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Iwahori-Hecke Algebras of Type A, Bitraces and Symmetric Functions

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1 The bitrace of the regular representation of $\mathcal{H}_n(q)$

Compositions and partitions

We use the notation $\lambda \models n$ to indicate that λ is a composition of n; that is, $\lambda = (\lambda_1, \lambda_2, \dots)$ where the parts, λ_i, are nonnegative for all i and $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ if λ is a partition of n, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. The length $\ell(\lambda)$ is the number of nonzero parts of λ. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ are compositions such that $\lambda_i \leq \mu_i$ for $1 \leq i \leq \ell$, then we write $\lambda \subseteq \mu$ and denote their difference or skew shape by μ/λ. In general, we adopt the notation of [Mac] for partitions and symmetric functions.

The Iwahori-Hecke Algebra

Let S_n denote the symmetric group on $\{1, 2, \dots, n\}$, and let $q \in \mathbb{C}$ such that $q \neq 0$ and q is not a root of unity. The Iwahori-Hecke algebra $\mathcal{H}_n(q)$ corresponding to S_n is the algebra over $\mathbb{C}$ given by generators $1, T_1, T_2, \dots, T_{n-1}$ and relations

$$\begin{aligned} T_iT_j &= T_jT_i, && \text{if } |i-j| > 1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && \text{for } 1 \leq i \leq n-2, \\ T_i^2 &= (q-1)T_i + q, && \text{for } 1 \leq i \leq n-1. \end{aligned} \tag{1.1}$$

Let $s_i = (i, i+1) \in S_n$ denote the simple transposition that exchanges i and $i+1$. Given a reduced word $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in S_n$, let $T_w = T_{i_1}T_{i_2}\cdots T_{i_k} \in \mathcal{H}_n(q)$. The element T_w is well-defined (independent of choice of the reduced word for w). The elements T_w, $w \in S_n$, form a basis of $\mathcal{H}_n(q)$.

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The irreducible representations of $\mathcal{H}_n(q)$ are labeled by the partitions $\lambda \vdash n$, and their traces χ_q^λ are the irreducible characters of $\mathcal{H}_n(q)$. A *character* of $\mathcal{H}_n(q)$ is a linear map $\chi: \mathcal{H}_n(q) \to \mathbb{C}$ which satisfies $\chi(ab) = \chi(ba)$ for all $a, b \in \mathcal{H}_n(q)$. Let $\gamma_r = (1, 2, \ldots, r) \in S_r$ in cycle notation, and for a composition $\mu = (\mu_1, \ldots, \mu_\ell) \models n$ define $\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \in S_{\mu_1} \times \cdots \times S_{\mu_\ell}$. Any character of $\mathcal{H}_n(q)$ is completely determined by its values on the elements T_{γ_μ}, $\mu \vdash n$ (see [Ca] and [Ra1]).

The bitrace

Let $x, y \in S_n$ and define

$$\mathrm{btr}(T_x, T_y) = \sum_{z \in S_n} T_x T_z T_y \big|_{T_z}, \tag{1.2}$$

where $T_x T_z T_y|_{T_z}$ denotes the coefficient of the basis element T_z when $T_x T_z T_y$ is expanded in terms of the basis T_w, $w \in S_n$. If $x \in S_n$, let L_x and R_x denote the linear transformations of $\mathcal{H}_n(q)$ induced by the action of T_x on $\mathcal{H}_n(q)$ by left multliplication and by right multiplication, respectively. If $x, y \in S_n$, then L_x and R_y commute and

$$\mathrm{btr}(T_x, T_y) = \mathrm{Tr}(L_x R_y). \tag{1.3}$$

Left and right multiplication make $\mathcal{H}_n(q)$ into a bimodule and, by double centralizer theory, we have

$$\mathcal{H}_n(q) \cong \bigoplus_{\lambda \vdash n} H_\lambda \otimes H^\lambda,$$

as $\mathcal{H}_n(q)$-bimodules, where H_λ is the irreducible left $\mathcal{H}_n(q)$-module labeled by λ, and H^λ is the irreducible right $\mathcal{H}_n(q)$-module labeled by λ. Taking traces on both sides of this identity gives

$$\mathrm{btr}(T_x, T_y) = \sum_{\lambda \vdash n} \chi_q^\lambda(T_x) \chi_q^\lambda(T_y). \tag{1.4}$$

This formula is an $\mathcal{H}_n(q)$ analogue of the second orthogonality relation for the irreducible characters of the symmetric group S_n.

Keeping in mind that any character of $\mathcal{H}_n(q)$ is completely determined by its values on the elements T_{γ_μ}, $\mu \vdash n$, we define

$$\mathrm{btr}(\mu, \nu) = \mathrm{btr}(T_{\gamma_\mu}, T_{\gamma_\nu}) \tag{1.5}$$

for any two compositions $\mu, \nu \models n$.

An inner product on $\mathcal{H}_n(q)$

Suppose that q is a prime power and let $\mathbb{F}_q$ be the finite field with q elements. Let B be the subgroup of the general linear group $GL_n(\mathbb{F}_q)$ consisting of upper-triangular matrices. Let 1_B^G be the $GL_n(\mathbb{F}_q)$-module which, as a vector space, is the linear span of the cosets in G/B and where the G-action on cosets is by left multiplication. There is a natural action of $\mathcal{H}_n(q)$ on 1_B^G and

$$\mathcal{H}_n(q) \cong \mathrm{End}_G(1_B^G).$$

Let $w \in S_n$. Then the trace of the action of T_w on 1_B^G is given by the formula

$$\mathrm{tr}(T_w) = \begin{cases} [n]!, & \text{if } w \text{ is the identity,} \\ 0, & \text{otherwise,} \end{cases}$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [n][n-1]\cdots[2][1]$. Define a bilinear form on $\mathcal{H}_n(q)$ by

$$\langle a, b \rangle = \frac{1}{[n]!}\,\mathrm{tr}(ab), \quad \text{for } a, b \in \mathcal{H}_n(q).$$

Note that the inner product $\langle a, b\rangle$ is the coefficient of 1 in the product ab. The dual basis to the basis T_w, $w \in S_n$, with respect to the inner product $\langle , \rangle$, is the basis $q^{-\ell(w)}T_{w^{-1}}$, $w \in S_n$.

Very general arguments [CR, 9.17], which work for any semisimple algebra, combined with the computation of the generic degrees in type A ([Ca2, 13.5] or [Hf, 3.4.14]), will show that

$$\sum_{w \in S_n} \chi^\lambda(T_w)\chi^\mu(q^{-\ell(w)}T_{w^{-1}}) = \delta_{\lambda\mu} n! \frac{q^{-n(\lambda)}H_\lambda(q)}{h(\lambda)}, \tag{1.6}$$

where

$$h(\lambda) = \prod_{x \in \lambda} h(x), \quad H_\lambda(q) = \prod_{x\in\lambda} \frac{1-q^{h(x)}}{1-q}, \quad n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i, \quad \text{and}$$

if $x \in \lambda$ is the box in position (i, j) in λ, then $h(x) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length at x. Formula (1.6) is the $\mathcal{H}_n(q)$-analogue of the first orthogonality relation for the irreducible characters of the symmetric group S_n.

For any element $x \in S_n$, define

$$[T_x] = \sum_{w \in S_n} T_w T_x q^{-\ell(w)} T_{w^{-1}}.$$

This is some sort of analogue of a conjugacy class sum in the group algebra of S_n. If $x, y \in S_n$,

$$\begin{aligned}\langle T_x, [T_y]\rangle &= \sum_{w\in S_n} \langle T_x, T_w T_y q^{-\ell(w)} T_{w^{-1}}\rangle = \sum_{w\in S_n} \frac{1}{[n]!} q^{-\ell(w)} \operatorname{tr}(T_x T_w T_y T_{w^{-1}}) \\ &= \sum_{w\in S_n} \langle T_x T_w T_y, q^{-\ell(w)} T_{w^{-1}}\rangle = \sum_{w\in S_n} T_x T_w T_y\big|_{T_w},\end{aligned}$$

and thus

$$\langle T_x, [T_y]\rangle = \langle [T_x], T_y\rangle = \operatorname{btr}(T_x, T_y). \tag{1.7}$$

Specializing q to 1

For each $\mu \vdash n$, the character $\chi_q^\lambda(T_{\gamma_\mu})$ is a polynomial in q with integer coefficients and

$$\chi_q^\lambda(T_{\gamma_\mu})\big|_{q=1} = \chi^\lambda(\mu)$$

where $\chi^\lambda(\mu)$ denotes the irreducible character of the symmetric group S_n corresponding to the partition λ evaluated at a permutation of cycle type μ. It follows from (1.4) and the second orthogonality relation for the characters of the symmetric group that

$$\operatorname{btr}(\mu,\nu)\big|_{q=1} = \sum_{\lambda\vdash n} \chi^\lambda(\mu)\chi^\lambda(\nu) = \delta_{\mu\nu} z_\mu, \quad \text{where } z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

if μ is the partition $\mu = (1^{m_1} 2^{m_2} \cdots)$.

Symmetric functions

Let $x_1, x_2, \ldots, x_n$ be commuting variables. Define $q_0(x_1, x_2, \ldots, x_n; q) = 1$ and for $r > 0$, define $q_r(x_1, x_2, \ldots, x_n; q)$ by the generating function

$$\prod_{i=1}^n \frac{1 - x_i z}{1 - q x_i z} = 1 + (q-1) \sum_{r>0} q_r(x; q) z^r.$$

For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, define $q_\mu(x; q) = q_{\mu_1} q_{\mu_2} \cdots q_{\mu_\ell}$. From [Ra1], [VK], [KW] we know that if $\mu \models n$,

$$q_\mu(x; q) = \sum_{\lambda\vdash n} \chi_q^\lambda(T_{\gamma_\mu}) s_\lambda(x), \tag{1.8}$$

where $s_\lambda(x)$ is the Schur function corresponding to λ (see [Mac]). There is a standard inner product on the ring of symmetric functions given by $\langle s_\mu, s_\nu\rangle = \delta_{\mu\nu}$ for all partitions μ, ν. It follows from (1.8) and (1.4) that

$$\operatorname{btr}(\mu,\nu) = \langle q_\mu(x; q), q_\nu(x; q)\rangle. \tag{1.9}$$

Summary

Proposition 1.10. Let $\mu, \nu \models n$. With notation as above,

$$\mathrm{btr}(\mu,\nu) = \sum_{\lambda \vdash n} \chi_q^\lambda(T_{\gamma_\mu})\chi_q^\lambda(T_{\gamma_\nu}) = \sum_{w\in S_n} T_{\gamma_\mu} T_w T_{\gamma_\nu}\Big|_{T_w} = \langle q_\mu(x;q), q_\nu(x;q)\rangle = \langle T_{\gamma_\mu}, [T_{\gamma_\nu}]\rangle$$

and, for partitions $\mu, \nu \vdash n$,

$$\mathrm{btr}(\mu,\nu)\big|_{q=1} = \delta_{\mu\nu} z_\mu,$$

where $z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ if μ is the partition $\mu = (1^{m_1} 2^{m_2} \cdots)$. □

2 The main theorem and corollaries

The following theorem is the main result of this paper and will be proved in Section 3.

Theorem 2.1. Let $\mu, \nu \models n$, $\mu = (\mu_1, \ldots, \mu_\ell)$, and $\nu = (\nu_1, \ldots, \nu_m)$. Then

$$\mathrm{btr}(\mu,\nu) = (q-1)^{-\ell(\mu)-\ell(\nu)} \sum_M \mathrm{wt}(M),$$

where the sum is over all $\ell \times m$ nonnegative integer matrices with row sums $\mu_1, \ldots, \mu_\ell$, column sums $\nu_1, \ldots, \nu_m$, and

$$\mathrm{wt}(M) = \prod_{x \in \mathcal{P}(M)} (q-1)^2 [x]_{q^2},$$

where $\mathcal{P}(M)$ is the multiset of nonzero entries x in the matrix M and $[x]_{q^2} = 1 + q^2 + q^4 + \cdots + q^{2(x-1)}$. □

The trace of the regular representation of $\mathcal{H}_n(q)$

Our main theorem has the following immediate corollary. This result has been obtained in the paper [RR] by a different method.

Corollary 2.2 [RR]. The trace of the regular representation of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$ is given by

$$\mathrm{Tr}(T_{\gamma_\mu}) = (q-1)^{n-\ell(\mu)} \frac{n!}{\mu_1!\mu_2!\cdots\mu_\ell!}, \quad \text{for all compositions } \mu = (\mu_1, \ldots, \mu_\ell) \models n. \quad \square$$

Proof. It follows from (1.3) that the trace of the regular representation is given by the formula $\mathrm{Tr}(T_{\gamma_\mu}) = \mathrm{btr}(\mu, (1^n))$. Applying Theorem 2.1, we find that the sum is over all nonzero matrices with column sums (1^n), and these are precisely the set of matrices which have exactly one 1 in each column and all the rest 0's. The weight of such a matrix is $(q-1)^{2n}$, and the number of such matrices is $n!/(\mu_1!\mu_2!\cdots\mu_\ell!)$. ■

Inner products of symmetric functions

For a nonnegative integer r, define the symmetric function t_r by the formula

$$\sum_{r\geq 0} t_r(x;q)z^r = \prod_i \frac{(1-qx_iz)^2}{(1-q^2x_iz)(1-x_iz)}; \tag{2.3}$$

and, for a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, define $t_\mu(x;q) = t_{\mu_1} t_{\mu_2} \cdots t_{\mu_\ell}$.

Corollary 2.4. If $\mu, \nu \models n$, then

$$\mathrm{btr}(\mu,\nu) = (q-1)^{-\ell(\mu)-\ell(\nu)}\ t_\mu(x;q)\big|_{m_\nu},$$

where $t_\mu(x;q)|_{m_\nu}$ denotes the coefficient of the monomial symmetric function m_ν in the symmetric function t_μ. □

Proof. We have

$$\frac{1}{1-q^2x} - \frac{1}{1-x} = (q^2-1)x + (q^4-1)x^2 + \cdots.$$

So

$$\left(\frac{1}{q^2-1}\right)\frac{(q^2-1)x}{(1-q^2x)(1-x)} = [1]_{q^2}x + [2]_{q^2}x^2 + \cdots,$$

and thus

$$\frac{(1-qx)^2}{(1-q^2x)(1-x)} = 1 + \left(\frac{(q-1)^2}{q^2-1}\right)\frac{(q^2-1)x}{(1-q^2x)(1-x)} = 1 + \sum_{k\geq 1}(q-1)^2[k]_{q^2}x^k.$$

The result now follows from the interpretation of the bitrace as a weighted sum over nonnegative integer matrices. ■

Corollary 2.5. Let $\mu, \nu \models n$ and let q_μ and t_μ be the symmetric functions defined in (1.8) and (2.3), respectively. Then

$$\langle q_\mu(x;q), q_\nu(x;q)\rangle = (q-1)^{-\ell(\mu)-\ell(\nu)}\langle t_\mu(x;q), h_\nu(x)\rangle,$$

where $h_\nu(x)$ is the homogeneous symmetric function and $\langle , \rangle$ is the inner product on symmetric functions that makes the Schur functions orthonormal. □

Proof. This result follows immediately from Corollary 2.4 by noting that the homogeneous symmetric functions h_μ are the dual basis to the monomial symmetric functions m_μ with respect to the inner product $\langle , \rangle$. ■

Specializations of $\langle q_\mu, q_\nu\rangle$

Define $\tilde{q}_0(x;q,t) = 1$ and, for positive integers r, define symmetric functions $\tilde{q}_r(x;q,t)$ by the formula

$$(q-t)\sum_{r\geq 0}\tilde{q}_r(x;q,t)z^r = \prod_i \frac{(1-tx_iz)}{(1-qx_iz)}. \tag{2.6}$$

For a composition $\mu = (\mu_1,\mu_2,\ldots,\mu_\ell)$, define $\tilde{q}_\mu(x;q,t) = \tilde{q}_{\mu_1}\tilde{q}_{\mu_2}\cdots\tilde{q}_{\mu_\ell}$. These symmetric functions differ from the symmetric functions $q_\mu(x;q)$ only by a change in normalization. On the other hand, they have the advantage that one can specialize either q, or t, or both as follows:

(a) $\tilde{q}_\mu(x;q,0) = q^{|\mu|-\ell(\mu)}h_\mu(x)$, where h_μ is the homogeneous symmetric function;

(b) $\tilde{q}_\mu(x;0,t) = (-t)^{|\mu|-\ell(\mu)}e_\mu(x)$, where e_μ is the elementary symmetric function;

(c) $\tilde{q}_\mu(x;q,q) = q^{|\mu|-\ell(\mu)}p_\mu(x)$, where p_μ is the power symmetric function.

The combinatorics of the symmetric functions $\tilde{q}_\mu(x;q,t)$ is studied in depth in [RRW]. The appropriate modifications to Theorem 2.1 give

$$\langle\tilde{q}_\mu,\tilde{q}_\nu\rangle = (q-t)^{-\ell(\mu)-\ell(\nu)}\sum_M \widetilde{\mathrm{wt}}(M), \quad \text{where } \widetilde{\mathrm{wt}}(M) = \prod_x (q-t)^2 t^{2(x-1)}[x]_{q^2t^{-2}},$$

where the sum is over all nonnegative integer matrices M with row sums μ and column sums ν, the product is over all nonzero entries x in the matrix M, and $t^{2(x-1)}[x]_{q^2t^{-2}} = t^{2(x-1)} + q^2t^{2(x-2)} + \cdots + q^{2(x-2)}t^2 + q^{2(x-1)}$. By specializing q and t, we have new proofs of the following well-known formulas ([Mac, I (6.6) (iv), (6.7)(ii), (4.7)]):

(2.7a) $\langle e_\mu, e_\nu\rangle$ is the number of nonnegative integer matrices with row sums μ and column sums ν,

(2.7b) $\langle h_\mu, h_\nu\rangle$ is the number of nonnegative integer matrices with row sums μ and column sums ν,

(2.7c) $\langle p_\mu, p_\nu\rangle = \delta_{\mu\nu}z_\mu$, where $z_\mu = 1^{m_1}m_1!2^{m_2}m_2!\cdots$ if μ is the partition $\mu = (1^{m_1}2^{m_2}\cdots)$.

The adjoint of multiplication by $\tilde{q}_r$

If f is a symmetric function, define $f^\perp$ to be the adjoint of multiplication by f, with respect to the inner product $\langle,\rangle$, i.e.,

$$\langle fg_1, g_2\rangle = \langle g_1, f^\perp g_2\rangle \quad \text{for all symmetric functions } g_1, g_2.$$

In Section 3 we will prove the following recursion rule for the bitrace.

Proposition 2.8. Let $\mu, \nu \models n$ and $\nu = (\nu_1, \ldots, \nu_\ell)$. Define $\nu' = (\nu_1, \ldots, \nu_{\ell-1})$. Then

$$\mathrm{btr}(\mu,\nu) = \sum_{\alpha} (q-1)^{s(\alpha,\mu)}\, \mathrm{btr}(\mu/\alpha, \nu')\, \mathrm{btr}(\alpha, (\nu_\ell)),$$

where the sum is over all compositions $\alpha \models \nu_\ell$ such that $\alpha \subseteq \mu$ and $s(\alpha,\mu) = \mathrm{Card}\big(\{k \mid 0 < \alpha_k < \mu_k\}\big)$. $\square$

It follows from Theorem 2.1 that if $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a composition of n, then

$$\mathrm{btr}(\alpha, (n)) = (q-1)^{\ell(\alpha)-1} \prod_{\alpha_i \neq 0} [\alpha_i]_{q^2}.$$

Combining this formula with Proposition 2.8 and 1.9 gives the following corollary, where we have done the necessary modifications to use $\tilde{q}_\mu$ instead of q_μ.

Corollary 2.9. Let r be a positive integer and let μ be a composition. Let $\tilde{q}_\mu(x; q, t)$ be the symmetric function defined in (2.6) and, if α is a composition contained in μ, let $s(\alpha,\mu)$ be as given in Proposition 2.8. Then

$$\tilde{q}_r^\perp \tilde{q}_\mu = \sum_{\alpha \models r} f(\alpha,\mu) \tilde{q}_{\mu/\alpha}, \qquad \text{where } f(\alpha,\mu) = (q-t)^{\ell(\alpha)-1+s(\alpha,\mu)} \prod_{\alpha_i \neq 0} t^{2(\alpha_i - 1)} [\alpha_i]_{q^2 t^{-2}}. \quad \square$$

By specializing q and t, we get the following results:

(a) $e_r^\perp e_\mu = \sum_{\alpha \models r} e_{\mu/\alpha}$,
(b) $h_r^\perp h_\mu = \sum_{\alpha \models r} h_{\mu/\alpha}$, and
(c) $p_r^\perp p_\mu = z_\mu z_\nu^{-1} p_\nu$, if r is a part of μ, and ν is the partition obtained by removing one part of size r from μ.

The result in (c) is well-known (see [Mac, I, §5, Ex. 3c]) and the results in (a) and (b) can also be deduced directly from (2.7a) and (2.7b), above.

3 A recurrence relation for the bitrace

The Roichman formula

The starting point for the proof of our main result is a recent formula of Y. Roichman [Ro] which expresses the irreducible character of the Iwahori-Hecke algebra as a weighted sum over standard tableaux. Let $\mu, \lambda \vdash n$ be partitions of n and let Q be a standard tableau of shape λ. Then the μ-*Roichman weight* of Q is

$$\mathrm{rwt}_q^\mu(Q) = \prod_{\substack{i=1\\ i\notin B(\mu)}}^{n} f_\mu(i,Q), \quad \text{where } B(\mu) = \{\mu_1+\mu_2+\cdots+\mu_r \mid 1 \le r \le \ell(\mu)\}, \text{ and}$$

$$f_\mu(i,Q) = \begin{cases} -1, & \text{if } i+1 \text{ is southwest of } i \text{ in } Q,\\ 0, & \text{if } i+1 \text{ is northeast of } i \text{ in } Q,\ i+1 \notin B(\mu),\\ & \text{and } i+2 \text{ is southwest of } i+1 \text{ in } Q,\\ q, & \text{otherwise.} \end{cases}$$

In the definition of the Roichman weight, our notation for partitions and their Ferrers diagrams are as in [Mac]: "northeast" means weakly north and strictly east, and "southwest" means strictly south and weakly west.

Theorem 3.1 [Ro]. If $\lambda \vdash n$ and $\mu \models n$, then

$$\chi_q^\lambda(T_{\gamma_\mu}) = \sum_Q \mathrm{rwt}_q^\mu(Q),$$

where χ_q^λ is the irreducible character of $\mathcal{H}_n(q)$ indexed by the partition λ, and the sum is taken over all standard tableaux Q of shape λ. □

An elementary proof of (the type A case) Roichman's theorem was given in [Ra2]. One of the ideas of [Ra2] was to convert the Roichman weight to a weight on sequences as follows. A sequence $w_1, w_2, \ldots, w_r$ of elements of $\{1, 2, \ldots, n\}$ has weight

$$\mathrm{wt}(w_1, w_2, \ldots, w_r) = \begin{cases} 1, & \text{if } r = 1 \text{ or the sequence is empty;}\\ (-1)^{t-1}q^{r-t}, & \text{if } w_1 < w_2 < \cdots < w_t > w_{t+1} > \cdots > w_r;\\ 0, & \text{otherwise.} \end{cases}$$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a composition of n and $w \in S_n$ is a permutation, define (w, λ) to be the injective λ-tableau obtained by filling in the boxes of λ with $w(1), w(2), \ldots, w(n)$ from left-to-right and top-to-bottom. Define

$$\mathrm{wt}_\lambda(w) = \text{the product of the weights of the rows of } (w,\lambda) \text{ and}$$
$$\mathrm{wt}^\lambda(w) = \mathrm{wt}_\lambda(w^{-1}).$$

For $w \in S_n$, write $w = [w_1, w_2, \ldots, w_n]$ if $w(i) = w_i$ for each $1 \le i \le n$. If $\lambda = (4, 3, 2)$ and $w = [2, 7, 5, 1, 9, 8, 3, 4, 6]$, then $w^{-1} = [4, 1, 7, 8, 3, 9, 2, 6, 5]$,

$$(w,\lambda) = \begin{matrix} 2 & 7 & 5 & 1\\ 9 & 8 & 3 & \\ 4 & 6 & & \end{matrix}, \qquad (w^{-1},\lambda) = \begin{matrix} 4 & 1 & 7 & 8\\ 3 & 9 & 2 & \\ 6 & 5 & & \end{matrix},$$

$\mathrm{wt}_\lambda(w) = (-q^2)(q^2)(-1) = q^4$, and $\mathrm{wt}^\lambda(w) = 0(-q)q = 0$.

The connection between this definition and the Roichman weight of a tableaux Q is via Robinson-Schensted-Knuth (RSK) column insertion. (The original references for the RSK insertion scheme are [Sz], [Sch], and [Kn]; for an expository treatment see [Sa].) Applying RSK insertion on the sequence w produces a pair (P, Q) of standard tableaux of the same shape $\lambda \vdash n$, where P is the result of insertion and Q is the so-called "recording tableau."

(a) RSK column insertion is a bijection between S_n and the set of all pairs of standard tableaux (P, Q) having the same shape $\lambda \vdash n$.

(b) If applying RSK insertion to $w \in S_n$ produces the pair (P, Q), then applying RSK insertion to w^{-1} produces (Q, P) ([Scü], [Sa]).

(c) We have $\mathrm{rwt}_q^\mu(Q) = \mathrm{wt}_\mu(w)$, where Q is the recording tableau produced by column insertion of the sequence $w = [w_1, \dots, w_n]$ (cf. [Ra2]).

Lemma 3.2. If $\mu, \nu \models n$, then

$$\mathrm{btr}(\mu, \nu) = \sum_{w \in S_n} \mathrm{wt}_\mu(w)\, \mathrm{wt}^\nu(w). \qquad \square$$

Proof. By (1.4) and Theorem 3.1, we have

$$\mathrm{btr}(\mu, \nu) = \sum_{\lambda \vdash n} \chi_q^\lambda(T_{\gamma_\mu})\chi_q^\lambda(T_{\gamma_\nu}) = \sum_{\lambda \vdash n} \left(\sum_Q \mathrm{rwt}_q^\mu(Q)\right)\left(\sum_P \mathrm{rwt}_q^\nu(P)\right),$$

where the sums are over all standard tableaux Q (resp., P) of shape λ. If $w \in S_n$, let $(P(w), Q(w))$ be the pair of tableaux obtained by performing RSK insertion of the sequence w. Since RSK insertion is a bijection between S_n and all pairs of tableaux (P, Q) having the same shape λ, as λ runs over all partitions of n, we have

$$\begin{aligned}\mathrm{btr}(\mu, \nu) &= \sum_{\lambda \vdash n} \sum_{P,Q} \mathrm{rwt}_q^\mu(Q)\, \mathrm{rwt}_q^\nu(P) = \sum_{w \in S_n} \mathrm{rwt}_q^\mu(Q(w))\, \mathrm{rwt}_q^\nu(P(w)) \\ &= \sum_{w \in S_n} \mathrm{rwt}_q^\mu(Q(w))\, \mathrm{rwt}_q^\nu(Q(w^{-1})) = \sum_{w \in S_n} \mathrm{wt}_\mu(w)\, \mathrm{wt}^\nu(w). \qquad \blacksquare\end{aligned}$$

Proof of Theorem 2.1

Let $\mathcal{C}_n$ denote the set of compositions of n. For $(w, \mu) \in S_n \times \mathcal{C}_n$, let $(\hat{w}, \lambda) \in S_{n-m} \times \mathcal{C}_{n-m}$ be the injective λ-tableau obtained by deleting $\{n-m+1, \dots, n\}$ from (w, μ) and left justifying the resulting tableau. Let $(w/\hat{w}, \mu/\lambda)$ be the diagram obtained by deleting $\{1, 2, \dots, n-m\}$ from (w, μ). Reading the elements of $((w/\hat{w}), \mu/\lambda)$ from left to right and top to bottom, we can view $w/\hat{w}$ as a permutation in the symmetric group S'_m on $\{n-m+1, n-m+2, \dots, n\}$.

We write $(w, \mu) \to ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda))$. As an example, let $m = 6$, $\mu = (4, 3, 2, 2)$, and $w = [2, 7, 6, 1, 9, 8, 3, 11, 10, 4, 5] \in \mathcal{S}_{11}$. Then the deletion of $\{6, 7, 8, 9, 10, 11\}$ from

$$(w, \mu) = \begin{array}{cccc} 2 & 7 & 6 & 1 \\ 9 & 8 & 3 & \\ 11 & 10 & & \\ 4 & 5 & & \end{array} \qquad \text{is} \qquad ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda)),$$

where

$$(\hat{w}, \lambda) = \begin{array}{cc} 2 & 1 \\ 3 & \\ & \\ 4 & 5 \end{array} \qquad \text{and} \qquad (w/\hat{w}, \mu/\lambda) = \begin{array}{ccc} & 7 & 6 \\ 9 & 8 & \\ 11 & 10 & \\ & & \end{array} .$$

Thus, $\hat{w} = [2, 1, 3, 4, 5] \in \mathcal{S}_5$, $\lambda = (2, 1, 0, 2)$, and $w/\hat{w} = [7, 6, 9, 8, 11, 10] \in \mathcal{S}'_6$.

Lemma 3.3. Assume that $(w, \mu) \to ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda))$ denotes the deletion of $\{n - m + 1, \ldots, n\}$. If $\mathrm{wt}_\mu(w) \neq 0$, then

(a) in each row of (w, μ), the elements from $\{n - m + 1, \ldots, n\}$ appear in a contiguous block;

(b) $\mathrm{wt}_\lambda(\hat{w}) \neq 0$ (thus the rows of $(\hat{w}, \lambda)$ form up-down sequences).

(c) $\mathrm{wt}_{\mu/\lambda}(w/\hat{w}) \neq 0$ (thus the rows of $(w/\hat{w}, \mu/\lambda)$ form up-down sequences).

(d) In each row of (w, μ), the elements from $\{n - m + 1, \ldots, n\}$ appear either immediately to the left or immediately to the right of the largest element from $\{1, 2, \ldots, n - m\}$.

□

Proof.

(a) If $\mathrm{wt}_\mu(w) \neq 0$, then within each row of (w, μ) the elements from $\{n - m + 1, \ldots, n\}$ must appear in a contiguous block; otherwise we go down from elements of $\{n - m + 1, \ldots, n\}$ to elements of $\{1, \ldots, n - m\}$ and back up to elements of $\{n - m + 1, \ldots, n\}$. This down-up configuration would give a zero in the weight of that row.

(b)–(c) If either $(\hat{w}, \lambda)$ or $(w/\hat{w}, \mu/\lambda)$ contains a down-up subsequence in one of its rows, then, since the elements from $\{n - m + 1, \ldots, n\}$ are contiguous in that row of (w, μ), there is necessarily a down-up sequence in that row of (w, μ). Thus, $\mathrm{wt}_\mu(w) = 0$.

(d) Suppose that $(\hat{w}, \lambda)$ and $(w/\hat{w}, \mu/\lambda)$ are given. Consider the places where the elements in the kth row of $(w/\hat{w}, \mu/\lambda)$ can be inserted into the kth row of $(\hat{w}, \lambda)$ to form an injective tableau (w, μ) such that $\mathrm{wt}_\mu(w) \neq 0$.

(i) If $\lambda_k = 0$, then row k of (w, μ) is equal to row k of $(w/\hat{w}, \mu/\lambda)$.

(ii) If $\lambda_k = \mu_k$, then row k of (w, μ) is equal to row k of $(\hat{w}, \lambda)$.

(iii) Assume that $0 < \lambda_k < \mu_k$, let $a_1 < a_2 < \cdots < a_{t-1} < a_t > a_{t+1} > \cdots > a_r$ be the kth row of $(\hat{w}, \lambda)$, and let $b_1 < b_2 < \cdots < b_t > b_{t+1} > \cdots > b_s$

be the kth row of $w/\hat{w}$. Then, keeping in mind that all of the b's are bigger than the peak a_t, we see that the only two possible kth rows of (w, μ) are

$$\begin{aligned}
&\text{(L)}\quad a_1 < a_2 < \cdots < a_{t-1} < \underbrace{b_1 < b_2 < \cdots < b_t > b_{t+1} > \cdots > b_s} \\
&\qquad > a_t > a_{t+1} > \cdots > a_r, \\
&\text{(R)}\quad a_1 < a_2 < \cdots < a_{t-1} < a_t < \underbrace{b_1 < b_2 < \cdots < b_t > b_{t+1} > \cdots > b_s} \\
&\qquad > a_{t+1} > \cdots > a_r.
\end{aligned}$$

■

In the proof of Lemma 2.4 (d), the insertion in the case of (L) is a *left insertion*, and the insertion in the case of (R) is a *right insertion*. Each $(w, \mu) \to ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda))$ with $\mathrm{wt}_\mu(\hat{w}) \neq 0$ gives rise to a unique sequence $I = (I_1, I_2, \ldots, I_{\ell(\mu)})$, where for each nonempty row k of μ we have

$$I_k = \begin{cases} T, & \text{if } \lambda_k = 0 \text{ or } \lambda_k = \mu_k, \\ L, & \text{if in row } k \text{ a left insertion takes } ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda)) \text{ to } (w, \mu), \\ R, & \text{if in row } k \text{ a right insertion takes } ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda)) \text{ to } (w, \mu). \end{cases}$$

In our example, the insertion sequence is $I = (R, L, T, T)$.

Given compositions $\mu \models n$ and $\lambda \models (n-m)$ with $\lambda \subseteq \mu$, we define the following sets:

$$\begin{aligned}
W_n^{\mu\to\lambda} &= \{w \in S_n \mid \mathrm{wt}_\mu(w) \neq 0 \text{ and } (w, \mu) \to ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda)) \text{ for some } \hat{w} \in S_{n-m}\}, \\
W_{n-m}^{\lambda} &= \{x \in S_{n-m} \mid \mathrm{wt}_\lambda(x) \neq 0\}, \\
W_m^{\mu/\lambda} &= \{y \in S'_m \mid \mathrm{wt}_{\mu/\lambda}(y) \neq 0\}, \\
I(\mu, \lambda) &= \left\{ (I_1, I_2, \ldots, I_{\ell(\mu)}) \;\middle|\; \begin{array}{ll} I_k \in \{T\}, & \text{if } \lambda_k = 0 \text{ or } \lambda_k = \mu_k, \text{ and} \\ I_k \in \{L, R\}, & \text{if } 0 < \lambda_k < \mu_k \end{array} \right\}.
\end{aligned}$$

Then we have a bijection

$$\begin{array}{ccc} W_n^{\mu\to\lambda} & \longrightarrow & W_{n-m}^{\lambda} \times W_m^{\mu/\lambda} \times I(\mu, \lambda), \\ w & \longmapsto & (\hat{w}, w/\hat{w}, I). \end{array} \tag{3.4}$$

Lemma 3.5. Let $\mu, \nu \models n$ with $\nu = (\nu_1, \ldots, \nu_\ell)$. Let $\nu' = (\nu_1, \ldots, \nu_{\ell-1})$ and $m = \nu_\ell$. Assume that $\mathrm{wt}_\mu(\hat{w}) \neq 0$ and let

$$(w, \mu) \to ((\hat{w}, \lambda), (w/\hat{w}, \mu/\lambda), I)$$

denote the deletion of $\{n-m+1, \ldots, n\}$ from (w, μ). Then

(a) $\mathrm{wt}_\mu(w) = (-1)^{R(I)} q^{L(I)}\, \mathrm{wt}_\lambda(\hat{w})\, \mathrm{wt}_{\mu/\lambda}(w/\hat{w})$, where $L(I)$ is the number of Ls in the insertion sequence I and $R(I)$ is the number of Rs in I, and

(b) $\mathrm{wt}^\nu(w) = \mathrm{wt}^{\nu'}(\hat{w})\, \mathrm{wt}^{(m)}(w/\hat{w})$. □

Proof. (a) If $\lambda_k = 0$, the weight of row k is the weight of row k of $(w/\hat{w}, \mu/\lambda)$. If $\lambda_k = \mu_k$, the weight of row k is the weight of row k of $(\hat{w}, \lambda)$. If $0 < \lambda_k < \mu_k$, then we are either in the situation of (L) or (R) (as in the proof of Lemma 3.3). In case (L), an extra $>$ is introduced and the weight of row k in w is q times the product of the weights in row k of $\hat{w}$ and $w/\hat{w}$. In case (R), an extra $<$ is added and the weight of row k in w is -1 times the product of the weights in row k of $\hat{w}$ and $w/\hat{w}$. The corollary is now proved by taking the product of the weights of each row.

(b) During the deletion process, when we break w into $\hat{w}$ and $w/\hat{w}$, we maintain the relative positions of the elements $1, 2, \dots, n-m$ and maintain the relative positions of the elements $n-m+1, n-m+2, \dots, n$. The last row of the tableau (w^{-1}, ν) contains the positions of $n-m+1, n-m+2, \dots, n$ in w. Relative to one another, these positions are the same in w as they are in $w/\hat{w}$. Thus the weight of the last row of (w^{-1}, ν) equals $\mathrm{wt}^{(m)}(w/\hat{w})$. The rows before the last row of the tableau (w^{-1}, ν) contain the positions of $1, 2, \dots, n-m$ in w and they are the same relative to one another as in $\hat{w}$. Thus the product of the weights on the rows before the last row equals $\mathrm{wt}^{\nu'}(\hat{w})$. ■

Proposition 3.6. Let $\mu, \nu \models n$, $\nu = (\nu_1, \dots, \nu_\ell)$, and $\nu' = (\nu_1, \dots, \nu_{\ell-1})$. Then

$$\mathrm{btr}(\mu, \nu) = \sum_{\substack{\lambda \models (n-\nu_\ell) \\ \lambda \subseteq \mu}} (q-1)^{s(\lambda,\mu)}\, \mathrm{btr}(\lambda, \nu')\, \mathrm{btr}(\mu/\lambda, (\nu_\ell))$$

where the sum is over all compositions λ of $n - \nu_\ell$ that are contained in μ and

$$s(\lambda, \mu) = \mathrm{Card}\big(\{k \mid 0 < \lambda_k < \mu_k\}\big).$$ □

Proof. Let $m = \nu_\ell$. When we compute the bitrace, we will sum over only the $w \in S_n$ with $\mathrm{wt}_\mu(w) \neq 0$, and we use Lemma 3.2, the bijection 3.4, and Lemma 3.5 as follows:

$$\begin{aligned}
\mathrm{btr}(\mu, \nu) &= \sum_{w \in S_n} \mathrm{wt}_\mu(w)\, \mathrm{wt}^\nu(w) \\
&= \sum_{\substack{\lambda \models (n-m) \\ \lambda \subseteq \mu}} \sum_{w \in W_n^{\mu \to \lambda}} \mathrm{wt}_\mu(w)\, \mathrm{wt}^\nu(w) \\
&= \sum_{\substack{\lambda \models (n-m) \\ \lambda \subseteq \mu}} \sum_{w \in W_n^{\mu \to \lambda}} (-1)^{R(I)} q^{L(I)}\, \mathrm{wt}_\lambda(\hat{w})\, \mathrm{wt}_{\mu/\lambda}(w/\hat{w})\, \mathrm{wt}^{\nu'}(\hat{w})\, \mathrm{wt}^{(m)}(w/\hat{w}) \\
&= \sum_{\substack{\lambda \models (n-m) \\ \lambda \subseteq \mu}} \sum_{x \in W_{n-m}^\lambda} \sum_{y \in W_m^{\mu/\lambda}} \sum_{I \in I(\mu,\lambda)} (-1)^{R(I)} q^{L(I)}\, \mathrm{wt}_\lambda(x)\, \mathrm{wt}_{\mu/\lambda}(y)\, \mathrm{wt}^{\nu'}(x)\, \mathrm{wt}^{(m)}(y) \\
&= \sum_{\substack{\lambda \models (n-m) \\ \lambda \subseteq \mu}} \sum_{x \in W_{n-m}^\lambda} \mathrm{wt}_\lambda(x)\, \mathrm{wt}^{\nu'}(x) \sum_{y \in W_m^{\mu/\lambda}} \mathrm{wt}_{\mu/\lambda}(y)\, \mathrm{wt}^{(m)}(y) \sum_{I \in I(\mu,\lambda)} (-1)^{R(I)} q^{L(I)} \\
&= \sum_{\substack{\lambda \models (n-m) \\ \lambda \subseteq \mu}} \mathrm{btr}(\lambda, \nu')\, \mathrm{btr}(\mu/\lambda, (m)) \sum_{I \in I(\mu,\lambda)} (-1)^{R(I)} q^{L(I)}.
\end{aligned}$$

In each row k where $0 < \lambda_k < \mu_k$, there are two possibilities in making the insertion sequence I. The left insertions give a multiple of q and the right insertions give a multiple of -1. Thus,

$$\sum_{I \in I(\mu,\lambda)} (-1)^{R(I)} q^{L(I)} = (q-1)^{s(\lambda,\mu)}. \qquad \blacksquare$$

Proposition 3.7.

(a) $\mathrm{btr}((n),(n)) = [n]_{q^2}$.

(b) $\mathrm{btr}(\alpha,(n)) = (q-1)^{\ell(\alpha)-1} \prod_{\alpha_i \neq 0} [\alpha_i]_{q^2}$ if α is the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$. □

Proof. (a) The elements $w \in S_n$ such that $\mathrm{wt}_{(n)}(w)\,\mathrm{wt}^{(n)}(w) \neq 0$ are

$$w^{(r)} = [1, 2, \ldots, r, n, n-1, n-2, \ldots, r+2, r+1],$$

where $0 \leq r \leq n-1$ and the case $r = 0$ is to be interpreted as meaning $w(1) = n$. Observe that $(w^{(r)})^{-1} = w^{(r)}$; hence

$$\mathrm{wt}_{(n)}(w^{(r)})\,\mathrm{wt}^{(n)}(w^{(r)}) = \left((-1)^r q^{n-r-1}\right)^2 = q^{2(n-r-1)}$$

for $0 \leq r \leq n-1$. Thus,

$$\mathrm{btr}((n),(n)) = \sum_{w \in S_n} \mathrm{wt}_{(n)}(w)\,\mathrm{wt}^{(n)}(w) = \sum_{r=0}^{n-1} q^{2(n-r-1)} = [n]_{q^2}.$$

(b) First note that $\mathrm{btr}(\alpha,(n)) = \mathrm{btr}((n),\alpha)$. Now use Proposition 3.6 and part (a). ■

Now we complete the proof of Theorem 2.1.

Proof. Let $\mu, \nu \models n$ and suppose that $\nu = (\nu_1, \ldots, \nu_\ell)$. By induction on Proposition 3.6, we have that

$$\mathrm{btr}(\mu,\nu) = \sum_L \prod_{k=1}^{\ell(\nu)} (q-1)^{s(\lambda^{(k)},\lambda^{(k-1)})}\,\mathrm{btr}(\lambda^{(k)}/\lambda^{(k-1)},(\nu_k)),$$

where the sum is over all sequences

$$L = (\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(\ell)} = \mu) \tag{3.8}$$

of compositions such that $|\lambda^{(i)}/\lambda^{(i-1)}| = \nu_i$ for each $1 \leq i \leq \ell$. Note also that $\mathrm{btr}(\lambda^{(k)}/\lambda^{(k-1)},(\nu_k))$ is determined by Proposition 3.7 (b).

We can encode each sequence $L = (\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(\ell)} = \mu)$ appearing in the sum in (3.8) as an $\ell(\mu) \times \ell(\nu)$ matrix of nonnegative integers M by defining its (i,k)-entry to be

$$(M)_{ik} = \lambda_i^{(k)} - \lambda_{i-1}^{(k)}.$$

In other words, the composition $\lambda^{(k)} - \lambda^{(k-1)}$ runs down the kth column of the matrix M. The matrix M has nonnegative integer entries and has row sums given by the vector μ and column sums given by the vector ν; this encoding procedure defines a bijection between the sequences L appearing in (3.8), and the nonnegative integer matrices M with row sums μ and column sums ν.

Let $\mathcal{P}(M)$ denote the multiset of nonzero entries in M. Notice that

$$\begin{aligned}\ell(\lambda^{(k)}/\lambda^{(k-1)}) &= \text{the number of nonzero entries in column } k \text{ of } M,\\ s(\lambda^{(k-1)},\lambda^{(k)}) &= \text{the number of nonzero entries in column } k \text{ of } M\\ &\quad\ \text{which are not preceded in their row by all zeros}\end{aligned}$$

where, in the second case, we assume that the 0th column is a column of all zeros. Thus,

$$\prod_{k=1}^{\ell}(q-1)^{s(\lambda^{(k-1)},\lambda^{(k)})} = \prod_{\substack{\text{columns}\\ \text{of} M}} (q-1)^{s(\lambda^{(k-1)},\lambda^{(k)})} = (q-1)^{-\ell(\mu)} \prod_{x\in\mathcal{P}(M)} (q-1),$$

and

$$\prod_{k=1}^{\ell}(q-1)^{\ell(\lambda^{(k)}/\lambda^{(k-1)})} = \prod_{x\in\mathcal{P}(M)} (q-1).$$

It follows that

$$\begin{aligned}&\prod_{k=1}^{\ell}(q-1)^{s(\lambda^{(k)},\lambda^{(k-1)})}\ \mathrm{btr}(\lambda^{(k)}/\lambda^{(k-1)},(\nu_k))\\ &= (q-1)^{-\ell(\mu)}\left(\prod_{x\in\mathcal{P}(M)}(q-1)\right)\left(\prod_{k=1}^{\ell}(q-1)^{\ell(\alpha^{(k)})-1}\prod_{\alpha_i^{(k)}\neq 0}[\alpha_i^{(k)}]_{q^2}\right)\\ &= (q-1)^{-\ell(\mu)-\ell(\nu)}\prod_{x\in\mathcal{P}(M)}(q-1)^2[x]_{q^2}\end{aligned}$$

where, for simplicity of notation, we have let $\alpha^{(k)} = \lambda^{(k)}/\lambda^{(k-1)}$. ■

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