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Bitraces for $GL_n(\mathbb{F}_q)$ and the Iwahori-Hecke algebra of type A_{n-1}

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ABSTRACT

Let $H_n(q)$ be the Iwahori–Hecke algebra of the symmetric group S_n. Let $\mathbb{F}_q$ be a finite field with q elements and let B be the Borel subgroup of upper triangular matrices in the general linear group $G = GL_n(\mathbb{F}_q)$. Let 1_B^G denote the trivial representation of B induced to G. Then $H_n(q)$ has a natural action on 1_B^G that commutes with the G-action, and we define the bitrace $\mathrm{btr}(g, a)$ to be the trace of $g \in G$ and $a \in H_n(q)$ acting simultaneously in 1_B^G. For partitions, μ, ν of n, let T_μ be a standard basis element of $H_n(q)$ corresponding to the S_n-conjugacy class μ, and let u_ν be a unipotent element of G with Jordan block structure ν. We give a combinatorial formula for $\mathrm{btr}(u_\nu, T_\mu)$ as a weighted sum of column strict tableaux of shape ν and content μ. This bitrace also essentially counts $\mathbb{F}_q$-rational points in the intersection of a conjugacy class with a Schubert cell, provides a new proof of the Frobenius formula for characters of $H_n(q)$, and gives a natural pairing between the conjugacy classes of S_n and the unipotent classes in G.

0. INTRODUCTION

Motivation for this paper

In the original work of Frobenius, where he determined the irreducible characters of the symmetric groups, one of the key features was a formula

$$(0.1) \qquad p_\mu = \sum_{\lambda \vdash n} \chi_{S_n}^\lambda(\mu) s_\lambda,$$

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which says that the transition matrix between the power sum symmetric functions and the Schur functions is the character table of the symmetric group. Later Schur explained that this formula was coming from the simultaneous trace of two commuting actions:

(a) $GL_r(\mathbb{C})$ acting on $V^{\otimes n}$, where $V = \mathbb{C}^r$ is the natural representation of $GL_r(\mathbb{C})$, and

(b) the symmetric group S_n acting on $V^{\otimes n}$ by permuting the tensor factors.

If $g \in GL_r(\mathbb{C})$ and $\gamma \in S_n$ is a permutation of cycle type μ, then the trace of $g\gamma$ on $V^{\otimes n}$ is

$$(0.2) \qquad \mathrm{btr}_{V^{\otimes n}}(g, \gamma) = p_\mu(x_1, \ldots, x_r),$$

where p_μ is a power sum symmetric formula and $x_1, \ldots, x_r$ are the eigenvalues of g. The 'bitrace' $\mathrm{btr}_{V^{\otimes n}}$ is simultaneously a trace on $GL_r(\mathbb{C})$ and a trace on S_n. Combined with the fact that $GL_r(\mathbb{C})$ and S_n are full centralizers of each other on $V^{\otimes n}$, formula (0.2) forces formula (0.1), where the Schur function s_λ is an irreducible character of $GL_r(\mathbb{C})$.

The same technique works well for computing the irreducible characters of the Iwahori–Hecke algebra of type A_{n-1} (see [Ra1]). The quantum group $U_q(\mathfrak{gl}_r)$ takes the place of $GL_r(\mathbb{C})$, the Iwahori–Hecke algebra $H_n(q)$ takes the place of S_n and both algebras are centralizers of each other on $V^{\otimes n}$ where V is the 'natural' representation of the quantum group. A bitrace computation analogous to (0.2) yields an analogue of the Frobenius formula (0.1). This generalization of Frobenius' formula was independently obtained (using several different methods) by Gyoja [Gy], King–Wybourne [KW], Ram [Ra1], Vershik–Kerov [VK], and Ueno–Shibukawa [US].

A priori, it is not clear how to generalize the Frobenius–Schur approach to compute the irreducible characters of Iwahori–Hecke algebras of other types. This paper is a first step in this direction. We have computed a new bitrace for the Iwahori–Hecke algebra of type A. Although the Frobenius–Schur bitrace does not make sense for Iwahori–Hecke algebras of general type, the bitrace that we study in this paper does, and it is our hope that the methods of this paper will eventually yield general formulas for computing the irreducible characters of all the Iwahori–Hecke algebras.

Let $\mathbb{F}_q$ be a field with q elements. We compute the bitrace of $GL_n(\mathbb{F}_q)$ and the Iwahori–Hecke algebra of type A_{n-1} acting on the representation 1_B^G, where B is the subgroup of upper triangular matrices in $GL_n(\mathbb{F}_q)$ and 1_B^G is the trivial representation of B induced to $GL_n(\mathbb{F}_q)$. In Section 3 we derive a combinatorial formula for this bitrace as a weighted sum of column strict tableaux.

In Section 4 we show that our new formula is equivalent to the original 'Frobenius formula' for the irreducible characters of the Iwahori–Hecke algebras of type A. This lends credibility to our idea that this type of bitrace formula will be a good tool for computing the irreducible characters of Iwahori–Hecke algebras of general type. In Section 2, we derive several other interpretations of the bitrace.

In general type the situation will be the following: Let G be a finite Chevalley group and let B be a Borel subgroup of G. Let 1_B^G be the trivial representation of B induced to G. The Iwahori–Hecke algebra H acts on 1_B^G and commutes with the action of G. These two actions are full centralizers of each other, and one would like to give a formula for the bitrace (simultaneous trace) of these two actions. With this more general goal in mind we have, when possible, presented our type A computations using algebraic group notations in order to indicate how the computation might be done in general type.

Some of the results of Kawanaka [Ka] are related to the results in this paper. In particular, interpretations (2) and (3) and special case (3) which we have given in Section 2 appear in [Ka] Lemma 3.6, the remark after formula (7.1), and formula (7.1), respectively.

We thank F. Digne for conversations which elucidated formula (2.3), T. Springer for explaining the relationship of formula (2.3) to the variety $\bar{\mathcal{F}}_{u,w}$ in Section 2 and pointing us to the paper of Kawanaka [Ka], and I.G. Macdonald for invaluable help with the symmetric function approach in Section 4. We also thank J. Ramagge for energetic assistance with some messy calculations that we were doing at the beginning of this project.

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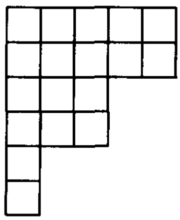
Remarks on the results in this paper

(1) Surprisingly, the matrix of polynomials determined by the bitraces is upper triangular with respect to appropriate orderings on the unipotent classes of $GL_n(\mathbb{F}_q)$ and on the conjugacy classes of the symmetric group. This fact makes one wonder whether a similar phenomenon might occur in general type. Is there an ordering on the unipotent classes of G (possibly by closure relations as in [Ca2]) and an ordering on the conjugacy classes of the Weyl group such that the matrix of the bitraces is upper triangular? If so, there is a natural correspondence between unipont classes in G and conjugacy classes in W. It is possible that this is an incarnation of the map from the unipotent classes to the conjugacy classes of the Weyl group which was given by Kazhdan and Lusztig [KL]?

(2) A consequence of our calculations in Section 4 has been to show that the matrix of Kostka–Foulkes polynomials appears naturally as the portion of the character table of $GL_n(\mathbb{F}_q)$ formed by the characters which appear in 1_B^G evaluated at the unipotent classes of $GL_n(\mathbb{F}_q)$ (see Theorem 4.9 (c)). This formula is well-known and appears, for example, in [Lu], formula (2.2). This interpretation of the Kostka–Foulkes polynomials indicates a natural generalization of these polynomials to other Lie types, one which, to our knowledge, has not been studied by combinatorialists.

Statement of the main result

We will use the notations for partitions and tableaux from [Mac]. A partition λ of n, denoted $\lambda \vdash n$, is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ where the parts λ_i are nonnegative integers satisfying $\sum_i \lambda_i = n$. We identify λ with its Ferrers diagram, given by λ_i left justified boxes in the ith row. For example, the Ferrers diagram of $\lambda = (553311)$ is



The length $\ell(\lambda)$ is the number of nonzero parts of λ, $m_i(\lambda)$ is the number of parts of λ that are equal to i, and

$$(0.3) \qquad n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i.$$

If ν and ρ are partitions and $\rho_i \leq \nu_i$ for each i, then we say that $\rho \subseteq \nu$. The skew diagram $\nu/\rho = (\nu_1 - \rho_1, \nu_2 - \rho_2, \dots)$ consists of the boxes in ν which are not in ρ. A horizontal strip is a skew diagram ν/ρ which has at most one box in each column. If $\rho \subseteq \nu$ and ν/ρ is a horizontal strip, define

$$(0.4) \qquad \mathrm{wt}(\nu/\rho) = \prod_{i \in I} (1 - q^{-m_i(\nu)}),$$

where I is the set of i such that ν/ρ has a box in column i and does not have a box in column $i+1$.

A *column strict tableau* of shape ν is a filling of boxes in the Ferrers diagram of ν with positive integers such that the rows of T are weakly increasing and the columns of T are strictly increasing. The content of T is the sequence $\mu = (\mu_1, \mu_2, \dots)$ where μ_i is the number of i's in T. For example

$$T = \begin{array}{llllllllllllll} 1&1&1&1&1&1&1&1&2&2&3&4&4&4 \\ 2&2&2&2&2&2&3&3&4&4&5&5&5 \\ 3&3&3&3&6&6 \\ 4&4&5&5 \\ 5&5&6 \end{array}$$

is a column strict tableau of shape $\nu = (14, 13, 6, 4, 3)$ and content $\mu = (8, 8, 7, 7, 7, 3)$. We will identify T with the sequence of partitions $T = (\emptyset = \nu^{(0)} \subseteq \nu^{(1)} \subseteq \dots \subseteq \nu^{(r)} = \nu)$ where $\nu^{(i)}$ is the partition containing the numbers $\leq i$. For each i, $\nu^{(i)}/\nu^{(i-1)}$ is a horizontal strip. The weight of T is given by

$$(0.5) \qquad \mathrm{wt}(T) = \prod_i \mathrm{wt}(\nu^{(i)}/\nu^{(i-1)}),$$

where the weights $\mathrm{wt}(\nu^{(i)}/\nu^{(i-1)})$ are as defined in (0.4). In our example, $\mathrm{wt}(\nu^{(5)}/\nu^{(4)}) = (1 - q^{-1})(1 - q^{-2})$.

Let B be the Borel subgroup of upper triangular matrices in $GL_n(\mathbb{F}_q)$ and let 1_B^G denote the trivial representation of B induced to G. There is an action of the Iwahori–Hecke algebra of type A_{n-1}, $H_n(q)$, on 1_B^G which commutes with G (see (1.2)). Let μ, ν be partitions of n, let u_ν be a unipotent element of $GL_n(\mathbb{F}_q)$ such that the sizes of the blocks of the Jordan normal form of u_ν are given by ν, and let T_{γ_μ} be a standard basis element of $H_n(q)$ corresponding to the S_n-conjugacy class μ. The following theorem is the main result of this paper. It is proved in Section 3.

Theorem 3.4. *Let $\nu, \mu \vdash n$. Then, with notations as in the previous paragraph, the bitrace of $T_{\gamma_\mu} \in H_n(q)$ and $u_\nu \in G$ on 1_B^G is given by*

$$\mathrm{btr}(u_\nu, T_{\gamma_\mu}) = \frac{q^{n+n(\nu)}}{(q-1)^{\ell(\mu)}} \sum_T \mathrm{wt}(T),$$

where the sum is over all column strict tableaux T of shape ν and content μ and $\mathrm{wt}(T)$ *is given by* (0.5).

1. PRELIMINARIES

The representation 1_B^G

Let $\mathbb{F}_q$ be the finite field with q elements and let $G = GL_n(\mathbb{F}_q)$. Fix B to be the Borel subgroup of upper triangular matrices in G. Let 1_B^G denote the trivial representation of B induced to G. Then

$$1_B^G = \mathbb{C}\text{-span}\{gB \mid gB \in G/B\}$$

is the vector space of formal linear combinations of cosets of B with the action of G given by left multiplication.

The matrices which have the property that

(1.1) *the rightmost nonzero entry in each row is a* 1 *and is the first nonzero entry in its column*

form a set of coset representatives of the cosets in G/B. See the example for $GL_4(\mathbb{F}_q)$, below. This is justified by the fact that every matrix can be 'row reduced' to one of this form by right multiplying by elements of B. The row reduction can be done inductively, one row at a time, where in the ith row one uses the leftmost nonzero entry which has all zeros above it to zero out all the entries to its right.

The permutation $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ is the longest element of the Weyl group $W = S_n$, and $B^- = w_0 B w_0$ is the subgroup of lower triangular matrices in G. If $w \in W$, the B-cosets in the set $B^- w B$ are the ones which have representatives such that the rightmost nonzero entries in each row form the permutation matrix w. For example, in $GL_4(\mathbb{F}_q)$, the B cosets in the sets B^-wB are represented as follows:

$$B^-1B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} B \,\middle|\, \alpha, \beta, \gamma \in \mathbb{F}_q \right\},$$

$$B^-s_1B = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \alpha & \beta & 1 \end{pmatrix} B \,\middle|\, \alpha, \beta \in \mathbb{F}_q, \right\}, \quad B^-s_2B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 0 & 1 \\ \beta & 1 & 0 \end{pmatrix} B \,\middle|\, \alpha, \beta \in \mathbb{F}_q \right\},$$

$$B^-s_2s_1B = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1 \\ 1 & 0 & 0 \end{pmatrix} B \,\middle|\, \alpha \in \mathbb{F}_q, \right\}, \quad B^-s_1s_2B = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} B \,\middle|\, \alpha \in \mathbb{F}_q \right\},$$

$$B^-w_0B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B.$$

(Analogous coset representatives can be found in general Lie type, see [Snb] p. 34, Theorem 4′.)

A *flag* is a sequence $f = (V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n)$ of subspaces of $\mathbb{F}_q^n$ such that $\dim(V_i) = i$ for $0 \le i \le n$. Let

$$\mathcal{F} = \{f = (V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n) \mid \dim(V_i) = i\}.$$

There is a left action of G on $\mathcal{F}$ given by

$$g(V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n) = (gV_0 \subseteq gV_1 \subseteq \ldots \subseteq gV_n), \quad \text{for all } g \in G.$$

Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{F}_q^n$, i.e., e_i is the column vector with a 1 in the ith row and all other entries 0. The Borel subgroup B is the stabilizer of the standard flag

$$f_1 = (0, \mathrm{span}\{e_1\}, \mathrm{span}\{e_1, e_2\}, \ldots, \mathrm{span}\{e_1, e_2, \ldots, e_n\}),$$

so $\mathcal{F}$ can be identified with the quotient G/B,

$$\begin{aligned} G/B &\longleftrightarrow \mathcal{F} \\ gB &\longleftrightarrow gf_1. \end{aligned}$$

Note that $gf_1 = (V_0 \subseteq \ldots \subseteq V_n)$ where V_i is the span of the first i columns of g.

The Iwahori–Hecke algebra

Iwahori [Iw] began the study of the Hecke algebra of the pair (G, B), i.e., the subalgebra of the group algebra $\mathbb{C}[G]$ given by

$$H_n(q) = \left\{ \sum_{g \in G} a_g g \,\middle|\, a_g \in \mathbb{C}, \text{ and } a_g = a_h \text{ if } BgB = BhB \right\}.$$

The Weyl group $W = S_n$ is the symmetric group of permutation matrices in

$GL_n(\mathbb{F}_q)$. The elements of W form a set of representatives of the B-double cosets in G. Thus $G = \bigcup_{w \in W} BwB$, and the elements

$$T_w = \frac{1}{|B|} \sum_{x \in BwB} x, \qquad w \in W,$$

form a basis of $H_n(q)$. There is a right action of $H_n(q)$ on 1_B^G defined by

$$(1.2) \qquad (gB)T_w = \sum_{\substack{hB \in G/B \\ hB \in gBwB}} hB,$$

for $g \in G$. This action commutes with the action of G since G acts on the left and $H_n(q)$ acts on the right. In fact,

(1.3) *The action of* G *generates the centralizer algebra* $\mathrm{End}_{H_n(q)}(1_B^G)$, *and* $H_n(q) \cong \mathrm{End}_G(1_B^G)$.

Let $s_i = (i, i+1)$ be the transposition in S_n which switches i and $i+1$ and let $T_i = T_{s_i}$. The elements T_i, $1 \le i \le n-1$, generate $H_n(q)$ and satisfy the relations

$$\begin{aligned} T_iT_j &= T_jT_i, && \text{if } |i-j| > 1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && \text{for } 1 \le i \le n-2, \\ T_i^2 &= (q-1)T_i + q, && \text{for } 1 \le i \le n-1. \end{aligned}$$

For more details on the construction of the Iwahori–Hecke algebra, see [CR]. For each $1 \le i \le n-1$ and each $t \in \mathbb{F}_q$, let

$$x_{\alpha_i}(t) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & t & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

be the matrix with 1s on the diagonal, t in the $(i+1, i)$ entry and zeros everywhere else. The double coset Bs_iB is a union of the coset s_iB with the cosets $x_{\alpha_i}(t)B$, where t is in the multiplicative group $\mathbb{F}_q^\times$. Using this, it follows from (1.2) that

$$(1.4) \qquad (gB)T_i = gs_iB + \sum_{t \ne 0} gx_{\alpha_i}(t)B.$$

If we write this equation in terms of flags we get

$$(1.5) \qquad (V_0 \subseteq \ldots \subseteq V_n)T_i = \sum_{W \ne V_i} (V_0 \subseteq \ldots \subseteq V_{i-1} \subseteq W \subseteq V_{i+1} \subseteq \ldots \subseteq V_n),$$

where the sum is over all subspaces $V_i \subseteq W \subseteq V_{i+1}$ such that $\dim(W) = i$ and $W \ne V_i$.

Definition. *Let $u \in G$ and $h \in H_n(q)$. The trace of the action of uh on 1_B^G is*

$$\text{(2.1)} \qquad \operatorname{btr}(u,h) = \sum_{gB \in G/B} u(gB)h|_{gB},$$

where $u(gB)h|_{gB}$ denotes the coefficient of gB in $u(gB)h$.

Interpretations of the bitrace

(1) By double centralizer theory, we have the following decomposition of 1_B^G as a $G \times H_n(q)$-bimodule,

$$1_B^G \cong \bigoplus_{\lambda} G^\lambda \otimes H^\lambda,$$

where G^λ is an irreducible G-module and H^λ is an irreducible $H_n(q)$-module. By taking traces on both sides of this isomorphism we get

$$\text{(2.2)} \qquad \operatorname{btr}(g,h) = \sum_{\lambda} \chi_G^\lambda(g)\chi_H^\lambda(h),$$

where χ_G^λ and χ_H^λ denote irreducible characters of G and $H_n(q)$, respectively.
(2) For $u \in G$ and a basis element $T_w \in H_n(q)$, we use formula (1.2) to get

$$\begin{aligned}
\operatorname{btr}(u, T_w) &= \sum_{gB \in G/B} u(gB)T_w|_{gB} \\
&= \sum_{gB \in G/B} (gB)T_w|_{u^{-1}gB} \\
&= \sum_{gB \in G/B} \sum_{\substack{hB \in G/B \\ hB \in gBwB}} hB|_{u^{-1}gB} \\
&= \operatorname{Card}\{gB \in G/B \mid u^{-1}gB \in gBwB\} \\
&= \operatorname{Card}\{gB \in G/B \mid g^{-1}u^{-1}g \in gBwB\}.
\end{aligned}$$

If $g \in G$ such that $g^{-1}u^{-1}g \in BwB$, then gb satisfies $(gb)^{-1}u^{-1}(gb) \in BwB$ for all $b \in B$. Thus, if we let $C_{u^{-1}}$ and $Z_{u^{-1}}$ denote the conjugacy class and the centralizer of u^{-1} in G, respectively, then

$$\begin{aligned}
\operatorname{btr}(u, T_w) &= \frac{1}{|B|}\operatorname{Card}\{g \in G \mid g^{-1}u^{-1}g \in BwB\} \\
&= \frac{1}{|B|}|C_{u^{-1}} \cap BwB|\,|Z_{u^{-1}}|.
\end{aligned}$$

This gives the following expression for the bitrace

$$\text{(2.3)} \qquad \operatorname{btr}(u, T_w) = \frac{|G|}{|B|}\frac{|C_{u^{-1}} \cap BwB|}{|C_{u^{-1}}|},$$

where $C_{u^{-1}}$ is the conjugacy class of u^{-1} in G.

(3) Let $w \in W$ be a permutation. A pair of flags $(f, f') \in \mathcal{F} \times \mathcal{F}$ is said to be

in relative position w if there is a $g \in G$ such that $(gf, gf') = (f_1, wf_1)$, where f_1 is the standard flag. The computation in the previous paragraph included the equation $\mathrm{btr}(u, T_w) = \mathrm{Card}\{gB \in G/B \mid u^{-1}gB \in gBwB\}$ which is equivalent to

$$\mathrm{btr}(u, T_w) = \mathrm{Card}\{f \in \mathcal{F} \mid (uf, f) \text{ is in relative position } w\}.$$

Let $\bar{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and consider the variety $\bar{\mathcal{F}}$ of flags in the vector space $\bar{\mathbb{F}}_q^n$. Then

$$\bar{\mathcal{F}}_{u,w} = \{f \in \bar{\mathcal{F}} \mid (uf, f) \text{ is in relative position } w\}$$

is an algebraic subvariety of the flag variety $\bar{\mathcal{F}}$. It follows that

$$\text{(2.4)} \qquad \mathrm{btr}(u, T_w) = \text{number of } \mathbb{F}_q\text{-rational points in } \bar{\mathcal{F}}_{u,w}.$$

(4) In Section 4 we shall show that the bitrace $\mathrm{btr}(u, h)$ can be given purely in terms of symmetric functions. The formula is

$$\text{(2.5)} \qquad \mathrm{btr}(u, T_{\gamma_\mu}) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} \langle \omega(q_\mu)(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}} \rangle,$$

for unipotent elements $u \in GL_n(\mathbb{F}_q)$ in the conjugacy class labeled by $\tilde{\nu}$. Section 4 gives a thorough discussion of the objects in this formula. In particular, the symmetric functions q_μ, $\tilde{Q}_{\tilde{\nu}}$ are defined in (4.10) and (4.2) respectively, the inner product is as defined in (4.3), and T_{γ_μ} is defined at the beginning of Section 3.

Special cases

(1) It follows from (2.3) that when $g = 1$ we have

$$\text{(2.6)} \qquad \mathrm{btr}(1, T_w) = \begin{cases} P_W(q) & \text{if } w = 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where} \quad P_W(q) = \sum_{w \in W} q^{\ell(w)},$$

is the Poincaré polynomial of W. Setting $q = 1$ in this formula shows that $\mathrm{btr}(1, T_w)$ is a generalization of the trace of the regular representation of W.

(2) It follows from (2.5), [Mac] Chapter IV, §6, Example 1 and [Mac] Chapter III, §7, that when $w = 1$ and $u \in G$ is a unipotent element,

$$\text{(2.7)} \qquad \mathrm{btr}(u, T_1) = Q^{\nu}_{(1^n)}(q),$$

where ν is the partition determined by the sizes of the blocks in the Jordan normal form of u, and $Q^{\nu}_{\mu}(q)$ is the Green polynomial.

(3) It follows from formula (7.1) in [Ka] that if u is a unipotent element of $GL_n(\mathbb{F}_q)$ which has a single Jordan block in its Jordan normal form then

$$\text{(2.8)} \qquad \mathrm{btr}(u, T_w) = q^{\ell(w)},$$

where $\ell(w)$ is the length of $w \in S_n$ (the minimum number of factors needed to write w as a product of the transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$.) We shall not prove this formula in this paper. Although it can be derived easily from our general result, Kawanaka's proof gives better insight into what actually makes this case work the way it does.

In this section we shall derive a formula for the bitrace $\mathrm{btr}(u, h)$ for all cases where u is a unipotent element in $GL_n(\mathbb{F}_q)$. The main theorem, Theorem (3.4), is that this bitrace can be written as a weighted sum of column strict tableaux.

Reduction to a function on a pair of partitions

A matrix $u \in G$ is unipotent if $u - 1$ is nilpotent, where 1 is the identity matrix in G. Our goal is to compute the bitrace $\mathrm{btr}(u, h)$ where $u \in G$ is unipotent and $h \in H_n(q)$. We have the following facts:

(a) Each unipotent $u \in G$ is conjugate to a unipotent u_ν, $\nu = (\nu_1, \nu_2, \ldots) \vdash n$, in Jordan normal form with Jordan blocks of sizes ν_i (see [Mac] Chapter IV, §2, Example 1 and 3).

(b) Let $\gamma_r = (1, 2, \ldots, r) \in S_r$ in cycle notation and for a composition $\mu = (\mu_1, \ldots, \mu_\ell) \models n$ define $\gamma_\mu = \gamma_{\mu_1} \times \ldots \times \gamma_{\mu_\ell} \in S_{\mu_1} \times \ldots \times S_{\mu_\ell}$. Any character of $H_n(q)$ is completely determined by its values on the elements T_{γ_μ}, $\mu \vdash n$ (see [Ca1] and [Ra1]).

In view of (a) and (b) it is sufficient to compute the bitrace

$$\mathrm{btr}(\nu, \mu) := \mathrm{btr}(u_\nu, T_{\gamma_\mu}), \quad \text{for } \mu, \nu \vdash n.$$

We will obtain a combinatorial formula (Theorem 3.4) which expresses $\mathrm{btr}(\nu, \mu)$ as a weighted sum of column strict tableaux of shape ν and content μ.

Computing btr(ν, (n))

For $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \vdash n$, define the Young subgroup $S_\mu = S_{\mu_1} \times \ldots \times S_{\mu_\ell} \subseteq S_n$. Recall from Section 1 that $B^- = w_0 B w_0$.

Lemma 3.1. *Let $g_1 B, g_2 B \in B^- w B$. If $(g_1 B) T_{\gamma_\mu}|_{g_2 B} \neq 0$, then w is a minimal length conjugacy class representative of S_n / S_μ.*

Proof. By Bourbaki [Bou] Chapter IV, §2, #1 (3′),

$$\begin{aligned}
(B^- w B)(B s_i B) &= w_0 (B w_0 w B)(B s_i B) \\
&= \begin{cases} w_0 B w_0 w s_i B, & \text{if } \ell(w_0 w) < \ell(w_0 w s_i), \\ w_0 B w_0 w B \cup w_0 B w_0 w s_i B, & \text{if } \ell(w_0 w) > \ell(w_0 w s_i), \end{cases} \\
&= \begin{cases} B^- w s_i B, & \text{if } \ell(w_0 w) < \ell(w_0 w s_i), \\ B^- w B \cup B^- w s_i B, & \text{if } \ell(w_0 w) > \ell(w_0 w s_i), \end{cases} \\
&= \begin{cases} B^- w s_i B, & \text{if } \ell(w) > \ell(w s_i), \\ B^- w B \cup B^- w s_i B, & \text{if } \ell(w) < \ell(w s_i). \end{cases}
\end{aligned}$$

It follows that $(gB) T_{\gamma_\mu}$ only contains terms in $B^- w B$ if $\ell(w s_i) > \ell(w)$ for all the s_i such that T_{s_i} is a factor of T_{γ_μ}. Since the factors of T_{γ_μ} are the $s_i \in S_\mu$ it follows that w must be a minimal length coset representative of S_n / S_μ. $\square$

Proposition 3.2. *Let $T_{\gamma_{(n)}} = T_{n-1} \dots T_1 \in H_n(q)$, let $u \in G$ be unipotent, and let ν be the partition of n determined by the sizes of the blocks in the Jordan normal form of u. Then*

$$\mathrm{btr}(\nu, (n)) = \begin{cases} q^{n-1}, & \text{if } \nu = (n), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Assume that $gB \in B^- wB$ and that $u \in B^-$ (since u is conjugate to a lower triangular matrix). Then $u(gB) \in B^- wB$ and, by Lemma 3.1,

$$(gB)T_{\gamma_{(n)}}|_{u^{-1}gB} \neq 0 \quad \text{implies} \quad w = 1,$$

since 1 is the only minimal length coset representative for S_n/S_n. So

$$\mathrm{btr}(\nu, (n)) = \sum_{gB \in G/B} u(gB)T_{\gamma_{(n)}}|_{gB} = \sum_{gB \in B^-1B} (gB)T_{n-1} \dots T_1|_{u^{-1}gB}.$$

The cosets gB in B^-1B are the ones with coset representatives given by lower triangular matrices with 1s on the diagonal:

$$g = \begin{pmatrix} 1 & & & & & \\ a_{2,1} & 1 & & & & \\ a_{3,1} & a_{3,2} & 1 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & \dots & a_{n,n-2} & a_{n,n-1} & 1 \end{pmatrix}.$$

By formula (1.4), we have $(gB)T_{n-1} \dots T_1|_{u^{-1}gB} \neq 0$ implies that

$$gx_{\alpha_{n-1}}(t_{n-1}) \dots x_{\alpha_1}(t_1)B = u^{-1}gB, \qquad \text{for some } t_1, \dots, t_n \neq 0.$$

Since g, $x_{\alpha_{n-1}}(t_{n-1}) \dots x_{\alpha_1}(t_1)$, and u^{-1} are all lower triangular matrices with 1s on the diagonal, and these matrices are coset representatives of the cosets in B^-1B, it follows that we must have

$$(*) \qquad gx_{\alpha_{n-1}}(t_{n-1}) \dots x_{\alpha_1}(t_1) = u^{-1}g.$$

This implies that u^{-1} is conjugate to $x_{\alpha_{n-1}}(t_{n-1}) \dots x_{\alpha_1}(t_1)$ and thus (by [Ca2], Proposition 5.1.3) that the Jordan block structure of u is given by the partition (n). So

$$\mathrm{btr}(\nu, (n)) = 0, \qquad \text{unless } \nu = (n).$$

Now assume the Jordan normal form of u is (n). Since btr is a trace on G we may assume that $u^{-1} = x_{\alpha_{n-1}}(1) \dots x_{\alpha_1}(1)$. Then, by explicitly computing the matrices $gx_{\alpha_{n-1}}(t_{n-1}) \dots x_{\alpha_1}(t_1)$ and $u^{-1}g$, we easily check that equation $(*)$ implies that $t_i = 1$, for all $1 \leq i \leq n-1$, and that

$$g = \begin{pmatrix} 1 & & & & & \\ b_1 & 1 & & & & \\ b_2 & b_1 & 1 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & \dots & b_2 & b_1 & 1 \end{pmatrix},$$

for some $b_1, \dots, b_{n-1} \in \mathbb{F}_q$. Since there are q^{n-1} such matrices it follows that $\operatorname{btr}(\nu, (n)) = q^{n-1}$. $\square$

The main theorem

Fix a unipotent element u of G and let $W \subseteq V$ be a subspace of V which is fixed by u. The *type* of W is the partition ρ determined by the sizes of the Jordan blocks of the restricted transformation $u|_W$ and the *co-type* is the partition π determined by sizes of the Jordan blocks of the unipotent transformation of V/W induced by the action of u. If $\dim(W) = n - d$, then $\rho \vdash (n-d)$ and $\pi \vdash d$.

Proposition 3.3 (Macdonald, [Mac], Chapter II, (4.12) and (4.13)). *Let* $1 \leq d \leq n$, $\nu \vdash n$, $\rho \vdash (n-d)$, *and let* $g^{\nu}_{\rho,\pi}(q)$ *denote the number of subspaces* $W \subseteq V_n$ *for which the type and co-type of* W *(determined by* u_ν*) are* ρ *and* π*, respectively. Then*

$$g^{\nu}_{\rho,(d)}(q) = \begin{cases} \frac{q^{n(\nu)-n(\rho)}}{1-q^{-1}} \operatorname{wt}(\nu, \rho), & \text{if } \nu/\rho \text{ is a horizontal strip of length } d, \\ 0, & \text{otherwise.} \end{cases}$$

where the weight $\operatorname{wt}(\nu/\rho)$ *is as given in* (0.5).

The following theorem is the main result of this paper.

Theorem 3.4. *Let* $\nu, \mu \vdash n$. *Then the bitrace of* $T_{\gamma_\mu} \in H_n(q)$ *and* $u_\nu \in G$ *on* 1_B^G *is given by*

$$\operatorname{btr}(\nu, \mu) = \frac{q^{n+n(\nu)}}{(q-1)^{\ell(\mu)}} \sum_T \operatorname{wt}(T),$$

where the sum is over all column strict tableaux T *of shape* ν *and content* μ *and* $\operatorname{wt}(T)$ *is given by* (0.5).

Proof. The proof uses induction on n. Let $\ell = \ell(\mu)$, $d = \mu_\ell$, and $\mu^* = (\mu_1, \mu_2, \dots, \mu_{\ell-1})$.

Suppose that $f = (V_0 \subseteq \dots \subseteq V_{n-d} \subseteq \dots \subseteq V_n)$. Formula (1.5) shows that the action of T_{γ_μ} leaves the vector space V_{n-d} fixed, and thus $V'_{n-d} = V_{n-d}$ for all flags $f' = (V'_0 \subseteq \dots \subseteq V'_n)$ which appear in the expansion of fT_{γ_μ}. If $u_\nu f T_{\gamma_\mu}|_f \neq 0$ then $u_\nu^{-1} f = (V'_0 \subseteq \dots \subseteq V'_n)$ with $V'_{n-d} = V_{n-d}$, since $u_\nu f T_{\gamma_\mu}|_f = f T_{\gamma_\mu}|_{u_\nu^{-1} f}$. Thus

$$u_\nu f T_{\gamma_\mu}|_f \neq 0 \text{ implies that the action of } u_\nu \text{ on } f \text{ fixes } V_{n-d}.$$

Using this we get

$$\begin{aligned}
\operatorname{btr}(\nu,\mu) &= \sum_{f=(V_0\subseteq\cdots\subseteq V_n)} u_\nu T_{\gamma_\mu} f|_f \\
&= \sum_{\substack{W\subseteq V\\ \dim(W)=n-d}} \sum_{f_1=(V_0\subseteq\cdots\subseteq V_{n-d}=W)} \sum_{f_2=(W=V_{n-d}\subseteq\cdots\subseteq V_n)} u_\nu(f_1,f_2)T_{\gamma_\mu}|_{(f_1,f_2)} \\
&= \sum_{\substack{W\subseteq V\\ \dim(W)=n-d}} \left(\sum_{f_1=(V_0\subseteq\cdots\subseteq V_{n-d}=W)} u_\nu f_1 T_{\gamma_{\mu^*}}|_{f_1}\right)\left(\sum_{f_2=(W\subseteq V_{n-d+1}\subseteq\cdots\subseteq V_n)} u_\nu f_2 T_{\gamma_{(1^{n-d}d)}}|_{f_2}\right)
\end{aligned}$$

We use induction to evaluate the first factor and get

$$\begin{aligned}
\sum_{f_1=(V_0\subseteq\cdots\subseteq V_{n-d}=W)} u_\nu f_1 T_{\gamma_{\mu^*}}|_{f_1} &= \sum_{f_1=(V_0\subseteq\cdots\subseteq V_{n-d}=W)} u_\rho f_1 T_{\gamma_{\mu^*}}|_{f_1} = \operatorname{btr}(\rho,\mu^*) \\
&= \frac{q^{n-d+n(\rho)}}{(q-1)^{\ell(\mu^*)}}\sum_{T'} \operatorname{wt}(T')
\end{aligned}$$

where ρ is the type of u_ν on W and the sum on the right hand side is over column strict tableaux T' of shape ρ and content μ^*.

Let $\bar{u}_\nu$ be the transformation of V_n/W induced by the action of u_ν. Then we use Proposition 3.3 to evaluate the second factor and get

$$\begin{aligned}
\sum_{f_2=(W=V_{n-d}\subseteq\cdots\subseteq V_n)} u_\nu f_2 T_{\gamma_{(1^{n-d}d)}}|_{f_2} &= \sum_{\bar{f}_2=(W/W=V_{n-d}/W\subseteq\cdots\subseteq V_n/W)} \bar{u}_\nu \bar{f}_2 T_{\gamma_d}|_{\bar{f}_2} \\
&= \sum_{f'=(0\subseteq V_1'\subseteq\cdots\subseteq V_d'=\mathbb{F}_q^d)} \bar{u}_\nu f' T_{\gamma_d}|_{f'} \\
&= \begin{cases} q^{d-1}, & \text{if } u_\nu \text{ has cotype } (d) \text{ on } (0\subseteq W\subseteq V_n), \\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Let $X^\nu_{\rho,(d)}$ denote the set of subspaces $W\subseteq V_n$ such that u_ν has type $\rho\vdash(n-d)$ and cotype (d). Then, using Proposition 3.2, $\operatorname{btr}(\nu,\mu)$ is equal to

$$\begin{aligned}
&\sum_{\substack{W\subseteq V\\ \dim(W)=n-d}} \left(\sum_{f_1=(V_0\subseteq\cdots\subseteq V_{n-d}=W)} u_\nu f_1 T_{\gamma_{\mu^*}}|_{f_1}\right)\left(\sum_{f_2=(W=V_{n-d}\subseteq\cdots\subseteq V_n)} u_\nu f_2 T_{\gamma_{(1^{n-d}d)}}|_{f_2}\right) \\
&= \sum_{\rho\vdash n-d} \sum_{W\in X^\nu_{\rho,(d)}} \left(\frac{q^{n-d+n(\rho)}}{(q-1)^{\ell(\mu^*)}}\sum_{T'}\operatorname{wt}(T')\right) q^{d-1} \\
&= \sum_{\substack{\rho\vdash n-d\\ \nu/\rho \text{ a horiz. strip}}} g^\nu_{\rho,(d)}(q)\left(\frac{q^{n-d+n(\rho)}}{(q-1)^{\ell(\mu^*)}}\sum_{T'}\operatorname{wt}(T')\right) q^{d-1} \\
&= \sum_{\substack{\rho\vdash n-d\\ \nu/\rho \text{ a horiz. strip}}} \left(\frac{q^{n(\nu)-n(\rho)}}{1-q^{-1}}\prod_{j\in I}(1-q^{-m_j(\nu)})\right)\left(\frac{q^{n-d+n(\rho)}}{(q-1)^{\ell(\mu^*)}}\sum_{T'}\operatorname{wt}(T')\right) q^{d-1} \\
&= \frac{q^{n+n(\nu)}}{(q-1)^{\ell(\mu)}}\sum_{T}\operatorname{wt}(T),
\end{aligned}$$

where I is the set of i such that ν/ρ has a box in column i and does not have a box in column $i+1$ (see (0.4)), and the last sum over column strict tableaux T of shape ν and weight μ. $\square$

4. THE FROBENIUS FORMULA

In this section we reinterpret the bitrace in terms of symmetric functions. There are two important points here: Theorem 4.11 says that the combinatorial formula for the bitrace given in Theorem 3.4 is equivalent to the 'Frobenius formula' for the characters of the Iwahori–Hecke algebra, and Corollary 4.12 shows that the values of the bitrace are the values of certain inner products of symmetric functions.

Characters and conjugacy classes of $GL_n(\mathbb{F}_q)$

We will work with the irreducible characters of $GL_n(\mathbb{F}_q)$ using symmetric functions and the notations of [Mac], Chapter IV. Let us summarize briefly these notations.

Let $\bar{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$. Let M denote the multiplicative group of $\bar{\mathbb{F}}_q$, and let M_n be the multiplicative group of $\mathbb{F}_{q^n} \subseteq \bar{\mathbb{F}}_q$ so that

$$M = \bigcup_{n \geq 1} M_n \quad \text{and} \quad \text{if } m \text{ divides } n, \text{ then } M_m \subseteq M_n.$$

Let L_n be the group of complex characters of M_n and let $L = \bigcup_{n \geq 1} L_n$. We have $L_m \subseteq L_n$ if m divides n. For $\xi \in L_n$ and $x \in M_n$, define

$$\langle \xi, x \rangle_n = \xi(x).$$

The Frobenius maps are given by

$$\begin{array}{rcl} F: M & \to & M \\ x & \mapsto & x^q \end{array} \quad \text{and} \quad \begin{array}{rcl} F: L & \to & L \\ \xi & \mapsto & \xi^q. \end{array}$$

Let Φ be the set of F orbits in M, Θ be the set of F-orbits in L, and $\mathcal{P}$ be the set of partitions. If $f \in \Phi$ and $\phi \in \Theta$ then $d(f)$ and $d(\phi)$ are the number of elements of f and ϕ respectively. For maps $\tilde{\mu}: \Phi \to \mathcal{P}$ and $\tilde{\lambda}: \Theta \to \mathcal{P}$ we define

$$\|\tilde{\mu}\| = \sum_{f \in \Phi} d(f)|\tilde{\mu}(f)| \quad \text{and} \quad \|\tilde{\lambda}\| = \sum_{\phi \in \Theta} d(\phi)|\tilde{\lambda}(\phi)|,$$

respectively. With these notations
(a) the conjugacy classes of $GL_n(\mathbb{F}_q)$ are indexed by maps of the form

$$\tilde{\mu}: \Phi \to \mathcal{P}, \text{ such that } \|\tilde{\mu}\| = n,$$

(b) the irreducible representations of $GL_n(\mathbb{F}_q)$ are indexed by maps of the form

$$\tilde{\lambda}: \Theta \to \mathcal{P}, \text{ such that } \|\tilde{\lambda}\| = n.$$

For $\tilde{\mu}: \Phi \to \mathcal{P}$, the *support* of $\tilde{\mu}$ is the set of $f \in \Phi$ for which $\tilde{\mu}(f) \neq 0$.

Let A_n be the space of complex-valued class functions on $GL_n(\mathbb{F}_q)$. Define an inner product on the vector space

$$A = \bigoplus_{n\geq 0} A_n = \mathbb{C}\text{-span}\{\chi^{\tilde{\lambda}}\} \quad \text{by} \quad \langle \chi^{\tilde{\lambda}}, \chi^{\tilde{\mu}} \rangle = \delta_{\tilde{\lambda}\tilde{\mu}},$$

where $\chi^{\tilde{\lambda}}$ denotes the irreducible character of $GL_n(\mathbb{F}_q)$ corresponding to the map $\tilde{\lambda}: \Theta \to \mathcal{P}$.

Let V be a $GL_k(\mathbb{F}_q)$-module and let W be a $GL_\ell(\mathbb{F}_q)$-module. Then $V \otimes W$ is a $GL_k(\mathbb{F}_q) \times GL_\ell(\mathbb{F}_q)$-module. Inside of $GL_{k+\ell}(\mathbb{F}_q)$, identify $GL_k(\mathbb{F}_q) \times GL_\ell(\mathbb{F}_q)$ with the Levi subgroup

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in GL_k(\mathbb{F}_q), B \in GL_\ell(\mathbb{F}_q) \right\}$$

of the parabolic subgroup

$$P = \left\{ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \mid A \in GL_k(\mathbb{F}_q), B \in GL_\ell(\mathbb{F}_q) \right\}.$$

Extend $V \otimes W$ to a P-module by letting the matrices in the unipotent radical

$$U = \left\{ \begin{pmatrix} I_k & * \\ 0 & I_\ell \end{pmatrix} \right\}, \qquad \text{where } I_j \text{ denotes the } j \times j \text{ identity matrix,}$$

act trivially on $V \otimes W$. If χ_V is the character of V and χ_W is the character of W, define

$$\chi_V \circ \chi_W = \mathrm{Ind}_P^G(\chi_{V\otimes W}),$$

where $\mathrm{Ind}_P^G(\chi_{V\otimes W})$ is the character of $V \otimes W$ induced from P to G. The operation $\circ$ makes A into an associative ring.

Symmetric functions

For each $\phi \in \Theta$, let Y_ϕ be a set of variables. Define

$$S_{\tilde{\lambda}} = \prod_{\phi\in\Theta} s_{\tilde{\lambda}(\phi)}(Y_\phi),$$

where $s_\lambda(Y)$ denotes the Schur function in the variables in the set Y. Define an inner product on the ring

$$\tilde{\Lambda} = \mathbb{C}\text{-span}\{S_{\tilde{\lambda}}\} \quad \text{by} \quad \langle S_{\tilde{\lambda}}, S_{\tilde{\mu}} \rangle = \delta_{\tilde{\lambda}\tilde{\mu}}.$$

For each $f \in \Phi$, let X_f be a set of variables. We shall assume that any symmetric function in the variables in the sets X_f, $f \in \Phi$, can be written as a symmetric function in the variables in the sets Y_ϕ, $\phi \in \Theta$, by using the 'Fourier transform' formula in (4.5), below. Let P_λ be the Hall–Littlewood symmetric functions defined in [Mac], Chapter III (2.1) and define

$$\tilde{P}_{\tilde{\mu}} = \prod_{f\in\Phi} \tilde{P}_{\tilde{\mu}(f)}(f), \quad \text{where} \quad \tilde{P}_\lambda(f) = q^{-d(f)n(\lambda)} P_\lambda(X_f; q^{-d(f)}).$$

The following theorem connects the characters of the groups $GL_n(\mathbb{F}_q)$ to the symmetric functions $S_{\tilde{\lambda}}$ and $\tilde{P}_{\tilde{\mu}}$.

Theorem 4.1 (Macdonald, [Mac] Chapter IV, (4.1) and (6.8)). *Let $\chi^{\tilde{\lambda}}$ be the irreducible character of $GL_n(\mathbb{F}_q)$ indexed by $\tilde{\lambda}$ and let $\pi_{\tilde{\nu}}$ be the characteristic function of the conjugacy class of $GL_n(\mathbb{F}_q)$ which is indexed by $\tilde{\nu}$. The characteristic map*

$$\begin{aligned} \mathrm{ch}: \quad A &\longrightarrow \tilde{\Lambda} \\ \chi^{\tilde{\lambda}} &\longmapsto S_{\tilde{\lambda}} \\ \pi_{\tilde{\nu}} &\longmapsto \tilde{P}_{\tilde{\nu}} \end{aligned}$$

is an isometric isomorphism.

Let Q_λ be the symmetric functions (also called Hall–Littlewood symmetric functions) defined in [Mac], Chapter III (2.11) and define

$$\tilde{Q}_{\tilde{\nu}} = \prod_{f\in\Phi} \tilde{Q}_{\tilde{\nu}(f)}(f), \quad \text{where} \quad \tilde{Q}_\lambda(f) = q^{d(f)(|\lambda|+n(\lambda))} Q_\lambda(X_f; q^{-d(f)}). \tag{4.2}$$

By [Mac], Chapter IV (4.7), the $\tilde{P}_{\tilde{\mu}}$ and the $\tilde{Q}_{\tilde{\nu}}$ are dual bases of $\tilde{\Lambda}$,

$$\langle \tilde{P}_{\tilde{\mu}}, \tilde{Q}_{\tilde{\nu}}\rangle = \delta_{\tilde{\mu}\tilde{\nu}}, \quad \text{and thus, from Theorem 4.1,} \quad \chi^{\tilde{\lambda}}(g) = \langle S_{\tilde{\lambda}}, \tilde{Q}_{\tilde{\nu}}\rangle, \tag{4.3}$$

when $\chi^{\tilde{\lambda}}$ is the irreducible character of $GL_n(\mathbb{F}_q)$ indexed by $\tilde{\lambda}$ and g is an element of the conjugacy class labeled by $\tilde{\nu}$.

Let $p_k(X_f)$ denote the kth power sum symmetric function in the variables in the set X_f and let $p_k(Y_\phi)$ be the kth power sum symmetric function in the variables in the set Y_ϕ. For $x \in M$ and $\xi \in L$, define

$$\begin{aligned} \tilde{p}_n(x) &= \begin{cases} p_{n/d(f)}(X_f), & \text{if } n \text{ is a multiple of } d(f), \\ 0, & \text{otherwise,} \end{cases} \\ \tilde{p}_n(\xi) &= \begin{cases} p_{n/d(\phi)}(Y_\phi), & \text{if } n \text{ is a multiple of } d(\phi), \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{4.4}$$

where $f \in \Phi$ and $\phi \in \Theta$ are the F-orbits of x and ξ, respectively. Then $\tilde{p}_n(x) = 0$ if $x \notin M_n$ and $\tilde{p}_n(x) = \tilde{p}_n(y)$ if x and y are in the same F-orbit. Similarly, $\tilde{p}_n(\xi) = 0$ if $\xi \notin L_n$ and $\tilde{p}_n(\xi) = \tilde{p}_n(\eta)$ if ξ and η are in the same F-orbit. With these notations, the relationship between the variables in the sets X_f, $f \in \Phi$, and the variables in the sets Y_ϕ, $\phi \in \Theta$, is determined by the formulas

$$\begin{aligned} \tilde{p}_n(\xi) &= (-1)^{n-1} \sum_{x\in M_n} \langle \xi, x\rangle_n \tilde{p}_n(x), \\ \tilde{p}_n(x) &= (-1)^{n-1}(q^n-1)^{-1} \sum_{\xi\in L_n} \overline{\langle \xi, x\rangle_n} \tilde{p}_n(\xi), \end{aligned} \tag{4.5}$$

for each $n \geq 0$, each $\xi \in L_n$, and each $x \in M_n$. The second equation follows from the first using orthogonality of characters of the group M_n (which has order $q^n - 1$).

Let $u \in G$ be unipotent and let $\nu = (\nu_1, \nu_2, \ldots) \vdash n$ be the partition determined by the sizes of the blocks in the Jordan normal form of u. Then u is in the conjugacy class labeled by the map $\tilde{\nu}\colon \Phi \to \mathcal{P}$ given by

$$\tilde{\nu}(f) = \begin{cases} \nu, & \text{if } f = \{1\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

There is an involutive automorphism of the ring of symmetric functions $\omega(p_r) = (-1)^{r-1} p_r$, where p_r is the rth power sum symmetric function. By [Mac], Chapter I (3.8),

$$\omega(s_\lambda) = s_{\lambda'}, \tag{4.7}$$

where λ' is the partition conjugate to λ.

The following technical lemma is similar to the observation in the last line of [Mac], Chapter IV, §6, Example 1.

Lemma 4.8 (Technical lemma). *Let ϕ_1 be the F-orbit of the trivial character $\hat{1} \in L_1$ and let $f_1 = \{1\}$ be the F-orbit of the element $1 \in \mathbb{F}_q^\times$. Let $\nu \vdash n$ and let $\tilde{\nu}\colon \Phi \to \mathcal{P}$ be the map given in* (4.6). *Then, for any symmetric function f,*

$$\langle f(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}} \rangle = \langle \omega(f)(X_{f_1}), \tilde{Q}_{\tilde{\nu}} \rangle.$$

Proof. Note that $\hat{1} \in L_1$ for all n and $d(\hat{1}) = 1$, since $\phi_1 = \{\hat{1}\}$. Thus $p_r(\phi_1) = \tilde{p}_r(\hat{1})$ and $p_\mu(\phi_1) = \tilde{p}_\mu(\hat{1})$, for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. By (4.5) and (4.4)

$$\begin{aligned}
\langle p_\mu(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}} \rangle &= \left\langle \prod_{i=1}^{\ell} \tilde{p}_{\mu_i}(\hat{1}), \tilde{Q}_{\tilde{\nu}} \right\rangle \\
&= \left\langle \prod_{i=1}^{\ell} (-1)^{\mu_i - 1} \sum_{x \in M_{\mu_i}} \langle \hat{1}, x \rangle_{\mu_i} \tilde{p}_{\mu_i}(x), \tilde{Q}_{\tilde{\nu}} \right\rangle \\
&= \left\langle \prod_{i=1}^{\ell} (-1)^{\mu_i - 1} \sum_{x \in M_{\mu_i}} \langle \hat{1}, x \rangle_{\mu_i} p_{\mu_i / d(f_x)}(X_{f_x}), \tilde{Q}_{\tilde{\nu}} \right\rangle,
\end{aligned}$$

where f_x is the F-orbit of x. Since the right hand side of the inner product only involves the variables in the set X_{f_1}, it follows from (4.3) that the only term on the left hand side which gives a nonzero contribution is $x = 1$. In other words, if the variables are different, then $\tilde{\mu}$ and $\tilde{\nu}$ have different support, so $\delta_{\tilde{\mu}\tilde{\nu}} = 0$. Thus,

$$\langle p_\mu(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}}(X_{f_1}) \rangle = \left\langle \prod_{i=1}^{\ell} (-1)^{\mu_i - 1} \langle \hat{1}, 1 \rangle_{\mu_i} p_{\mu_i}(X_{f_1}), \tilde{Q}_{\tilde{\nu}} \right\rangle = \langle \omega(p_\mu)(X_{f_1}), \tilde{Q}_{\tilde{\nu}} \rangle.$$

□

The following proposition gives information about the irreducible $GL_n(\mathbb{F}_q)$ characters which appear in the decomposition of 1_B^G.

Theorem 4.9. *Let ϕ_1 be the F-orbit of the trivial character $\hat{1} \in L_1$.*

(a) *Let μ be a partition of n and let P_μ be the parabolic subgroup of $GL_n(\mathbb{F}_q)$*

consisting of block upper triangular matrices where the diagonal blocks have sizes $\mu_1, \mu_2, \ldots$. *Let* $1_{P_\mu}^G$ *be the trivial representation of* P_μ *induced to G. The characteristic of this character is*

$$\mathrm{ch}(1_{P_\mu}^G) = e_\mu(Y_{\phi_1}),$$

where $e_\mu(Y_{\phi_1})$ *is the elementary symmetric function in the variables in the set* Y_{ϕ_1}.

(b) *If* $\lambda \vdash n$ *we let* χ_H^λ *be the irreducible character of* $H_n(q)$ *which, at* $q = 1$, *specializes to the irreducible character* χ_S^λ *of the symmetric group* S_n *which is indexed by* λ. *Let* χ_G^λ *be the irreducible character of* $GL_n(\mathbb{F}_q)$ *such that* $\mathrm{ch}(\chi_G^\lambda) = s_{\lambda'}(Y_{\phi_1})$. *Then*

$$\mathrm{btr}(g,h) = \sum_{\lambda \vdash n} \chi_H^\lambda(h)\chi_G^\lambda(g),$$

for all $g \in G$ *and* $h \in H_n(q)$.

(c) *If* $\nu \vdash n$ *and* u_ν *is a unipotent of* $GL_n(\mathbb{F}_q)$ *such that Jordan normal form of* u_ν *has blocks of sizes* ν_i *then*

$$\chi_G^\lambda(u_\nu) = q^{n(\nu)} K_{\lambda\nu}(q^{-1}),$$

where $K_{\lambda\nu}(q^{-1})$ *is the Kostka–Foulkes polynomial* (see [Mac], Chapter III, §6) *defined by*

$$q^{n(\nu)} K_{\lambda\nu}(q^{-1}) = \langle s_\lambda(X_{f_1}), \tilde{Q}_{\tilde{\nu}} \rangle$$

and $\tilde{\nu}$ *is the map defined in* (4.6).

Proof. (a) Since ch is a ring isomorphism it is sufficient to show that the characteristic of the trivial character of $GL_k(\mathbb{F}_q)$ is $e_k(Y_{\phi_1})$. This will be accomplished by showing that

$$\sum_{k \geq 0} e_k(Y_{\phi_1}) t^k = \sum_{\tilde{\mu}} t^{\|\tilde{\mu}\|} \tilde{P}_{\tilde{\mu}}.$$

The proof of this identity is similar to the argument in [Mac], Chapter IV, §6, p. 285-286 except that we use the homomorphism

$$\gamma(\tilde{p}_n(\xi)) = \begin{cases} t^n(-1)^{n-1}, & \text{if } \xi = \hat{1}, \\ 0, & \text{otherwise,} \end{cases}$$

in place of δ and we invoke [Mac], Chapter III, §2 Example 1 instead of invoking Chapter I, §3, Example 2.

In greater detail: For $c \in \tilde{\Lambda}$, write $c = \sum c_\mu \tilde{P}_{\tilde{\mu}}$ with $c_\mu \in \mathbb{C}$, and define $\bar{c} = \sum \bar{c}_\mu \tilde{P}_{\tilde{\mu}}$. Since γ sends $\tilde{p}_n(\xi)$ to 0 unless $\xi = \hat{1}$, we know that $\gamma(\overline{S_{\tilde{\lambda}}}) = 0$ unless the support of $\tilde{\lambda}$ is $\{\phi_1\}$. If $\mu \vdash k$, then let $z_\mu = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!$. By the orthogonality of characters of the symmetric group S_k, we have, for $\lambda \vdash k$,

$$\gamma(s_\lambda(Y_{\phi_1})) = \sum_{\mu \vdash k} \chi_{S_k}^\lambda(\mu) \frac{\gamma(p_\mu(Y_{\phi_1}))}{z_\mu} = \sum_{\mu \vdash k} \chi_{S_k}^\lambda(\mu) t^k \frac{\chi^{(1^k)}(\mu)}{z_\mu} = \begin{cases} t^k, & \text{if } \lambda = (1^k), \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\sum_{\tilde{\lambda}} S_{\tilde{\lambda}}\gamma(\bar{S}_{\tilde{\lambda}}) = \sum_{k\geq 0} e_k(Y_{\phi_1})t^k.$$

Now, using (4.5),

$$\begin{aligned}\gamma(\tilde{p}_n(x)) &= (-1)^{n-1}(q^n-1)^{-1}\sum_{\xi\in L_n}\overline{\langle\xi,x\rangle}\gamma(\tilde{p}_n(\xi))\\ &= (-1)^{n-1}(q^n-1)^{-1}\cdot 1\cdot t^n(-1)^{n-1}\\ &= \frac{t^n}{q^n-1},\end{aligned}$$

for $x\in M_n$. Hence, for all $f\in\Phi$,

$$\gamma(p_n(f)) = t^{d(f)n}(q_f^n-1)^{-1} = \sum_{i\geq 1} t^{d(f)n}q_f^{-in},$$

where $q_f = q^{d(f)}$. So $p_n(f) = p_n(t^{d(f)}q_f^{-1}, t^{d(f)}q_f^{-2},\ldots)$, and we get that

$$\begin{aligned}\gamma(\tilde{Q}_\mu(f)) &= \gamma(q_f^{|\mu|+n(\mu)}Q_\mu(X_f;q_f^{-1}))\\ &= q_f^{|\mu|+n(\mu)}Q_\mu(t^{d(f)}q_f^{-1}, t^{d(f)}q_f^{-2},\ldots;q_f^{-1})\\ &= q_f^{|\mu|+n(\mu)}t^{d(f)|\mu|}q_f^{-|\mu|}q_f^{-n(\mu)} = t^{d(f)|\mu|},\end{aligned}$$

by [Mac], Chapter III, §2, Example 1. Thus, by [Mac], Chapter IV (4.7),

$$\sum_{k\geq 0} e_k(Y_{\phi_1})t^k = \sum_{\tilde{\lambda}} S_{\tilde{\lambda}}\gamma(\bar{S}_{\tilde{\lambda}}) = \sum_{\tilde{\mu}} \tilde{P}_{\tilde{\mu}}\gamma(\tilde{Q}_{\tilde{\mu}}) = \sum_{\tilde{\mu}} t^{\|\tilde{\mu}\|}\tilde{P}_{\tilde{\mu}},$$

as desired.

(b) By [Mac], Chapter I, §6,

$$e_\mu(Y_{\phi_1}) = \sum_{\lambda\vdash n} K_{\lambda\mu}s_{\lambda'}(Y_{\phi_1}),$$

where $K_{\lambda\mu}$ is the number of column strict tableaux of shape λ and content μ. So we have that

$$1_{P_\mu}^G = \sum_{\lambda\vdash n} K_{\lambda\mu}\chi_G^\lambda.$$

On the other hand, the remark after (7.8) in [Mac], Chapter I, says that

$$1_{S_\mu}^{S_n} = \sum_{\lambda\vdash n} K_{\lambda\mu}\chi_S^\lambda,$$

where $S_\mu = S_{\mu_1}\times\ldots\times S_{\mu_\ell}$. Thus we may use [CR] 68.24 and 68.26, [Ca2] 10.1.14, and double centralizer theory (2.2) to conclude that

$$\mathrm{btr}(g,T_w) = \sum_{\lambda\vdash n}\chi_H^\lambda(T_w)\chi_G^\lambda(g).$$

(c) By (4.3) and (4.6),

$$\chi_G^\lambda(u_\nu) = \langle s_{\lambda'}(Y_{\phi_1}),\tilde{Q}_{\tilde{\nu}}\rangle,$$

and, by the technical lemma and (4.7), $\langle s_{\lambda'}(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}}\rangle = \langle \omega(s_{\lambda'})(X_{f_1}), \tilde{Q}_{\tilde{\nu}}\rangle = \langle s_\lambda(X_{f_1}), \tilde{Q}_{\tilde{\nu}}\rangle$. $\square$

The Frobenius formula

If Y is a set of variables we define symmetric functions $q_r(Y;t)$ via the generating function

$$\sum_{r\geq 0} q_r(Y;t)u^r = \prod_{y_i\in Y} \frac{1-y_i tu}{1-y_i u},$$

and define $q_\mu = q_{\mu_1}\dots q_{\mu_\ell}$, for a partition $\mu = (\mu_1,\dots,\mu_\ell)$. The 'Frobenius formula' for the characters of $H_n(q)$ is

$$\text{(4.10)}\qquad \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} q_\mu(Y;q^{-1}) = \sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu}) s_\lambda(Y).$$

This formula was discovered independently by Gyoja [Gy], King–Wybourne [KW], Ram [Ra1], Vershik–Kerov [VK], and Ueno–Shibukawa [US]. It has been applied to give combinatorial formulas for the irreducible characters of $H_n(q)$, see [Ra2].

Theorem 4.11. *The Frobenius formula* (4.10) *and the bitrace formula, Theorem* 3.4, *are equivalent in the sense that each can be derived from the other.*

Proof. (a) Let us first show that (4.10) implies Theorem 3.4.

$$\begin{aligned}
\mathrm{btr}(u_\nu, T_{\gamma_\mu}) &= \sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})\chi_G^\lambda(u_\nu), \quad \text{by Theorem 4.9 (b)},\\
&= \sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})\langle s_\lambda(X_{f_1}), \tilde{Q}_{\tilde{\nu}}\rangle, \quad \text{by Theorem 4.9 (c)},\\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}\langle q_\mu(X_{f_1};q^{-1}), \tilde{Q}_{\tilde{\nu}}\rangle, \quad \text{by (4.10)},\\
&= \frac{q^{|\mu|+|\nu|+n(\nu)}}{(q-1)^{\ell(\mu)}}\langle q_\mu(X_{f_1};q^{-1}), Q_\nu(X_{f_1};q^{-1})\rangle, \quad \text{by (4.2)}.
\end{aligned}$$

It follows from [Mac], Chapter III (5.10) that

$$\langle q_\mu(x;q^{-1}), Q_\nu(x;q^{-1})\rangle = \frac{1}{q^{|\nu|}}\sum_T \mathrm{wt}(T),$$

(In doing the conversion from [Mac], Chapter III (5.10) to the formula above it is important to note that the inner product which is used in Chapter III of [Mac] differs from the inner product used in Chapter IV by a factor of $q^{|\nu|}$.) Thus

$$\mathrm{btr}(u_\nu, T_{\gamma_\mu}) = \frac{q^{|\mu|+n(\nu)}}{(q-1)^{\ell(\mu)}}\sum_T \mathrm{wt}(T).$$

(b) To show that Theorem 3.4 implies (4.10) we reorder the equalities in the proof of (a),

$$\begin{aligned}
\sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})\langle s_\lambda(X_{f_1}), \tilde{Q}_{\tilde{\nu}}\rangle &= \sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})\chi_G^\lambda(u_\nu) \\
&= \mathrm{btr}(u_\nu, T_{\gamma_\mu}) \\
&= \frac{q^{|\nu|+n(\nu)}}{(q-1)^{\ell(\mu)}}\sum_T \mathrm{wt}(T) \\
&= \frac{q^{|\mu|+|\nu|+n(\nu)}}{(q-1)^{\ell(\mu)}}\langle q_\mu(X_{f_1};q^{-1}), Q_\nu(X_{f_1};q^{-1})\rangle \\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}\langle q_\mu(f_1;q^{-1}), \tilde{Q}_{\tilde{\nu}}\rangle.
\end{aligned}$$

Since the $\tilde{Q}_{\tilde{\nu}} = q^{|\nu|+n(\nu)}Q_\nu(X_{f_1};q^{-1})$ form a basis of the ring of symmetric functions in the variables in the set X_{f_1} we conclude that

$$\sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})s_\lambda(X_{f_1}) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}} q_\mu(X_{f_1};q^{-1}). \quad \square$$

Corollary 4.12. *Let $\mu, \nu \vdash n$. Let u_ν be a unipotent element of $GL_n(\mathbb{F}_q)$ such that the sizes of the blocks in the Jordan normal form of u_ν are given by ν and let T_{γ_μ} be the element of the Iwahori–Hecke algebra $H_n(q)$ defined at the beginning of Section 3. Then*

$$\mathrm{btr}(u_\nu, T_{\gamma_\mu}) = \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}\langle \omega(q_\mu)(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}}\rangle,$$

where q_μ is as defined in (4.10), *ω is the involution on symmetric functions given in* (4.7), *$\tilde{\nu}$ is as given in* (4.6), *and $\tilde{Q}_{\tilde{\nu}}$ is as defined in* (4.2).

Proof. By Theorem 4.9 (b) and (4.10),

$$\begin{aligned}
\mathrm{btr}(u_\nu, T_{\gamma_\mu}) &= \sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})\chi_G^\lambda(u_\nu) = \sum_{\lambda\vdash n} \chi_H^\lambda(T_{\gamma_\mu})\langle s_{\lambda'}(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}}\rangle \\
&= \frac{q^{|\mu|}}{(q-1)^{\ell(\mu)}}\langle \omega(q_\mu)(Y_{\phi_1}), \tilde{Q}_{\tilde{\nu}}\rangle. \quad \square
\end{aligned}$$

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