SEMINORMAL REPRESENTATIONS OF WEYL GROUPS AND IWAHORI–HECKE ALGEBRAS

ARUN RAM

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0. *Introduction*

The purpose of this paper is to describe a general procedure for computing analogues of Young's seminormal representations of the symmetric groups. The method is to generalize the Jucys–Murphy elements in the group algebras of the symmetric groups to arbitrary Weyl groups and Iwahori–Hecke algebras. The combinatorics of these elements allow one to compute irreducible representations explicitly and often very easily. In this paper we do these computations for Weyl groups and Iwahori–Hecke algebras of types A_n, B_n, D_n, G_2. Although these computations are within reach for types F_4, E_6, and E_7, we shall, in view of the length of the current paper, postpone this to another work.

In reading this paper, I would suggest that the reader begin with § 3, the symmetric group case, and go back and pick up the generalities from §§ 1 and 2 as they are needed. This will make the motivation for the material in the earlier sections much more clear and the further examples in the later sections very easy.

The realization that the Jucys–Murphy elements for Weyl groups and Iwahori–Hecke algebras come from the very natural central elements in (2.1) and Proposition 2.4 is one of the main points of this paper. There is a simple concrete connection (Proposition 2.8) between Jucys–Murphy type elements in Iwahori–Hecke algebras and Jucys–Murphy elements in group algebras of Weyl groups. I know that the analogues of the Jucys–Murphy elements in Weyl groups of types B and D will be new to some of the experts and known to others. These Jucys–Murphy elements for types B and D are not new; similar elements appear in the paper of Cherednik [**7**], but I was not able to recognize them there until they were pointed out to me by M. Nazarov. I extend my thanks to him for this. Some people were asking me for Jucys–Murphy elements in type G_2 as late as June 1995. In July 1995 I was told that it was not known how to quantize the elements of Cherednik, that is, to find analogues of them in the Iwahori–Hecke algebras of types B and D. Of course, this had been done already in 1974, by Hoefsmit.

I have chosen to state my results in terms of the general mechanism of path algebras which I have defined in § 1. This is a technique which I learned from H. Wenzl during our work on the paper [**30**]. It is a well-known method in several fields (with many different terminologies). I shall mention here only a few of the

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many possible references to these ideas: the book by Goodman, de la Harpe and Jones [**16**], the book by Chen [**6**], the paper of Sunder [**33**], and the paper of Gelfand and Tseitlin [**15**].

The theory and the method which I have applied in this paper also work for studying the representations of centralizer algebras such as the Brauer algebra and the Birman–Wenzl–Murakami algebra. In the case of the Brauer centralizer algebra, the analogues of the Jucys–Murphy elements are due to M. L. Nazarov [**28**], and in the case of the Birman–Wenzl–Murakami algebra to N. Reshetikhin [**31**] and R. Leduc and the author [**22**]. In the theory of centralizer algebras the Jucys–Murphy elements 'come from' the Casimir element of the centralizing Lie algebra or quantum group.

History. Alfred Young [**35**] wrote down several ways of describing the irreducible representations of the symmetric group, one of which gives explicit matrices for the images of the simple transpositions. These are the seminormal representations of the symmetric group. In 1974 Hoefsmit [**17**] completed his Ph.D. thesis in which he wrote down analogues of Young's irreducible seminormal representations for the Iwahori–Hecke algebras of types A, B, and D. Hoefsmit's thesis was never published and these representations were independently rediscovered by H. Wenzl [**34**] (in the type A case).

Jucys and Murphy inserted a new and beautiful feature into Young's theory by writing down elements, which, in Young's irreducible seminormal representations of the symmetric group, are always diagonal matrices. Even better, the diagonal entries of these matrices have an easy combinatorial description. They showed that Young's seminormal representations could be reconstructed from the knowledge of these special elements.

The original work of Jucys ([**18–20**] 1966, 1971, 1974) was published mostly in Lithuanian physics journals and was not read by many in the western mathematical community. Only when the work of Murphy ([**24**, **25**] 1981, 1983) appeared did these elements begin to receive wider attention. Cherednik [**7**] gave analogues of the Jucys–Murphy elements for the Weyl groups of types B and D, but his paper was read by almost no-one in the seminormal representations camp since his paper was written from the point of view of constructing monodromy representations. Hoefsmit ([**17**] 1974), unknowingly, had analogues of these elements in the Iwahori–Hecke algebras of type B but since his thesis was never published, these elements remained largely unknown. Hoefsmit's construction in type B easily generalizes to Iwahori–Hecke algebras of types A and D. In the period 1985–1995, Dipper, James, Murphy, and Pallikaros ([**10–12**] 1986, 1987, 1992, [**26**, **27**] 1992, 1995, [**13**] 1995, [**29**] 1995) have done a lot of work on representations of Iwahori–Hecke algebras and have produced analogues of the Jucys–Murphy elements for Iwahori–Hecke algebras of types A and B. Their version of these elements in type A was written in such a way that it was not clear to anybody that they were the same as the elements that were in Hoefsmit's thesis!

More recently, there have been new 'Hecke algebras' which have been discovered by Ariki [**1**], Ariki and Koike [**2**], and Broué and Malle [**5**], which are similar to the Iwahori–Hecke algebras of types B and D. Ariki and Koike [**2**] and Ariki [**1**] have shown that Hoefsmit's constructions can be extended to give seminormal representations of these algebras as well. Ariki, Koike, and Broué and Malle have given essentially the same analogues of the Jucys–Murphy elements as Hoefsmit had.

Acknowledgements. I thank especially Curtis Greene for asking me questions and for his encouragement of this work and for his company on a wonderful trip through northern Italy when he explained to me the story of Jucys–Murphy elements for the symmetric group. I thank Rob Leduc for pushing and wanting to find explicit representations of the Brauer and Birman–Wenzl algebras in early 1994. It was while we were doing this work [**22**] that we first saw analogues of Jucys–Murphy elements coming out of quantum groups. I thank Persi Diaconis, G. Benkart, L. Solomon, and A. N. Kirillov for their interest and for many conversations during which many parts of this work and this subject solidified for me. I am indebted to P. Orlik and H. Terao for introducing me to the work of Brieskorn and Saito and Deligne on braid groups.

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1. *What is a seminormal representation?*

For convenience and simplicity we shall work over the field $\mathbb{C}$ of complex numbers. Let $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$ be a chain of finite groups. Let V^λ be an irreducible module for G. Upon restriction to the group G_{n-1} the module V^λ decomposes as a direct sum

$$V^\lambda \cong V^{\mu_1} \oplus \ldots \oplus V^{\mu_k}$$

of irreducible modules for G_{n-1}. Similarly each of these summands decomposes into irreducible submodules on restriction to G_{n-2}, and so on.

A *seminormal basis* of V^λ is a basis $B^\lambda = \{v_L\}$ of V^λ that explicitly realizes these decompositions, that is, there is a partition of B^λ into subsets $B^{\mu_1}, \ldots, B^{\mu_k}$ such that if $V^{\mu_i} = \mathbb{C}\text{-span}(B^{\mu_i})$ then

$$V^\lambda = V^{\mu_1} \oplus \ldots \oplus V^{\mu_k}$$

as G_{n-1}-modules (note that here there is an $=$ sign rather than only $\cong$). Further, we require that each of the subsets B^{μ_i} is partitioned into subsets which realize the decomposition upon restricting to G_{n-2}, and so on, all the way down the chain. Thus, to specify a seminormal basis one must give, not only the basis of V^λ but also the series of partitions. The resulting representation

$$\rho^\lambda\colon\ G \to M_{d_\lambda}(\mathbb{C}), \quad d_\lambda = \dim(V^\lambda),$$

of G, which is specified by V^λ and the basis B^λ, is a *seminormal representation of G with respect to the chain $G_0 \subseteq \ldots \subseteq G_n = G$.*

The concepts of seminormal bases and seminormal representations apply equally well to any chain of split semisimple algebras $\mathbb{C} \cong H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = H$.

The graph Γ. Let $\hat{G}_i$ be an index set for the irreducible representations of G_i.

Define non-negative integers c_μ^λ, where $\mu \in \hat{G}_{i-1}$, $\lambda \in \hat{G}_i$, by the restriction rule from G_i to G_{i-1},

$$V^\lambda \downarrow^{G_i}_{G_{i-1}} \cong \bigoplus_{\mu \in \hat{G}_{i-1}} c_\mu^\lambda V^\mu.$$

In other words, upon restriction from G_i to G_{i-1} the irreducible module V^μ, with $\mu \in \hat{G}_{i-1}$, appears in the irreducible G_i-module V^λ, with $\lambda \in \hat{G}_i$, with multiplicity c_μ^λ.

Define a graph Γ with

(1.1) vertices labelled by the elements of the sets $\hat{G}_i$, and such that $\mu \in \hat{G}_{i-1}$ and $\lambda \in \hat{G}_i$ are connected by c_μ^λ edges.

The graph Γ encodes the restriction rules for the chain $\{1\} = G_0 \subseteq \ldots \subseteq G_n = G$. We shall assume that the unique element in $\hat{G}_0$ is denoted $\emptyset$.

Let $\mu \in \hat{G}_r$ and $\lambda \in \hat{G}_s$ where $r < s$. A *path* from μ to λ is a sequence of $s - r$ edges connecting μ to λ,

$$L = \left(\mu = \lambda^{(r)} \xrightarrow{e_r} \lambda^{(r+1)} \xrightarrow{e_{r+1}} \ldots \xrightarrow{e_{s-1}} \lambda^{(s)} = \lambda \right),$$

such that $\lambda^{(i)} \in \hat{G}_i$ for $r \leqslant i \leqslant s$. We distinguish paths which 'travel' from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ along different edges. We use the following notation:

$\mathscr{L}(\lambda \to \mu)$ is the set of paths from λ to μ;

$\mathscr{L}(\lambda)$ is the set of paths from $\emptyset$ to λ;

$\mathscr{L}(\lambda \to s)$ is the set of paths from λ to any element $\mu \in \hat{G}_s$;

$\mathscr{L}(m)$ is the set of paths from $\emptyset$ to any element $\lambda \in \hat{G}_m$;

$\mathscr{L} = \mathscr{L}(n)$, where n is the total number of groups in the chain $G_0 \subseteq \ldots \subseteq G_n$;

$\Omega(\lambda)$ is the set of pairs (S, T) of paths such that $S, T \in \mathscr{L}(\lambda)$;

$\Omega(m)$ is the set of pairs (S, T) of paths such that $S, T \in \mathscr{L}(\lambda)$ for some $\lambda \in \hat{G}_m$.

In general, by 'a path in Γ' we shall mean an element $L = (\lambda^{(0)} \to \ldots \to \lambda^{(n)}) \in \mathscr{L}$.

Path algebras. For each $0 \leqslant m \leqslant n$ define a *path algebra* P_m over $\mathbb{C}$ (see [**16**]) with basis E_{ST}, where $(S, T) \in \Omega(m)$, and multiplication given by

$$E_{ST} E_{PQ} = \delta_{TP} E_{SQ}. \tag{1.2}$$

To avoid confusion with another type of path algebra used in other parts of representation theory, note that the multiplication in the path algebra does not involve composition of paths. We have $P_0 \simeq \mathbb{C}$. Each of the algebras P_m is isomorphic to a direct sum of matrix algebras

$$P_m \simeq \bigoplus_{\lambda \in \hat{G}_m} M_{d_\lambda}(\mathbb{C}),$$

where $M_d(\mathbb{C})$ denotes the algebra of $d \times d$ matrices with entries from $\mathbb{C}$ and $d_\lambda = \mathrm{Card}(\mathscr{L}(\lambda))$. For each $\lambda \in \hat{G}_m$ define a P_m-module by defining

$$V^\lambda = \mathbb{C}\text{-span}\{v_L \mid L \in \mathscr{L}(\lambda)\} \quad \text{and} \quad E_{ST} v_L = \delta_{TL} v_S, \tag{1.3}$$

for all paths $S, T, L \in \mathscr{L}(\lambda)$. The P_m-modules V^λ, with $\lambda \in \hat{G}_m$, realize all of the irreducible P_m-modules.

Given a path $T=(\lambda\to\dots\to\mu)$ from λ to μ and a path $S=(\mu\to\dots\to\nu)$ from μ to ν define

$$T*S=(\lambda\to\dots\to\mu\to\dots\to\nu)$$

to be the concatenation of the two paths. Let $r<s$ and for each $\lambda\in\hat{G}_r$ and each pair $(P,Q)\in\Omega(\lambda)$ view the element $E_{PQ}\in P_r$ as an element of P_s by the formula

$$E_{PQ}=\sum_{T\in\mathscr{L}(\lambda\to s)}E_{P*T,Q*T}. \tag{1.4}$$

This defines, in particular, an inclusion of P_{m-1} into P_m for every $m>0$. Let $\lambda\in\hat{G}_m$ and let V^λ be the irreducible representation of P_m corresponding to λ as given in (1.3). Then the restriction of V^λ to P_{m-1} decomposes as

$$V^\lambda\big\downarrow^{P_m}_{P_{m-1}}\simeq\bigoplus V^\mu,$$

where the sum is over all edges $\mu\xrightarrow{e}\lambda$ that connect an element $\mu\in\hat{G}_{m-1}$ to the $\lambda\in\hat{G}_m$. The basis vectors v_L form a seminormal basis of the P_m-module V^λ.

Constructing seminormal representations of $\mathbb{C}G$. As above, let

$$\{1\}=G_0\subseteq\dots\subseteq G_n=G \tag{1.5}$$

be a chain of finite groups, let Γ be the graph which describes the restriction rules for the inclusions in (1.5) and let P_m, for $0\leqslant m\leqslant n$, denote the corresponding path algebras. By construction, the path algebras P_n have natural seminormal representations (1.3) with respect to the inclusions $P_0\subseteq P_1\subseteq\dots\subseteq P_n$. Thus, we should try to find an isomorphism

$$\begin{aligned}\Phi\colon\ P_n&\simeq\mathbb{C}G\\ E_{ML}&\mapsto e_{ML}\end{aligned}\qquad\text{such that } \Phi(P_i)=\mathbb{C}G_i, \tag{1.6}$$

for all $0\leqslant i\leqslant n$. Given such an isomorphism, irreducible seminormal representations are given by the modules V^λ in (1.3) where the action of an element $g\in\mathbb{C}G$ is given by

$$gv_L=\Phi^{-1}(g)v_L, \tag{1.7}$$

for all $g\in G$.

Suppose that, for each $1\leqslant k\leqslant n$,

$$Z_k=\{z_{k,j}\}_{1\leqslant j\leqslant r_k} \tag{1.8}$$

is a set of central elements in the group algebra $\mathbb{C}G_k$.

LEMMA 1.9. *Let $z_{k,j}$ be a central element in $\mathbb{C}G_k$. Let*

$$L=(\lambda^{(0)}\to\dots\to\lambda^{(n)})\in\mathscr{L}(n)$$

be a path in the graph Γ and let $\chi^{\lambda^{(k)}}$ be the irreducible character of G_k indexed by the element $\lambda^{(k)}\in\hat{G}_k$. For any choice of isomorphism Φ between the path algebra P_n and $\mathbb{C}G$ as in (1.6),

$$z_{k,j}v_L=c_{k,j}(\lambda^{(k)})v_L,\quad\textit{where } c_{k,j}(\lambda^{(k)})=\frac{\chi^{\lambda^{(k)}}(z_{k,j})}{\chi^{\lambda^{(k)}}(1)}.$$

Proof. By Schur's lemma any central element $z_{k,j} \in \mathbb{C}G_k$ must act by a scalar multiple of the identity in every irreducible representation of G_k. Specifically, $z_{k,j}$ acts by the scalar $c_{k,j}(\mu)$ in the irreducible G_k-module indexed by μ. Each of the basis vectors v_L, $L = (\lambda^{(0)} \to \dots \to \lambda^{(n)})$ is in an irreducible P_k-module which is isomorphic to the irreducible P_k-module $V^{\lambda^{(k)}}$ indexed by $\lambda^{(k)} \in \hat{G}_k$. It follows that, for any choice of the isomorphism Φ: $P_n \to \mathbb{C}G$ in (1.6), we must have

$$z_{k,j} v_L = \Phi^{-1}(z_{k,j}) v_L = c_{k,j}(\lambda^{(k)}) v_L.$$

REMARK 1.10. The set $Z = \bigcup_{k=0}^{n} Z_k$ is a set of elements of the group algebra $\mathbb{C}G$ that all commute with each other. They generate a commutative subalgebra T of $\mathbb{C}G$. The subalgebra T acts diagonally on the basis v_L, that is, for each $t \in T$ and each $L \in \mathscr{L}$,

$$t v_L = c(t, L) v_L$$

for some constant $c(t, L) \in \mathbb{C}$.

For each $\mu \in \hat{G}_k$ let $c_k(\mu)$ be the ordered r_k-tuple $c_k(\mu) = (c_{k,j}(\mu))_{1 \leqslant j \leqslant r_k}$. Define the *weight* of a path $L = (\lambda^{(0)}, \dots, \lambda^{(n)})$ in Γ to be the n-tuple

$$\text{wt}(L) = (c_0(\lambda^{(0)}), \dots, c_n(\lambda^{(n)})). \tag{1.11}$$

The following proposition shows that in many cases the isomorphism Φ in (1.6) can be determined more or less explicitly.

PROPOSITION 1.12. *The choice of an isomorphism* Φ *as in* (1.6) *is determined by the choice of elements*

$$\Phi(E_{ML}) = e_{ML} \in \mathbb{C}G$$

for each pair of paths L, M in Γ. *Assume that each path in* Γ *is distinguished by its weight, that is, if L and M are paths in* Γ *and* $L \neq M$, *then* $\text{wt}(L) \neq \text{wt}(M)$.

(a) *For each path L in* Γ *the element e_{LL} is determined uniquely by the elements $z_{k,j} \in Z_k$ and the constants $c_{k,j}(\mu)$, where $\mu \in \hat{G}_k$ for $0 \leqslant k \leqslant n$.*

(b) *If M and L are paths in* Γ *such that $M \neq L$ then e_{ML} is determined up to a constant by the elements $z_{k,j} \in Z_K$ and the constants $c_{k,j}(\mu)$, where $\mu \in \hat{G}_k$ and $0 \leqslant k \leqslant n$.*

Proof. Let $L = (\lambda^{(0)}, \dots, \lambda^{(n)})$ be a path in Γ. For each $0 \leqslant k \leqslant n$ and each $1 \leqslant j \leqslant r_k$, let

$$p_{k,j}(\lambda^{(k)}) = \prod_{c_{k,j}(\mu) \neq c_{k,j}(\lambda^{(k)})} \frac{z_{k,j} - c_{k,j}(\mu)}{c_{k,j}(\lambda^{(k)}) - c_{k,j}(\mu)},$$

where the product is over all $c_{k,j}(\mu)$, with $\mu \in \hat{G}_k$, such that $c_{k,j}(\mu) \neq c_{k,j}(\lambda^{(k)})$. There may be elements $\mu \in \hat{G}_k$ such that $\mu \neq \lambda^{(k)}$ but such that $c_{k,j}(\mu) = c_{k,j}(\lambda^{(k)})$. These $\mu \in \hat{G}_k$ are not included in the product. It follows from Lemma 1.9 that if $M = (\mu^{(0)} \to \dots \to \mu^{(n)})$ is a path in Γ then, for any isomorphism Φ as in (1.6),

$$\Phi^{-1}(p_{k,j}(\lambda^{(k)})) v_M = \begin{cases} v_M & \text{if } c_{k,j}(\mu^{(k)}) = c_{k,j}(\lambda^{(k)}), \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$e_{LL} = \prod_{k,j} p_{k,j}(\lambda^{(k)}).$$

If $M = (\mu^{(0)} \to \dots \to \mu^{(n)})$ is a path in Γ then

$$\Phi^{-1}(e_{LL})v_M = \delta_{LM}v_M = E_{LL}v_M,$$

since, if $L \neq M$ then $\mathrm{wt}(L) \neq \mathrm{wt}(M)$. The result follows since Φ is injective.

(b) Assume that M and L are paths in Γ such that $M \neq L$. Let $a \in \mathbb{C}G$ such that $e_{MM}ae_{LL} \neq 0$. Then e_{ML} must be a constant times the element $e_{MM}ae_{LL} \in \mathbb{C}G$. Since the elements e_{LL} and e_{MM} are completely determined by the elements $z_{k,j}$ and the constants $c_{k,j}(\mu)$, it follows that the elements e_{ML} are determined (up to a constant) by them.

REMARK 1.13. Suppose that

$$\begin{aligned} \Phi\colon P_n &\to \mathbb{C}G \\ E_{ML} &\mapsto e_{ML} \end{aligned} \quad \text{and} \quad \begin{aligned} \Phi'\colon P_n &\to \mathbb{C}G \\ E_{ML} &\mapsto e'_{ML} \end{aligned}$$

are two isomorphisms between the path algebra P_n and $\mathbb{C}G$. Let $\kappa_{ML} \in \mathbb{C}$ be such that $e_{ML} = \kappa_{ML}e'_{ML}$. The constants κ_{ML} must satisfy the relations

$$\kappa_{ML}\kappa_{LM} = 1 \quad \text{and} \quad \kappa_{ML}\kappa_{LN} = \kappa_{MN},$$

for all choices of paths $M, L, N \in \mathscr{L}$. These relations follow from the relations $e_{ML}e_{PS} = \delta_{LP}e_{MS}$ in (1.2).

EXAMPLE 1.14. Suppose that $\{1\} = G_0 \subseteq \dots \subseteq G_n = G$ is a chain of finite groups such that, for each $1 \leqslant i \leqslant n$, the restriction rules describing the decomposition of irreducible G_i-representations into irreducible G_{i-1}-representations are multiplicity free. For each $1 \leqslant i \leqslant n$, let Z_i be the set of sums of elements in each conjugacy class of G_i. Clearly Z_i is a set of central elements in $\mathbb{C}G_i$. This is an example of a situation in which the paths in the graph Γ are distinguished by their weights.

2. *Weyl groups and Iwahori–Hecke algebras*

The branching rules for the chains of Weyl groups

$$\begin{aligned} &S_1 \subseteq S_2 \subseteq \dots \subseteq S_n, \\ &WB_1 \subseteq WB_3 \subseteq \dots \subseteq WB_n, \\ &WB_2 \subseteq WB_3 \subseteq WF_4, \\ &WD_5 \subseteq WE_6 \subseteq WE_7, \end{aligned}$$

are all multiplicity free. Thus, the Weyl groups S_n, WB_n, WF_4, WE_6, and WE_7 all fall into the situation of Example 1.14 and one can use the sets Z_k consisting of all conjugacy class sums and Proposition 1.12 to compute all the irreducible representations of these Weyl groups and their corresponding Iwahori–Hecke algebras. (In the $WE_7 \subseteq WE_8$ case the branching rule has multiplicities at most 2 and could be treated in a similar fashion to Example 1.14 except that one would also have to use some additional elements from the centralizer of the WE_7 action on irreducible WE_8 representations.) We shall show in the remainder of this paper that one can use much smaller sets for the Z_k and obtain the same results in a quicker way. We shall obtain 'seminormal' representations of the Weyl groups and Iwahori–Hecke algebras of type D_n by using the representation theory for type B_n; see § 5.

Weyl groups. Let $\varnothing = R_0 \subseteq \ldots \subseteq R_n = R$ be a chain of root systems and let

$$\{1\} = W_0 \subseteq \ldots \subseteq W_n$$

be the corresponding chain of Weyl groups. Let R_k^+ denote the set of positive roots in the root system R_k and, for each $\alpha \in R_k^+$ let s_α denote the element of W_k which is the reflection in the hyperplane perpendicular to α.

(2.1a) If all roots in R_k are the same length then the set of elements $\{s_\alpha \mid \alpha \in R_k^+\}$ is a conjugacy class in W_k. It follows that

$$z_{k,\ell} = \sum_{\alpha \in R_k^+} s_\alpha$$

is a central element of $\mathbb{C}W_k$. (The index ℓ here simply denotes that this is a sum over the 'long' roots in R_k^+.)

(2.1b) If the roots in R_k are not all the same length then there are two lengths of roots. Let $R_{k,s}^+$ be the set of short positive roots and let $R_{k,\ell}^+$ be the set of long positive roots in R_k. The sets $\{s_\alpha \mid \alpha \in R_{k,s}^+\}$ and $\{s_\alpha \mid \alpha \in R_{k,\ell}^+\}$ are conjugacy classes in W and the elements

$$z_{k,s} = \sum_{\alpha \in R_{k,s}^+} s_\alpha \quad \text{and} \quad z_{k,\ell} = \sum_{\alpha \in R_{k,\ell}^+} s_\alpha,$$

are central elements in $\mathbb{C}W_k$.

(2.1c) If the longest element $w_{k,0}$ in the Weyl group W_k acts as -1 in the reflection representation of W_k then the element

$$z_{k,0} = w_{k,0}$$

is central in W_k.

For each $0 \leqslant k \leqslant n$, let Z_k denote the set of central elements in $\mathbb{C}W_k$ which are determined by (2.1). Depending on which cases apply, the set Z_k contains 1, 2, or 3 elements. In view of Lemma 1.9, we define, for each irreducible character χ of the Weyl group W_k and each $z_{k,j} \in Z_k$, a constant

$$c_{k,j}(\chi) = \frac{\chi(z_{k,j})}{\chi(1)}. \tag{2.2}$$

We shall use the central elements in the sets Z_k (with some slight modification in the D_n case) to compute seminormal representations for the Weyl groups of types A_{n-1}, B_n, D_n, G_2.

REMARK 2.3. In my view, the central elements in (2.1) are the appropriate generalization of Jucys–Murphy elements to arbitrary Weyl groups (or Coxeter groups). Remarks 3.6 and 4.6 illustrate this idea in special cases.

Iwahori–Hecke algebras. Let Δ be a Dynkin diagram of a finite Weyl group, let R be the corresponding root system, and let W be the corresponding Weyl group. Let $\{\alpha_i \mid i \in \Delta\}$ be the set of simple roots in R, indexed by the nodes in the Dynkin diagram Δ. The Iwahori–Hecke algebra $H(p^2, q^2)$ corresponding to the Weyl group W is the algebra over $\mathbb{C}(p, q)$ generated by elements T_i, with $i \in \Delta$, and relations

(a) $T_iT_jT_iT_j \ldots = T_jT_iT_jT_i \ldots$, where each side contains m_{ij} factors and m_{ij} is the order of the element $s_{\alpha_i}s_{\alpha_j}$ in the Weyl group W,

(b) $T_i^2 = \begin{cases} (p-p^{-1})T_i + 1 & \text{if } \alpha_i \text{ is a short root,} \\ (q-q^{-1})T_i + 1 & \text{if } \alpha_i \text{ is a long root.} \end{cases}$

If all roots in R are the same length then we make the convention, for the purposes of the definition of the Iwahori–Hecke algebra, that all roots in R are long and we simply define the Iwahori–Hecke algebra as an algebra $H(q^2)$ over $\mathbb{C}(q)$.

It is a standard fact ([**3**, Chapter IV, § 2, Ex. 23–24] or [**8**]) that the Iwahori–Hecke algebra $H(q^2)$ corresponding to a Weyl group W is split-semisimple and its irreducible representations can be indexed by the same set $\hat{W}$ that indexes the irreducible representations of W. Following the standard notation, we see that if $w \in W$ then $T_w = T_{i_1} \ldots T_{i_p}$ where $w = s_{i_1} \ldots s_{i_p}$ is a reduced expression for $w \in W$. The element $T_w \in H$ is well defined and does not depend on the choice of the reduced expression for w.

Let $\emptyset = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_n = \Delta$ be a chain of Dynkin diagrams of finite Weyl groups. These Dynkin diagrams correspond to a chain of root systems

$$R_0 \subseteq R_1 \subseteq \ldots \subseteq R_n = R$$

and to a chain of Weyl groups $W_0 \subseteq W_1 \subseteq \ldots \subseteq W_n = W$ such that, for each $1 \leqslant i \leqslant n$, the group W_{i-1} is a parabolic subgroup of W_i. Let

$$\mathbb{C} \cong H_0 \subseteq \ldots \subseteq H_n = H$$

be the corresponding chain of Iwahori–Hecke algebras.

PROPOSITION 2.4. *Let H_k be the Iwahori–Hecke algebra corresponding to a finite Weyl group W_k. Let $w_{k,0}$ denote the longest element in the Weyl group W_k.*

(a) *The element $T_{w_{k,0}}^2$ is central in $H_k(p^2, q^2)$. If ρ is an irreducible representation of $H_k(p^2, q^2)$ corresponding to the irreducible character χ of the Weyl group W_k, then*

$$\rho(T_{w_{k,0}}^2) = p^{2c_{k,s}(\chi)} q^{2c_{k,\ell}(\chi)}\, \mathrm{Id},$$

where $c_{k,s}(\chi)$ and $c_{k,\ell}(\chi)$ are the constants given in (2.2).

(b) *If $w_{k,0} = -1$ in the reflection representation of W_k, then $T_{w_{k,0}}$ is a central element in $H_k(p^2, q^2)$. If ρ is an irreducible representation of $H_k(p^2, q^2)$ corresponding to the irreducible character χ of the Weyl group W_k, then*

$$\rho(T_{w_{k,0}}) = c_{k,0}(\chi) p^{c_{k,s}(\chi)} q^{c_{k,\ell}(\chi)}\, \mathrm{Id},$$

where $c_{k,s}(\chi)$, $c_{k,\ell}(\chi)$, and $c_{k,0}(\chi)$ are the constants given in (2.2).

Proof. (a) By a theorem of Breiskorn and Saito [**4**] and Deligne [**9**], the element $T_{w_{k,0}}^2$ is central in the generalized braid group. Thus $T_{w_{k,0}}^2$ is central in

$H_k(p^2, q^2)$ and it follows that $T^2_{w_{k,0}}$ acts by a constant in every irreducible representation. The constant is computed by writing $T_{w_{k,0}}$ as a product of generators and taking the determinant of both sides of the equation

$$T^2_{w_{k,0}} = (T_{i_1} \dots T_{i_N})^2 = p^{c_1} q^{c_2} \,\mathrm{Id}.$$

It remains only to note that the number of short roots in R_k is the same as the number of factors T_{i_j} in the product $T_{w_{k,0}} = T_{i_1} \dots T_{i_N}$ such that α_{i_j} is a short root in R_k, and that the analogous result holds for the number of long roots in R_k.

(b) The result of Brieskorn and Saito and Deligne says that $T_{w_{k,0}}$ is central in the braid group when $w_{k,0} = -1$ in the reflection representation of the Weyl group W_k. It follows that $T_{w_{k,0}}$ is central in $H_k(p^2, q^2)$. The eigenvalues of $T_{w_{k,0}}$ must be square roots of the eigenvalues of $T^2_{w_{k,0}}$ and they must specialize to the eigenvalues of $w_{k,0}$ when $p = q = 1$. The result now follows from (a), (2.1)(c), and the definition of the constant $c_{k,0}(\chi)$.

Remark 2.5. The above proposition is somewhat folklore in the subject of Iwahori–Hecke algebras. The argument given here appears in Propositions 26 and 27 of Kilmoyer's thesis [**21**] and also appears in complete detail in the recent paper of Geck and Michel [**14**].

Let $w_{k,0}$ be the longest element in the Weyl group W_k and define $Z_k = \{z_k\}$ where

$$z_k = \begin{cases} T_{w_{k,0}} & \text{if } w_{k,0} = -1 \text{ in the reflection representation of } W_k, \\ T^2_{w_{k,0}} & \text{otherwise.} \end{cases} \tag{2.6}$$

We shall use these central elements (with some slight modification in the D_n case) to compute seminormal representations for the Iwahori–Hecke algebras of types A_{n-1}, B_n, D_n, G_2. The cases F_4, E_6 and E_7 will be treated in future work.

Remark 2.7. In my view, the central elements in Proposition 2.4 are the analogues of the Jucys–Murphy elements for the Iwahori–Hecke algebras. Remarks 3.17 and 4.22 illustrate this idea in special cases.

The following proposition describes concretely the connection between the central elements of $H_k(p^2, q^2)$ in Proposition 2.4 and the central elements in $\mathbb{C}W_k$ given in (2.1).

Proposition 2.8. *If $x \in H_k(p^2, q^2)$, use the notation $[x]_{q=1}$ to denote the value of x when q is specialized to 1. Then*

$$\left[\frac{([T_{w_{k,0}}]_{p=1})^2 - 1}{q - q^{-1}}\right]_{q=1} = \sum_{\alpha \in R^+_{k,s}} s_\alpha = z_{k,s},$$

$$\left[\frac{([T_{w_{k,0}}]_{q=1})^2 - 1}{p - p^{-1}}\right]_{p=1} = \sum_{\alpha \in R^+_{k,\ell}} s_\alpha = z_{k,\ell},$$

$$[T_{w_{k,0}}]_{p=q=1} = w_{k,0} = z_{k,0},$$

where $z_{k,s}$, $z_{k,\ell}$, and $z_{k,0}$ are the central elements of $\mathbb{C}W_k$ given in (2.1).

Proof. Let us assume, for convenience, that all roots in R_k are the same length. In this case we have only one indeterminate q and we are working in the Iwahori–Hecke algebra $H_k(q^2)$. The proof is similar in the general case.

If $T_{i_1} \dots T_{i_N}$ is a reduced expression for $T_{w_{k,0}}$ then so is $T_{i_N} \dots T_{i_1}$. We can expand $(T_{w_{k,0}})^2$ by using the relation $T_i^2 = (q - q^{-1})T_i + 1$ to obtain

$$\begin{aligned} T_{w_{k,0}}^2 &= T_{i_1} \dots T_{i_N} T_{i_N} \dots T_{i_1} \\ &= 1 + (q - q^{-1}) \sum_{j=1}^{N} T_{i_1} \dots T_{i_{j-1}} T_{i_j} T_{i_{j-1}} \dots T_{i_1} \\ &\qquad + \text{terms divisible by } (q - q^{-1})^2. \end{aligned}$$

It follows that

$$\left[\frac{T_{w_{k,0}}^2 - 1}{q - q^{-1}}\right]_{q=1} = \sum_{j=1}^{N} s_{i_1} \dots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \dots s_{i_1}.$$

The result now follows from [**3**, Chapter VI, § 1, Corollary 2].

Remark 2.9. In Proposition 2.8 we have been carefree about the process of specializing p and q to 1. Of course this really should be done properly. One must define a $\mathbb{Z}$-form of the Iwahori–Hecke algebra $H_k(p^2, q^2)$ as an algebra over $\mathscr{A} = \mathbb{Z}[q, q^{-1}, p, p^{-1}]$ and only specialize, by an appropriate tensor product $\mathbb{Z} \otimes_{\mathscr{A}} H_k(p^2, q^2)$, elements x which are in the $\mathbb{Z}$-form of $H_k(p^2, q^2)$. This is standard and it is clear that the elements $(T_{w_{k,0}}^2 - 1)/(q - q^{-1})$ in (the proof of) Proposition 2.8 are elements in the $\mathbb{Z}$-form of $H_k(q^2)$.

3. *Type A_{n-1}, the symmetric group S_n*

The Weyl group. The Weyl group of the root system A_{n-1} is the symmetric group S_n of permutations of $\{1, 2, \dots, n\}$. The simple transpositions

$$s_i = (i-1, i) \quad \text{for } 2 \leqslant i \leqslant n,$$

generate S_n and these elements satisfy the relations

$$\begin{aligned} s_i s_j &= s_j s_i \quad \text{for } |i - j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for } 2 \leqslant i \leqslant n-1, \\ s_i^2 &= 1 \quad \text{for } 2 \leqslant i \leqslant n. \end{aligned} \tag{3.1}$$

Partitions and standard tableaux. As in [**23**], we shall identify each partition λ with its Ferrers diagram and say that a box b in λ is in position (i, j) in λ if b is in row i and column j of λ. The rows and columns of λ are labelled in the same way as for matrices. We shall write $|\lambda| = n$ if λ is a partition with n boxes. We shall often refer to partitions as *shapes.*

A *standard tableau* L of shape λ is a filling of the Ferrers diagram of λ with the numbers $1, 2, \dots, n$ such that the numbers are increasing left to right across the rows of L and increasing down the columns of L. For any shape λ, let $\mathscr{L}(\lambda)$ denote the set of standard tableaux of shape λ and, for each standard tableau L, let $L(k)$ denote the box containing k in L. For example, Fig. 3.2 illustrates a standard tableau of shape (332).

1	2	4
3	5	6
7	8	

FIG. 3.2

The chain $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_{n-1}$. The chain of root systems $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_{n-1}$ corresponds to the chain of Weyl groups

$$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n,$$

where S_k denotes the symmetric group of permutations of $1, 2, \ldots, k$. The irreducible representations of the symmetric group S_k are indexed by the partitions λ such that $|\lambda| = k$. The restriction rule from S_k to S_{k-1} is given by

$$V^\lambda \downarrow^{S_k}_{S_{k-1}} \cong \bigoplus_{\mu \in \lambda^-} V^\mu,$$

where the sum is over all partitions μ of $k-1$ that are obtained from λ by removing one box. For the chain $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n$, the graph Γ defined in (1.1) is the Young lattice. For $n = 5$, Γ is as in Fig. 3.3. A path $(\lambda^{(0)} \to \ldots \to \lambda^{(n)})$ in Γ is naturally identified with the standard tableau L of shape $\lambda^{(n)}$ which has i in the box which is added to obtain $\lambda^{(i)}$ from $\lambda^{(i-1)}$.

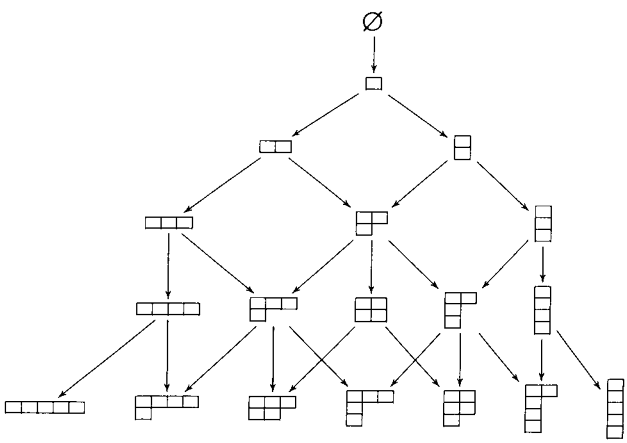

FIG. 3.3

Jucys–Murphy elements. Following (2.1), let us compute the sets Z_k for this case. Write permutations in the symmetric groups S_k in cycle notation. In the root system A_{k-1}, all roots are of one length and the longest element

$$w_{k,0} = (1, k)(2, k-1) \ldots$$

of the Weyl group S_k does not act by -1 in the reflection representation. For each $1 \leqslant k \leqslant n$, the set Z_k contains a single element $z_{k,\ell}$, which is the central element of $\mathbb{C}S_k$ given by

$$z_{k,\ell} = \sum_{\alpha \in A_{k-1}^+} s_\alpha = \sum_{1 \leqslant i < j \leqslant k} (i, j). \tag{3.4}$$

(The ℓ here is spurious; as in (2.1a) it only indicates the fact that in the root system A_{k-1} all roots are long.) Since the elements $z_{k,\ell}$, for $1 \leqslant k \leqslant n$, all commute with each other in $\mathbb{C}S_n$, it follows that the elements

$$m_k = z_{k,\ell} - z_{k-1,\ell} = \sum_{i=2}^{k} (i-1, k) \quad \text{for } 2 \leqslant k \leqslant n, \tag{3.5}$$

all commute with each other in $\mathbb{C}S_n$.

Remark 3.6. The elements m_k, for $2 \leqslant k \leqslant n$, are the elements defined by Jucys [**18**–**20**] and Murphy [**24**–**26**].

Weights. The *content* of a box b in a shape λ is given by

$$\mathrm{ct}(b) = j - i, \quad \text{if } b \text{ is in position } (i, j) \text{ in } \lambda. \tag{3.7}$$

It follows immediately from [**23,** I, § 7, Ex. 7, and I, § 1, Ex. 3] that, for each $1 \leqslant k \leqslant n$ and for each partition μ such that $|\mu| = k$,

$$c_{k,\ell}(\mu) = \chi^\mu(z_{k,\ell})/\chi^\mu(1) = \sum_{b \in \mu} \mathrm{ct}(b), \tag{3.8}$$

where χ^μ denotes the character of the irreducible representation of the symmetric group S_k labelled by the partition μ. Following (1.11), we see that the *weight* of a standard tableau $L = (\lambda^{(1)} \to \ldots \to \lambda^{(n)})$, where $|\lambda^{(k)}| = k$, is

$$\mathrm{wt}(L) = (c_{1,\ell}(\lambda^{(1)}), \ldots, c_{n,\ell}(\lambda^{(n)})).$$

Note that $\mathrm{wt}(L)$ is completely determined by the n-tuple

$$\widetilde{\mathrm{wt}}(L) = (\mathrm{ct}(L(1)), \ldots, \mathrm{ct}(L(n))), \quad \text{since } c_{k,\ell}(\lambda^{(k)}) = \sum_{i=1}^{k} \mathrm{ct}(L(i)).$$

Proposition 3.9. *Each standard tableau $L = (\lambda^{(1)} \to \ldots \to \lambda^{(n)})$ is determined uniquely by its weight.*

Proof. Two boxes b and b' in a partition λ have the same content only if they lie on the same diagonal. It follows easily that, if $\lambda^{(i)}$ is a partition, then each of the boxes b that can be added to $\lambda^{(i)}$ to get a new partition has a different content $\mathrm{ct}(b)$. Thus, the shape $\lambda^{(i+1)}$ in a standard tableau L is completely determined by the previous shape $\lambda^{(i)}$ and the content $\mathrm{ct}(b)$ of the added box b. It follows that a standard tableau L is completely determined by $\widetilde{\mathrm{wt}}(L)$ and therefore by its weight $\mathrm{wt}(L)$.

Proposition 1.12 and Proposition 3.9 together show that the seminormal representations of S_n corresponding to the chain of groups $\{1\} = S_0 \subseteq \ldots \subseteq S_n$ are essentially determined by the elements $z_{k,\ell}$ in (3.4) and the constants $c_{k,\ell}(\mu)$ in (3.8). It follows that we should be able to determine seminormal representations of the group S_n from the elements m_k and the constants $\mathrm{ct}(b)$. This is done in Theorems 3.12 and 3.14 below.

Seminormal representations. Let $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_n$ be the path algebras, defined in (1.2), which are associated to the diagram Γ which describes the restriction rules for the chain $S_1 \subseteq \ldots \subseteq S_n$. For each partition λ of size n, let

$$V^\lambda = \mathbb{C}\text{-span}\{v_L,\ L \in \mathscr{L}(\lambda)\}, \tag{3.10}$$

so that the vectors v_L, indexed by standard tableaux L of shape λ, form a seminormal basis of the P_n-module V^λ. It follows from Lemma 1.9, that for any choice of an isomorphism Φ between the path algebra P_n and $\mathbb{C}S_n$ such that $\Phi(P_k) = \mathbb{C}S_k \subseteq \mathbb{C}S_n$ for all $1 \leqslant k \leqslant n$, we have that

$$z_{k,\ell} v_L = c_{k,\ell}(\lambda^{(k)}) v_L,$$

if $L = (\lambda^{(1)} \to \ldots \to \lambda^{(n)})$. If m_k is as in (3.5), then

$$m_k v_L = \mathrm{ct}(L(k)) v_L,$$

for a standard tableau $L = (\lambda^{(1)} \to \ldots \to \lambda^{(n)})$.

For each $2 \leqslant k \leqslant n$ and each standard tableau L of size n, define

$$(s_k)_{LL} = \frac{1}{\mathrm{ct}(L(k)) - \mathrm{ct}(L(k-1))}. \tag{3.11}$$

In the interests of space we shall not give the proof of the following theorems here. The proofs are essentially the same as the proofs which are given for Theorem 4.15 and Theorem 4.18.

THEOREM 3.12 (Young [**35**]). *Let λ be a partition such that $|\lambda| = n$. Define an action of each generator $s_2, \ldots, s_n$ of the symmetric group S_n on V^λ by defining*

$$s_i v_L = (s_i)_{LL} v_L + (1 + (s_i)_{LL}) v_{s_i L} \quad \textit{for } 2 \leqslant i \leqslant n, \tag{3.13}$$

where $s_i L$ is the same standard tableau as L except that the positions of i and $i-1$ are switched in $s_i L$. If $s_i L$ is not standard, then we define $v_{s_i L} = 0$. This action extends to a well-defined action of S_n on V^λ.

THEOREM 3.14 (Young [**35**]). *The S_n modules V^λ defined in Theorem 3.12, where λ runs over all partitions such that $|\lambda| = n$, form a complete set of non-isomorphic irreducible modules for the symmetric group S_n and, for each λ, the basis $\{v_L \mid L \in \mathscr{L}(\lambda)\}$ is a seminormal basis of the S_n-module V^λ.*

Iwahori–Hecke algebras $HA_{n-1}(q^2)$

Let q be an indeterminate. The Iwahori–Hecke algebra $HA_{k-1}(q^2)$ corresponding to the root system A_{k-1} is the associative algebra with 1 over the field $\mathbb{C}(q)$ given by generators $T_2, T_3, \ldots, T_k$ and relations

$$\begin{aligned} T_i T_j &= T_j T_i \quad \text{for } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad \text{for } 2 \leqslant i \leqslant k-1, \\ T_i^2 &= (q - q^{-1}) T_i + 1 \quad \text{for } 2 \leqslant i \leqslant k. \end{aligned} \tag{3.15}$$

Analogues of Jucys–Murphy elements. For each $2 \leqslant k \leqslant n$, define

$$M_k = T_k \ldots T_3 T_2 T_2 T_3 \ldots T_k. \tag{3.16}$$

In type A_{k-1} the longest element $w_{k,0}$ of the group S_k does not act by -1 in the reflection representation. Following Proposition 2.4, we define sets $Z_k = \{z_k\}$, for $2 \leqslant k \leqslant n$, where z_k is the central element of $HA_{k-1}(q^2)$ given by

$$z_k = T^2_{w_{k,0}} = M_k M_{k-1} \dots M_2.$$

Since the elements z_k, for $2 \leqslant k \leqslant n$, all commute in $HA_{n-1}(q^2)$, it follows that the elements M_k, for $2 \leqslant k \leqslant n$, all commute with each other.

REMARK 3.17. Using the relation $T_i^2 = (q - q^{-1})T_i + 1$, an easy computation shows that

$$\frac{M_k - 1}{q - q^{-1}} = \sum_{i=2}^{k} T_{(i-1,k)}, \tag{3.18}$$

where $T_{(i-1,k)} = T_k T_{k-1} \dots T_{i+1} T_i T_{i+1} \dots T_k$. The elements in (3.18) are elements used by Dipper, James, and Murphy in their work on Iwahori–Hecke algebras of type A, see [**10**, **11**, **26**, **27**]. It is clear that (3.18) gives a q-analogue of the Jucys–Murphy elements in (3.5).

Seminormal representations. Let $P_1 \subseteq P_2 \subseteq \dots \subseteq P_n$ be the path algebras (over the field $\mathbb{C}(q)$ instead of $\mathbb{C}$), defined in (1.2), which are associated to the diagram Γ which describes the restriction rules for the chain $S_1 \subseteq \dots \subseteq S_n$. For each partition λ of size n, let

$$V^\lambda = \mathbb{C}(q)\text{-span}\{v_L \mid L \in \mathscr{L}(\lambda)\}, \tag{3.19}$$

so that the vectors v_L, indexed by standard tableaux L of shape λ, form a seminormal basis of the P_n-module V^λ. It follows from Lemma 1.9, that for any choice of an isomorphism Φ between the path algebra P_n and $HA_{n-1}(q^2)$ such that $\Phi(P_k) = HA_{k-1}(q^2) \subseteq HA_{n-1}(q^2)$ for all $1 \leqslant k \leqslant n$, we have

$$z_k v_L = T^2_{w_{k,0}} v_L = q^{c_{k,\ell}(\lambda^{(k)})} v_L,$$

if $L = (\lambda^{(1)} \subseteq \dots \subseteq \lambda^{(n)})$ and $c_{k,\ell}(\lambda^{(k)})$ is as given in (3.8). Thus,

$$M_k v_L = T_k \dots T_3 T_2 T_2 T_3 \dots T_k v_L = T^2_{w_{k,0}} T^{-2}_{w_{k-1,0}} = q^{2\,\mathrm{ct}(L(k))} v_L, \tag{3.20}$$

if $L = (\lambda^{(1)} \to \dots \to \lambda^{(n)})$ is a standard tableau. For each $2 \leqslant k \leqslant n$ and each standard tableau L of size n, define

$$(T_k)_{LL} = (q - q^{-1}) \Big/ \left(1 - \frac{\mathrm{CT}(L(k-1))}{\mathrm{CT}(L(k))}\right) \tag{3.21}$$

where $\mathrm{CT}(b) = q^{2\mathrm{ct}(L(k))}$. In the interests of space we shall not give the proof of the following theorems here. The proofs are essentially the same as the proofs which are given for Theorem 4.26 and Theorem 4.28.

THEOREM 3.22 (Hoefsmit [**17**], Wenzl [**34**]). *Let λ be a partition such that $|\lambda| = n$. Define an action of each generator $T_2, \dots, T_n$ of the Iwahori–Hecke algebra $HA_{n-1}(q^2)$ on V^λ by defining*

$$T_i v_L = (T_i)_{LL} v_L + (q^{-1} + (T_i)_{LL}) v_{s_i L} \quad \textit{for } 2 \leqslant i \leqslant n,$$

where $s_i L$ is the same standard tableau as L except that the positions of i and $i-1$

are switched in s_iL. If s_iL is not standard, then we define $v_{s_iL} = 0$. This action extends to a well-defined action of $HA_{n-1}(q^2)$ on V^λ.

THEOREM 3.23 (Hoefsmit [**17**], Wenzl [**34**]). *The $HA_{n-1}(q^2)$-modules V^λ defined in Theorem 3.22, where λ runs over all partitions such that $|\lambda| = n$, form a complete set of non-isomorphic irreducible modules for the Iwahori–Hecke algebra $HA_{n-1}(q^2)$ and, for each λ, the basis $\{v_L \mid L \in \mathcal{L}(\lambda)\}$ is a seminormal basis of the $HA_{n-1}(q^2)$-module V^λ.*

4. *Type B_n with $n \geqslant 2$*

The Weyl group. The Weyl group WB_n of type B_n is the group of signed permutations of $1, 2, ..., n$. More specifically, WB_n consists of all permutations π of $\{-n, ..., -1, 1, ..., n\}$ such that $\pi(-k) = -\pi(k)$ for all $1 \leqslant k \leqslant n$. We represent elements of WB_n in cycle notation as permutations of $\{-n, ..., -1, 1, ..., n\}$. The elements

$$s_1 = (1, -1), \quad \text{and} \quad s_i = (i-1, i)(-(i-1), -i) \ \text{ for } 2 \leqslant i \leqslant n,$$

generate WB_n and satisfy the relations

$$\begin{aligned} s_i s_j &= s_j s_i \quad \text{for } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for } 2 \leqslant i \leqslant n-1, \\ s_1 s_2 s_1 s_2 &= s_2 s_1 s_2 s_1, \\ s_i^2 &= 1 \quad \text{for } 1 \leqslant i \leqslant n. \end{aligned} \tag{4.1}$$

Double partitions and standard tableaux. A *double partition* of size n,

$$\lambda = (\alpha, \beta),$$

is an ordered pair of partitions α and β such that $|\alpha| + |\beta| = n$. We shall often refer to double partitions as *shapes*. A *standard tableau* $L = (L^\alpha, L^\beta)$ of shape $\lambda = (\alpha, \beta)$ is a filling of the Ferrers diagram of λ with the numbers $1, 2, ..., n$ such that the numbers are increasing left to right across the rows of L^α and L^β and increasing down the columns of L^α and L^β. For any shape λ, let $\mathcal{L}(\lambda)$ denote the set of standard tableaux of shape λ and, for each standard tableau L, let $L(k)$ denote the box containing k in L. For example, Fig. 4.2 shows a standard tableau of shape $((332), (411))$.

$$\begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 7 & 11 & 13 \\ \hline 8 & 12 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 5 & 6 & 10 \\ \hline 9 \\ \cline{1-1} 14 \\ \cline{1-1} \end{array}$$

FIG. 4.2

The chain $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n$. By convention we let $B_0 = \emptyset$ be the empty root system and $B_1 = A_1$. The chain of root systems $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n$ corresponds to the chain of Weyl groups

$$\{1\} \subseteq WB_1 \subseteq WB_2 \subseteq \ldots \subseteq WB_n, \tag{4.3}$$

where WB_k denotes the hyperoctahedral group of signed permutations of $1, 2, \ldots, k$.

The irreducible representations of the symmetric group WB_k are indexed by double partitions $\lambda = (\alpha, \beta)$ such that $|\lambda| = |\alpha| + |\beta| = k$. The restriction rule from WB_k to WB_{k-1} is given by

$$V^{(\alpha, \beta)}\big\downarrow^{WB_k}_{WB_{k-1}} \cong \bigoplus_{(\mu, \nu) \in (\alpha, \beta)^-} V^{(\mu, \nu)}$$

where the sum is over all double partitions (μ, ν) of size $k-1$ that are obtained from (α, β) by removing one box. If we define the graph Γ as in (1.1) for the chain in (4.3), then a path $(\lambda^{(0)} \to \ldots \to \lambda^{(n)})$ in Γ is naturally identified with the standard tableau L of shape $\lambda^{(n)}$ which has i in the box which is added to obtain $\lambda^{(i)}$ from $\lambda^{(i-1)}$. The graph Γ for the case of the chain $\{1\} \subseteq WB_1 \subseteq WB_2 \subseteq WB_3$ is displayed in Fig. 4.4.

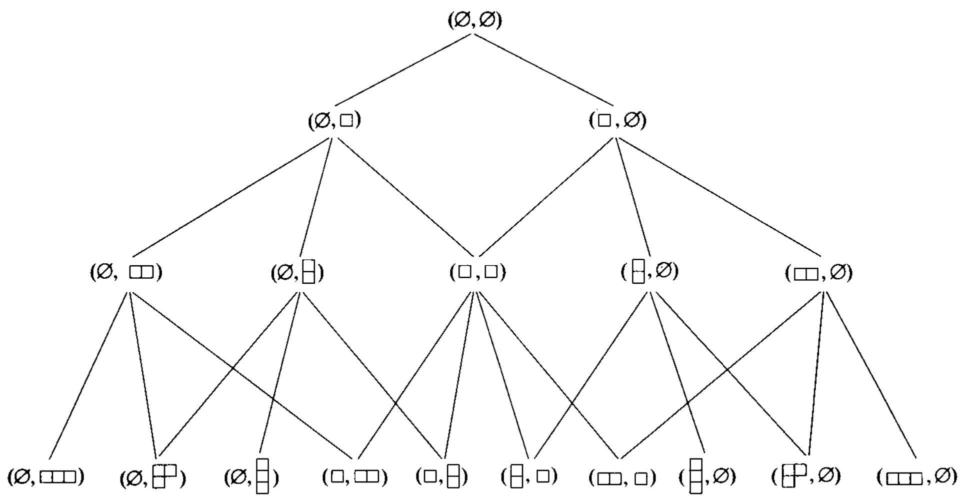

FIG. 4.4

Analogues of Jucys–Murphy elements. Following (2.1), let us compute the sets Z_k for this case. In the root system B_k, with $k \geqslant 2$, we have both long and short roots and the longest element $w_{k,0} = (1, -1)(2, -2) \ldots (k, -k)$ of the Weyl group WB_k acts by -1 in the reflection representation. For each $1 \leqslant k \leqslant n$, let

$$Z_k = \{z_{k,s}, z_{k,\ell}, z_{k,0}\},$$

where

$$
\begin{aligned}
z_{k,s} &= \sum_{\alpha \in (B_k)_s^+} s_\alpha = \sum_{i=1}^{k} (i, -i),\\
z_{k,\ell} &= \sum_{\alpha \in (B_k)_\ell^+} s_\alpha = \sum_{1 \leqslant i < j \leqslant k} (i, j) + (i, -j)(-i, j), \qquad (4.5)\\
z_{k,0} &= w_{k,0} = (1, -1)(2, -2) \ldots (k, -k).
\end{aligned}
$$

In (4.5) the sets $(B_k)_s^+$ and $(B_k)_\ell^+$ are, respectively, the sets of short and long positive roots in the root system B_k. Since the elements $z_{k,j}$, with $j \in \{s, \ell, 0\}$ and $1 \leqslant k \leqslant n$, all commute with each other in $\mathbb{C}WB_n$, it follows that the elements

$$
\begin{aligned}
m_{k,s} &= z_{k,s} - z_{k-1,s} = (k, -k), \quad \text{for } 1 \leqslant k \leqslant n,\\
m_{k,\ell} &= z_{k,\ell} - z_{k-1,\ell}\\
&= \sum_{i=2}^{k} (i-1, k) + (i-1, -k)(-(i-1), k), \quad \text{for } 2 \leqslant k \leqslant n,
\end{aligned}
$$

all commute with each other in $\mathbb{C}WB_n$.

REMARK 4.6. The elements $m_{k,s}$ and $m_{k,\ell}$ are the appropriate B_n-analogues of the Jucys–Murphy elements (3.5) for the symmetric group. Cherednik [**7**] has used a linear combination of $m_{k,s}$ and $m_{k,\ell}$ as an analogue of the Jucys–Murphy element.

Weights. The *sign* and the *content* of a box b in a shape (α, β) are given respectively by

$$
\begin{aligned}
\operatorname{sgn}(b) &= \begin{cases} 1 & \text{if } b \in \alpha,\\ -1 & \text{if } b \in \beta, \end{cases} \qquad (4.7)\\
\operatorname{ct}(b) &= j - i \quad \text{if } b \text{ is in position } (i, j) \text{ (in either } \alpha \text{ or } \beta).
\end{aligned}
$$

PROPOSITION 4.8. *Fix $1 \leqslant k \leqslant n$ and let $z_{k,s}$, $z_{k,\ell}$, and $z_{k,0}$ be the central elements in $\mathbb{C}WB_k$ given in* (4.5). *Let (μ, ν) be a double partition such that $|\mu| + |\nu| = k$. Let $\chi^{(\mu,\nu)}$ be the character of the irreducible representation of WB_k indexed by the double partition (μ, ν). Then, in the notation of Lemma* 1.9 *and* (4.7),

$$
\begin{aligned}
c_{k,s}(\mu, \nu) &= \chi^{(\mu,\nu)}(z_{k,s})/\chi^{(\mu,\nu)}(1) = \sum_{b \in (\mu,\nu)} \operatorname{sgn}(b),\\
c_{k,\ell}(\mu, \nu) &= \chi^{(\mu,\nu)}(z_{k,\ell})/\chi^{(\mu,\nu)}(1) = 2 \sum_{b \in (\mu,\nu)} \operatorname{ct}(b),\\
c_{k,0}(\mu, \nu) &= \chi^{(\mu,\nu)}(z_{k,0})/\chi^{(\mu,\nu)}(1) = \prod_{b \in (\mu,\nu)} \operatorname{sgn}(b).
\end{aligned}
$$

Proof. Fix k and a double partition (μ, ν) such that $|\mu| + |\nu| = k$. Let $a = |\mu|$ and $b = |\nu|$. Let WB_a be the subgroup of WB_k of signed permutations of

$\{1, 2, \ldots, a\}$ and let WB_b be the subgroup of WB_k of signed permutations of $\{a+1, \ldots, k\}$. Let S_a and S_b be the symmetric groups of permutations of $\{1, \ldots, a\}$ and $\{a+1, \ldots, k\}$ respectively.

Let V^μ be the irreducible module for the symmetric group S_a which is labelled by the partition μ. Extend this module to the group WB_a by letting the signs, that is, the element $(1, -1)$ act trivially. Let V^ν be the irreducible module for the symmetric group S_b which is labelled by ν. Extend this module to the group WB_b by letting the element $(a+1, -(a+1))$ act by -1 on V^ν. Then V^μ and V^ν are irreducible modules for the groups WB_a and WB_b respectively; they are the modules that are ordinarily denoted by $V^{(\mu, \emptyset)}$ and $V^{(\emptyset,\nu)}$ respectively. It follows from this construction of $V^{(\mu,\emptyset)}$ and $V^{(\emptyset,\nu)}$ that

$$\begin{aligned}
&\chi^{(\mu,\emptyset)}((1,2)) = \chi^\mu((1,2)),\\
&\chi^{(\mu,\emptyset)}((1,-1)) = \chi^\mu(1),\\
&\chi^{(\mu,\emptyset)}((1,-1)(2,-2)\ldots(a,-a)) = \chi^\mu(1),\\
&\chi^{(\emptyset,\nu)}((a+1,a+2)) = \chi^\nu((a+1,a+2)),\\
&\chi^{(\emptyset,\nu)}((a+1,-(a+1)) = -\chi^\nu(1),\\
&\chi^{(\emptyset,\nu)}((a+1,-(a+1)(a+2,-(a+2))\ldots(k,-k)) = (-1)^b\chi^\nu(1),
\end{aligned} \tag{4.9}$$

where χ^μ and χ^ν denote the irreducible characters of the symmetric groups S_a and S_b labelled by μ and ν, respectively.

It is well known that the induced module

$$V^{(\mu,\emptyset)} \otimes V^{(\emptyset,\nu)}\uparrow^{WB_k}_{WB_a\times WB_b} \cong V^{(\mu,\nu)}$$

is a realization of the irreducible WB_k-module indexed by the double partition (μ, ν). (I believe that this fact is originally due to Specht [**32**].) Given this realization we can write down its character $\chi^{(\mu,\nu)}$ explicitly by using the standard formula for induced characters:

$$\chi^{(\mu,\nu)}(w) = \sum_{g_i^{-1}wg_i \in WB_a\times WB_b} \chi^{(\mu,\emptyset)}(g_i^{-1}wg_i)\chi^{(\emptyset,\nu)}(g_i^{-1}wg_i), \tag{4.10}$$

where the sum is over coset representatives g_i of $WB_k/(WB_a \times WB_b)$ such that $g_i^{-1}wg_i \in WB_a \times WB_b$.

Using (4.9) and (4.10), we have

$$\begin{aligned}
&\frac{\chi^{(\mu,\nu)}(\sum_{\alpha\in(B_k)_\ell^+} s_\alpha)}{\chi^{(\mu,\nu)}(1)}\\
&\quad= \frac{k(k-1)\chi^{(\mu,\nu)}((1,2))}{\chi^{(\mu,\nu)}(1)}\\
&\quad= \frac{k(k-1)(\binom{k-2}{a-2}\chi^{(\mu,\emptyset)}((1,2))\chi^{(\emptyset,\nu)}(1) + \binom{k-2}{a}\chi^{(\mu,\emptyset)}(1)\chi^{(\emptyset,\nu)}((a+1,a+2)))}{\binom{k}{a}\chi^{(\mu,\emptyset)}(1)\chi^{(\emptyset,\nu)}(1)}\\
&\quad= k(k-1)\Big(\frac{a(a-1)}{k(k-1)}\frac{\chi^\mu((1,2))}{\chi^\mu(1)} + \frac{b(b-1)}{k(k-1)}\frac{\chi^\nu((a+1,a+2))}{\chi^\nu(1)}\Big)\\
&\quad= 2\Big(\frac{\chi^\mu(\sum_{\alpha\in A_{a-1}^+} s_\alpha)}{\chi^\mu(1)} + \frac{\chi^\nu(\sum_{\alpha\in A_{b-1}^+} s_\alpha)}{\chi^\nu(1)}\Big).
\end{aligned}$$

The value for $c_{k,\ell}(\mu, \nu)$ is obtained from this last equation and (3.8). A similar calculation gives

$$\begin{aligned}
&\frac{\chi^{(\mu,\nu)}(\sum_{\alpha\in(B_k)_s^+} s_\alpha)}{\chi^{(\mu,\nu)}(1)}\\
&\quad=\frac{k\chi^{(\mu,\nu)}((1,-1))}{\chi^{(\mu,\nu)}(1)}\\
&\quad=\frac{k(\binom{k-1}{a-1}\chi^{(\mu,\emptyset)}((1,-1))\chi^{(\emptyset,\nu)}(1)+\binom{k-1}{a}\chi^{(\mu,\emptyset)}(1)\chi^{(\emptyset,\nu)}((a+1,-(a+1))))}{\binom{k}{a}\chi^{(\mu,\emptyset)}(1)\chi^{(\emptyset,\nu)}(1)}\\
&\quad=\frac{a\chi^{\mu}((1,-1))}{\chi^{\mu}(1)}+\frac{b\chi^{\nu}((a+1,-(a+1)))}{\chi^{\nu}(1)}\\
&\quad=a-b.
\end{aligned}$$

Since $\operatorname{sgn}(b)=1$ for all boxes $b\in\mu$, $\operatorname{sgn}(b)=-1$ for all boxes $b\in\nu$, and $|\mu|=a$ and $|\nu|=b$, the formula for $c_{k,s}(\mu,\nu)$ follows. The formula for $c_{k,0}(\mu,\nu)$ is obtained in a similar fashion.

Following (1.11), we define the *weight* of a standard tableau

$$L=(\lambda^{(1)}\to\dots\to\lambda^{(n)}),$$

where $|\lambda^{(k)}|=k$, to be

$$\operatorname{wt}(L)=(c_1(\lambda^{(1)}),\dots,c_n(\lambda^{(n)})),$$

where, for a double partition (μ,ν), $c_k(\mu,\nu)$ is the triple

$$c_k(\mu,\nu)=(c_{k,s}(\mu,\nu),c_{k,\ell}(\mu,\nu),c_{k,0}(\mu,\nu)),$$

determined by Proposition 4.8. Note that $\operatorname{wt}(L)$ is completely determined by the n-tuples

$$\widetilde{\operatorname{wt}}_1(L)=(\operatorname{ct}(L(1)),\dots,\operatorname{ct}(L(n)))$$

and

$$\widetilde{\operatorname{wt}}_2(L)=(\operatorname{sgn}(L(1)),\dots,\operatorname{sgn}(L(n))),$$

since $c_{k,s}(\lambda^{(k)})=\sum_{i=1}^k\operatorname{sgn}(L(i))$, $c_{k,\ell}(\lambda^{(k)})=\sum_{i=1}^k\operatorname{ct}(L(i))$, and

$$c_{k,0}(\lambda^{(k)})=\prod_{i=1}^k\operatorname{sgn}(L(i)).$$

PROPOSITION 4.11. *Each standard tableau $L=(\lambda^{(1)}\to\dots\to\lambda^{(n)})$ is determined uniquely by its weight.*

Proof. Suppose that the weight $\operatorname{wt}(L)$ of a standard tableau L is given but that we do not know L. The vector $\operatorname{wt}(L)$ uniquely determines the vectors $\widetilde{\operatorname{wt}}_1(L)$ and $\widetilde{\operatorname{wt}}_2(L)$. We want to show that the tableau L can be reconstructed from $\widetilde{\operatorname{wt}}_1(L)$ and $\widetilde{\operatorname{wt}}_2(L)$. Assume that we have reconstructed L up to the ith step, that is, assume that we know $\lambda^{(0)},\dots,\lambda^{(i)}$, but that we do not yet know $\lambda^{(i+1)}$. Suppose that $\lambda^{(i)}$ is the double partition $(\mu^{(i)},\nu^{(i)})$.

We need to figure out from $\widetilde{\operatorname{wt}}_1(L)$ and $\widetilde{\operatorname{wt}}_2(L)$ where to add the box to get $\lambda^{(i+1)}=(\mu^{(i+1)},\nu^{(i+1)})$. The entry $\operatorname{sgn}(L(i+1))$ in $\widetilde{\operatorname{wt}}_2(L)$ tells us whether we must add the box to the partition $\mu^{(i)}$ or to the partition $\nu^{(i)}$. As in the proof of

Proposition 3.9, the entry $\mathrm{ct}(L(i+1))$ from $\widetilde{\mathrm{wt}}_1(L)$ indicates the position where this box must be added. It follows that $\lambda^{(i+1)}$ is uniquely determined. Thus L is completely determined by $\mathrm{wt}(L)$.

Proposition 1.12 and Proposition 4.11 together show that the seminormal representations of WB_n corresponding to the chain of groups

$$\{1\} = WB_0 \subseteq \ldots \subseteq WB_n$$

are essentially determined by the elements $z_{k,j}$ in (4.5) and the constants $c_{k,j}(\mu)$ in Proposition 4.8. It follows that we should be able to determine seminormal representations of the group WB_n from the elements $m_{k,j}$ and the constants $\mathrm{ct}(b)$ and $\mathrm{sgn}(b)$. This is done in Theorems 4.15 and 4.18 below.

Seminormal representations. Let $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_n$ be the path algebras, defined in (1.2), which are associated to the diagram Γ which describes the restriction rules for the chain $WB_1 \subseteq \ldots \subseteq WB_n$. For each double partition (α, β) such that $|\alpha| + |\beta| = n$, let

$$V^{(\alpha,\beta)} = \mathbb{C}\text{-span}\{v_L \mid L \in \mathcal{L}(\alpha, \beta)\}, \tag{4.12}$$

so that the vectors v_L, indexed by standard tableaux L of shape (α, β), form a seminormal basis of the P_n-module $V^{(\alpha,\beta)}$. It follows from Lemma 1.9, that for any choice of an isomorphism Φ between the path algebra P_n and $\mathbb{C}WB_n$ such that $\Phi(P_k) = \mathbb{C}WB_k \subseteq \mathbb{C}WB_n$ for all $1 \leqslant k \leqslant n$,

$$z_{k,s}v_L = c_{k,s}(\lambda^{(k)})v_L, \quad z_{k,\ell}v_L = c_{k,\ell}(\lambda^{(k)})v_L, \quad z_{k,0}v_L = c_{k,0}(\lambda^{(k)})v_L,$$

if $L = (\lambda^{(1)} \to \ldots \to \lambda^{(n)})$. Thus, by Proposition 4.8,

$$m_{k,s}v_L = \mathrm{sgn}(L(k))v_L \quad \text{and} \quad m_{k,\ell}v_L = \mathrm{ct}(L(k))v_L, \tag{4.13}$$

if $L = (\lambda^{(1)} \to \ldots \to \lambda^{(n)})$.

For each $2 \leqslant k \leqslant n$ and each standard tableau L of length n, define

$$(s_k)_{LL} = \frac{1 + \mathrm{sgn}(L(k))\mathrm{sgn}(L(k-1))}{\mathrm{ct}(L(k)) - \mathrm{ct}(L(k-1))}. \tag{4.14}$$

THEOREM 4.15 (Young [**35**]). *Let (α, β) be a double partition such that $|\alpha| + |\beta| = n$. Define an action of each generator $s_1, \ldots, s_n$ of WB_n on $V^{(\alpha,\beta)}$ by defining*

$$\begin{aligned} &s_1 v_L = \mathrm{sgn}(L(1))v_L, \\ &s_i v_L = (s_i)_{LL}v_L + (1 + (s_i)_{LL})v_{s_iL} \quad \textit{for } 2 \leqslant i \leqslant n, \end{aligned} \tag{4.16}$$

where s_iL is the same standard tableau as L except that the positions of i and $i-1$ are switched in s_iL. If s_iL is not standard, then we define $v_{s_iL} = 0$. This action extends to a well-defined action of WB_n on $V^{(\alpha,\beta)}$.

Proof. We shall show that the action of s_i, for $1 \leqslant i \leqslant n$, on $V^{(\alpha,\beta)}$ is essentially forced by the formulas in (4.13). We shall prove this for $i = n$. The proof for $i < n$ is similar. Note that the formula for the action of s_1 follows immediately from the formula for the action of $m_{1,s}$ in (4.13).

For any two standard tableaux M and L of shape (α, β) let $(s_n)_{ML}$ be the coefficient of v_M in $s_n v_L$.

Step 1. Let $L = (\lambda^{(0)} \to \dots \to \lambda^{(n)})$ be a standard tableau of shape (α, β). For each $0 \leqslant k \leqslant n$ and each $j \in \{s, \ell, 0\}$ with $1 \leqslant j \leqslant r_k$, let

$$p_{k,j}(\lambda^{(k)}) = \prod_{c_{k,j}(\mu) \neq c_{k,j}(\lambda^{(k)})} \frac{z_{k,j} - c_{k,j}(\mu)}{c_{k,j}(\lambda^{(k)}) - c_{k,j}(\mu)},$$

as in the proof of Proposition 1.12. Define

$$(*) \qquad p_{L[n-2]} = \prod_{k=1}^{n-2} \prod_{j \in \{s, \ell, 0\}} p_{k,j}(\lambda^{(k)}).$$

If $M = (\mu^{(0)} \to \dots \to \mu^{(n)})$ is another standard tableau of shape (α, β) then

$$p_{L[n-2]} v_M = \begin{cases} v_M & \text{if } \mu^{(k)} = \lambda^{(k)} \text{ for } 1 \leqslant k \leqslant n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $\lambda^{(n-2)}$ and $\lambda^{(n)} = (\alpha, \beta)$ only differ by two boxes, there are only two tableaux M such that $\mu^{(k)} = \lambda^{(k)}$ for all $1 \leqslant k \leqslant n-2$. These two tableaux are $s_n L$ and L itself. It follows that

$$p_{L[n-2]} v_M = \begin{cases} v_M & \text{if } M = s_n L \text{ or } M = L, \\ 0 & \text{otherwise.} \end{cases}$$

Since each of the elements $z_{k,s}$, $z_{k,\ell}$, $z_{k,0}$ appearing in the product $(*)$ is an element of WB_{n-2}, it follows that $p_{L[n-2]}$ commutes with s_n in WB_n. Thus

$$(s_n)_{LL} v_L + (s_n)_{s_n L, L} v_{s_n L} = p_{L[n-2]} s_n v_L = s_n p_{L[n-2]} v_L = s_n v_L.$$

It follows that $(s_n)_{ML} = 0$ unless $M = s_n L$ or $M = L$.

Step 2. A direct computation shows that

$$s_n m_{n-1,\ell} = m_{n,\ell} s_n - 1 - m_{n,s} m_{n-1,s}.$$

Let both sides act on v_L and take the coefficient of v_L in the result. Then, using (4.13), we have

$$(s_n)_{LL} \operatorname{ct}(L(n-1)) = \operatorname{ct}(L(n))(s_n)_{LL} - 1 - \operatorname{sgn}(L(n)) \operatorname{sgn}(L(n-1)).$$

It follows that $(s_n)_{LL}$ is as given in (4.14).

Step 3. Consider the equation $s_n^2 = 1$. Let both sides act on v_L and take the coefficient of v_L in the result. We get the equation $(s_n)_{LL}^2 + (s_n)_{LM}(s_n)_{ML} = 1$, where $M = s_n L$. It follows that

$$(4.17) \qquad (s_n)_{LM}(s_n)_{ML} = (1 + (s_n)_{LL})(1 - (s_n)_{LL}).$$

By Proposition 1.12(b), the values of $(s_n)_{LM}$ and $(s_n)_{ML}$ are determined only up to a constant and we may choose them to be anything such that the equation in (4.17) holds. Note that this is consistent with the definition of the action in the statement of the theorem since $1 - (s_n)_{s_n L, s_n L} = 1 + (s_n)_{LL}$.

THEOREM 4.18 (Young [**35**]). *The WB_n-modules $V^{(\alpha,\beta)}$ defined in Theorem* 4.15, *where (α, β) runs over all ordered pairs of partitions such that $|\alpha| + |\beta| = n$, form a complete set of non-isomorphic irreducible modules for the Weyl group WB_n and,* *for each* (α, β), *the basis* $\{v_L \mid L \in \mathcal{L}(\alpha, \beta)\}$ *is a seminormal basis of the* WB_n*-module* $V^{(\alpha,\beta)}$.

Proof. This now follows immediately by induction on n. Indeed, $V^{(\alpha,\beta)}$ is the unique WB_n-module such that

(1) the equations for the action of $z_{n,s}$, $z_{n,\ell}$, $z_{n,0}$ are as in (4.13), and

(2) on restriction to WB_{n-1} we have

$$V^{(\alpha,\beta)}\big\downarrow^{WB_n}_{WB_{n-1}} \cong \bigoplus_{(\mu,\nu)\in(\alpha,\beta)^-} V^{(\mu,\nu)},$$

where the sum is over all double partitions (μ, ν) of size $n-1$ that are obtained from (α, β) by removing one box.

Iwahori–Hecke algebras $HB_n(p^2, q^2)$

Let p and q be indeterminates. The Iwahori–Hecke algebra $HB_k(p^2, q^2)$ corresponding to the root system WB_k is the associative algebra with 1 over the field $\mathbb{C}(p, q)$ given by generators $T_1, T_2, \ldots, T_k$ and relations

$$\begin{aligned} T_iT_j &= T_jT_i \quad \text{for } |i-j|>1,\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for } 2\leqslant i\leqslant k-1,\\ T_1T_2T_1T_2 &= T_2T_1T_2T_1,\\ T_1^2 &= (p-p^{-1})T_1+1,\\ T_i^2 &= (q-q^{-1})T_i+1 \quad \text{for } 1\leqslant i\leqslant k. \end{aligned} \tag{4.19}$$

Analogues of Jucys–Murphy elements. For each $1\leqslant k\leqslant n$, define

$$M_k = T_k \ldots T_2T_1T_2 \ldots T_k. \tag{4.20}$$

The longest element $w_{k,0}$ in the Weyl group WB_k acts by -1 in the reflection representation. Following (1.8) and Proposition 2.4, we define sets $Z_k = \{z_k\}$ for $1\leqslant k\leqslant n$, where z_k is the central element of $HB_k(p^2, q^2)$ given by

$$z_k = T_{w_{k,0}} = M_kM_{k-1} \ldots M_2M_1. \tag{4.21}$$

Since the elements z_k, for $1\leqslant k\leqslant n$, all commute in $HB_n(p^2, q^2)$, it follows that the elements M_k, for $1\leqslant k\leqslant n$, all commute with each other.

REMARK 4.22. The elements M_k appear in Hoefsmit's thesis [**17**, Proposition 3.3.3] and also in work of Ariki and Koike [**2**], Ariki [**1**], Broué and Malle [**4**], and Dipper, James and Murphy [**13**]. These elements can be viewed as the quantized versions of the elements in (4.13).

Seminormal representations. Let $P_1\subseteq P_2\subseteq \ldots \subseteq P_n$ be the path algebras (over $\mathbb{C}(p, q)$ instead of $\mathbb{C}$), defined in (1.2), which are associated to the diagram Γ which describes the restriction rules for the chain $WB_1\subseteq \ldots \subseteq WB_n$. For each double partition (α, β) such that $|\alpha|+|\beta| = n$, let

$$V^{(\alpha,\beta)} = \mathbb{C}(p, q)\text{-span}\{v_L \mid L\in\mathcal{L}(\alpha, \beta)\}, \tag{4.23}$$

so that the vectors v_L, indexed by standard tableaux L of shape (α, β), form a seminormal basis of the P_n-module $V^{(\alpha,\beta)}$. It follows from Lemma 1.9 that, for any choice of an isomorphism Φ between the path algebra P_n and $HB_n(p^2, q^2)$ such that $\Phi(P_k) = HB_k(p^2, q^2) \subseteq HB_n(p^2, q^2)$ for all $1 \leqslant k \leqslant n$,

$$z_k v_L = T_{w_{k,0}} v_L = c_{k,0}(\lambda^{(k)}) p^{c_{k,s}(\lambda^{(k)})} q^{c_{k,\ell}(\lambda^{(k)})} v_L,$$

if $L = (\lambda^{(1)} \to \dots \to \lambda^{(n)})$. Thus, by Proposition 4.8,

$$\begin{aligned} M_k v_L &= T_{k-1} \dots T_2 T_1 T_2 \dots T_{k-1} v_L = T_{w_{k,0}} T^{-1}_{w_{k-1,0}} v_L \\ &= \operatorname{sgn}(L(k)) p^{\operatorname{sgn}(L(k))} q^{2\,\operatorname{ct}(L(k))} v_L, \end{aligned} \tag{4.24}$$

if $L = (\lambda^{(1)} \to \dots \to \lambda^{(n)})$ is a standard tableau. For each $2 \leqslant k \leqslant n$ and each standard tableau L of size n, define

$$(T_k)_{LL} = (q - q^{-1}) \Big/ \left(1 - \frac{\mathrm{CT}(L(k-1))}{\mathrm{CT}(L(k))}\right), \tag{4.25}$$

where $\mathrm{CT}(b) = \operatorname{sgn}(L(k)) p^{\operatorname{sgn}(L(k))} q^{2\,\operatorname{ct}(L(k))}$, if b is a box in a shape $\lambda = (\alpha, \beta)$.

THEOREM 4.26. *Let (α, β) be a double partition such that $|\alpha| + |\beta| = n$. Define an action of each generator $T_1, \dots, T_n$ of $HB_n(p^2, q^2)$ on $V^{(\alpha,\beta)}$ by defining*

$$\begin{aligned} T_1 v_L &= \mathrm{CT}(L(1)) v_L, \\ T_i v_L &= (T_i)_{LL} v_L + (q^{-1} + (T_i)_{LL}) v_{s_i L} \quad \textit{for } 2 \leqslant i \leqslant n, \end{aligned} \tag{4.27}$$

where $s_i L$ is the same standard tableau as L except that the positions of i and $i-1$ are switched in $s_i L$. If $s_i L$ is not standard, then we define $v_{s_i L} = 0$. This action extends to a well-defined action of $HB_n(p^2, q^2)$ on $V^{(\alpha,\beta)}$.

Proof. The proof is similar to the proof of Theorem 4.15, in all essential aspects. We shall only give the details for Step 2.

Step 2. It is immediate from the definition of M_k in (4.20) that $M_n = T_n M_{n-1} T_n$, which can be rewritten as

$$T_n^{-1} = M_n^{-1} T_n M_{n-1}.$$

Rewrite T_n^{-1} as $T_n - (q - q^{-1})$, let both sides act on v_L and take the coefficient of v_L in the result. Using (4.24), we get

$$(T_n)_{LL} - (q - q^{-1}) = \mathrm{CT}(L(n))^{-1} (T_n)_{LL}\, \mathrm{CT}(L(n-1)).$$

It follows that $(T_n)_{LL}$ is as given in (4.25).

As in the case of Weyl group WB_n, Theorem 4.18, the following result follows easily.

THEOREM 4.28 (Hoefsmit [**17**, Theorem 2.2.14]). *The $HB_n(p^2, q^2)$-modules $V^{(\alpha,\beta)}$ defined in Theorem 4.26, where (α, β) runs over all ordered pairs of partitions such that $|\alpha| + |\beta| = n$, form a complete set of non-isomorphic* *irreducible modules for the Iwahori–Hecke algebra* $HB_n(p^2, q^2)$. *For each* (α, β), *the basis* $\{v_L \mid L \in \mathscr{L}(\alpha, \beta)\}$ *is a seminormal basis of the* $HB_n(p^2, q^2)$*-module* $V^{(\alpha,\beta)}$.

5. *Type* D_n *with* $n \geqslant 4$

The Weyl group. The Weyl group WD_n of type D_n is the group of signed permutations of $\{1, 2, ..., n\}$ with an even number of signs. More specifically, WD_n consists of all permutations π of $\{-n, ..., -1, 1, ..., n\}$ such that $\pi(-k) = -\pi(k)$ for all $1 \leqslant k \leqslant n$, and an even number of the elements of $\{\pi(1), \pi(2), ..., \pi(n)\}$ are negative. We represent elements of WD_n in cycle notation as permutations of $\{-n, ..., -1, 1, ..., n\}$.

The elements

$$\tilde{s}_1 = (1, -2)(2, -1), \quad \text{and} \quad \tilde{s}_i = (i-1, i)(-(i-1), -i) \ \text{ for } 2 \leqslant i \leqslant n,$$

generate WD_n and satisfy the relations

$$\begin{aligned} \tilde{s}_i\tilde{s}_j &= \tilde{s}_j\tilde{s}_i \quad \text{for } |i-j| > 1 \quad \text{and } i, j > 1, \\ \tilde{s}_1\tilde{s}_j &= \tilde{s}_j\tilde{s}_1 \quad \text{if } j \neq 3, \\ \tilde{s}_1\tilde{s}_3\tilde{s}_1 &= \tilde{s}_3\tilde{s}_1\tilde{s}_3, \\ \tilde{s}_i\tilde{s}_{i+1}\tilde{s}_i &= \tilde{s}_{i+1}\tilde{s}_i\tilde{s}_{i+1} \quad \text{for } 2 \leqslant i \leqslant n-1, \\ \tilde{s}_i^2 &= 1 \quad \text{for } 1 \leqslant i \leqslant n. \end{aligned} \tag{5.1}$$

The Weyl group WD_n can be realized as a normal subgroup of the Weyl group WB_n of index 2 by defining

$$\tilde{s}_1 = s_1 s_2 s_1, \quad \text{and} \quad \tilde{s}_i = s_i \ \text{ for } 2 \leqslant i \leqslant n, \tag{5.2}$$

where s_i, for $1 \leqslant i \leqslant n$, are as in (4.1).

Double partitions and standard tableaux. We shall use the same notation for partitions, double partitions, shapes, and tableaux as in § 4. For each standard tableau $L = (L^\alpha, L^\beta)$ of shape (α, β) define σL to be the standard tableau of shape (β, α) given by $\sigma L = (L^\beta, L^\alpha)$,

$$\begin{aligned} \sigma\colon \mathscr{L}(\alpha, \beta) &\to \mathscr{L}(\beta, \alpha), \\ (L^\alpha, L^\beta) &\mapsto (L^\beta, L^\alpha). \end{aligned} \tag{5.3}$$

The map σ is an involution on the set of standard tableaux whose shape is a double partition.

Which chain? One finds that it is more natural to use the representation theory of the Weyl groups WB_n and the fact that WD_n is a normal subgroup of index 2 in WB_n rather than to try to choose an appropriate chain of root systems leading up to D_n. The reason for this is that one wants to have an approach that treats all of the groups WD_n, for $n \geqslant 4$, uniformly. Otherwise one must distinguish the cases when n is even and when n is odd. In the end we shall find a set of commuting elements in the group algebra of WD_n, analogues of the Jucys–Murphy elements, which determine a complete set of irreducible representations.

Representations. We shall retain the notation from § 4 for the sign and the content of a box in a double partition. Let $\lambda = (\alpha, \beta)$ be a double partition such that $|\alpha| + |\beta| = n$. As in (4.12), let

$$V^{(\alpha,\beta)} = \mathbb{C}\text{-span}\{v_L \mid L \in \mathscr{L}(\alpha, \beta)\}, \tag{5.4}$$

so that the vectors v_L form a basis of the vector space $V^{(\alpha,\beta)}$ indexed by standard tableaux L of shape (α, β).

For each standard tableau L, define

$$(s_k)_{LL} = \frac{1 + \mathrm{sgn}(L(k))\mathrm{sgn}(L(k-1))}{\mathrm{ct}(L(k)) - \mathrm{ct}(L(k-1))}, \quad \text{for } 2 \leq k \leq n, \tag{5.5}$$

as in (4.14). Recall, Theorem 4.15, that there is an action of WB_n on the vector space $V^{(\alpha,\beta)}$. Restricting this action to WD_n gives

$$\begin{aligned} \tilde{s}_1 v_L &= s_1 s_2 s_1 v_L = (s_2)_{LL} v_L - (1 + (s_2)_{LL}) v_{s_2 L}, \\ \tilde{s}_i v_L &= s_i v_L = (s_i)_{LL} v_L + (1 + (s_i)_{LL}) v_{s_i L} \quad \text{for } 2 \leq i \leq n, \end{aligned} \tag{5.6}$$

for each $L \in \mathscr{L}(\alpha, \beta)$, where, as in the case of type B_n, we define $v_{s_i L} = 0$ if $s_i L$ is not standard. In deriving the first formula of (5.6) it is helpful to note that $\mathrm{sgn}(L(1)) = \pm 1$, and $\mathrm{sgn}(s_2 L(1)) = -\mathrm{sgn}(L(1))$ if $s_2 L$ is standard.

Now suppose n is even, and let α be a partition such that $2|\alpha| = n$. Define

$$\begin{aligned} V^{(\alpha,\alpha)^+} &= \mathbb{C}\text{-span}\{v_L^+ = v_L + v_{\sigma L} \mid L \in \mathscr{L}(\alpha, \alpha)\} \subseteq V^{(\alpha,\alpha)}, \\ V^{(\alpha,\alpha)^-} &= \mathbb{C}\text{-span}\{v_L^- = v_L - v_{\sigma L} \mid L \in \mathscr{L}(\alpha, \alpha)\} \subseteq V^{(\alpha,\alpha)}. \end{aligned} \tag{5.7}$$

The following (well-known) results follow easily from Clifford theory [**8**] since WD_n is a subgroup of index 2 in WB_n and σ commutes with the action of WD_n on the vectors v_L, where $L \in \mathscr{L}$.

PROPOSITION 5.8. (a) *For each pair of partitions (α, β) such that $|\alpha| + |\beta| = n$, $V^{(\alpha,\beta)}$ and $V^{(\beta,\alpha)}$ are isomorphic WD_n-modules.*

(b) *For each partition α such that $2|\alpha| = n$, the subspaces $V^{(\alpha,\alpha)^\pm}$ are WD_n-submodules of $V^{(\alpha,\alpha)}$, and*

$$V^{(\alpha,\alpha)} \cong V^{(\alpha,\alpha)^+} \oplus V^{(\alpha,\alpha)^-},$$

as WD_n-modules.

THEOREM 5.9 (Young [**35**]). *The modules $V^{(\alpha,\beta)}$, where (α, β) runs over all unordered pairs of partitions such that $\alpha \neq \beta$ and $|\alpha| + |\beta| = n$ and, when n is even, the modules $V^{(\alpha,\alpha)^+}$ and $V^{(\alpha,\alpha)^-}$, where α runs over all partitions such that $2|\alpha| = n$, form a complete set of non-isomorphic irreducible modules for WD_n.*

REMARK 5.10. The involution σ on standard tableaux in (5.3) is a realization of the module isomorphism between the WD_n-modules $V^{(\alpha,\beta)}$ and $V^{(\beta,\alpha)}$, which, in turn, comes from the automorphism of the Dynkin diagram of type D_n.

Instead of defining $V^{(\alpha,\alpha)^{\pm}}$ as in (5.7), let us define them as the quotient spaces

$$V^{(\alpha,\alpha)^{+}} = \frac{V^{(\alpha,\alpha)}}{\langle v_L = v_{\sigma L}\rangle} \quad \text{and} \quad V^{(\alpha,\alpha)^{-}} = \frac{V^{(\alpha,\alpha)}}{\langle v_L = -v_{\sigma L}\rangle}, \tag{5.11}$$

where σ is the involution given in (5.3) and $\langle v_L = v_{\sigma L}\rangle$ and $\langle v_L = -v_{\sigma L}\rangle$ denote the subspaces spanned by the vectors $v_L - v_{\sigma L}$ and $v_L + v_{\sigma L}$ respectively. Clearly the two definitions of the modules $V^{(\alpha,\alpha)^{\pm}}$ are equivalent, the first represents $V^{(\alpha,\alpha)^{\pm}}$ as subspaces of $V^{(\alpha,\alpha)}$, and the second as quotient spaces of $V^{(\alpha,\alpha)}$. The only difference is that for some computations the quotient module approach is easier; one may compute the action as in the formulas in (5.6) and then apply the relations $v_L = \pm v_{\sigma L}$.

Analogues of Jucys–Murphy elements. We have the following theorem.

THEOREM 5.12. *Define elements m_k, for $2 \leqslant k \leqslant n$, in the group algebra of the Weyl group WD_n by*

$$\tilde{m}_{k,1} = (1, -1)(k, -k) \quad \textit{and} \quad \tilde{m}_{k,2} = \sum_{i=2}^{k} (i-1, k) + (i-1, -k)(-(i-1), k),$$

where elements of WD_n are written in cycle notation as permutations of $\{-n, \ldots, -1, 1, \ldots, n\}$. Then the elements $\tilde{m}_{k,1}$ and $\tilde{m}_{k,2}$ all commute with each other in $\mathbb{C}WD_n$ and they act in the representations $V^{(\alpha,\beta)}$ and $V^{(\alpha,\alpha)^{\pm}}$ from (5.6) *and* (5.7) *by*

$\tilde{m}_{k,1} v_L = \mathrm{sgn}(L(1))\mathrm{sgn}(L(k))V_L$, *for all standard tableaux L, and*
$\tilde{m}_{k,1} v_L^{\pm} = \mathrm{sgn}(L(1))\mathrm{sgn}(L(k))v_L^{\pm}$, *for all standard tableaux L of shape* (α, α),
$\tilde{m}_{k,2} v_L = \mathrm{ct}(L(k))V_L$, *for all standard tableaux L, and*
$\tilde{m}_{k,2} v_L^{\pm} = \mathrm{ct}(L(k))v_L^{\pm}$, *for all standard tableaux L of shape* (α, α).

Proof. This follows immediately from (4.13) once one notices that

$$\tilde{m}_{k,1} = m_{1,s} m_{k,s} \quad \text{and} \quad \tilde{m}_{k,2} = m_{k,\ell},$$

where $m_{k,s}$ and $m_{k,\ell}$ are as in (4.5).

Weights. If L is a standard tableau of shape (α, β), define

$$\widetilde{\mathrm{wt}}_1(L) = (\mathrm{ct}(L(1)), \ldots, \mathrm{ct}(L(n))),$$

and

$$\widetilde{\mathrm{wt}}_2'(L) = (\mathrm{sgn}(L(1))^2, \ldots, \mathrm{sgn}(L(1))\mathrm{sgn}(L(n))),$$

where sgn and ct are as given in (4.7).

LEMMA 5.13. *If L is a standard tableau then there is only one other standard tableau L' such that $\widetilde{\mathrm{wt}}_1(L') = \widetilde{wt}_1(L)$ and $\widetilde{\mathrm{wt}}_2'(L') = \widetilde{\mathrm{wt}}_2'(L)$. This standard tableau is $L' = \sigma L$, where σ is the involution defined in* (5.3).

Proof. Let $\widetilde{\mathrm{wt}}_1$ and $\widetilde{\mathrm{wt}}_2$ be as defined in Proposition 4.11. It follows from Proposition 4.11 that $\widetilde{\mathrm{wt}}_1(L)$ and $\widetilde{\mathrm{wt}}_2(L)$ uniquely determine L. Since $\widetilde{\mathrm{wt}}_2'(L)$ is always either $+1$ or -1 times every entry in the sequence $\widetilde{\mathrm{wt}}_2(L)$, it follows that there can be at most two standard tableaux L and L' with the same weights $\widetilde{\mathrm{wt}}_1(L')=\widetilde{\mathrm{wt}}_1(L)$ and $\widetilde{\mathrm{wt}}'_2(L')=\widetilde{\mathrm{wt}}'_2(L)$. On the other hand, it is immediate that one always has that $\widetilde{\mathrm{wt}}'_2(\sigma L)=\widetilde{\mathrm{wt}}'_2(L)$ since $\mathrm{sgn}(\sigma L(k))=-\mathrm{sgn}(L(k))$ for all k.

Iwahori–Hecke algebras $HD_n(q^2)$

Let q be an indeterminate. The Iwahori–Hecke algebra $HD_n(q^2)$ of type D_n is the associative algebra with 1 over the field $\mathbb{C}(q)$ given by generators $\tilde{T}_1, \tilde{T}_2, ..., \tilde{T}_n$ and relations

$$\begin{aligned}
\tilde{T}_i\tilde{T}_j &= \tilde{T}_j\tilde{T}_i \quad \text{for } |i-j|>1,\ i,j>1,\\
\tilde{T}_1\tilde{T}_j &= \tilde{T}_j\tilde{T}_1 \quad \text{if } j\neq 3,\\
\tilde{T}_1\tilde{T}_3\tilde{T}_1 &= \tilde{T}_3\tilde{T}_1\tilde{T}_3,\\
\tilde{T}_i\tilde{T}_{i+1}\tilde{T}_i &= \tilde{T}_{i+1}\tilde{T}_i\tilde{T}_{i+1} \quad \text{for } 2\leqslant i\leqslant n-1,\\
\tilde{T}_i^2 &= (q-q^{-1})\tilde{T}_i+1 \quad \text{for } 1\leqslant i\leqslant n.
\end{aligned} \tag{5.14}$$

Let $HB_n(1, q^2)$ be the algebra over $\mathbb{C}(q)$ defined by generators $T_1, ..., T_n$ and relations as in (4.19) except with $p=1$. Define

$$\tilde{T}_1 = T_1T_2T_1, \quad \text{and} \quad \tilde{T}_i = T_i \ \text{for } 2\leqslant i\leqslant n. \tag{5.15}$$

Then one checks that with these definitions the $\tilde{T}_i$ satisfy the relations in (5.14). The elements $\tilde{T}_i$, for $1\leqslant i\leqslant n$, generate a subalgebra of the algebra $HB_n(1, q^2)$ which is isomorphic to the algebra $HD_n(q^2)$.

Representations. One derives the representation theory of the Iwahori–Hecke algebra $HD_n(q^2)$ using the results from §4 and the fact that $HD_n(q^2)$ is a subalgebra of the Iwahori–Hecke algebra $HB_n(1, q^2)$. The procedure is exactly as for the case (5.4)–(5.9) of the Weyl groups $WD_n \subseteq WB_n$.

Let $V^{(\alpha,\beta)}$ be as in (4.23). As in (4.25), for $2\leqslant k\leqslant n$ and each standard tableau L, define

$$(\tilde{T}_k)_{LL} = (T_k)_{LL} = (q-q^{-1})\Big/\left(1-\frac{\mathrm{CT}(L(k-1))}{\mathrm{CT}(L(k))}\right) \tag{5.16}$$

where $\mathrm{CT}(b)=\mathrm{sgn}(L(k))q^{2\,\mathrm{ct}(L(k))}$, for a box b in a shape $\lambda=(\alpha,\beta)$. Restricting the action (4.27) of $HB_n(1, q^2)$ to $HD_n(q^2)$ gives

$$\begin{aligned}
\tilde{T}_1v_L &= T_1T_2T_1v_L = (\tilde{T}_2)_{LL}v_L-(q^{-1}+(\tilde{T}_2)_{LL})v_{s_2L},\\
\tilde{T}_iv_L &= T_iv_L = (\tilde{T}_i)_{LL}v_L+(q^{-1}+(\tilde{T}_i)_{LL})v_{s_iL} \quad \text{for } 2\leqslant i\leqslant n,
\end{aligned} \tag{5.17}$$

for each $L\in\mathscr{L}(\alpha,\beta)$, where, as in the case of type B_n, we define $v_{s_iL}=0$ if s_iL is not standard. In deriving the first formula of (5.17) it is helpful to note that $\mathrm{CT}(L(1))=\pm 1$, and $\mathrm{CT}(s_2L(1))=-\mathrm{CT}(L(1))$ if s_2L is standard.

If n is even, and α is a partition such that $2|\alpha|=n$, define $V^{(\alpha,\alpha)^\pm}$ as in (5.7) except over the field $\mathbb{C}(q)$. The following results can be proved by 'setting $q=1$' and then using Proposition 5.8 and Theorem 5.9.

PROPOSITION 5.18 (Hoefsmit [**17**, Lemmas 2.3.3 and 2.3.5]). (*a*) *For each pair of partitions (α,β) such that $|\alpha|+|\beta|=n$, $V^{(\alpha,\beta)}$ and $V^{(\beta,\alpha)}$ are isomorphic $HD_n(q^2)$-modules.*

(b) *For each partition* α *such that* $2\,|\alpha| = n$, *the subspaces* $V^{(\alpha,\alpha)^{\pm}}$ *are* $HD_n(q^2)$-*submodules of* $V^{(\alpha,\alpha)}$, *and*

$$V^{(\alpha,\alpha)} \cong V^{(\alpha,\alpha)^{+}} \oplus V^{(\alpha,\alpha)^{-}},$$

as $HD_n(q^2)$-*modules.*

THEOREM 5.19 (Hoefsmit [**17**, Theorem 2.3.9]). *The modules* $V^{(\alpha,\beta)}$, *where* (α, β) *runs over all unordered pairs of partitions such that* $\alpha \neq \beta$ *and* $|\alpha| + |\beta| = n$ *and, when n is even, the modules* $V^{(\alpha,\alpha)^{+}}$ *and* $V^{(\alpha,\alpha)^{-}}$, *where* α *runs over all partitions such that* $2\,|\alpha| = n$, *form a complete set of non-isomorphic irreducible modules for* $HD_n(q^2)$.

Analogues of Jucys–Murphy elements. Define

$$\begin{aligned} &\tilde{M}_1 = 1, \quad \tilde{M}_2 = \tilde{T}_2\tilde{T}_1, \quad \text{and} \\ &\tilde{M}_k = \tilde{T}_k\tilde{T}_{k-1} \ldots \tilde{T}_3\tilde{T}_2\tilde{T}_1\tilde{T}_3\tilde{T}_4 \ldots \tilde{T}_{k-1}\tilde{T}_k, \quad \text{for } 3 \leqslant k \leqslant n. \end{aligned} \tag{5.20}$$

If w_0 is the longest element of the Weyl group WD_n, then $\tilde{T}_{w_0} = \tilde{M}_n\tilde{M}_{n-1} \ldots \tilde{M}_1$ is the corresponding element in the Iwahori–Hecke algebra $HD_n(q^2)$.

THEOREM 5.21. *The action of the element* $\tilde{M}_k$ *in the irreducible representations given by Theorem* 5.19 *is*

$$\tilde{M}_k v_L = \mathrm{CT}(L(1))\mathrm{CT}(L(k))v_L, \quad \textit{for all standard tableaux } L,$$

and

$$\tilde{M}_k v_L^{\pm} = \mathrm{CT}(L(1))\mathrm{CT}(L(k))v_L^{\pm}, \quad \textit{for all standard tableaux } L \textit{ of shape } (\alpha, \alpha).$$

Proof. Let M_k be the elements of $HB_n(1, q^2)$ given by (4.20), and use the imbedding of $HD_n(q^2)$ into $HB_n(1, q^2)$. The case $k = 1$ is trivial, since $\mathrm{CT}(L(1)) = \pm 1$. For $k = 2$, observe that $\tilde{M}_2 = \tilde{T}_2\tilde{T}_1 = T_2T_1T_2T_1 = M_2M_1$. For $3 \leqslant k \leqslant n$, note that T_1 commutes with $T_3, T_4, \ldots$ in $HB_n(1, q^2)$, and thus

$$\begin{aligned} \tilde{M}_k &= \tilde{T}_k\tilde{T}_{k-1} \ldots \tilde{T}_3\tilde{T}_2\tilde{T}_1\tilde{T}_3\tilde{T}_4 \ldots \tilde{T}_{k-1}\tilde{T}_k \\ &= T_kT_{k-1} \ldots T_3T_2(T_1T_2T_1)T_3T_4 \ldots T_{k-1}T_k \\ &= T_kT_{k-1} \ldots T_3T_2T_1T_2T_3T_4 \ldots T_{k-1}T_kT_1 = M_kM_1. \end{aligned}$$

The result now follows from the definition of the action of $HB_n(1, q^2)$ and of $HD_n(q^2)$ on irreducible modules and (4.24).

6. *Type* G_2

The chain $A_0 \subseteq A_1 \subseteq G_2$. The Weyl group WG_2 is the dihedral group of order 12. The group WG_2 can be presented by generators s_1, s_2 and relations

$$s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1, \quad \text{and} \quad s_i^2 = 1 \text{ for } i = 1, 2.$$

The irreducible representations of the dihedral group WG_2 can be indexed by the labels

$$\hat{G}_2 = \{\phi_{1,0}, \phi_{1,6}, \phi'_{1,3}, \phi''_{1,3}, \phi_{2,1}, \phi_{2,2}\}$$

and the character table of the group WG_2 is as shown in Table 1.

TABLE 1

	1	s_1	s_2	s_1s_2	$s_1s_2s_1s_2$	$s_1s_2s_1s_2s_1s_2$
$\phi_{1,0}$	1	1	1	1	1	1
$\phi_{1,6}$	1	-1	-1	1	1	1
$\phi'_{1,3}$	1	1	-1	-1	1	-1
$\phi''_{1,3}$	1	-1	1	-1	1	-1
$\phi_{2,1}$	2	0	0	1	-1	-2
$\phi_{2,2}$	2	0	0	-1	-1	2

The chain of root systems $A_0 \subseteq A_1 \subseteq G_2$ corresponds to the chain of Weyl groups $S_1 \subseteq S_2 \subseteq WG_2$, where S_1 and S_2 are symmetric groups. The graph Γ, as defined in (1.1), corresponding to the inclusion $S_1 \subseteq S_2 \subseteq WG_2$ is given by Fig. 6.1. We have indexed the representations of the symmetric groups S_1 and S_2 by partitions as in § 3.

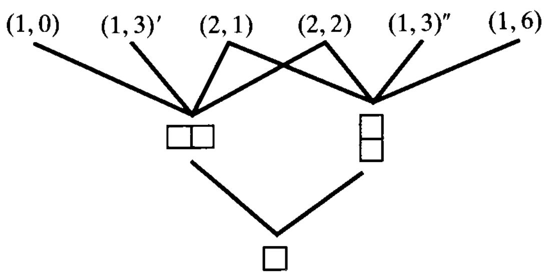

FIG. 6.1

Analogues of Jucys–Murphy elements. Following (1.8) and (2.1), let us compute the sets Z_k for this case. In the root system A_1 all roots are the same length and the longest element $w_{1,0}$ in the Weyl group S_2 acts by -1 in the reflection representation. In the root system G_2 we have both long and short roots and the longest element $w_{2,0} = s_1s_2s_1s_2s_1s_2$ of the Weyl group WG_2 acts by -1 in the reflection representation. Let

$$Z_1 = \{z_{1,0}\} \quad \text{and} \quad Z_2 = \{z_{2,s}, z_{2,\ell}, z_{2,0}\},$$

where

$$\begin{aligned} z_{1,0} &= w_{1,0} = s_1, \\ z_{2,s} &= \sum_{\alpha \in (G_2)_s^+} s_\alpha = s_1 + s_2s_1s_2 + s_1s_2s_1s_2s_1, \\ z_{2,\ell} &= \sum_{\alpha \in (G_2)_\ell^+} s_\alpha = s_2 + s_1s_2s_1 + s_2s_1s_2s_1s_2, \\ z_{2,0} &= w_{2,0} = s_1s_2s_1s_2s_1s_2. \end{aligned} \tag{6.2}$$

In (6.2) the sets $(G_2)_s^+$ and $(G_2)_l^+$ are, respectively, the sets of short and long positive roots in the root system G_2. The elements $z_{k,j}$ in (6.2) are the appropriate G_2-analogues of the Jucys–Murphy elements in (3.5).

Weights. Following Lemma 1.9 and (2.2), we use the character table of WG_2 to compute

$$\begin{aligned}
&c_{2,s}(\phi_{1,0}) = 3, && c_{2,\ell}(\phi_{1,0}) = 3, && c_{2,0}(\phi_{1,0}) = 1,\\
&c_{2,s}(\phi_{1,6}) = -3, && c_{2,\ell}(\phi_{1,6}) = -3, && c_{2,0}(\phi_{1,6}) = 1,\\
&c_{2,s}(\phi'_{1,3}) = 3, && c_{2,\ell}(\phi'_{1,3}) = -3, && c_{2,0}(\phi'_{1,3}) = -1,\\
&c_{2,s}(\phi''_{1,3}) = -3, && c_{2,\ell}(\phi''_{1,3}) = 3, && c_{2,0}(\phi''_{1,3}) = -1,\\
&c_{2,s}(\phi_{2,1}) = 0, && c_{2,\ell}(\phi_{2,1}) = 0, && c_{2,0}(\phi_{2,1}) = -1,\\
&c_{2,s}(\phi_{2,2}) = 0, && c_{2,l}(\phi_{2,2}) = 0, && c_{2,0}(\phi_{2,2}) = 1,\\
&c_{1,0}((2)) = 1, && c_{1,0}((1^2)) = -1,
\end{aligned}$$

so that $c_{k,j}(\mu) = \chi^\mu(z_{k,j})/\chi^\mu(1)$ where χ^μ denotes the irreducible character labelled by μ. The *weight* of a path $L = (\square \to \lambda^{(1)} \to \lambda^{(2)})$ in the graph Γ is

$$\mathrm{wt}(L) = (c_{1,0}(\lambda^{(1)}), c_{2,s}(\lambda^{(2)}), c_{2,\ell}(\lambda^{(2)}), c_{2,0}(\lambda^{(2)})). \tag{6.3}$$

PROPOSITION 6.4. *Each path $L = (\square \to \lambda^{(1)} \to \lambda^{(2)})$ in Γ is distinguished by its weight, that is, if L and M are paths in Γ and $L \neq M$ then* $\mathrm{wt}(L) \neq \mathrm{wt}(M)$.

Proof. This follows easily by a direct check.

Proposition 1.12 and Proposition 6.4 together show that the seminormal representations of WG_2 corresponding to the chain of groups $\{1\} \subseteq S_2 \subseteq WG_2$ are essentially determined by the elements $z_{k,j}$ in (6.2) and the constants $c_{k,j}(\mu)$ which appear in (6.3). These representations are given in Theorem 6.7 below.

Seminormal representations. Let $P_1 \subseteq P_2$ be the path algebras, defined in § 1, which are associated to the diagram Γ in Fig. 6.1. For each $\lambda \in \hat{G}_2$, let

$$V^\lambda = \mathbb{C}\text{-span}\{v_L \mid L \in \mathcal{L}(\lambda)\}, \tag{6.5}$$

so that the vectors v_L, indexed by the paths $L = (\square \to \lambda^{(1)} \to \lambda^{(2)} = \lambda)$ in Γ which end at λ, form a seminormal basis of the irreducible P_2-module V^λ. It follows from Lemma 1.9, that for any choice of an isomorphism Φ between the path algebra P_2 and the $\mathbb{C}WG_2$ such that $\Phi(P_1) = \mathbb{C}S_2 \subseteq \mathbb{C}WG_2$,

$$\begin{aligned}
z_{1,0}v_L &= c_{1,0}(\lambda^{(1)})v_L, & z_{2,s}v_L &= c_{2,s}(\lambda^{(2)})v_L,\\
z_{2,0}v_L &= c_{2,0}(\lambda^{(2)})v_L, & z_{2,\ell}v_L &= c_{2,\ell}(\lambda^{(2)})v_L,
\end{aligned} \tag{6.6}$$

if $L = (\square \to \lambda^{(1)} \to \lambda^{(2)})$ is a path in Γ.

THEOREM 6.7. *Irreducible seminormal representations of the Weyl group WG_2 with respect to the chain $S_1 \subseteq S_2 \subseteq WG_2$ can be given by*

$$\begin{aligned}
\phi_{1,0}(s_1) &= (1), & \phi_{1,6}(s_1) &= (-1),\\
\phi_{1,0}(s_2) &= (1), & \phi_{1,6}(s_2) &= (-1),\\
\phi'_{1,3}(s_1) &= (1), & \phi''_{1,3}(s_1) &= (-1),\\
\phi'_{1,3}(s_2) &= (-1), & \phi''_{1,3}(s_2) &= (1),\\
\phi_{2,1}(s_1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \phi_{2,2}(s_1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
\phi_{2,1}(s_2) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}, & \phi_{2,2}(s_2) &= \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\end{aligned}$$

Proof. For any two paths M and L in the graph Γ which end at the same label λ, let $(s_k)_{ML}$ denote the coefficient of v_M in $s_k v_L$. The constant $(s_k)_{ML}$ is a matrix entry in the matrix for s_k in the irreducible representation labelled by λ. It follows from Proposition 1.12(a) and Proposition 6.4 that the diagonal entries of these matrices must be determined by the equations in (6.6) and that the off-diagonal entries are determined up to a constant.

The matrices giving the one-dimensional representations are easily obtained from the relations in (6.6). Let us explain how one derives the matrices for the two-dimensional case.

(a) The matrices for s_1 are determined by (6.6).

(b) From the definitions (6.2), one gets easily that

$$z_{2,\ell} = s_2 + z_{1,\ell} s_2 z_{1,\ell} + z_{2,0} z_{1,\ell}.$$

Let both sides of this equation act on v_L and take the coefficient of v_L to get, via (6.6), the equation

$$c_{2,\ell}(\lambda^{(2)}) = (s_2)_{LL} + c_{1,\ell}(\lambda^{(1)})(s_2)_{LL} c_{1,\ell}(\lambda^{(1)}) + c_{2,0}(\lambda^{(2)}) c_{1,\ell}(\lambda^{(1)}).$$

It follows that

$$(s_2)_{LL} = \frac{c_{2,\ell}(\lambda^{(2)}) - c_{2,0}(\lambda^{(2)}) c_{1,0}(\lambda^{(1)})}{1 + c_{1,0}(\lambda^{(1)})^2}.$$

All of the diagonal entries in the matrices for s_2 are determined by this formula and the values in (6.3).

(c) Let both sides of the equation $s_2^2 = 1$ act on the vector v_L and take the coefficient of v_L in the result. One gets the equation

$$(s_2)_{LM}(s_2)_{ML} + (s_2)_{LL}^2 = 1,$$

where M is the path to λ in Γ which is not L. It follows that

$$(s_2)_{LM}(s_2)_{ML} = (1 + (s_2)_{LL})(1 - (s_2)_{LL}).$$

Because of the freedom in the choice of the off-diagonal entries, Proposition 1.12(b), it follows that we may choose $(s_2)_{ML} = 1 + (s_2)_{LL} = 1 - (s_2)_{MM}$.

The Iwahori–Hecke algebra $HG_2(p^2, q^2)$

Let p, q be indeterminates. The Iwahori–Hecke algebra $HG_2(p^2, q^2)$ of type G_2 is the associative algebra with 1 over the field $\mathbb{C}(p, q)$ given by generators T_1, T_2

and relations

$$T_1T_2T_1T_2T_1T_2 = T_2T_1T_2T_1T_2T_1, \tag{6.8}$$
$$T_1^2 = (p-p^{-1})T_1 + 1 \quad \text{and} \quad T_2^2 = (q-q^{-1})T_2 + 1.$$

Analogues of Jucys–Murphy elements. If $w_{1,0}$ is the longest element of the Weyl group $S_2 = WA_1$ and $w_{2,0}$ is the longest element of the Weyl group WG_2, then the corresponding elements in the Iwahori–Hecke algebras $HA_1(p^2)$ and $HG_2(p^2, q^2)$ are given by

$$z_1 = T_{w_{1,0}} = T_1 \quad \text{and} \quad z_2 = T_{w_{2,0}} = T_1T_2T_1T_2T_1T_2.$$

Following (1.8) and Proposition 2.4, define sets $Z_k = \{z_k\}$, for $k = 1, 2$.

Seminormal representations. Let $P_1 \subseteq P_2$ be the path algebras over the field $\mathbb{C}(p, q)$ which are associated to the diagram Γ in Fig. 6.1. For each $\lambda \in \hat{G}_2$, let

$$V^\lambda = \mathbb{C}(p, q)\text{-span}\{v_L \mid L \in \mathscr{L}(\lambda)\}, \tag{6.9}$$

so that the vectors v_L, indexed by the paths $L = (\square \to \lambda^{(1)} \to \lambda^{(2)} = \lambda)$ in Γ which end at λ, form a seminormal basis of the irreducible P_2-module V^λ. It follows from Lemma 1.9, that for any choice of an isomorphism Φ between the path algebra P_2 and $HG_2(p^2, q^2)$ such that $\Phi(P_1) = \mathbb{C}HA_1(p^2) \subseteq HG_2(p^2, q^2)$,

$$z_1 v_L = p^{c_{1,0}(\lambda^{(1)})} v_L \quad \text{and} \quad z_2 v_L = c_{2,0}(\lambda^{(2)}) p^{c_{2,s}(\lambda^{(2)})} q^{c_{2,\ell}(\lambda^{(2)})} v_L, \tag{6.10}$$

if $L = (\square \to \lambda^{(1)} \to \lambda^{(2)})$ is a path in Γ.

THEOREM 6.11. *Irreducible seminormal representations of $HG_2(p^2, q^2)$ are given explicitly by*

$$\begin{aligned}
&\phi_{1,0}(T_1) = (p), && \phi_{1,0}(T_2) = (q),\\
&\phi_{1,6}(T_1) = (-p^{-1}), && \phi_{1,6}(T_2) = (-q^{-1}),\\
&\phi'_{1,3}(T_1) = (p), && \phi'_{1,3}(T_2) = (-q^{-1}),\\
&\phi''_{1,3}(T_1) = (-p^{-1}), && \phi''_{1,3}(T_2) = (q),\\
&\phi_{2,1}(T_1) = \begin{pmatrix} p & 0 \\ 0 & -p^{-1} \end{pmatrix}, && \phi_{2,1}(T_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\\
&\phi_{2,2}(T_1) = \begin{pmatrix} p & 0 \\ 0 & -p^{-1} \end{pmatrix}, && \phi_{2,2}(T_2) = \begin{pmatrix} x & y \\ z & w \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
a &= \frac{1 + p^{-1}(q - q^{-1})}{p + p^{-1}}, & x &= \frac{-1 + p^{-1}(q - q^{-1})}{p + p^{-1}},\\
b &= q - a, & y &= q - x,\\
c &= q^{-1} + a, & z &= q^{-1} + x,\\
d &= (q - q^{-1}) - a, & w &= (q - q^{-1}) - x.
\end{aligned}$$

Proof. The proof is entirely similar to the proof of Theorem 6.7. Let us explain only how to get the entries in the matrices $\phi_{2,1}(T_2)$ and $\phi_{2,2}(T_2)$. Let $\lambda = (2, 1)$ or

$\lambda=(2,2)$ and suppose that L and M are the two paths to λ in Γ. From the definition of the element z_2 we get

$$T_1^{-1}T_2^{-1}T_1^{-1}z_2=T_2T_1T_2.$$

By rewriting T_2^{-1} as $T_2-(q-q^{-1})$ we have

$$T_1^{-1}T_2T_1^{-1}z_2-(q-q^{-1})T_1^{-2}z_2=T_2T_1T_2.$$

Taking the (L,M) entry of each side of the above equation gives

$$\begin{aligned}(T_1)_{LL}^{-1}(T_2)_{LM}(T_1)_{MM}^{-1}c_{2,0}(\lambda)-0\\ =(T_2)_{LL}(T_1)_{LL}(T_2)_{LM}+(T_2)_{LM}(T_1)_{MM}(T_2)_{MM}.\end{aligned}$$

Since these representations are irreducible, it follows that $(T_2)_{LM}\neq 0$. Dividing by $(T_2)_{LM}$ and using the fact that $(T_1)_{LL}=p$ and $(T_1)_{MM}=-p^{-1}$, we get the equation

(i) $$-c_{2,0}(\lambda)=p(T_2)_{LL}-p^{-1}(T_2)_{MM}.$$

Now, the equation $T_2^2=(q-q^{-1})T_2+1$ forces the trace of the matrix of T_2 to be $q-q^{-1}$, and so

(ii) $$(T_2)_{LL}+(T_2)_{MM}=q-q^{-1}.$$

The values for x and a in the statement of the theorem now follow easily from (i) and (ii). The values of the off-diagonal entries are determined (up to a constant, see Proposition 1.12(a)) by the equation

$$(T_2)_{LM}(T_2)_{ML}=(q^{-1}+(T_2)_{LL})(q-(T_2)_{MM}).$$

This equation is obtained by taking the (L,L) entry in the equation

$$T_2^2=(q-q^{-1})T_2+1.$$

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Department of Mathematics
Fine Hall
Princeton University
Princeton
New Jersey 08544
U.S.A.

E-mail: rama@math.princeton.edu