{g}$ is an algebra over $\mathbb{C}[[h]]$ where h is an indeterminate. They have many similar properties. Most of the theorems about the structure of the algebra $\mathfrak{U}_h\mathfrak{g}$ have analogues for the case of the algebra $U_q\mathfrak{g}$. The category of modules for $U_q\mathfrak{g}$ is very similar to the category of module for the enveloping algebra $\mathfrak{U}\mathfrak{g}$. One should note, however, in contrast to Chapt. VI Theorem (1.1) which says that $\mathfrak{U}_h\mathfrak{g} \cong \mathfrak{U}\mathfrak{g}[[h]]$, it is *not* true that $U_q\mathfrak{g}$ is isomorphic to $\mathfrak{U}\mathfrak{g}$, even if $k = \mathbb{C}$ and $q \in k$. This fact complicates many of the proofs when one is trying to generalize results from the classical case of $\mathfrak{U}\mathfrak{g}$ to the quantum case $U_q\mathfrak{g}$.

3. Integral forms of the quantum group

There are two different commonly used integral forms of a $\mathbb{C}(q)$-algebra $U_q\mathfrak{g}$, the "non-restricted integral form" $U_{\mathcal{A}}\mathfrak{g}$ and the "restricted integral form" $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$. Let us begin by defining integral forms precisely.

(3.1) Definition of integral forms

Let q be an indeterminate and let U_q be an algebra over $\mathbb{C}(q)$, the field of rational functions in q. An *integral form* of U_q is a $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_{\mathcal{A}}$ of U_q such that the map

$$U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(q) \longrightarrow U_q$$

is an isomorphism of $\mathbb{C}(q)$ algebras. In other words, upon extending scalars from $\mathbb{Z}[q, q^{-1}]$ to $\mathbb{C}(q)$ the algebra $U_{\mathcal{A}}$ turns into U_q.

(3.2) Motivation for integral forms

The purpose of defining integral forms of algebras is that we can use them to specialize the variable q to certain elements of $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$, etc. Let $U_{\mathcal{A}}$ be an integral form of an algebra U_q over $\mathbb{C}(q)$ and let $\eta \in \mathbb{C}$, $\eta \neq 0$. The *specialization at* $q = \eta$ (over $\mathbb{C}$) of $U_{\mathcal{A}}$ is the algebra over $\mathbb{C}$ given by

$$U_\eta = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}, \quad \text{where the equation} \quad qc = \eta c$$

describes how $\mathbb{C}$ is an $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$-module. Similarly, we can define specializations of $U_{\mathcal{A}}$ over any field. With this last definition in mind we see that one could regard an integral form of U_q as an $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_{\mathcal{A}}$ such that U_q is the specialization of $U_{\mathcal{A}}$ over $\mathbb{C}(q)$ at $q = q$.

(3.3) Definition of the non-restricted integral form of the quantum group

Let q be an indeterminate and let $k = \mathbb{C}(q)$ be the field of rational functions in q. Let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group. For each $1 \leq i \leq r$, define elements

$$[K_i; 0]_{q^{d_i}} = \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}.$$

The *non-restricted integral form of* $U_q\mathfrak{g}$ is the $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_{\mathcal{A}}\mathfrak{g}$ of $U_q\mathfrak{g}$ generated by the elements

$$F_1, F_2, \ldots, F_r, \quad K_1^{\pm 1}, K_2^{\pm 1}, \ldots, K_r^{\pm 1}, \quad [K_1; 0], [K_2; 0], \ldots, [K_r; 0], \quad E_1, E_2, \ldots, E_r.$$

The Hopf algebra structure on $U_q\mathfrak{g}$ restricts to a well defined Hopf algebra structure on $U_{\mathcal{A}}\mathfrak{g}$.

(3.4) Definition of the restricted integral form of the quantum group

Let q be an indeterminate and let $k = \mathbb{C}(q)$ be the field of rational functions in q. Let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group. The *restricted integral form of* $U_q\mathfrak{g}$ is the $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ subalgebra $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$ of $U_q\mathfrak{g}$ generated by the elements $K_1^{\pm 1}, K_2^{\pm 1}, \ldots, K_r^{\pm 1}$, and the elements

$$F_i^{(\ell)} = \frac{F_i^\ell}{[\ell]_{q^{d_i}}!}, \quad \text{and} \quad E_i^{(\ell)} = \frac{E_i^\ell}{[\ell]_{q^{d_i}}!}, \quad \text{for all } 1 \le i \le r \text{ and all } \ell \ge 1.$$

(The notation for the q-factorials is as in (1.5).) The Hopf algebra structure on $U_q\mathfrak{g}$ restricts to a well defined Hopf algebra structure on $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$. It is nontrivial to prove that $U_{\mathcal{A}}^{\mathrm{res}}\mathfrak{g}$ is an integral form of $U_q\mathfrak{g}$.

VI. Modules for quantum groups

The isomorphism theorem in (1.1) is found (with proof) in [D2] p. 330-331. The proof of this theorem uses several cohomological facts,

$$H^2(\mathfrak{g}, \mathfrak{U}\mathfrak{g}) = 0, \quad H^1(\mathfrak{h}, \mathfrak{U}\mathfrak{g}/(\mathfrak{U}\mathfrak{g})^{\mathfrak{h}}) = 0, \quad \text{and} \quad H^1(\mathfrak{g}, \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}) = 0.$$

The correspondence theorem in (1.3) is also found in [D2] p.331. All of the results in section 2 can be found, with detailed proofs, in [Ja] Chapt. 5.

1. Finite dimensional $\mathfrak{U}_h\mathfrak{g}$-modules

(1.1) As algebras, $\mathfrak{U}_h\mathfrak{g} \cong \mathfrak{U}\mathfrak{g}[[h]]$

The algebra $\mathfrak{U}\mathfrak{g}[[h]]$ is just the enveloping algebra of the Lie algebra $\mathfrak{g}$ except over the ring $\mathbb{C}[[h]]$ (and then h-adically completed) instead of over the field $\mathbb{C}$. It acts exactly like the algebra $\mathfrak{U}\mathfrak{g}$, the only difference is that we have extended coefficients.

The following theorem says that the algebra $\mathfrak{U}_h\mathfrak{g}$ and the algebra $\mathfrak{U}\mathfrak{g}[[h]]$ are exactly the same! In fact we have already seen that this must be so, since $\mathfrak{U}\mathfrak{g}$ has no deformations as an algebra (Chapt. III Theorem (2.6)). One might ask: If $\mathfrak{U}_h\mathfrak{g}$ and $\mathfrak{U}\mathfrak{g}[[h]]$ are the same then what is big deal about quantum groups? The answer is: They are the same as algebras but they are *not* the same when you look at them as Hopf algebras.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to $\mathfrak{g}$. Then there is an isomorphism of algebras*

$$\varphi : \mathfrak{U}_h\mathfrak{g} \longrightarrow \mathfrak{U}\mathfrak{g}[[h]], \qquad \textit{such that}$$

$$\textit{(a)}\ \ \varphi = \mathrm{id}_{\mathfrak{U}\mathfrak{g}} \pmod h, \quad \textit{and}$$

$$\textit{(b)}\ \ \varphi\big|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}},$$

where, in the second condition, $\mathfrak{h} = \mathbb{C}$-span$\{H_1, \ldots, H_r\} \subseteq \mathfrak{U}_h\mathfrak{g}$.

(1.2) Definition of weight spaces in a $\mathfrak{U}_h\mathfrak{g}$ module

A *finite dimensional $\mathfrak{U}_h\mathfrak{g}$-module* is a $\mathfrak{U}_h\mathfrak{g}$-module that is a finitely generated free module as a $\mathbb{C}[[h]]$-module. If M is a finite dimensional $\mathfrak{U}_h\mathfrak{g}$-module and $\mu \in \mathfrak{h}^*$ define the *μ-weight space of M* to be the subspace

$$M_\mu = \{m \in M \mid am = \mu(a)m, \quad \text{for all } a \in \mathfrak{h}\}.$$

The *dimension* of the weight space M_μ is the number of elements in a basis for it, as a $\mathbb{C}[[h]]$-module.

(1.3) Classification of modules for $\mathfrak{U}_h\mathfrak{g}$

Theorem (1.1) says that $\mathfrak{U}_h\mathfrak{g}$ and $\mathfrak{U}\mathfrak{g}[[h]]$ are the same as algebras. Since the category of finite dimensional modules for an algebra depends only on its algebra structure it follows immediately that the category of finite dimensional modules for $\mathfrak{U}_h\mathfrak{g}$ is the same as the category of modules for $\mathfrak{U}\mathfrak{g}[[h]]$.

Theorem. *There is a one to one correspondence between the isomorphism classes of finite dimensional $\mathfrak{U}_h\mathfrak{g}$-modules and the isomorphism classes of finite dimensional $\mathfrak{U}\mathfrak{g}$-modules given by*

$$\begin{array}{ccc} \mathfrak{U}_h\mathfrak{g}\text{-}modules & \stackrel{1-1}{\longleftrightarrow} & \mathfrak{U}\mathfrak{g}\text{-}modules \\ \\ M & \longleftrightarrow & M/hM \\ V[[h]] & \longleftrightarrow & V \end{array}$$

where the $\mathfrak{U}_h\mathfrak{g}$ module structure on $V[[h]]$ is defined by the composition

$$\mathfrak{U}_h\mathfrak{g} \xrightarrow{\sim} \mathfrak{U}\mathfrak{g}[[h]] \longrightarrow \mathrm{End}(V[[h]]).$$

It follows from condition (b) of Theorem (1.1) that, under the correspondence in the Theorem above, weight spaces of $\mathfrak{U}_h\mathfrak{g}$-modules are taken to weight spaces of $\mathfrak{U}\mathfrak{g}$-modules and their dimension remains the same. Furthermore, irreducible finite dimensional $\mathfrak{U}\mathfrak{g}$-modules correspond taken to indecomposable $\mathfrak{U}_h\mathfrak{g}$-modules and vice versa. (Note that $hV[[h]]$ is always a $\mathfrak{U}_h\mathfrak{g}$-submodule of the $\mathfrak{U}_h\mathfrak{g}$-module $V[[h]]$.)

The previous theorem combined with Chapt. II Theorem (2.5) gives the following corollary.

Corollary. *Let $P^+ = \sum_{i=1}^r \mathbb{N}\omega_i$ be the set of dominant integral weights for $\mathfrak{g}$, as in (2.4). For every $\lambda \in P^+$ there is a unique (up to isomorphism) finite dimensional indecomposable $\mathfrak{U}_h\mathfrak{g}$-module $L(\lambda)$ corresponding to λ.*

2. Finite dimensional $U_q\mathfrak{g}$-modules

The category of finite dimensional modules for the rational form $U_q\mathfrak{g}$ of the quantum group is slightly different from the category of finite dimensional modules for $\mathfrak{U}_h\mathfrak{g}$. The construction of the finite dimensional irreducible modules for $U_q\mathfrak{g}$ is similar to the construction of these modules in the case of the Lie algebra $\mathfrak{g}$. Let us describe how this is done.

(2.1) Construction of the Verma module $M(\lambda)$ and the simple module $L(\lambda)$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group over a field k and with $q \in k$. We shall assume that

$$\text{char } k \neq 2, 3 \text{ and } q \text{ is not a root of unity in } k.$$

Let $\lambda \in P$ be an element of the weight lattice for $\mathfrak{g}$. The *Verma module* $M(\lambda)$ is be the $U_q\mathfrak{g}$-module generated by a single vector v_λ where the action of $U_q\mathfrak{g}$ satisfies the relations

$$E_i v_\lambda = 0, \quad \text{and} \quad K_i v_\lambda = q^{(\lambda,\alpha_i)} v_\lambda, \quad \text{for all } 1 \le i \le r.$$

The map

$$\begin{array}{ccc} U_q\mathfrak{n}^- & \longrightarrow & M(\lambda) \\ y & \longmapsto & y v_\lambda \end{array}$$

is a vector space isomorphism.

The module $M(\lambda)$ has a unique maximal proper submodule. For each $\lambda \in P$ define

$$L(\lambda) = \frac{M(\lambda)}{N}$$

where N is the maximal proper submodule of the Verma module $M(\lambda)$.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the corresponding rational form of the quantum group over a field k with $q \in k$. Assume that* char $k \neq 2, 3$ *and that q is not a root of unity in k. Let $\lambda \in P$ be an element of the weight lattice of $\mathfrak{g}$ and let $L(\lambda)$ be the $U_q\mathfrak{g}$-module defined above.*

(a) The module $L(\lambda)$ is a simple $U_q\mathfrak{g}$-module.

(b) The module $L(\lambda)$ is finite dimensional if and only if λ is a dominant integral weight.

(2.2) Twisting $L(\lambda)$ to get $L(\lambda, \sigma)$

Let Q be the root lattice corresponding to $\mathfrak{g}$ as given in II (2.6) and let $\sigma: Q \to \{\pm 1\}$ be a group homomorphism. The homomorphism σ induces an automorphism $\sigma: U_q\mathfrak{g} \to U_q\mathfrak{g}$ of $U_q\mathfrak{g}$ defined by

$$\begin{array}{cccc} \sigma: & U_q\mathfrak{g} & \longrightarrow & U_q\mathfrak{g} \\ \\ & E_i & \longmapsto & \sigma(\alpha_i)E_i \\ & F_i & \longmapsto & F_i \\ \\ & K_i^{\pm 1} & \longmapsto & \sigma(\pm\alpha_i)K_i^{\pm 1}, \end{array}$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots for $\mathfrak{g}$. Let $\lambda \in P$ be an element of the weight lattice and let $L(\lambda)$ be the irreducible $U_q\mathfrak{g}$-module defined in (2.1). Define a $U_q\mathfrak{g}$-module $L(\lambda, \sigma)$ by defining

(a) $L(\lambda, \sigma) = L(\lambda)$ as vector spaces,

(b) $U_q\mathfrak{g}$ acts on $L(\lambda, \sigma)$ by the formulas

$$u \star m = \sigma(u)m, \quad \text{for all } u \in U_q\mathfrak{g},\ m \in L(\lambda),$$

where σ is the automorphism of $U_q\mathfrak{g}$ defined above.

(2.3) Classification of finite dimensional irreducible modules for $U_q\mathfrak{g}$

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the rational form of the quantum group over a field k. Assume that* char $k \neq 2, 3$ *and $q \in k$ is not a root of unity in k. Let P^+ be the set of dominant integral weights for $\mathfrak{g}$ and let Q be the root lattice for $\mathfrak{g}$ (see II (2.6)).*

(a) *Let $\lambda \in P^+$ and let $\sigma: Q \to \{\pm 1\}$ be a group homomorphism. The modules $L(\lambda, \sigma)$ defined in (2.2) are all finite dimensional irreducible $U_q\mathfrak{g}$-modules.*

(b) *Every finite dimensional $U_q\mathfrak{g}$-module is isomorphic to $L(\lambda, \sigma)$ for some $\lambda \in P^+$ and some group homomorphism $\sigma: Q \to \{\pm 1\}$.*

(2.4) Weight spaces for $U_q\mathfrak{g}$-modules

Retain the notations and assumptions from (2.3) and let $(,)$ be the inner product on $\mathfrak{h}^*_{\mathbb{R}}$ defined in II (2.7). Let M be a finite dimensional $U_q\mathfrak{g}$-module. Let $\sigma: Q \to \{\pm 1\}$ be a group homomorphism and let $\lambda \in P$. The (λ, σ)-*weight space* of M is the vector space

$$M_{(\lambda,\sigma)} = \{m \in M \mid K_i m = \sigma(\alpha_i) q^{(\lambda, \alpha_i)} m \text{ for all } 1 \leq i \leq r.\}$$

The following proposition is analogous to Chapt. II Proposition (2.4).

Proposition. *Every finite dimensional $U_q\mathfrak{g}$-module is a direct sum of its weight spaces.*

The following theorem says that the dimensions of the weight spaces of irreducible $U_q\mathfrak{g}$-modules coincide with the dimensions of the weight space of corresponding irreducible modules for the Lie algebra $\mathfrak{g}$.

Theorem. *Let $\lambda \in P^+$ be a dominant integral weight and let σ be a group homomorphism $\sigma: Q \to \{\pm 1\}$. Let V^λ be the simple $\mathfrak{g}$-module indexed by the λ and let $L(\lambda, \sigma)$ be the irreducible $U_\mathfrak{g}$-module indexed by the pair (λ, σ). Then, for all $\mu \in P$ and all group homomorphisms $\tau: Q \to \{\pm 1\}$,*

$$\dim_k\big(L(\lambda,\sigma)_{\mu,\sigma}\big) = \dim_{\mathbb{C}}\big((V^\lambda)_\mu\big) \qquad \textit{and} \qquad \dim_k\big(L(\lambda,\sigma)_{\mu,\tau}\big) = 0, \quad \textit{if } \sigma \neq \tau.$$

VII. Properties of quantum groups

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to $\mathfrak{g}$ that was defined in V (1.3). We shall often use the presentation of $\mathfrak{U}_h\mathfrak{g}$ given in V (1.6). In this chapter we shall describe some of the structure which quantum groups have. In many cases this structure is similar to the structure of the enveloping algebra $\mathfrak{U}\mathfrak{g}$.

The proofs of the triangular decomposition and the grading on the quantum group given in §1 can be found in [Ja] 4.7 and 4.21. The proof of the statements in (2.1) and (2.3), concerning the pairing $\langle , \rangle$, can be found in [Ja] 6.12, 6.18, 8.28, and 6.22. The statement in (2.2) follows from Chapt I, Prop. (5.5). The theorem giving the existence and uniqueness of the $\mathcal{R}$-matrix is stated in [D2] p.329 and the uniqueness is proved there. The existence follows from (7.4); see also [Lu] Theorem 4.1.2. The properties of the $\mathcal{R}$-matrix stated in (3.3) are proved in [D2] Prop. 3.1 and Prop. 4.2. Proofs of the statements in the section on the Casimir element can be found in [D2] Prop 2.1, Prop 3.2 and Prop. 5.1.

Theorem (5.2a) is proved in [Ja] 8.15-8.17 and [Lu] 39.2.2. Theorem (5.2b) is a non-trivial, but very natural, extension of well known results which appear, for example, in [Ja] Chapt. 8. The proof is a combination of the methods used in [CP] 8.2B and [Ja] 8.4 and a calculation similar to that in the proof of [Ja] Lemma 8.3. The properties of the element T_{w_0} given in (5.3) are proved in the following places: The formula for $\sigma(T_{w_0})T_{w_0}$ is proved in [CP] 8.2.4; The formula for $T_{w_0}^{-1}$ is proved by a method similar to [Ja] 8.4; The formula for $\Delta_h(T_{w_0})$ is proved in [CP] 8.3.11 and the remainder of the formulas are proved in [CP] 8.2.3.

The construction of the Poincaré-Birkhoff-Witt basis of $\mathfrak{U}_h\mathfrak{g}$ given in section 6 appears in detail in [Ja] 8.18-8.30. The statement that $\mathfrak{U}_h\mathfrak{g}$ is almost a quantum double, Theorem (7.3), appears in [D1] §13, and an outline of the proof can be found in [CP] 8.3. The proof of Theorem (8.4) can be gleaned from a combination of [Ja] 6.11 and 6.18. Both of the books [Lu] and [Jo] also contain this fact.

1. Triangular decomposition and grading

(1.1) Triangular decomposition of $\mathfrak{U}_h\mathfrak{g}$

The triangular decomposition of the quantum group $\mathfrak{U}_h\mathfrak{g}$ is analogous to the triangular decomposition of the Lie algebra $\mathfrak{g}$ and the triangular decomposition of the enveloping algebra $\mathfrak{U}\mathfrak{g}$ given in II (2.3) and II (4.2).

Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be* *the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). Define*

$$\mathfrak{U}_h\mathfrak{n}^- = \textit{subalgebra of } \mathfrak{U}_h\mathfrak{g} \textit{ generated by } X_1^-, X_2^-, \ldots, X_r^-,$$

$$\mathfrak{U}_h\mathfrak{h} = \textit{subalgebra of } \mathfrak{U}_h\mathfrak{g} \textit{ generated by } H_1, H_2, \ldots, H_r,$$

$$\mathfrak{U}_h\mathfrak{n}^+ = \textit{subalgebra of } \mathfrak{U}_h\mathfrak{g} \textit{ generated by } X_1^+, X_2^+, \ldots, X_r^+.$$

The map

$$\begin{array}{ccc} \mathfrak{U}_h\mathfrak{n}^- \otimes \mathfrak{U}_h\mathfrak{h} \otimes \mathfrak{U}_h\mathfrak{n}^+ & \longrightarrow & \mathfrak{U}_h\mathfrak{g} \\ u^- \otimes u^0 \otimes u^+ & \longmapsto & u^-u^0u^+ \end{array}$$

is an isomorphism of vector spaces.

(1.2) The grading on $\mathfrak{U}_h\mathfrak{n}^+$ and $\mathfrak{U}_h\mathfrak{n}^-$

The gradings on the positive part $\mathfrak{U}_h\mathfrak{n}^+$ and on the negative part $\mathfrak{U}_h\mathfrak{n}^-$ of the quantum group $\mathfrak{U}_h\mathfrak{g}$ are exactly analogous to the gradings on the postive part $\mathfrak{U}\mathfrak{n}^+$ and the negative part $\mathfrak{U}\mathfrak{n}^-$ of the enveloping algebra $\mathfrak{U}\mathfrak{g}$ which are given in II (4.3).

Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). Let $\alpha_1, \ldots, \alpha_r$ be the simple roots for $\mathfrak{g}$ and let*

$$Q^+ = \sum_i \mathbb{N}\alpha_i, \qquad \textit{where } \mathbb{N} = \mathbb{Z}_{\geq 0}.$$

For each element $\nu = \sum_{i=1}^r \nu_i\alpha_i \in Q^+$ define

$$(\mathfrak{U}_h\mathfrak{n}^+)_\nu = \textit{span-}\{X_{i_1}^+ \cdots X_{i_p}^+ \mid X_{i_1}^+ \cdots X_{i_p}^+ \ \textit{ has } \nu_j\textit{-factors of type } X_j^+\}$$

$$(\mathfrak{U}_h\mathfrak{n}^-)_\nu = \textit{span-}\{X_{i_1}^- \cdots X_{i_p}^- \mid X_{i_1}^- \cdots X_{i_p}^- \ \textit{ has } \nu_j\textit{-factors of type } X_j^-\}.$$

Then

$$\mathfrak{U}_h\mathfrak{n}^- = \bigoplus_{\nu \in Q^+} (\mathfrak{U}_h\mathfrak{n}^-)_\nu \quad \textit{and} \quad \mathfrak{U}_h\mathfrak{n}^+ = \bigoplus_{\nu \in Q^+} (\mathfrak{U}_h\mathfrak{n}^+)_\nu,$$

as vector spaces.

2. The inner product $\langle , \rangle$

In some sense the nonnegative part $\mathfrak{U}_h\mathfrak{b}^+$ of the quantum group is the dual of the nonpositive part $\mathfrak{U}_h\mathfrak{b}^-$ of the quantum group. This is reflected in the fact that there is a nondegenerate bilinear pairing between the two. Later we shall see that this pairing can be extended to a pairing on all of $\mathfrak{U}_h\mathfrak{g}$. The extended pairing is an analogue of the Killing form on $\mathfrak{g}$ in two ways:

(1) it is an ad-invariant form on $\mathfrak{U}_h\mathfrak{g}$, and

(2) upon restriction to $\mathfrak{g}$ it coincides $\pmod h$ with the Killing form.

(2.1) The pairing between $\mathfrak{U}_h\mathfrak{b}^-$ and $\mathfrak{U}_h\mathfrak{b}^+$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). Define

$$\mathfrak{U}_h\mathfrak{b}^- = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^-, X_2^-, \ldots, X_r^- \text{ and } H_1, \ldots, H_r,$$

$$\mathfrak{U}_h\mathfrak{b}^+ = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X_1^+, X_2^+, \ldots, X_r^+ \text{ and } H_1, \ldots, H_r.$$

Theorem.

(1) There is a unique $\mathbb{C}[[h]]$-bilinear pairing

$$\langle,\rangle : \mathfrak{U}_h\mathfrak{b}^- \times \mathfrak{U}_h\mathfrak{b}^+ \longrightarrow \mathbb{C}[[h]] \quad \textit{which satisfies}$$

(a) $\langle 1, 1\rangle = 1$,

(b) $\langle H_i, H_j\rangle = \dfrac{\alpha_j(H_i)}{d_j}$,

(c) $\langle X_i^-, X_j^+\rangle = \delta_{ij}\dfrac{1}{e^{d_ih} - e^{-d_ih}}$,

(d) $\langle ab, c\rangle = \langle a\otimes b, \Delta_h(c)\rangle$, *for all* $a, b \in \mathfrak{U}_h\mathfrak{b}^-$ *and* $c \in \mathfrak{U}_h\mathfrak{b}^+$,

(e) $\langle a, bc\rangle = \langle \Delta_h^{\mathrm{op}}(a), b\otimes c\rangle$, *for all* $a \in \mathfrak{U}_h\mathfrak{b}^-$ *and* $b, c \in \mathfrak{U}_h\mathfrak{b}^+$.

(2) The pairing $\langle,\rangle$ is nondegenerate.

(3) The pairing $\langle,\rangle$ respects the gradings on $\mathfrak{U}_h\mathfrak{n}^+$ and $\mathfrak{U}_h\mathfrak{n}^-$ in the following sense:

(a) Let $\mu, \nu \in Q^+$.

$$\textit{If } \mu \neq \nu \textit{ then } \big\langle (\mathfrak{U}_h\mathfrak{b}^-)_\mu, (\mathfrak{U}_h\mathfrak{b}^+)_\nu \big\rangle = 0.$$

(b) Let $\nu \in Q^+$. The restriction of the pairing $\langle,\rangle$ to $(\mathfrak{U}_h\mathfrak{n}^-)_\nu \times (\mathfrak{U}_h\mathfrak{n}^+)_\nu$ is a nondegenerate pairing

$$\langle,\rangle : (\mathfrak{U}_h\mathfrak{n}^-)_\nu \times (\mathfrak{U}_h\mathfrak{n}^+)_\nu \to \mathbb{C}[[h]].$$

If θ is the Cartan involution of $\mathfrak{U}_h\mathfrak{g}$ as given in V (1.6) and S_h is the antipode of $\mathfrak{U}_h\mathfrak{g}$ then

$$\langle \theta(u^+), \theta(u^-)\rangle = \langle u^-, u^+\rangle \quad \text{and} \quad \langle S_h(u^-), S_h(u^+)\rangle = \langle u^-, u^+\rangle,$$

for all $u^- \in \mathfrak{U}_h\mathfrak{b}^-$ and $u^+ \in \mathfrak{U}_h\mathfrak{b}^+$.

(2.2) Extending the pairing to an ad-invariant pairing on $\mathfrak{U}_h\mathfrak{g}$

The triangular decomposition (1.1) of $\mathfrak{U}_h\mathfrak{g}$ says that $\mathfrak{U}_h\mathfrak{g} \cong \mathfrak{U}_h\mathfrak{n}^- \otimes \mathfrak{U}_h\mathfrak{h} \otimes \mathfrak{U}_h\mathfrak{n}^+$ and that

$$\text{every element } u \in \mathfrak{U}_h\mathfrak{g} \text{ can be written in the form } u^-u^0u^+,$$

where $u^- \in \mathfrak{U}_h\mathfrak{n}^-$, $u^0 \in \mathfrak{U}_h\mathfrak{h}$, and $u^+ \in \mathfrak{U}_h\mathfrak{n}^+$. We can use this to extend the pairing defined in (2.1) to a pairing

$$\langle , \rangle\colon \mathfrak{U}_h\mathfrak{g} \times \mathfrak{U}_h\mathfrak{g} \longrightarrow \mathbb{C}[[h]] \quad \text{defined by the formula}$$

$$\langle u_1^-u_1^0u_1^+, u_2^-u_2^0u_2^+ \rangle = \langle u_1^-, S_h(u_2^0u_2^+)\rangle\langle u_2^-, S_h^{-1}(u_1^0u_1^+)\rangle,$$

for all $u_1^-, u_2^- \in \mathfrak{U}_h\mathfrak{n}^-$, $u_1^0, u_2^0 \in \mathfrak{U}_h\mathfrak{h}$, and $u_1^+, u_2^+ \in \mathfrak{U}_h\mathfrak{n}^+$, where S_h is the antipode of $\mathfrak{U}_h\mathfrak{g}$. Then

$$\langle \mathrm{ad}_u(v_1), v_2\rangle = \langle v_1, \mathrm{ad}_{S_h(u)}(v_2)\rangle, \quad \text{for all } u, v_1, v_2 \in \mathfrak{U}_h\mathfrak{g},$$

This formula says that the extended pairing $\langle , \rangle$ is an ad-invariant pairing as defined in I (5.5). The pairing $\langle , \rangle\colon \mathfrak{U}_h\mathfrak{g} \times \mathfrak{U}_h\mathfrak{g} \to \mathbb{C}[[h]]$ is *not* symmetric, see I (5.5).

(2.3) Duality between matrix coefficients for representations and $U_q\mathfrak{g}$.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $U_q\mathfrak{g}$ be the rational form of the quantum group over a field k, where char $k \neq 2, 3$ and $q \in k$ is not a root of unity. Let Q be the root lattice for $\mathfrak{g}$.

Theorem. *Let M be a finite dimensional $U_q\mathfrak{g}$ module such that all weights λ of M satisfy $2\lambda \in Q$. Then, for each pair $n^* \in M^*$ and $m \in M$ there is a unique element $u \in U_q\mathfrak{g}$ such that*

$$n^*(vm) = \langle v, u\rangle, \quad \text{for all } v \in U_q\mathfrak{g},$$

where $\langle , \rangle$ is the bilinear form on $U_q\mathfrak{g}$ given by (2.1) after making the substitutions in V (2.2).

The function

$$c_{m,n^*}\colon U_q\mathfrak{g} \to \mathbb{C}(q) \quad \text{defined by} \quad c_{m,n^*}(v) = \langle n^*, vm\rangle$$

is the (m, n^*)-*matrix coefficient of v acting on M.* The above theorem gives a duality between matrix coefficient functions and $U_q\mathfrak{g}$. It also says that every element of $U_q\mathfrak{g}$ is determined by how it acts on finite dimensional $U_q\mathfrak{g}$-modules.

3. The universal $\mathcal{R}$-matrix

(3.1) Motivation for the $\mathcal{R}$-matrix

The following theorem states that there is an element $\mathcal{R}$ such that the pair $(\mathfrak{U}_h\mathfrak{g}, \mathcal{R})$ is a quasitriangular Hopf algebra. In particular, this implies that the category of finite dimensional modules for the quantum group $\mathfrak{U}_h\mathfrak{g}$ is a braided SRMCwMFF.

(3.2) Existence and uniqueness of $\mathcal{R}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Recall the Killing form on $\mathfrak{g}$ from II (1.6).

Let $\{\tilde{H}_i\}$ be an orthonormal basis of $\mathfrak{h}$ with respect to the Killing form and define

$$t_0 = \sum_{i=1}^{r} \tilde{H}_i \otimes \tilde{H}_i.$$

If $\nu \in Q^+$ (see (1.2)) and $\nu = \sum_{i=1}^r \nu_i \alpha_i$ where $\alpha_1, \ldots, \alpha_r$ are the simple roots, define n_ν to be the smallest number of positive roots $\alpha > 0$ whose sum is equal to ν.

The element $\mathcal{R}$ is not quite an element of $\mathfrak{U}_h\mathfrak{g} \otimes \mathfrak{U}_h\mathfrak{g}$ so we have to make the tensor product just a tiny bit bigger. To do this we let $\mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g}$ denote the h-adic completion of the tensor product $\mathfrak{U}_h\mathfrak{g} \otimes \mathfrak{U}_h\mathfrak{g}$, see III §1.

Theorem. *There exists a unique invertible element $\mathcal{R} \in \mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g}$ such that*

$$\mathcal{R}\Delta_h(a)\mathcal{R}^{-1} = \Delta_h^{\mathrm{op}}(a), \quad \textit{for all } a \in \mathfrak{U}_h\mathfrak{g}, \textit{ and}$$

$$\mathcal{R} \textit{ has the form} \quad \mathcal{R} = \sum_{\nu \in Q^+} \exp\left\{h\left(t_0 + \tfrac{1}{2}(H_\nu \otimes 1 - 1 \otimes H_\nu)\right)\right\} P_\nu, \qquad \textit{where}$$

$P_\nu \in (\mathfrak{U}_h\mathfrak{n}^-)_\nu \otimes (\mathfrak{U}_h\mathfrak{n}^+)_\nu$,

$H_\nu = \sum_{i=1}^r \nu_i H_i, \quad$ *if* $\nu = \sum_i \nu_i \alpha_i$,

P_ν is a polynomial in $X_i^+ \otimes 1$ and $1 \otimes X_i^-$, $1 \le i \le r$, with coefficients in $\mathbb{C}[[h]]$, such that

the smallest power of h in P_ν with nonzero coefficient is h^{n_ν}.

(3.3) Properties of the $\mathcal{R}$-matrix

Recall V (1.6) that $\mathfrak{U}_h\mathfrak{g}$ is a Hopf algebra with comultiplication Δ_h, counit ϵ_h, and antipode S_h and that $\mathfrak{U}_h\mathfrak{g}$ comes with a Cartan involution θ. The following formulas describe the relationship between the $\mathcal{R}$-matrix and the Hopf algebra structure of $\mathfrak{U}_h\mathfrak{g}$. If $\mathcal{R} = \sum a_i \otimes b_i$ then let

$$\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1, \quad \mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \quad \text{and} \quad \mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i,$$

$$\text{and let} \qquad \mathcal{R}_{21} = \sum b_i \otimes a_i.$$

Let $\sigma: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ be the $\mathbb{C}$-algebra automorphism of $\mathfrak{U}_h\mathfrak{g}$ given by $\sigma(h) = -h$, $\sigma(X_i^{\pm}) = X_i^{\pm}$, and $\sigma(H_i) = H_i$. With these notations we have

$$(\Delta_h \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad \text{and} \quad (\mathrm{id} \otimes \Delta_h)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

$$(\epsilon_h \otimes \mathrm{id})(\mathcal{R}) = 1 = (\mathrm{id} \otimes \epsilon_h)(\mathcal{R}),$$

$$(S_h \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S_h^{-1})(\mathcal{R}) = \mathcal{R}^{-1} \quad \text{and} \quad (S_h \otimes S_h)(\mathcal{R}) = \mathcal{R},$$

$$(\theta \otimes \theta)(\mathcal{R}) = \mathcal{R}_{21} \quad \text{and} \quad (\sigma \otimes \sigma)(\mathcal{R}) = \mathcal{R}^{-1}.$$

4. An analogue of the Casimir element

(4.1) Definition of the element u

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group as presented in V (1.6). The antipode $S_h: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ is an antiautomorphism of $\mathfrak{U}_h\mathfrak{g}$, see I (2.1). This means that the map $S_h^2: \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ is an automorphism of $\mathfrak{U}_{\mathfrak{g}}$. The following theorem says that this automorphism is inner!

Theorem. *Let $\mathcal{R} \in \mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g}$ be the universal $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$ as defined in (3.2). Suppose that $\mathcal{R} = \sum a_i \otimes b_i$ and define $u = \sum S(b_i)a_i$. Then u is invertible and*

$$uau^{-1} = S_h^2(a), \quad \textit{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

(4.2) Properties of the element u.

The relationship of the element u to the Hopf algebra structure of $\mathfrak{U}_h\mathfrak{g}$ is given by the formulas

$$\Delta_h(u) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(u \otimes u), \quad S_h(u) = u, \quad \text{and} \quad \epsilon_h(u) = 1,$$

where $\mathcal{R}_{12} = \mathcal{R} = \sum a_i \otimes b_i$ is the universal $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$ given in (3.2), and $\mathcal{R}_{21} = \sum b_i \otimes a_i$. The inverse of the element u is given by

$$u^{-1} = \sum S_h^{-1}(d_j)c_j, \quad \text{where} \quad \mathcal{R}^{-1} = \sum c_j \otimes d_j.$$

(4.3) Why the element u is an analogue of the Casimir element

Let $\tilde\rho$ be the element of $\mathfrak{h}$ such that $\alpha_i(\tilde\rho) = 1$ for all simple roots α_i of $\mathfrak{g}$. An easy check on the generators of $\mathfrak{U}_h\mathfrak{g}$ shows that

$$e^{h\tilde\rho}ae^{-h\tilde\rho} = S_h^2(a), \quad \text{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

It follows that

$$\text{the element } e^{-h\tilde\rho}u = ue^{-h\tilde\rho} \text{ is a central element in } \mathfrak{U}_h\mathfrak{g}.$$

Any central element of $\mathfrak{U}_h\mathfrak{g}$ must act on each finite dimensional simple $\mathfrak{U}_h\mathfrak{g}$-module by a constant. For each dominant integral weight λ let $L(\lambda)$ be the finite dimensional simple $\mathfrak{U}_h\mathfrak{g}$-module indexed by λ (see VI (1.3)). As in II (4.5), let ρ be the element of $\mathfrak{h}_\mathbb{R}^*$ given by

$$\rho = \tfrac{1}{2}\sum_{\alpha>0}\alpha,$$

where the sum is over all positive roots for $\mathfrak{g}$. Then the element

$$e^{-h\tilde\rho}u \text{ acts on } L(\lambda) \text{ by the constant } q^{-(\lambda+\rho,\lambda+\rho)+(\rho,\rho)},$$

where $q = e^h$ and the inner product in the exponent of q is the inner product on $\mathfrak{h}_\mathbb{R}^*$ given in II (2.7). Note the analogy with II (4.5). It is also interesting to note that

$$(e^{-h\tilde\rho}u)^2 = uS_h(u).$$

5. The element T_{w_0}

(5.1) The automorphism $\phi\circ\theta\circ S_h$

Let W be the Weyl group corresponding to $\mathfrak{g}$ and let w_0 be the longest element of W (see II (2.8)). Let $s_1,\ldots,s_r$ be the simple reflections in W. For each $1\le i\le r$ there is a unique $1\le j\le r$ such that $w_0s_iw_0^{-1} = s_j$. The map given by

$$\phi(X_i^\pm) = X_j^\pm, \quad \text{and} \quad \phi(H_i) = H_j, \quad \text{where} \quad w_0s_iw_0^{-1} = s_j, \quad \text{for } 1\le i\le r,$$

extends to an automorphism of $\mathfrak{U}_h\mathfrak{g}$. Let $\tilde\theta$ be the anti-automorphism of $\mathfrak{U}_h\mathfrak{g}$ defined by $\tilde\theta(X_i^\pm) = X_i^\mp$ and $\tilde\theta(H_i) = H_i$. This is an analogue of the Cartan involution. Let S_h be the antipode of $\mathfrak{U}_h\mathfrak{g}$ as given in V (1.6). These are both anti-automorphisms of $\mathfrak{U}_h\mathfrak{g}$. The composition

$$(S_h\circ\tilde\theta\circ\phi)\colon \mathfrak{U}_h\mathfrak{g}\to\mathfrak{U}_h\mathfrak{g}$$

is an automorphism of $\mathfrak{U}_h\mathfrak{g}$. The following result says that this automorphism is inner.

(5.2) Definition of the element T_{w_0}

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Let $q = e^h$ and for each $1 \le i \le r$ let

$$E_i^{(r)} = \frac{(X_i^+)^r}{[r]_{q^{d_i}}!} \quad F_i^{(r)} = \frac{(X_i^+)^r}{[r]_{q^{d_i}}!} \quad \text{and} \quad K_i = e^{hd_iH_i},$$

where the notation for q-factorials is as in V (1.5). For each $1 \le i \le r$, define

$$T_i = \sum_{a,b,c \ge 0} (-1)^b q^{b-ac+(c+a-b)(c-a)} E_i^{(a)} F_i^{(b)} E_i^{(c)} K_i^{c-a},$$

where the sum is over all nonnegative integers a,b, and c.

Theorem.

(a) The elements T_i satisfy the relations

$$\underbrace{T_iT_jT_iT_j\cdots}_{m_{ij}\ \text{factors}} = \underbrace{T_jT_iT_jT_i\cdots}_{m_{ij}\ \text{factors}} \qquad \textit{for } i \ne j,$$

where the m_{ij} are as given in II (2.8).

(b) Let $w_0 = s_{i_1}\cdots s_{i_N}$ be a reduced word for the longest element of the Weyl group W, see II (2.8). Define

$$T_{w_0} = T_{i_1}\cdots T_{i_N}.$$

Then T_{w_0} is invertible and

$$T_{w_0} a T_{w_0}^{-1} = (S_h \circ \tilde{\theta} \circ \phi)(a), \qquad \textit{for all } a \in \mathfrak{U}_h\mathfrak{g}.$$

(5.3) Properties of the element T_{w_0}

Let $u \in \mathfrak{U}_h\mathfrak{g}$ be the analogue of the Casimir element for $\mathfrak{U}_h\mathfrak{g}$ as given in §4 and let σ be the $\mathbb{C}$-algebra automorphism of $\mathfrak{U}_h\mathfrak{g}$ given in (3.3). Let $\tilde{\sigma}$ be the $\mathbb{C}$-linear automorphism of $\mathfrak{U}_h\mathfrak{g}$ given by $\tilde{\sigma}(h) = -h$, $\tilde{\sigma}(X_i^{\pm}) = X_i^{\mp}$, and $\tilde{\sigma}(H_i) = -H_i$. Then

$$\sigma(T_{w_0})T_{w_0} = u \quad \text{and} \quad T_{w_0}^{-1} = \tilde{\sigma}(T_{w_0}).$$

The relationship between the element T_{w_0} and the Hopf algebra structure of $\mathfrak{U}_h\mathfrak{g}$ is given by the formulas

$$\Delta_h(T_{w_0}) = \mathcal{R}_{12}^{-1}(T_{w_0} \otimes T_{w_0}) = (T_{w_0} \otimes T_{w_0})\mathcal{R}_{21}^{-1}, \quad S_h(T_{w_0}) = T_{w_0}e^{h\tilde{\rho}}, \quad \text{and} \quad \epsilon_h(T_{w_0}) = 1,$$

where $\mathcal{R}_{12} = \mathcal{R} = \sum a_i \otimes b_i$ is the universal $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$ given in (3.2), and $\mathcal{R}_{21} = \sum b_i \otimes a_i$.

6. The Poincaré-Birkhoff-Witt basis of $\mathfrak{U}_h\mathfrak{g}$

(6.1) Root vectors in $\mathfrak{U}_h\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Let T_i be the elements of $\mathfrak{U}_h\mathfrak{g}$ given in (5.2). Define an automorphism $\tau_i\colon \mathfrak{U}_h\mathfrak{g} \to \mathfrak{U}_h\mathfrak{g}$ by

$$\tau_i(u) = T_i u T_i^{-1}, \qquad \text{for all } u \in \mathfrak{U}_h\mathfrak{g},$$

Let W be the Weyl group corresponding to $\mathfrak{g}$. Fix a reduced decomposition $w_0 = s_{i_1}\cdots s_{i_N}$ of the longest element $w_0 \in W$, see II (2.8). Define

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_N = s_{i_1}s_{i_2}\cdots s_{i_{N-1}}(\alpha_{i_N}).$$

The elements $\beta_1,\ldots,\beta_N$ are the positive roots $\mathfrak{g}$. Define elements of $\mathfrak{U}_h\mathfrak{g}$ by

$$X^{\pm}_{\beta_1} = X^{\pm}_{i_1}, \quad X^{\pm}_{\beta_2} = \tau_{i_1}(X^{\pm}_{i_2}), \quad \ldots, \quad X^{\pm}_{\beta_N} = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_{N-1}}(X^{\pm}_{i_N}).$$

These elements *depend* on the choice of the reduced decomposition. They are analogues of the elements X_β and $X_{-\beta}$ in $\mathfrak{U}\mathfrak{g}$ which are given in II (4.4).

(6.2) Poincaré-Birkhoff-Witt bases of $\mathfrak{U}_h\mathfrak{n}^-$, $\mathfrak{U}_h\mathfrak{h}$, and $\mathfrak{U}_h\mathfrak{n}^+$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Let $\mathfrak{U}_h\mathfrak{n}^-$, $\mathfrak{U}_h\mathfrak{h}$, and $\mathfrak{U}_h\mathfrak{n}^+$ be the subalgebras of $\mathfrak{U}_h\mathfrak{g}$ defined in (1.1). The following bases of $\mathfrak{U}_h\mathfrak{n}^-$, $\mathfrak{U}_h\mathfrak{h}$, $\mathfrak{U}_h\mathfrak{n}^+$, and $\mathfrak{U}_h\mathfrak{g}$ are analogues of the Poincaré-Birkhoff-Witt bases of $\mathfrak{U}\mathfrak{n}^-$, $\mathfrak{U}\mathfrak{h}$, and $\mathfrak{U}\mathfrak{n}^+$ which are given in II (4.4).

Theorem. *Let $X^{\pm}_{\beta_1},\ldots,X^{\pm}_{\beta_N}$ be the elements of $\mathfrak{U}_h\mathfrak{g}$ defined in (6.1). Then*

$$\{(X^+_{\beta_1})^{p_1}(X^+_{\beta_2})^{p_2}\cdots(X^+_{\beta_N})^{p_N} \mid p_1,\ldots,p_N \in \mathbb{Z}_{\geq 0}\} \quad \textit{is a basis of } \mathfrak{U}_h\mathfrak{n}^+,$$

$$\{(X^-_{\beta_1})^{n_1}(X^-_{\beta_2})^{n_2}\cdots(X^-_{\beta_1})^{n_N} \mid n_1,\ldots,n_N \in \mathbb{Z}_{\geq 0}\} \quad \textit{is a basis of } \mathfrak{U}_h\mathfrak{n}^-,$$

$$\{H_1^{s_1}H_2^{s_2}\cdots H_1^{s_r} \mid s_1,\ldots,s_N \in \mathbb{Z}_{\geq 0}\} \qquad \textit{is a basis of } \mathfrak{U}_h\mathfrak{h}.$$

(6.3) The PBW-bases of $\mathfrak{U}_h n^-$ and $\mathfrak{U}_h\mathfrak{n}^+$ are dual bases with respect to $\langle,\rangle$ (almost)

Recall the pairing between $\mathfrak{U}_h\mathfrak{b}^-$ and $\mathfrak{U}_h\mathfrak{b}^+$ given in (2.1).

Theorem. *Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of the longest element of the Weyl group and let β_j and $X^{\pm}_{\beta_j}$, $1 \le j \le N$, be the elements defined in (6.1). Let $p_1, \ldots, p_N, n_1, \ldots, n_N \in \mathbb{Z}_{\ge 0}$. Then*

$$\left\langle (X^-_{\beta_1})^{n_1}(X^-_{\beta_2})^{n_2}\cdots(X^-_{\beta_N})^{n_N}, (X^+_{\beta_1})^{p_1}(X^+_{\beta_2})^{p_2}\cdots(X^+_{\beta_N})^{p_N}\right\rangle = \prod_{j=1}^{N} \delta_{n_j,p_j}\left\langle (X^-_{i_j})^{n_j}, (X^+_{i_j})^{n_j}\right\rangle,$$

where δ_{n_j,p_j} is the Kronecker delta.

Furthermore, we have that, for each $1 \le i \le r$,

$$\left\langle (X^-_i)^n, (X^+_i)^n\right\rangle = (-1)^n q^{-d_i n(n-1)/2}\frac{[n]_{q^{d_i}}!}{(q^{d_i}-q^{-d_i})^n}, \qquad \text{where } q = e^h.$$

7. The quantum group is a quantum double (almost)

(7.1) The identification of $(\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}}$ with $\mathfrak{U}_h\mathfrak{b}^-$

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding quantum group as presented in V (1.6). Define

$$\mathfrak{U}_h\mathfrak{b}^- = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X^-_1, X^-_2, \ldots, X^-_r \text{ and } H_1, \ldots, H_r,$$

$$\mathfrak{U}_h\mathfrak{b}^+ = \text{subalgebra of } \mathfrak{U}_h\mathfrak{g} \text{ generated by } X^+_1, X^+_2, \ldots, X^+_r \text{ and } H_1, \ldots, H_r,$$

except let us distinguish the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^+$ from the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^-$ by writing H^+_i and H^-_i respectively, instead of just H_i in both cases.

The nondegeneracy of the pairing $\langle , \rangle$ between $\mathfrak{U}_h\mathfrak{b}^+$ and $\mathfrak{U}_h\mathfrak{b}^-$ (see (2.1)) shows that $\mathfrak{U}_h\mathfrak{b}^-$ is essentially the dual of $U_h\mathfrak{b}^+$. Furthermore, it follows from the conditions

$$\langle x_1x_2, y\rangle = \langle x_1\otimes x_2, \Delta_h(y)\rangle \quad \text{and} \quad \langle x, y_1y_2\rangle = \langle \Delta^{\mathrm{op}}(x), y_1\otimes y_2\rangle$$

that the multiplication in $\mathfrak{U}_h\mathfrak{b}^-$ is the adjoint of the comultiplication in $\mathfrak{U}_h\mathfrak{b}^+$ and the opposite of the comultiplication in $\mathfrak{U}_h\mathfrak{b}^-$ is the adjoint of the multiplication in $\mathfrak{U}_h\mathfrak{b}^+$. Thus (here we are fudging a bit since $\mathfrak{U}_h\mathfrak{b}^+$ is infinite dimensional),

$$\mathfrak{U}_h\mathfrak{b}^- \;\simeq\; (\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}} \qquad \text{as Hopf algebras,}$$

where $(\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}}$ is the Hopf algebra defined in I (5.2).

(7.2) Recalling the quantum double

Recall, from I (5.3), that the quantum double $D(A)$ of a finite dimensional Hopf algebra A is the new Hopf algebra

$$D(A) = \{\ a\alpha \ \mid \ a \in A,\ \alpha \in A^{*\mathrm{coop}}\} \cong A \otimes A^{*\mathrm{coop}}$$

with multiplication determined by the formulas

$$\begin{aligned}
\alpha a &= \sum_{\alpha,a} \langle \alpha_{(1)}, S^{-1}(a_{(1)})\rangle \langle \alpha_{(3)}, a_{(3)}\rangle a_{(2)}\alpha_{(2)}, \quad \text{and} \\
a\alpha &= \sum_{\alpha,a} \langle \alpha_{(1)}, a_{(1)}\rangle \langle \alpha_{(3)}, S^{-1}(a_{(3)})\rangle \alpha_{(2)} a_{(2)},
\end{aligned}$$

where, if Δ is the comultiplication in A and $A^{*\mathrm{coop}}$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \quad \text{and} \quad (\Delta \otimes \mathrm{id}) \circ \Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}.$$

The comultiplication $D(A)$ is determined by the formula

$$\Delta(a\alpha) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)},$$

where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)}$.

(7.3) The relation between $D(\mathfrak{U}_h\mathfrak{b}^+)$ and $\mathfrak{U}_h\mathfrak{g}$

With the definition of the quantum double in mind it is natural that we should define the quantum double of $\mathfrak{U}_h\mathfrak{b}^+$ to be the algebra

$$D(\mathfrak{U}_h\mathfrak{b}^+) = (\mathfrak{U}_h\mathfrak{b}^+)^{*\mathrm{coop}} \otimes \mathfrak{U}_h\mathfrak{b}^+ \cong \mathfrak{U}_h\mathfrak{b}^- \otimes \mathfrak{U}_h\mathfrak{b}^+$$

with multiplication and comultiplication given by the formulas in (7.2). The following theorem says that the quantum group $\mathfrak{U}_h\mathfrak{g}$ is almost the quantum double of $\mathfrak{U}_h\mathfrak{b}^+$, in other words, $\mathfrak{U}_h\mathfrak{g}$ is almost completely determined by pasting two copies of $\mathfrak{U}_h\mathfrak{b}^+$ together.

Theorem. *Let $(B_{ij}) = C^{-1}$ be the inverse of the Cartan matrix corresponding to $\mathfrak{g}$ and, for each $1 \le i \le r$, define*

$$H_i^* = \sum_{j=1}^r B_{ij} H_j \ \in \mathfrak{U}_h\mathfrak{g}.$$

(a) There is a surjective homomorphism $\phi\colon D(\mathfrak{U}_h\mathfrak{b}^+) \longrightarrow \mathfrak{U}_h\mathfrak{g}$ determined by

$$\begin{array}{rccc}
\phi\colon & D(\mathfrak{U}_h\mathfrak{b}^+) & \longrightarrow & \mathfrak{U}_h\mathfrak{g} \\
& X_i^+ & \longmapsto & X_i^+ \\
& H_i^+ & \longmapsto & H_i \\
& X_i^- & \longmapsto & X_i^- \\
& H_i^- & \longmapsto & H_i^*
\end{array} \qquad \textit{and thus} \qquad \frac{D(\mathfrak{U}_h\mathfrak{b}^+)}{\ker\phi} \cong \mathfrak{U}_h\mathfrak{g}.$$

(Recall (7.1) that we distinguish the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^+$ from the elements H_i which are in $\mathfrak{U}_h\mathfrak{b}^-$ by writing H_i^+ and H_i^- respectively, instead of just H_i in both cases.)
(b) The ideal $\ker\phi$ *is the ideal generated by the relations*

$$H_i^- \quad - \quad \left(\sum_{j=1}^{r} B_{ij}H_j^+\right), \qquad \text{where } 1 \le i \le r.$$

(7.4) Using the $\mathcal{R}$-matrix of $D(\mathfrak{U}_h\mathfrak{b}^+)$ to get the $\mathcal{R}$-matrix of $\mathfrak{U}_h\mathfrak{g}$
Recall (7.2) that the double $D(\mathfrak{U}_h\mathfrak{b}^+)$ comes with a natural universal $\mathcal{R}$-matrix given by

$$\tilde{\mathcal{R}} = \sum_i b_i \otimes b^i,$$

where the sum is over a basis $\{b_i\}$ of $\mathfrak{U}_h\mathfrak{b}^+$ and $\{b^i\}$ is the dual basis in $\mathfrak{U}_h\mathfrak{b}^-$ with respect to the form $\langle,\rangle$ given in (2.1). We have used the notation $\tilde{\mathcal{R}}$ here to distinguish it from the element $\mathcal{R}$ in Theorem (3.2). The element $\tilde{\mathcal{R}}$ is not exactly in the tensor product $D(\mathfrak{U}_h\mathfrak{b}^+) \otimes D(\mathfrak{U}_h\mathfrak{b}^+)$ but if we make the tensor product just a tiny bit bigger by taking the h-adic completion $D(\mathfrak{U}_h\mathfrak{b}^+)\hat{\otimes}D(\mathfrak{U}_h\mathfrak{b}^+)$ of $D(\mathfrak{U}_h\mathfrak{b}^+) \otimes D(\mathfrak{U}_h\mathfrak{b}^+)$ then we do have

$$\tilde{\mathcal{R}} \in D(\mathfrak{U}_h\mathfrak{b}^+)\hat{\otimes}D(\mathfrak{U}_h\mathfrak{b}^+).$$

The image of $\tilde{\mathcal{R}}$ under the homomorphism

$$\begin{array}{cccc} \phi\otimes\phi\colon & D(\mathfrak{U}_h\mathfrak{b}^+)\hat{\otimes}D(\mathfrak{U}_h\mathfrak{b}^+) & \longrightarrow & \mathfrak{U}_h\mathfrak{g}\hat{\otimes}\mathfrak{U}_h\mathfrak{g} \\ & \tilde{\mathcal{R}} & \longmapsto & \mathcal{R} \end{array}$$

coincides with the element $\mathcal{R}$ given in Theorem (3.2). This means that we actually get the element $\mathcal{R}$ in Theorem (3.2) for free by realising the quantum group as a quantum double (almost).

8. The quantum Serre relations occur naturally

In this section we will see that the most complicated of the defining relations in the quantum group can be obtained in quite a natural way. More specifically, the ideal generated by them is the radical of a certain bilinear form.

(8.1) Definition of the algebras $\mathbf{U}_h\mathfrak{b}^+$ and $\mathbf{U}_h\mathfrak{b}^-$
Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $C = (\alpha_j(H_i))_{1\le i,j\le r}$ be the corresponding Cartan matrix.

Let $\mathbf{U}_h\mathfrak{b}^+$ be the associative algebra over $\mathbb{C}[[h]]$ generated (as a complete $\mathbb{C}[[h]]$-algebra in the h-adic topology) by

$$H_1, H_2, \cdots, H_r, \quad X_1^+, X_2^+, \cdots, X_r^+,$$

with relations

$$[H_i, H_j] = 0, \quad \text{and} \quad [H_i, X_j^+] = \alpha_j(H_i)X_j^+, \quad \text{for all } 1 \le i, j \le r,$$

and define an algebra homomorphism $\Delta_h\colon \mathbf{U}_h\mathfrak{b}^+ \to \mathbf{U}_h\mathfrak{b}^+\hat{\otimes}\mathbf{U}_h\mathfrak{b}^+$ by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \text{and} \quad \Delta_h(X_i^+) = X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+,$$

where $\mathbf{U}_h\mathfrak{b}^+\hat{\otimes}\mathbf{U}_h\mathfrak{b}^+$ denotes the h-adic completion of the tensor product $\mathbf{U}_h\mathfrak{b}^+ \otimes_{\mathbb{C}[[h]]} \mathbf{U}_h\mathfrak{b}^+$.

Let $\mathbf{U}_h\mathfrak{b}^-$ be the associative algebra over $\mathbb{C}[[h]]$ generated (as a complete $\mathbb{C}[[h]]$-algebra in the h-adic topology) by

$$X_1^-, X_2^-, \cdots, X_r^-, \quad H_1, H_2, \cdots, H_r,$$

with relations

$$[H_i, H_j] = 0, \quad \text{and} \quad [H_i, X_j^-] = -\alpha_j(H_i)X_j^-, \quad \text{for all } 1 \le i, j \le r,$$

and define an algebra homomorphism $\Delta_h\colon \mathbf{U}_h\mathfrak{b}^- \to \mathbf{U}_h\mathfrak{b}^-\hat{\otimes}\mathbf{U}_h\mathfrak{b}^-$ by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \text{and} \quad \Delta_h(X_i^-) = X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-,$$

where $\mathbf{U}_h\mathfrak{b}^-\hat{\otimes}\mathbf{U}_h\mathfrak{b}^-$ denotes the h-adic completion of the tensor product $\mathbf{U}_h\mathfrak{b}^- \otimes_{\mathbb{C}[[h]]} \mathbf{U}_h\mathfrak{b}^-$.

(8.2) The difference between the algebras $\mathbf{U}_h\mathfrak{b}^\pm$ and the algebras $\mathfrak{U}_h\mathfrak{b}^\pm$

The algebras $\mathbf{U}_h\mathfrak{b}^+$ are much larger than the algebras $\mathfrak{U}_h\mathfrak{b}^+$ used in (2.1) since they have fewer relations between the $X_i^\pm$ generators.

(8.3) A pairing between $\mathbf{U}_h\mathfrak{b}^+$ and $\mathbf{U}_h\mathfrak{b}^-$

In exactly the same way that we had a pairing between $\mathfrak{U}_h\mathfrak{b}^+$ and $\mathfrak{U}_h\mathfrak{b}^-$ in (2.1), there is a unique $\mathbb{C}[[h]]$-bilinear pairing

$$\langle , \rangle : \mathbf{U}_h\mathfrak{b}^- \times \mathbf{U}_h\mathfrak{b}^+ \longrightarrow \mathbb{C}[[h]] \quad \text{which satisfies}$$

(a) $\langle 1, 1\rangle = 1$,

(b) $\langle H_i, H_j\rangle = \dfrac{\alpha_j(H_i)}{d_j}$,

(c) $\langle X_i^-, X_j^+\rangle = \delta_{ij}\dfrac{h}{e^{d_i h} - e^{-d_i h}}$,

(d) $\langle ab, c\rangle = \langle a \otimes b, \Delta_h(c)\rangle$, for all $a, b \in \mathbf{U}_h\mathfrak{b}^-$ and $c \in \mathbf{U}_h\mathfrak{b}^+$,

(e) $\langle a, bc\rangle = \langle \Delta_h^{\mathrm{op}}(a), b \otimes c\rangle$, for all $a \in \mathbf{U}_h\mathfrak{b}^-$ and $b, c \in \mathbf{U}_h\mathfrak{b}^+$.

(8.4) The radical of $\langle , \rangle$ is generated by the quantum Serre relations

Let $\mathfrak{r}^-$ and $\mathfrak{r}^+$ be the left and right radicals, respectively, of the form $\langle , \rangle$ defined in (8.3), i.e.

$$\mathfrak{r}^- = \{a \in \mathbf{U}_h\mathfrak{b}^- \mid \langle a, b\rangle = 0 \text{ for all } b \in \mathbf{U}_h\mathfrak{b}^+\}, \quad \text{and}$$
$$\mathfrak{r}^+ = \{b \in \mathbf{U}_h\mathfrak{b}^+ \mid \langle a, b\rangle = 0 \text{ for all } a \in \mathbf{U}_h\mathfrak{b}^-\}.$$

Theorem. *The sets $\mathfrak{r}^-$ and $\mathfrak{r}^+$ are the ideals of $\mathbf{U}_h\mathfrak{b}^-$ and $\mathbf{U}_h\mathfrak{b}^+$ generated by the elements*

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{e^{d_ih}} (X_i^-)^s X_j^- (X_i^-)^t, \qquad \text{for } i \neq j,$$

and

$$\sum_{s+t=1-\alpha_j(H_i)} (-1)^s \begin{bmatrix} 1-\alpha_j(H_i) \\ s \end{bmatrix}_{e^{d_ih}} (X_i^+)^s X_j^+ (X_i^+)^t, \qquad \text{for } i \neq j,$$

respectively.

It follows from this theorem that the quantum group $\mathfrak{U}_h\mathfrak{g}$ is determined by the algebras $\mathbf{U}_h\mathfrak{b}^+$, $\mathbf{U}_h\mathfrak{b}^-$ and the form $\langle , \rangle$. A construction of the quantum group along these lines would be very similar to the standard construction of Kac-Moody Lie algebras (see [K] §1.3).

VIII. Hall algebras

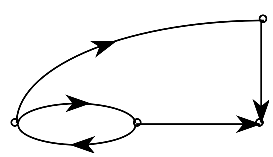
The results in §1 are outlined in [CP] §9.3D. The proof of Theorem (1.2) appears in [BGP] Theorem 3.1 and the proof of Theorem (1.4) appears in [Lu1] Prop. 5.7. The material in §2 is a combination of [Lu2] and [Lu] Part II. In particular, Theorem (2.7)(1) is proved in [Lu] 13.1.2, 12.3.2, and 9.2.7, Theorem (2.7)(2) is proved in [Lu] 13.1.5, 13.1.12e, 12.3.3, and 9.2.11, Theorem (2.7)(3) is proved in [Lu] 13.1.12d and 12.3.6. The statement about the symmetric form given in (2.6) is proved in [Lu] 12.2.2, 9.2.9 and the references given there. The proof of the isomorphism theorem in (2.8) is given in [Lu] 13.2.11 and in [Lu2] Th. 10.17. The material in §3 appears in [Lu1] §9. The isomorphism theorem in (3.4) is stated in [Lu1] 9.6.

1. Hall algebras

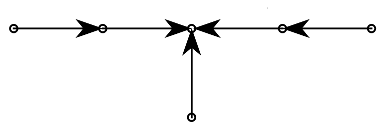
The Hall algebra is an algebra which has a basis labeled by representations of quivers and for which the structure constants with respect to this basis reflect the structure of these representations. The Hall algebra encodes a large amount of information about the representations of the quiver. Amazingly, this algebra is almost isomorphic to the nonnegative part of the quantum group.

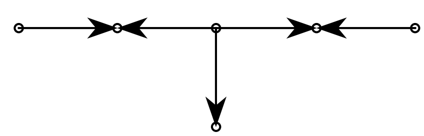
(1.1) Quivers

A *quiver* is an oriented graph Γ, i.e. a set of vertices and directed edges. The following is an example of a quiver.



Every Dynkin diagram if type A, D or E can be made into a quiver by orienting the edges. Note that there are many possible ways of orienting the edges of a Dynkin diagram in order to make a quiver. For example the quivers





are both obtained by orienting the edges of the Dynkin diagram of type E_6.

(1.2) Representations of a quiver

A *representation* R of a quiver Γ over a field k is a labeling of the graph Γ such that

(1) Each vertex $i \in \Gamma$ is labeled by a vector space R_i over k,

(2) Each edge $i \to j$ in Γ is labeled by a (vector space) homomorphism $\phi_{ij}\colon R_i \to R_j$.

Define morphisms of representations of quivers in the natural way and make the category of representations of the quiver Γ. The *dimension* of a representation R is the vector $\dim(R) = (d_i)$ where, for each vertex $i \in \Gamma$, $d_i = \dim(R_i)$. An *irreducible* representation of Γ is a representation R of Γ such that the only subrepresentations of R are 0 and R.

A representation R of a quiver Γ is *indecomposable* if it cannot be written as $R = S \oplus T$ where S and T are nonzero representations of Γ.

Theorem. *Let Γ be a quiver.*

(a) *There are a finite number of indecomposable representations of Γ if and only if Γ is an oriented Dynkin diagram of type A, D or E.*

(b) *If Γ is an oriented Dynkin diagram of type A, D or E then the indecomposable representations of Γ are in 1-1 correspondence with the positive roots for the Lie algebra $\mathfrak{g}$ corresponding to the Dynkin diagram.*

(1.3) Definition of the Hall algebra

Let Γ be a quiver and let $\mathbb{F}_q$ be a finite field with q elements. The *Hall algebra* or *Grothendieck ring* $R\Gamma$ of representations of Γ is the algebra over $\mathbb{C}$ with

(1) basis labeled by the isomorphism classes $[R]$ of representations of Γ over $\mathbb{F}_q$, and

(2) multiplication of two isomorphism classes $[R]$ and $[S]$ given by

$$[R]\cdot[S] = \sum_{[T]} C^T_{RS}[T] \qquad \text{where} \qquad C^T_{RS} = \mathrm{Card}\big(\{P \subseteq T \mid P \cong R,\ T/P \cong S\}\big).$$

(1.4) Connecting Hall algebras to the quantum group

Let Γ be a quiver which is obtained by orienting the edges of a Dynkin diagram of type A, D, or E, and let $\mathbb{F}_q$ be a finite field with q elements. Let us describe explicitly two types of indecomposable representations of Γ.

(1) Let i be a vertex of Γ. The representation

$$e_i \quad \text{given by} \quad V_j = \begin{cases} \mathbb{F}_q, & \text{if } j = i; \\ 0, & \text{if } j \neq i; \end{cases}$$

is an irreducible representation of Γ.

(2) Let $i \to j$ be an edge of Γ. The representation

$$e_{ij} \quad \text{given by} \quad V_\ell = \begin{cases} \mathbb{F}_q, & \text{if } \ell = i \text{ or } \ell = j; \\ 0, & \text{otherwise}; \end{cases} \qquad \text{and} \qquad \phi_{ij} = \mathrm{id}_{\mathbb{F}_q},$$

is an indecomposable (but not irreducible) representation of Γ.

The following relations hold in the Hall algebra $R\Gamma$,

$$e_{ij} = e_ie_j - e_je_i, \quad e_ie_{ij} = qe_{ij}e_i, \quad e_{ij}e_i = qe_je_{ij}, \quad \text{for each edge } i \to j \text{ in } \Gamma.$$

It is easier to prove the first relation by writing it in the form $e_ie_j = e_{ij} + e_je_i$. Combining the first two of these relations and the first and last of these relations respectively, gives the identities

$$e_i^2e_j - (q+1)e_ie_je_i + qe_je_i^2 = 0 \quad \text{and} \quad e_ie_j^2 - (q+1)e_je_ie_j + qe_j^2e_i = 0, \quad \text{respectively.}$$

We shall make the Hall algebra a bit bigger by adding the $K_i^{\pm1}$s that are in the quantum group $U_q\mathfrak{g}$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ and let $U_q\mathfrak{g}$ be the rational version of the quantum group with $k = \mathbb{C}$ and $q \in \mathbb{C}$ the number of elements in the field $\mathbb{F}_q$. Let $U_q\mathfrak{h}$ be the subalgebra of $U_q\mathfrak{g}$ generated by $K_1^{\pm1}, \ldots, K_r^{\pm1}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots corresponding to the Lie algebra $\mathfrak{g}$ (see II (2.6)). Define

$$\widetilde{R\Gamma} \;=\; \text{algebra generated by } R\Gamma \text{ and } K_1^{\pm1}, \ldots, K_r^{\pm1} \text{ with the additional relations}$$

$$K_i[R]K_i^{-1} = q^{(\alpha_i, d(R))}[R], \quad \text{for all } 1 \le i \le r \text{ and representations } R \text{ of } \Gamma,$$

where $d(R) = \sum_{j=1}^r \dim(R_j)\alpha_j$, and the inner product in the exponent of q is the inner product on $\mathfrak{h}_{\mathbb{R}}^*$ given in II (2.7).

Theorem. *Let Γ be a quiver which is obtained by orienting the edges of a Dynkin diagram of type A, D or E. Let $R\Gamma$ be the Hall algebra of representations of Γ over the finite field $\mathbb{F}_q$ with q elements and let $\widetilde{R\Gamma}$ be the extended Hall algebra defined above. Let $U_q\mathfrak{g}$ be the rational form of the quantum group with $k = \mathbb{C}$ which corresponds to the Dynkin diagram Γ and let*

$$U_q\mathfrak{b}^+ = \textit{subalgebra of } U_q\mathfrak{g} \textit{ generated by } K_1^{\pm1}, \ldots, K_r^{\pm1} \textit{ and } E_1, \ldots E_r.$$

Choose elements $z_1, \cdots, z_r \in \mathbb{Z}$ such that $z_i - z_j = 1$ if $i \to j$ is an edge in Γ. Then the homomorphism of algebras determined by

$$\begin{array}{ccc} U_q\mathfrak{b}^+ & \longrightarrow & \widetilde{R\Gamma} \\ \\ K_i^{\pm1} & \longmapsto & K_i^{\pm1} \\ E_i & \longmapsto & K_i^{z_i}e_i \end{array}$$

is an isomorphism.

2. An algebra of perverse sheaves

In this section we shall construct an algebra $\mathcal{K}$ from a Dynkin diagram Γ. There is a strong relationship between this algebra and the quantum group $U_q\mathfrak{g}$ where $\mathfrak{g}$ is the simple complex Lie algebra corresponding to the Dynkin diagram Γ.

The algebra $\mathcal{K}$ is graded,

$$\mathcal{K} = \bigoplus_{\nu \in Q^+} \mathcal{K}_\nu,$$

in the same way that the quantum group $U_q\mathfrak{n}^+$ is graded, see VII (1.2). The vector space $\mathcal{K}$ comes with natural shift maps $[n]$ which correspond to multiplication by q^n in the quantum group $U_q\mathfrak{b}^+$. The algebra $\mathcal{K}$ has a natural multiplication which comes from an induction functor and a natural "pseudo-comultiplication" which comes from a restriction functor. The multiplication and the pseudo-comultiplication turn out to be almost the same as the multiplication and the comultiplication on the quantum group $U_q\mathfrak{b}^+$. Lastly, the algebra $\mathcal{K}$ has a natural inner product $\{,\}$ that is related to the inner product $\langle,\rangle$ pairing $U_q\mathfrak{b}^-$ and $U_q b^+$, (see VII (2.1)).

In Theorem (2.8) we shall see that if we extend the algebra $\mathcal{K}$ a little bit, by adding the $K_i^{\pm 1}$'s that are in the quantum group $U_q\mathfrak{g}$ then we get an algebra $\widetilde{\mathcal{K}}$ such that

$$\widetilde{\mathcal{K}} \simeq U_q\mathfrak{b}^+.$$

This last fact is very similar to the case of the Hall algebra (1.4) where after extending the Hall algebra $R\Gamma$ by adding the $K_i^{\pm 1}$'s that are in the quantum group $U_q\mathfrak{g}$, we got an algebra $\widetilde{R\Gamma}$ which was also isomorphic to $U_q\mathfrak{b}^+$. We shall see in section 3 that this is not a coincidence, there is a concrete connection between $R\Gamma$ and the algebra $\mathcal{K}$. The advantage of working with the algebra $\mathcal{K}$ instead of the Hall algebra $R\Gamma$ is that $\mathcal{K}$ has more natural structure than $R\Gamma$, it has:

(a) a natural pseudo-comultiplication $r: \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$,

(b) a natural inner product $\{,\}: \mathcal{K} \times \mathcal{K} \to \mathbb{Z}((q))$,

(c) a natural involution $D: \mathcal{K} \to \mathcal{K}$,

(d) a natural basis coming from simple perverse sheaves.

The natural basis coming from simple perverse sheaves is called the *canonical basis*.

(2.1) Γ-graded vector spaces and the varieties E_V with G_V action

Let Γ be a quiver obtained by orienting the edges of a Dynkin diagram of type A, D or E. For convenience we label the vertices by $1, 2, \ldots, r$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ.

Let p be a positive prime integer and let $\overline{\mathbb{F}_p}$ be the algebraic closure of the finite field $\mathbb{F}_p$ with p elements. A Γ-*graded vector space* V over $\overline{\mathbb{F}_p}$ is a labeling of the graph Γ such that each vertex i is labeled by a vector space V_i over $\overline{\mathbb{F}_p}$. The *dimension* of a Γ-graded vector space V is the r-tuple of nonnegative integers $\dim(V) = (\dim(V_i))$. We shall identify dimensions of Γ-graded vector spaces with elements of

$$Q^+ = \sum_i \mathbb{N}\alpha_i \qquad \text{so that} \qquad \dim(V) = \sum_{i=1}^r \dim(V_i)\alpha_i,$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots for $\mathfrak{g}$ and $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Fix an element $\nu \in Q^+$ and a Γ-graded vector space V over $\overline{\mathbb{F}_p}$ such that $\dim(V) = \nu$. Define

$$G_V = \prod_i GL(V_i) \qquad \text{and} \qquad E_V = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j),$$

where the sum in the definition of E_V is over all edges of Γ. There is a natural action of G_V on E_V given by

$$g \cdot (\phi_{ij}) = (g_j \phi_{ij} g_i^{-1}), \quad \text{if } (\phi_{ij}) \in E_V \text{ and } g = (g_1, \cdots, g_r) \in G_V.$$

Let $x \in E_V$ and let W be a Γ-*graded subspace* of V, i.e. $W_i \subseteq V_i$ for all vertices i in Γ. The subspace W is x-*stable* if $xW_i \subseteq W_j$ for all edges $i \to j$ in Γ. We shall simply write $W \subseteq V$ if W is a Γ-graded subspace of V and $xW \subseteq W$ if W is x-stable.

(2.2) Definition of the categories $\mathcal{Q}_V$ and $\mathcal{Q}_T \otimes \mathcal{Q}_W$

The reader may skip this definition if it looks like too much to swallow. The only important thing at this stage is that $\mathcal{Q}_V$ is a category of objects and it is contained in a category called $D^b_c(E_V)$.

Let V be a Γ-graded vector space over $\overline{\mathbb{F}_p}$ and let E_V be the variety over $\overline{\mathbb{F}_p}$ defined in (2.1). Let $D^b_c(E_V)$ be the bounded derived category of $\overline{\mathbb{Q}}_l$-(constructible) sheaves on E_V, see IV (1.4). Recall that $D^b_c(E_V)$ comes endowed with shift functors IV (2.4),

$$\begin{array}{cccc} [n]\colon & D^b_c(E_V) & \longrightarrow & D^b_c(E_V) \\ & A & \longmapsto & A[n]. \end{array}$$

Define

$\mathcal{Q}_V$ = the full subcategory of $D^b_c(E_V)$ consisting of finite direct sums of simple perverse sheaves L such that some shift of L is a direct summand of $L_{\vec{\nu}}$ for some partition $\vec{\nu}$ of $\nu = \dim(V)$.

The complexes $L_{\vec{\nu}}$ are defined in (2.7). Let T and W be Γ-graded vector spaces over $\overline{\mathbb{F}_p}$. Define

$$\mathcal{Q}_T \otimes \mathcal{Q}_W = \text{the complexes } L \in D^b_c(E_T \times E_W) \text{ such that } L \cong \bigoplus_{i=1}^s A_i \otimes B_i,$$

for some $A_i \in \mathcal{Q}_T$, $B_i \in \mathcal{Q}_W$, and some positive integer s.

This is a subcategory of $D^b_c(E_T \times E_W)$.

(2.3) The Grothendieck group $\mathcal{K}$ associated to the categories $\mathcal{Q}_V$

Let $\nu \in Q^+$ and let V be a Γ-graded vector space of dimension ν. Let $\mathcal{Q}_V$ be as in (2.2). The important thing about $\mathcal{Q}_V$ at the moment is that it is a category related to E_V.

The *Grothendieck group* $\mathcal{K}(\mathcal{Q}_V)$ of the category $\mathcal{Q}_V$ is the $\mathbb{C}(q)$-module generated by the isomorphism classes of objects in $\mathcal{Q}_V$ with the addition operation given by the relations

$$[B_1 \oplus B_2] = [B_1] + [B_2], \quad \text{if } B_1, B_2 \in \mathcal{Q}_V,$$

and multiplication by q given by the relations

$$[B[n]] = q^n[B], \qquad \text{for } B \in \mathcal{Q}_V \text{ and } n \in \mathbb{Z},$$

where the map $B \to B[n]$ is the shift functor on $D^b_c(E_V)$, see IV (2.4). The structure of $\mathcal{K}(\mathcal{Q}_V)$ depends only on the element ν and so we shall often write $\mathcal{K}_\nu$ in place of $\mathcal{K}(\mathcal{Q}_V)$.

Define

$$\mathcal{K} = \bigoplus_{\nu \in Q^+} \mathcal{K}_\nu.$$

The group $\mathcal{K}$ is graded in the same way that $\mathfrak{U}_q\mathfrak{n}^+$ is graded, see VII (1.2).

(2.4) Definition of the multiplication in $\mathcal{K}$

Let V be a Γ-graded vector space. Let T and W be Γ-graded vector spaces such that

$$W \subseteq V \qquad \text{and} \qquad V/W \cong T.$$

If $x \in E_V$ such that $xW \subseteq W$ then let x_W be the linear transformation of W induced by the action of x on W and let x_T be the linear transformation of $T \cong V/W$ induced by the action of x on V/W. Define

$$\mathcal{S} = \{x \in E_V \mid xW \subseteq W\},$$
$$P = \{g \in G_V \mid gW \subseteq W\}, \qquad U = \{g \in P \mid g_W = \mathrm{id}_W, g_T = \mathrm{id}_T\}.$$

The groups P and U are subgroups of G_V. The group P is the stabilizer of W in G_V, it is a parabolic subgroup of G_V. The group U is the unipotent radical of P.

Let $\mathcal{Q}_T \otimes \mathcal{Q}_W$ be the subcategory of $D^b_c(E_T \otimes E_W)$ which is defined in (2.2). The diagram

$$\begin{array}{ccccccc} E_T \times E_W & \xleftarrow{p_1} & G_V \times_U \mathcal{S} & \xrightarrow{p_2} & G_V \times_P \mathcal{S} & \xrightarrow{p_3} & E_V \\ (x_T, x_W) & \longleftarrow & (g,x) & \longmapsto & (g,x) & \longmapsto & gx \end{array}$$

induces the diagram

$$\mathcal{Q}_T\otimes\mathcal{Q}_W \quad\longrightarrow\quad D^b_c(E_T\times E_W) \quad\xrightarrow{p_1^*}\quad D^b_c(G\times_U\mathcal{S}) \quad\xrightarrow{(p_2)_\flat}\quad D^b_c(G\times_P\mathcal{S}) \quad\xrightarrow{(p_3)_!}\quad D^b_c(E_V)$$

where the first map is the inclusion map.

Theorem. *Let V be a Γ-graded vector space and let E_V be the variety with the G_V action which is defined in (2.1). Let W and T be Γ-graded vector spaces such that $W\subseteq V$ and $V/W\cong T$. Let $\mathcal{Q}_T\otimes\mathcal{Q}_W$ and $\mathcal{Q}_V$ be the categories of complexes of sheaves on $E_T\times E_W$ and E_V, respectively, which are defined in (2.2). There is a well defined functor*

$$\begin{array}{rccc}\mathrm{Ind}^V_{T,W}\colon & \mathcal{Q}_T\otimes\mathcal{Q}_W & \longrightarrow & \mathcal{Q}_V\\ & A & \longmapsto & \big((p_3)_!(p_2)_\flat p_1^*A\big)[\dim(p_1)-\dim(p_2)]\end{array}$$

where p_1,p_2, and p_3 are as defined in the diagram above, $\dim(p_1)$ is the dimension of the fibers of the map p_1, and $\dim(p_2)$ is the dimension of the fibers of the map p_2.

The *multiplication* in $\mathcal{K}$ is defined by the formula

$$[A]\cdot[B]=\big[\mathrm{Ind}^V_{T,W}(A\otimes B)\big],\qquad \text{for } A\in\mathcal{Q}_T \text{ and } B\in\mathcal{Q}_W.$$

With this multiplication $\mathcal{K}$ becomes an algebra. The strange shift by $[\dim(p_1)-\dim(p_2)]$ in the definition of $\mathrm{Ind}^V_{T,W}$ is there to make the multiplication in $\mathcal{K}$ match up with the multiplication in the nonnegative part of the quantum group $U_q\mathfrak{b}^+$, see Theorem (2.8) below.

(2.5) Definition of the pseudo-comultiplication $r\colon\mathcal{K}\to\mathcal{K}\otimes\mathcal{K}$

Let V be a Γ-graded vector space. Let T and W be Γ-graded vector spaces such that

$$W\subseteq V \qquad\text{and}\qquad V/W\cong T.$$

If $x\in E_V$ such that $xW\subseteq W$ then let x_W be the linear transformation of W induced by the action of x on W and let x_T be the linear transformation of $T\cong V/W$ induced by the action of x on V/W.

Define

$$\mathcal{S}=\{x\in E_V\mid xW\subseteq W\}$$

and let $\mathcal{Q}_V$ be the subcategory of $D^b_c(E_V)$ which is defined in (2.2). The diagram

$$\begin{array}{ccccc}E_V & \xleftarrow{\iota} & \mathcal{S} & \xrightarrow{\kappa} & E_T\times E_W\\ x & \longleftarrow\!\shortmid & x & \longmapsto & (x_T,x_W)\end{array}$$

induces the diagram

$$\mathcal{Q}_V \quad\longrightarrow\quad D^b_c(E_V) \quad\xrightarrow{\iota^*}\quad D^b_c(\mathcal{S}) \quad\xrightarrow{\kappa_!}\quad D^b_c(E_T\times E_W)$$

where the first map is the inclusion map.

Theorem. *Let V be a Γ-graded vector space and let E_V be the variety with the G_V action which is defined in (2.1). Let W and T be Γ-graded vector spaces such that $W \subseteq V$ and $V/W \cong T$. Let $\mathcal{Q}_T \otimes \mathcal{Q}_W$ and $\mathcal{Q}_V$ be the categories of complexes of sheaves on $E_T \times E_W$ and E_V, respectively, which are defined in (2.2). There is a well defined functor*

$$\begin{array}{rccc} \mathrm{Res}^V_{T,W}\colon & \mathcal{Q}_V & \longrightarrow & \mathcal{Q}_T \otimes \mathcal{Q}_W \\ & B & \longmapsto & \big(\kappa_!\iota^* B\big)[\dim(p_1) - \dim(p_2) - 2\dim(G_V/P)] \end{array}$$

where p_1, p_2, κ, and ι are as defined above, $\dim(p_1)$ is the dimension of the fibers of the map p_1, $\dim(p_2)$ is the dimension of the fibers of the map p_2, and P is the parabolic subgroup of G_V defined in (2.4).

The *pseudo-comultiplication* on $\mathcal{K}$ is the map $r\colon \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ defined by

$$r([A]) = \big[\mathrm{Res}^V_{T,W}(A)\big], \qquad \text{if } A \in \mathcal{Q}_V.$$

The strange shift by $[\dim(p_1) - \dim(p_2) - 2\dim(G_V/P)]$ in the definition of $\mathrm{Res}^V_{T,W}$ is there to make the pseudo-comultiplication in $\mathcal{K}$ match up with the comultiplication in the nonnegative part of the quantum group $U_q\mathfrak{b}^+$, see Theorem (2.8) below.

(2.6) The symmetric form on $\mathcal{K}$

Recall that we write $\mathcal{K}_\nu$ in place of $\mathcal{K}(\mathcal{Q}_V)$ since the structure of $\mathcal{K}(\mathcal{Q}_V)$ depends only on ν. For each $\nu \in Q^+$, define a bilinear form

$$\{,\}_\nu\colon \mathcal{K}_\nu \times \mathcal{K}_\nu \to \mathbb{C}(q) \qquad \text{by defining}$$

$$\big\{[B_1],[B_2]\big\}_\nu = \sum_j q^{-j} \dim\big(\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^* B_1 \otimes t_\flat s^* B_2))\big),$$

for $B_1, B_2 \in \mathcal{Q}_V$. The vector spaces $\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^* B_1 \otimes t_\flat s^* B_2))$ are defined in (2.10) below. At this stage the important thing is that they depend only on B_1, B_2 and j.

Use the forms $\{,\}_\nu$, $\nu \in Q^+$, to define a bilinear form

$$\{,\}\colon \mathcal{K} \times \mathcal{K} \to \mathbb{Z}((q)) \qquad \text{on} \quad \mathcal{K} = \bigoplus_{\nu \in Q^+} \mathcal{K}_\nu \qquad \text{by setting}$$

$$\begin{array}{rll} \big\{\mathcal{K}_\mu, \mathcal{K}_\nu\big\} = 0, & & \text{if } \mu,\nu \in Q^+ \text{ such that } \mu \neq \nu, \text{ and} \\ \big\{x,y\big\} = \big\{x,y\big\}_\nu, & & \text{if } x,y \in \mathcal{K}_\nu. \end{array}$$

Theorem. *Let V be a Γ-graded vector space and let T and W be Γ-graded subspaces such that $W \subseteq V$ and $T \cong V/W$. Let $A \in \mathcal{Q}_T \otimes \mathcal{Q}_W$ and let $B \in \mathcal{Q}_V$. Then*

$$\{ \; A \; , \; \mathrm{Res}^V_{T,W}(B) \; \} = \{ \; \mathrm{Ind}^V_{T,W}(A) \; , \; B \; \}$$

The result in this theorem is an analogue of the property of the bilinear form $\langle , \rangle$ on the quantum group which is given in VII (2.1)(d).

(2.7) Definition of the elements $L_{\vec{\nu}} \in \mathcal{K}$

Let $\nu \in Q^+$ and let V be a Γ-graded subspace of dimension ν. A *partition* of ν is a sequence $\vec{\nu} = (\nu^1, \ldots, \nu^m)$ of elements of the root lattice Q such that

(1) each ν^j, $1 \le j \le m$, is a nonnegative integer multiple of a simple root, and

(2) $\nu^1 + \cdots + \nu^m = \nu$.

For example we might have $\vec{\nu} = (3\alpha_1, 2\alpha_3, 0, \alpha_1, 2\alpha_1)$ if $\nu = 6\alpha_1 + 2\alpha_3$. A *flag of type $\vec{\nu}$ in V* is a sequence

$$f = (V = V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V^{(m)} = 0)$$

of Γ-graded subspaces of V such that $\dim(V^{(\ell-1)}/V^{(\ell)}) = \nu^\ell$, for all $1 \le \ell \le m$.

Let $x \in E_V$. A flag f is *x-stable* if $xV^{(\ell)} \subseteq V^{(\ell)}$ for all $1 \le \ell \le m$. Define

$$\mathcal{F}_{\vec{\nu}} = \{(x, f) \mid x \in E_V, \; f \text{ is an } x\text{-stable flag of type } \vec{\nu} \text{ in } V\}.$$

The map

$$\begin{array}{ccc} \mathcal{F}_{\vec{\nu}} & \xrightarrow{\pi_{\vec{\nu}}} & E_V \\ (x,f) & \longmapsto & x \end{array} \qquad \text{induces a map} \qquad D^b_c(\mathcal{F}_{\vec{\nu}}) \xrightarrow{(\pi_{\vec{\nu}})_!} D^b_c(E_V).$$

Let $f(\vec{\nu}) = \dim(\mathcal{F}_{\vec{\nu}})$ and define

$$L_{\vec{\nu}} = \big((\pi_{\vec{\nu}})_! \mathbf{1}\big)[\dim(\mathcal{F}_{\vec{\nu}})], \qquad \text{i.e.}$$

$$\begin{array}{ccccc} D^b_c(\mathcal{F}_{\vec{\nu}}) & \xrightarrow{(\pi_{\vec{\nu}})_!} & D^b_c(E_V) & \xrightarrow{[\dim(\mathcal{F}_{\vec{\nu}})]} & D^b_c(E_V) \\ \mathbf{1} & \longmapsto & & \longmapsto & L_{\vec{\nu}} \end{array}$$

where $\mathbf{1}$ is the constant sheaf on $\mathcal{F}_{\vec{\nu}}$ and $[\dim(\mathcal{F}_{\vec{\nu}})]$ is a shift, see IV (2.4).

Theorem. *Let V be a Γ-graded vector space of dimension ν and let T and W be Γ-graded vector spaces such that $W \subseteq V$ and $T \cong V/W$.*

(1) Let $\vec{\tau}$ and $\vec{\omega}$ be partitions of $\dim(T)$ *and* $\dim(W)$*, respectively. Then*

$$\mathrm{Ind}^V_{T,W}(L_{\vec{\tau}} \otimes L_{\vec{\omega}}) = L_{\vec{\tau}\vec{\omega}},$$

where, if $\vec{\tau} = (\tau^1, \tau^2, \ldots, \tau^s)$ *and* $\vec{\omega} = (\omega^1, \ldots, \omega^t)$*, then* $\vec{\tau}\vec{\omega} = (\tau^1, \ldots, \tau^s, \omega^1, \ldots, \omega^t)$*.*

(2) Let $\vec{\nu}$ *be a partition of* $\dim(V)$*. Then*

$$\mathrm{Res}^V_{T,W} L_{\vec{\nu}} \cong \bigoplus_{\vec{\tau},\vec{\omega}} (L_{\vec{\tau}} \otimes L_{\vec{\omega}})[M'(\vec{\tau},\vec{\omega})],$$

where the sum is over all $\vec{\tau},\vec{\omega}$ *such that* $\vec{\tau}$ *is a partition of* $\dim(T)$*,* $\vec{\omega}$ *is a partition of* $\dim(W)$ *and* $\vec{\tau}+\vec{\omega}=\vec{\nu}$*. The positive integer* $M'(\vec{\tau},\vec{\omega})$ *is defined in (2.9) below.*

(3) Let $\nu = \alpha_i$ *be a simple root for* $\mathfrak{g}$ *and let* V *be a* Γ*-graded subspace such that* $\dim(V) = \alpha_i$*. Define* $L_i \in \mathcal{K}(\mathcal{Q}_V)$ *by* $L_i = L_{\vec{\nu}}$ *where* $\vec{\nu} = (\alpha_i)$*. Then*

$$\{\, [L_i]\ ,\ [L_i]\, \} \ = \ \frac{1}{1-q^2}.$$

(2.8) The connection between $\mathcal{K}$ and the quantum group

We shall make the algebra

$$\mathcal{K} = \bigoplus_{\nu\in Q^+} \mathcal{K}_\nu$$

a bit bigger by adding the $K_i^{\pm 1}$'s that are in the quantum group $U_q\mathfrak{g}$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ and let $U_q\mathfrak{g}$ be the rational version of the quantum group with $k = \mathbb{C}(q)$ where q is an indeterminate. Let $U_q\mathfrak{h}$ be the subalgebra of $U_q\mathfrak{g}$ generated by $K_1^{\pm1}, \ldots, K_r^{\pm1}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots corresponding to the Lie algebra $\mathfrak{g}$. Define

$$\widetilde{\mathcal{K}} \ = \ \text{algebra generated by } \mathcal{K} \text{ and } K_1^{\pm1}, \ldots, K_r^{\pm1} \text{ with the additional relations}$$

$$K_i x K_i^{-1} = q^{(\alpha_i,\nu)}x, \quad \text{for all } 1 \le i \le r \text{ and all } x \in \mathcal{K}_\nu,$$

where the inner product in the exponent of q is the inner product on $\mathfrak{h}^*_{\mathbb{R}}$ given in II (2.7).

Define a map $j^+ : \mathcal{K}\otimes\mathcal{K} \to \widetilde{\mathcal{K}}\otimes\widetilde{\mathcal{K}}$ by

$$j^+(x\otimes y) = xK_1^{\nu_1}\cdots K_r^{\nu_r}\otimes y, \quad \text{if } x\in\mathcal{K} \text{ and } y \in \mathcal{K}_\nu, \text{ where } \nu = \textstyle\sum_i \nu_i\alpha_i.$$

Use the map j^+ and the pseudo-comultiplication $r: \mathcal{K}\to\mathcal{K}\otimes\mathcal{K}$ defined in (2.5) to define a *coproduct* on $\widetilde{\mathcal{K}}$ by

$$\begin{array}{rccccc} \Delta: & \widetilde{\mathcal{K}} & \longrightarrow & \widetilde{\mathcal{K}}\otimes\widetilde{\mathcal{K}} & & \\ & & & & & \\ & K_i^{\pm1} & \longmapsto & K_i^{\pm1}\otimes K_i^{\pm1} & & \text{for } 1\le i\le r, \\ & x & \longmapsto & j^+r(x) & & \text{for } x\in\mathcal{K}, \end{array}$$

where $r: \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is the pseudo-comultiplication defined in (2.5). Then $\widetilde{\mathcal{K}}$ is a Hopf algebra!

Theorem. *Let L_i be as defined in Theorem (2.7b). The algebra homomorphism determined by*

$$\begin{array}{cccc} \mathcal{I}: & \widetilde{\mathcal{K}} & \longrightarrow & U_q\mathfrak{b}^+ \\ \\ & L_i & \longmapsto & E_i \\ & K_i^{\pm 1} & \longmapsto & K_i^{\pm 1} \end{array}$$

is an isomorphism of Hopf algebras.

(2.9) Dictionary between $\mathcal{K}$ and $U_q\mathfrak{b}^+$

Let us make a small dictionary between the algebra $\mathcal{K}$ and the quantum group $U_q\mathfrak{b}^+$. Our intent is to describe, conceptually, the correspondence between the structures inherent in the algebra $\mathcal{K}$ and the structures in the quantum group $U_q\mathfrak{b}^+$. The map $\mathcal{I}$ is the isomorphism given in Theorem (2.8).

$\widetilde{\mathcal{K}}$	is isomorphic to	$U_q\mathfrak{b}^+$.
$\widetilde{\mathcal{K}}$ is the algebra generated by $\mathcal{K}$ and the $K_i^{\pm 1}$s.	Similarly,	$U_q\mathfrak{b}^+$ is the algebra generated by $U_q\mathfrak{n}^+$ and the $K_i^{\pm 1}$s.
$\mathcal{K}$ is graded, $\mathcal{K} = \bigoplus_{\nu\in Q^+} \mathcal{K}_\nu$.	Similarly,	$U_q\mathfrak{n}^+$ is graded, $U_\mathfrak{n}^+ = \bigoplus_{\nu\in Q^+} (U_q\mathfrak{n}^+)_\nu$.
The shift functor $[n]$ gives rise to multiplication by q^n in $\mathcal{K}$	which corresponds to	multiplication by q^n in $U_q\mathfrak{b}^+$.
The functor $\mathrm{Ind}_{T,W}^V$	corresponds to	the multiplication in $U_q\mathfrak{n}^+$.
The functor $\mathrm{Res}_{T,W}^V$	corresponds to	the comultiplication in $U_q\mathfrak{b}^+$.
The inner product $\{,\}$	corresponds to	the bilinear form $\langle,\rangle$ pairing $U_q\mathfrak{b}^-$ and $U_q\mathfrak{b}^+$.
A partition $\vec{\nu} = (\nu_1\alpha_{i_1}, \ldots, \nu_l\alpha_{i_l})$ indexes $L_{\vec{\nu}}$	which maps, under $\mathcal{I}$, to	$E_{i_1}^{(\nu_1)} \ldots E_{i_l}^{(\nu_l)}$ where $E_i^{(n)} = E_i^n/[n]!$.
The Verdier duality functor D	corresponds to	the $\mathbb{C}$-algebra involution $^-: U_q\mathfrak{n}^+ \to U_q\mathfrak{n}^+$ which sends $q \mapsto q^{-1}$ and $E_i \mapsto E_i$.
The simple perverse sheaves in the various $\mathcal{Q}_V$	map, under $\mathcal{I}$, to	a canonical basis in $U_q\mathfrak{n}^+$.

(2.10) Definition of the constant $M'(\tau, \omega)$ which was used in (2.7)

Let V be a Γ-graded vector space and let T and W be Γ-graded subspaces such that $W \subseteq V$ and $T \cong V/W$. If $x \in E_V$ such that $xW \subseteq W$ then let x_W be the linear transformation of W induced by the action of x on W and let x_T be the linear transformation of $T \cong V/W$ induced by the action of x on V/W. Let $\vec{\nu}$ be a partition of $\dim(V)$. If

$$f = \big(\ V = V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V^{(m)} = 0\ \big)$$

is a flag of type $\vec{\nu}$ in V then define

$$f_W = \big(\ (V\cap W) = (V^0 \cap W) \supseteq (V^{(1)}\cap W) \supseteq \cdots \supseteq (V^{(m)}\cap W) = 0\ \big) \qquad \text{and}$$
$$f_T = \big(\ p(V) = p(V^{(0)}) \supseteq p(V^{(1)}) \supseteq \cdots \supseteq p(V^{(m)}) = 0\ \big) \qquad \text{where } p: V \to V/W$$

is the canonical projection.

Let $\vec{\tau}$ be a partition of $\dim(T)$ and let $\vec{\omega}$ be a partition of $\dim(W)$, such that $\vec{\tau}+\vec{\omega}=\vec{\nu}$. Define

$$\tilde{F}(\vec{\tau},\vec{\omega}) = \left\{ (x,f) \;\middle|\; xW \subseteq W, \begin{array}{l} f \text{ is an } x\text{-stable flag of type } \vec{\nu} \text{ in } V, \\ \text{and } f_W \text{ is a flag of type } \vec{\omega} \text{ in } W \end{array} \right\}.$$

Define a map

$$\begin{array}{rccc} \alpha\colon & \tilde{F}(\vec{\tau},\vec{\omega}) & \longrightarrow & \mathcal{F}_{\vec{\tau}} \times \mathcal{F}_{\vec{\omega}} \\ & (x,f) & \longmapsto & ((x_T,f_T),(x_W,f_W)) \end{array}$$

and define

$$M'(\tau,\omega) = \dim(p_1) - \dim(p_2) - 2\dim(G_V/P) + \dim(\mathcal{F}_{\vec{\nu}}) - \dim(\mathcal{F}_{\vec{\tau}}) - \dim(\mathcal{F}_{\vec{\omega}}) - 2\dim(\alpha).$$

where p_1 and p_2 are the maps given in (2.4), P is the parabolic subgroup of G_V defined in (2.4), and $\dim(p_1)$, $\dim(p_2)$ and $\dim(\alpha)$ are the dimensions of the fibers of the maps p_1, p_2, and α, respectively.

(2.11) Definition of the vector spaces $\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^* B_1 \otimes t_\flat s^* B_2))$ from (2.6)

Let Ω be a smooth irreducible algebraic variety with a free action of G_V such that the $\overline{\mathbb{Q}_l}$-cohomology of Ω is zero in degrees $1,2,\ldots,m$ where m is a large integer. Consider the diagram

$$\begin{array}{ccccc} E_V & \overset{s}{\longleftarrow} & \Omega \times E_V & \overset{t}{\longrightarrow} & G\backslash(\Omega \times E_V) \\ x & \longleftarrow\!\shortmid & (\omega,x) & \longmapsto & G_V(\omega,x) \end{array} \quad \text{and the diagram} \quad G_V\backslash(\Omega \times E_V) \overset{u}{\longrightarrow} \{\text{point}\}.$$

These diagrams induce diagrams

$$\begin{array}{l} D_c^b(E_V) \xrightarrow{s^*} D_c^b(\Omega \times E_V) \xrightarrow{t_\flat} D_c^b(G\backslash(\Omega \times E_V)) \qquad \text{and} \\ D_c^b(G_V\backslash(\Omega \times E_V)) \xrightarrow{u_!} D_c^b(\{\text{point}\}). \end{array}$$

With these notations one has that $\mathcal{H}^{j+2\dim(G\backslash\Omega)}(u_!(t_\flat s^* B_1 \otimes t_\flat s^* B_2))$ is a sheaf on the space $\{\text{point}\}$, i.e. a $\overline{\mathbb{Q}_l}$-vector space.

(2.12) Some remarks on Part II of Lusztig's book

The construction of the algebra $\mathcal{K}$ and the relationship between it and the quantum group is detailed in Lusztig's book [Lu]. Lusztig works in much more generality there.

(1) Lusztig allows Γ to be an arbitrary quiver, rather than just a quiver gotten by orienting a Dynkin diagram of type A, D or E. It does not require any more theory than what we have already outlined in order to define the algebra $\mathcal{K}$ in this more general setting.

(2) Lusztig wants to construct algebras $\mathcal{K}$ which will be isomorphic to the nonnegative parts of the quantum groups corresponding to general Dynkin diagrams. In order to do this he must first consider only diagrams with single bonds and then 'fold' the

diagram by analyzing the action of an automorphism of the diagram. The addition of the folding automorphism into the theory is a nontrivial extension of what we have developed in these notes.

(3) We have ignored the effect of the orientation of the quiver. If one wants to compare the algebras $\mathcal{K}$ that are obtained by orienting the same quiver in different ways one must analyze a Fourier-Deligne transform between these two different algebras. The amazing thing is that, after one extends the algebras by adding the $K_i^{\pm 1}$s that are in the quantum group, the two different algebras (from the different orientations) become isomorphic!

3. The connection between representations of quivers and perverse sheaves

(3.1) Correspondence between orbits and isomorphism classes of representations of Γ

Let Γ be a quiver obtained by orienting the edges of a Dynkin diagram of type A, D or E. For convenience we label the vertices by $1, 2, \ldots, r$. Let $\mathfrak{g}$ be the finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram given by Γ.

Let p be a positive prime integer and let $\overline{\mathbb{F}_p}$ be the algebraic closure of the finite field $\mathbb{F}_p$ with p elements. Fix an element $\nu \in Q^+$ (see VII (1.2)) and a Γ-graded vector space V over $\overline{\mathbb{F}_p}$ such that $\dim(V) = \nu$. Define

$$G_V = \prod_i GL(V_i) \qquad \text{and} \qquad E_V = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j),$$

where the sum in the definition of E_V is over all edges of Γ. The natural action of G_V on E_V is given by

$$g \cdot (\phi_{ij}) = (g_j \phi_{ij} g_i^{-1}), \quad \text{if } (\phi_{ij}) \in E_V \text{ and } g = (g_1, \cdots, g_r) \in G_V.$$

The group G_V is an algebraic group over $\overline{\mathbb{F}_p}$ and E_V is a variety over $\overline{\mathbb{F}_p}$ with a G_V action. Each element $(\phi_{ij}) \in E_V$ determines a representation of Γ of dimension $\dim(V)$. Each G_V-orbit in E_V determines an isomorphism class of representations of Γ. Let us make this correspondence precise.

An *orbit index for V* is a sequence of positive integers labeled by the positive roots

$$\vec{c} = (c_\alpha)_{\alpha \in R^+} \quad \text{such that} \quad \sum_{\alpha \in R^+} c_\alpha \alpha \;=\; \dim(V),$$

where R^+ is the set of positive roots for $\mathfrak{g}$. For each orbit index $\vec{c}$ for V define a representation of Γ by

$$R_{\vec{c}} = \bigoplus_{\alpha \in R^+} e_\alpha^{\oplus c_\alpha} \qquad \text{and let} \qquad \mathcal{O}_{\vec{c}} = \text{ the } G_V\text{-orbit in } E_V \text{ corresponding to } R_{\vec{c}},$$

where e_α is the indecomposable representation of Γ indexed by the positive root α, see Theorem (1.2b). Then we have a one-to-one correspondence

$$\begin{array}{ccc} G_V \text{ orbits in } E_V & \stackrel{1-1}{\longleftrightarrow} & \text{isomorphism classes of representations} \\ & & \text{of } \Gamma \text{ of dimension } \nu \\ & & \\ \mathcal{O}_{\vec{c}} & \longleftrightarrow & [R_{\vec{c}}] \end{array}$$

(3.2) Realizing the structure constants of the Hall algebra in terms of orbits

Let q be a power of the prime p. Since E_V is a variety over $\overline{\mathbb{F}_p}$ there is an action of the the qth power Frobenius map F on E_V, see [Ca] p. 503. If X is a subset of E_V then let X^F denote the set of points of X which are fixed under the action of the Frobenius map F.

Let T and W be Γ-graded vector spaces such that $W \subseteq V$ and $T \cong V/W$. Recall the diagram

$$\begin{array}{ccccccc} E_T \times E_W & \stackrel{p_1}{\longleftarrow} & G_V \times_U \mathcal{S} & \stackrel{p_2}{\longrightarrow} & G_V \times_P \mathcal{S} & \stackrel{p_3}{\longrightarrow} & E_V \\ (x_T, x_W) & \longleftarrow\!\shortmid & (g,x) & \longmapsto & (g,x) & \longmapsto & gx \end{array}$$

given in (2.4). Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be orbit indices for T, W and V, respectively. Then we have

$$\begin{array}{ccccccc} E_T \times E_W & \stackrel{p_1}{\longleftarrow} & G_V \times_U \mathcal{S} & \stackrel{p_2}{\longrightarrow} & G_V \times_P \mathcal{S} & \stackrel{p_3}{\longrightarrow} & E_V \\ \mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}} & \leftrightarrow & p_1^{-1}(\mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}}) & \longmapsto & p_2(p_1^{-1}(\mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}})) & & \\ & & & & p_3^{-1}(\mathcal{O}_{\vec{c}}) & \longleftrightarrow & \mathcal{O}_{\vec{c}} \end{array}$$

Let $M = R_{\vec{a}}$, $N = R_{\vec{b}}$ and $P = R_{\vec{c}}$ be the representations of Γ given in (3.1). By a direct count, we have

$$C^P_{M,N} = \mathrm{Card}\left(\left(\; p_2(p_1^{-1}(\mathcal{O}_{\vec{a}} \times \mathcal{O}_{\vec{b}})) \; \bigcap \; p_3^{-1}(\mathcal{O}_{\vec{c}}) \; \right)^F \right).$$

where $C^P_{M,N}$ are the structure coefficients of the Hall algebra $R\Gamma$ given in (1.3).

(3.3) Rewriting the Hall algebra in terms of functions constant on orbits

Let q be a power of the prime p. On any variety Y over $\overline{\mathbb{F}_p}$ there is an action of the the qth power Frobenius map F on E_V, see [Ca] p. 503. If X is a subset of Y then X^F denotes the set of points of X which are fixed under the action of the Frobenius map F.

Let l be a positive prime number, invertible in $\overline{\mathbb{F}_p}$. Let $\overline{\mathbb{Q}_l}$ be the algebraic closure of the field of l-adic numbers. Define

K_ν = the vector space of $\overline{\mathbb{Q}_l}$-valued functions on $(E_V)^F$ which are constant on the orbits $(\mathcal{O}_{\vec{c}})^F$ for all orbit indexes $\vec{c}$ for V.

Define

$$K = \bigoplus_{\nu \in Q^+} K_\nu,$$

where Q^+ is as in VII (1.2).

Define a multiplication on K as follows. Let T and W be Γ-graded vector spaces such that $W \subseteq V$ and $T \cong V/W$. Recall the diagram

$$\begin{array}{ccccccc} E_T \times E_W & \xleftarrow{p_1} & G_V \times_U \mathcal{S} & \xrightarrow{p_2} & G_V \times_P \mathcal{S} & \xrightarrow{p_3} & E_V \\ (x_T, x_W) & \longleftarrow\!\shortmid & (g,x) & \longmapsto & (g,x) & \longmapsto & gx \end{array}$$

given in (2.4). Let $\tau = \dim(T)$ and $\omega = \dim(W)$. Given $f_1 \in K_\tau$ and $f_2 \in K_\omega$ define a function $f_1 * f_2$ as follows:
If $x \in (E_V)^F$ then

$$(f_1 * f_2)(x) = \sum_{x_T, x_W} C^V_{T,W} f_1(x_T) f_2(x_W),$$

where the sum is over all $x_T \in (E_T)^F$ and $x_W \in (E_W)^F$, and

$$C^V_{T,W} = \frac{\mathrm{Card}(\ \{(y,f) \in (G_V \times_P \mathcal{S})^F \mid p_1(y,f) = (x_T, x_W), p_3(p_2(y,f)) = x\}\ )}{\mathrm{Card}((G_T)^F)\mathrm{Card}((G_W)^F)}.$$

Let $\vec{c}$ be an orbit index and let $\chi_{\vec{c}}$ be the characteristic function of the orbit $\mathcal{O}_{\vec{c}}$, i.e.

$$\text{for } x \in (E_V)^F, \qquad \chi_{\vec{c}}(x) = \begin{cases} 1, & \text{if } x \in (\mathcal{O}_{\vec{c}})^F, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from the observation in (3.2) that the map

$$\begin{array}{ccc} K & \longrightarrow & R\Gamma \\ \chi_{\vec{c}} & \longmapsto & [R_{\vec{c}}] \end{array}$$

is an isomorphism of algebras, where $R\Gamma$ is the Hall algebra defined in (1.3).

(3.4) The isomorphism between $\mathcal{K}$ and K

Let $\vec{a}$ be an orbit index and let $\mathcal{O}_{\vec{a}}$ be the corresponding G_V-orbit in E_V as defined in (3.1). Let $F_{\vec{c}}$ be the constant sheaf $\overline{\mathbb{Q}_l}$ on the orbit $\mathcal{O}_{\vec{c}}$ extended by 0 on the complement. This sheaf can be viewed as the complex of sheaves A, for which $A^0 = F_{\vec{c}}$ and $A^i = 0$, for all $i \neq 0$. In this way $F_{\vec{c}}$ can be viewed as an element of $\mathcal{Q}_V$, see IV (1.4), and the isomorphism class $[F_{\vec{c}}]$ of $F_{\vec{c}}$ is an element of $\mathcal{K}$.

Theorem. *Let $\mathcal{K}$ be the algebra defined in §2 and let K be the algebra defined in (3.3). For each orbit index $\vec{c}$ let $\mathcal{O}_{\vec{c}}$ be the corresponding G_V orbit in E_V, as given in (3.1), and let $\chi_{\vec{c}}$ be the characteristic function of the orbit $\mathcal{O}_{\vec{c}}$. The map*

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & K \\ [F_{\vec{c}}] & \longmapsto & \chi_{\vec{c}} \end{array}$$

is an isomorphism of algebras.

This theorem is a consequence of an analogue of the Grothendieck trace formula. The Grothendieck trace formula, [Ca] p. 504, is the formula

$$|X^F| = \sum_{i=0}^{2\dim(X)} (-1)^i \, \mathrm{Tr}(F, H_c^i(X, \mathbb{Q}_l)),$$

which describes the number of points of X which are fixed under a Frobenius map F in terms of the trace of the action of the Frobenius map on the l-adic cohomology $H_c^i(X, \mathbb{Q}_l)$ of the variety X.

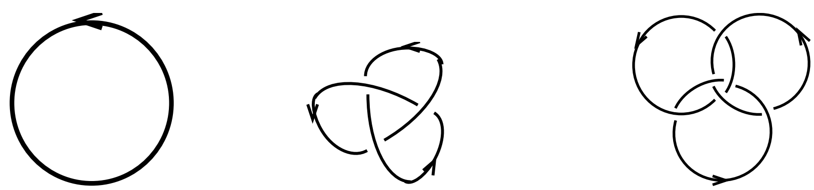
Theorems (3.4) and (2.8) together show that there is a natural connection between the algebra $\mathcal{K}$ and the Hall algebra $R\Gamma$ which was introduced in (1.3).

IX. Link invariants from quantum groups

The theorems of Alexander and Markov given in (1.4) and (1.5) are considered classical, they can be found in [Bi] Theorem 2.1 and Theorem 2.3, respectively. A sketch, with further references, of the proof of Theorem (1.7) can be found in [CP] 15.2. See [J] Prop. 6.2 for the proof of Theorem (1.2) and [Stb] Lemma 2.5 for the proof of Proposition (1.6).

(1.1) Knots, links and isotopy

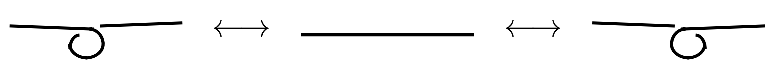
A *knot* is an imbedded circle in $\mathbb{R}^3$. By circle we mean an S^1 and imbedded is in the sense of differential geometry. A *link* is a disjoint union of imbedded circles in $\mathbb{R}^3$. A link is *oriented* if each connected component is oriented. We shall identify a link with its "picture in the plane".



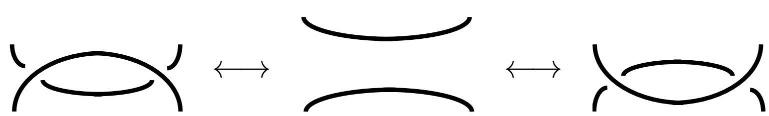
knot (unknot) knot (trefoil) link (Borromean rings)

The conceptual idea of when two links are the same is called ambient isotopy. More precisely, two oriented links L_1 and L_2 are *equivalent under ambient isotopy* if there is an orientation preserving diffeomorphism of $\mathbb{R}^3$ which takes L_1 to L_2. In terms of pictures in the plane L_1 and L_2 are equivalent under ambient isotopy if the picture for L_1 can be transformed into the picture for L_2 by a sequence of *Reidemeister moves*:

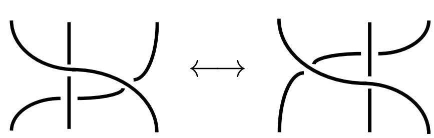
(R1)



(R2)



(R3)



These moves are applied locally to a region in the picture and all possible orientations of the strings are allowed. The equivalence relation on pictures in the plane gotten by only allowing moves (R2) and (R3) is called *regular isotopy.*

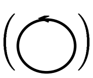
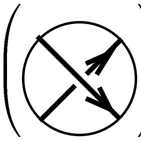
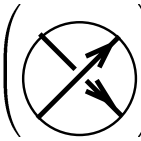
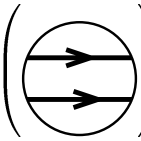
(1.2) Link invariants

Let S be a set. An *oriented link invariant* with values in S is a map

$$P : \mathcal{L} \longrightarrow S$$

from the set $\mathcal{L}$ of equivalence classes of oriented links under ambient isotopy to S.

Theorem. *There exists a unique oriented link invariant* $P : \mathcal{L} \longrightarrow \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ *such that*

P () $= 1$, *and* xP () $- x^{-1}P$ () $= yP$ ().

The unusual notation in the second relation indicates changes to the link in a local region.

The link invariant defined in the above Theorem is the *HOMFLY polynomial.* Other famous link invariants can be obtained in a similar fashion by specializing x and y, as follows:

Jones polynomial	$x = t^{-1}$	and	$y = t^{1/2} - t^{-1/2}$,
Conway polynomial	$x = 1$	and	$y = y$,
Alexander polynomial	$x = 1$	and	$y = t^{1/2} - t^{-1/2}$.

(1.3) Braids

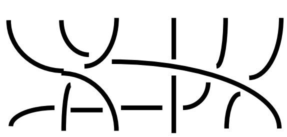
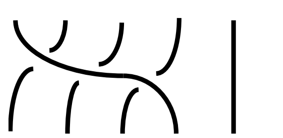
A *braid on m-strands* consists of two rows of m dots each, one above the other, and m strands in $\mathbb{R}^3$ such that

(1) each strand connects a dot in the top row to a dot in the bottom row,

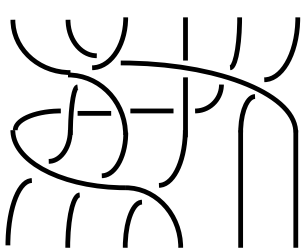
(2) the strands do not intersect,

(3) every dot is incident to exactly one strand.

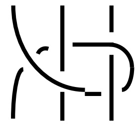
Composition of two braids b_1, b_2 on m-strands is given by identifying the bottom points of b_1 with the top points of b_2. The following are braids on 6 strands,

$b_1 =$  , $b_2 =$  ,

and the product $b_1 b_2$ is the braid

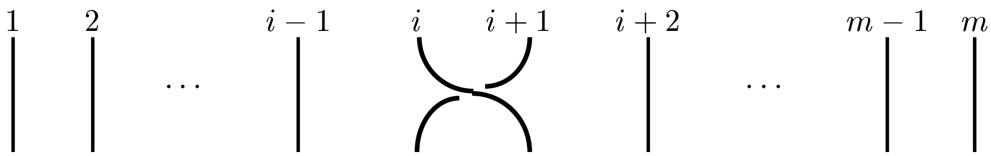
$b_1 b_2 =$  .

One should note that it is important to be careful in defining the word "strand" since the diagram



is not a legal braid.

The *braid group* $\mathcal{B}_m$ is the group of braids on m strands and it is a famous theorem of E. Artin that $\mathcal{B}_m$ has a presentation by generators

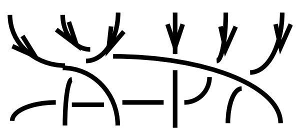
$g_i =$ 

for $1 \le i \le m-1$, and relations

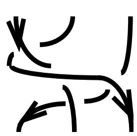
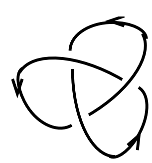
$$\begin{aligned} g_i g_j &= g_j g_i, && \text{if} \quad |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, && \text{for } \ 1 \le i \le m-2. \end{aligned}$$

(1.4) Every link is the closure of a braid

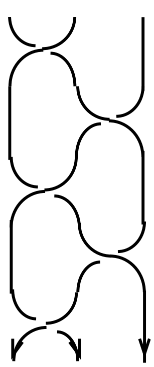
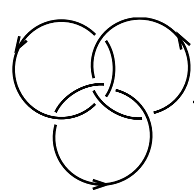
It will be convenient to "orient" the strands of a braid so that they "travel" from top to bottom.



The *closure* $(\hat{\beta}, m)$ of a braid $\beta \in \mathcal{B}_m$ on m-strands is the oriented link obtained by joining together (identifying) each dot in the top row to the corresponding dot in the bottom row. If

$\beta =$ , then $(\hat{\beta}, 3) =$ ,

and if

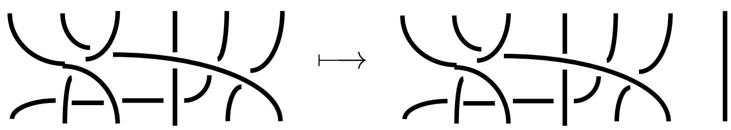
$\beta =$ , then $(\hat{\beta}, 3) =$ .

Theorem. *(Alexander) Every oriented link is the closure* $(\hat{\beta}, m)$ *of a braid* $\beta \in \mathcal{B}_m$ *for some* m.

(1.5) Markov equivalence

The braid group $\mathcal{B}_m$ can be embedded into the braid group $\mathcal{B}_{m+1}$ by adding a strand.

$$\mathcal{B}_6 \longrightarrow \mathcal{B}_7$$



Two braids $\beta_1 \in \mathcal{B}_m$ and $\beta_2 \in \mathcal{B}_n$ are *Markov equivalent* if they are equivalent under the equivalence relation on $\sqcup_m \mathcal{B}_m$ (disjoint union of $\mathcal{B}_m$) which is defined by the relations

(M1) $\beta' \sim \beta\beta'\beta^{-1}$, for all $\beta, \beta' \in \mathcal{B}_k$, and
(M2) $\beta \sim \beta g_k \sim \beta g_k^{-1}$, if $\beta \in \mathcal{B}_k$;

where in the relation (M2) the products βg_k and βg_k^{-1} are obtained by viewing β as an element of $\mathcal{B}_{k+1}$ under the imbedding $\mathcal{B}_k \subseteq \mathcal{B}_{k+1}$.

Theorem. *(Markov) Two braids* $\beta_1 \in \mathcal{B}_m$ *and* $\beta_2 \in \mathcal{B}_n$ *have equivalent closures* $(\hat{\beta}_1, m)$ *and* $(\hat{\beta}_2, n)$ *(under ambient isotopy) if and only if* β_1 *and* β_2 *are Markov equivalent.*

(1.6) Quantum dimensions and quantum traces

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the corresponding Drinfel'd-Jimbo quantum group. Let $\tilde{\rho}$ be the element of $\mathfrak{h}$ such that $\alpha_i(\tilde{\rho}) = 1$ for all simple roots α_i, see II (2.6).

Let V be a finite dimensional $\mathfrak{U}_h\mathfrak{g}$ module. The *quantum dimension* of V is

$$\dim_q(V) = \mathrm{Tr}_V(e^{h\tilde{\rho}}).$$

If $z \in \mathrm{End}_{\mathfrak{U}_h\mathfrak{g}}(V)$ then the *quantum trace* of z is

$$\mathrm{tr}_q(z) = \mathrm{Tr}_V(e^{h\tilde{\rho}}z)\ .$$

Proposition. *Let* $L(\lambda)$ *be the irreducible* $\mathfrak{U}_h\mathfrak{g}$*-module of highest weight* λ *as given in VI (1.3) and VI (2.3). Then*

$$\dim_q(L(\lambda)) = \prod_{\alpha>0} \frac{1-q^{(\lambda+\rho,\alpha)}}{1-q^{(\rho,\alpha)}}, \qquad \textit{where} \quad q = e^h,$$

$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the half sum of the positive roots, and the inner product $(,)$ on $\mathfrak{h}_{\mathbb{R}}^*$ is as given in II (2.7).

(1.7) Quantum traces give us link invariants!

Recall that $\mathfrak{U}_h\mathfrak{g}$ is a quasitriangular Hopf algebra and that therefore the category of finite dimensional $\mathfrak{U}_h\mathfrak{g}$-modules is a braided SRMCwMFF. Let

$$\check{R}_{VV} \;:\; V\otimes V \;\longrightarrow\; V\otimes V$$

be the braiding isomorphism from $V\otimes V$ to $V\otimes V$. It follows from the identity I (3.5) that the map

$$\begin{array}{rccc} \Phi\colon & \mathcal{B}_m & \longrightarrow & \mathrm{End}_{\mathfrak{U}_h\mathfrak{g}}(V^{\otimes m}) \\ & g_i & \longmapsto & \check{R}_i = \mathrm{id}^{\otimes(i-1)}\otimes \check{R}_{VV}\otimes \mathrm{id}^{\otimes m-(i+1)} \end{array}$$

is well defined and that $\Phi(\beta_1\beta_2) = \Phi(\beta_1)\Phi(\beta_2)$ for all braids $\beta_1, \beta_2 \in \mathcal{B}_m$.

Theorem. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let $\mathfrak{U}_h\mathfrak{g}$ be the correponding Drinfel'd-Jimbo quantum group. Let $L(\lambda)$ be an irreducible $\mathfrak{U}_h\mathfrak{g}$-module of highest weight λ (see VI (1.3) and VI (2.3)). Let ρ be the half sum of the positive roots and let $(,)$ be the inner product on $\mathfrak{h}_{\mathbb{R}}$ as given in II (2.7). For each braid β on m-strands define*

$$P(\hat{\beta}, m) \;=\; \left(\frac{1}{q^{<\lambda,\lambda+2\rho>}\,\dim_q(V)}\right)^m \mathrm{tr}_q(\Phi(\beta)),$$

where $q = e^h$. Then P is a well defined link invariant.

Remark. The above theorem gives the Jones polynomial when $\mathfrak{g} = \mathfrak{sl}_2$, the simple Lie algebra corresponding to the Dynkin diagram A_1, and $L(\lambda)$ is chosen to be the irreducible representation of $\mathfrak{U}_h\mathfrak{g}$ with highest weight $\lambda = \omega_1$.

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