

Representations of Rank Two Affine Hecke Algebras

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Abstract. This paper gives a complete analysis of the irreducible representations of affine Hecke algebras of rank two when the parameter q is not a root of unity. The irreducible representations are classified in terms of the Kazhdan-Lusztig classification and the relation to the Springer correspondence is given in detail. Each irreducible representation is constructed explicitly. These results are used in a crucial way in the classification of calibrated representations of general affine Hecke algebras as done in [?] and [?]. Though the same methods can be used to handle the very few root of unity cases ($q^{2k} = 1$, $k = 0, 1, 2$, and also $k = 3$ in type A_2 , $k = 4$ in type B_2 and $k = 3, 6$ in type G_2) which have different representations, this extra analysis is not completed in this paper.

This paper classifies and constructs explicitly all the irreducible representations of affine Hecke algebras of rank two root systems. The methods used to obtain this classification are primarily combinatorial and are, for the most part, an application of the methods used in [?]. I have made special effort to describe how the classification here relates to the classifications by Langlands parameters (coming from p -adic group theory) and by indexing triples (coming from a q -analogue of the Springer correspondence). There are several reasons for doing the details of this classification:

- (a) The proof of the one of the main results of [?] depends on this classification of representations for rank two affine Hecke algebras. Specifically, in the proof of Proposition 4.4 of [?], one needs to know exactly which weights can occur in calibrated representations. The

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reason that this naturally depends on a rank two classification is outlined in (d) below.

- (b) The examples here illustrate (and clarify) results of [?], [?], [?], [?], [?], [?], [?]-[?]. Much of the power of the combinatorial methods which are now available is evident from the calculations in this paper, especially when one compares with the effort needed in other sources (for example [?], Chapt. 11).
- (c) The explicit information here can be very useful for obtaining results on representations of p -adic groups (see, for example, [?]).
- (d) One hopes that eventually there will be a combinatorial construction of all irreducible representations of all affine Hecke algebras. I expect that such a construction will depend heavily on the rank two cases. This idea is analogous to the way that the rank two cases are the basic building blocks in the presentations of Coxeter groups by “braid” relations and the presentations of Kac-Moody Lie algebras (and quantum groups) by Serre relations.

The first section of this paper is a review of definitions and basic results about affine Hecke algebras and their representations. A few additional lemmas are proved in order to aid the proofs and constructions in later sections. The remainder of the sections detail the classification and construction of the irreducible representations of affine Hecke algebras of types A_1 , $A_1 \times A_1$, A_2 , C_2 and G_2 . In each case I have indicated how the results here relate to the “Langlands classification”, the classification of Kazhdan and Lusztig [?], and the results in [?].

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1. Definitions and preliminary results

The Weyl group. Let R be a reduced irreducible root system in \mathbb{R}^n , fix a set of positive roots R^+ and let $\{\alpha_1, \dots, \alpha_n\}$ be the corresponding simple roots in R . Let W be the Weyl group corresponding to R . Let s_i denote the simple reflection in W corresponding to the simple root α_i and recall that W can be presented by generators s_1, s_2, \dots, s_n and relations

$$\begin{aligned} s_i^2 &= 1, & \text{for } 1 \leq i \leq n, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}, & \text{for } i \neq j, \end{aligned}$$

where $m_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$, and $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$.

The Iwahori-Hecke algebra. Fix $q \in \mathbb{C}^*$ such that q is not a root of unity. The *Iwahori-Hecke algebra* H is the associative algebra over \mathbb{C} defined by generators T_1, T_2, \dots, T_n and relations

$$(1.1) \quad \begin{aligned} T_i^2 &= (q - q^{-1})T_i + 1, & \text{for } 1 \leq i \leq n, \\ \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}, & \text{for } i \neq j, \end{aligned}$$

where m_{ij} are the same as in the presentation of W . For $w \in W$ define $T_w = T_{i_1} \cdots T_{i_p}$ where $s_{i_1} \cdots s_{i_p} = w$ is a reduced expression for w . By [?], Ch. IV §2 Ex. 23, the element T_w does not depend on the choice of the reduced expression. The algebra H has dimension $|W|$ and the set $\{T_w\}_{w \in W}$ is a basis of H .

The group X . The *fundamental weights* are the elements $\omega_1, \dots, \omega_n$ of \mathbb{R}^n given by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The *weight lattice* is the W -invariant lattice in \mathbb{R}^n given by

$$P = \sum_{i=1}^n \mathbb{Z}\omega_i.$$

Let X be the abelian group P except written multiplicatively. In other words,

$$X = \{X^\lambda \mid \lambda \in P\}, \quad \text{and} \quad X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda, \quad \text{for } \lambda, \mu \in P.$$

Let $\mathbb{C}[X]$ denote the group algebra of X . There is a W -action on X given by $wX^\lambda = X^{w\lambda}$ for $w \in W$, $X^\lambda \in X$, which we extend linearly to a W -action on $\mathbb{C}[X]$.

The affine Hecke algebra. The *affine Hecke algebra* \tilde{H} associated to R and P is the algebra given by

$$\tilde{H} = \mathbb{C}\text{-span}\{T_w X^\lambda \mid w \in W, X^\lambda \in X\}$$

where the multiplication of the T_w is as in the Iwahori-Hecke algebra H , the multiplication of the X^λ is as in $\mathbb{C}[X]$ and we impose the relation

$$(1.2) \quad X^\lambda T_i = T_i X^{s_i \lambda} + (q - q^{-1}) \frac{X^\lambda - X^{s_i \lambda}}{1 - X^{-\alpha_i}}, \quad \text{for } 1 \leq i \leq n \text{ and } X^\lambda \in X.$$

This formulation of the definition of \tilde{H} is due to Lusztig [?] following work of Bernstein and Zelevinsky. The elements $T_w X^\lambda$, $w \in W$, $X^\lambda \in X$, form a basis of \tilde{H} .

Weights. Let

$$T = \{\text{group homomorphisms } t: X \rightarrow \mathbb{C}^*\}.$$

The *torus* T is an abelian group with a W -action given by $(wt)(X^\lambda) = t(X^{w^{-1}\lambda})$. For any element $t \in T$ define the *polar decomposition*

$$t = t_r t_c, \quad t_r, t_c \in T \text{ such that } t_r(X^\lambda) \in \mathbb{R}_{>0}, \text{ and } |t_c(X^\lambda)| = 1,$$

for all $X^\lambda \in X$. Let $Q^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$. There is a unique $\mu \in \mathbb{R}^n$ and a unique $\nu \in \mathbb{R}^n/Q^\vee$ such that

$$(1.3) \quad t_r(X^\lambda) = e^{\langle \mu, \lambda \rangle} \quad \text{and} \quad t_c(X^\lambda) = e^{2\pi i \langle \nu, \lambda \rangle}, \quad \text{for all } \lambda \in P.$$

In this way we identify the sets $T_r = \{t \in T \mid t = t_r\}$ and $T_c = \{t \in T \mid t = t_c\}$ with \mathbb{R}^n and \mathbb{R}^n/Q^\vee , respectively.

Central characters. Theorem 1.4. (*Bernstein, Zelevinsky, Lusztig [?], 8.1*) *The center of \tilde{H} is $\mathbb{C}[X]^W = \{f \in \mathbb{C}[X] \mid wf = f\}$.*

Since \tilde{H} has countable dimension, Dixmier's version of Schur's lemma implies that $Z(\tilde{H})$ acts on an irreducible \tilde{H} -module M by scalars. Let $t \in T$ be such that

$$pM = t(p)M, \quad \text{for all } p \in Z(\tilde{H}).$$

Since $Z(\tilde{H}) = \mathbb{C}[X(T)]^W$ it follows that $t(p(X)) = (wt)(p(X))$ for all $w \in W$. The W -orbit Wt of t is the *central character* of M . We shall often abuse notation and refer to any weight $s \in Wt$ as “the central character” of M .

Weight spaces. Let M be a finite dimensional \tilde{H} -module. For each $t \in T$ the *t -weight space* of M and the *generalized t -weight space* are the subspaces

$$M_t = \{m \in M \mid X^\lambda m = t(X^\lambda)m \text{ for all } X^\lambda \in X\} \quad \text{and}$$

$$M_t^{\text{gen}} = \{m \in M \mid \text{for each } X^\lambda \in X, (X^\lambda - t(X^\lambda))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\},$$

respectively. If $M_t^{\text{gen}} \neq 0$ then $M_t \neq 0$. In general $M \neq \bigoplus_{t \in T} M_t$, but we do have

$$M = \bigoplus_{t \in T} M_t^{\text{gen}}.$$

This is a decomposition of M into Jordan blocks for the action of $\mathbb{C}[X]$. The set of *weights* of M is the set

$$(1.5) \quad \text{supp}(M) = \{t \in T \mid M_t^{\text{gen}} \neq 0\}.$$

The calibration graph. Let $t \in T$. Define a graph $\Gamma(t)$ with

$$\begin{aligned} \text{Vertices:} & \quad Wt, \\ \text{Edges:} & \quad wt \longleftrightarrow s_i wt, \quad \text{if } (wt)(X^{\alpha_i}) \neq q^{\pm 2}. \end{aligned}$$

Proposition 1.6. ([?] Proposition 2.12) *Let M be a finite dimensional irreducible \tilde{H} -module with central character t . Then*

$$\dim(M_s^{\text{gen}}) = \dim(M_{s'}^{\text{gen}})$$

if s and s' are in the same connected component of the calibration graph $\Gamma(t)$.

If $t \in T$ define

$$(1.7) \quad P(t) = \{\alpha > 0 \mid t(X^\alpha) = q^{\pm 2}\} \quad \text{and} \quad Z(t) = \{\alpha > 0 \mid t(X^\alpha) = 1\}.$$

For each subset $J \subseteq P(t)$ define

$$(1.8) \quad \mathcal{F}^{(t,J)} = \{w \in W \mid R(w) \cap Z(t) = \emptyset, \quad R(w) \cap P(t) = J\},$$

where $R(w) = \{\alpha > 0 \mid w\alpha < 0\}$ is the *inversion set* of w . Define a *placed shape* to be a pair (t, J) such that $t \in T$, $J \subseteq P(t)$ and $\mathcal{F}^{(t,J)} \neq \emptyset$. The elements of the set $\mathcal{F}^{(t,J)}$ are called *standard tableaux* of shape (t, J) .

Proposition 1.9. ([?] Theorem 2.14) *Let $t \in T$. The connected components of the calibration graph $\Gamma(t)$ are the sets*

$$\{wt \mid w \in \mathcal{F}^{(t,J)}\}, \quad J \subseteq P(t), \quad \text{such that } \mathcal{F}^{(t,J)} \neq \emptyset.$$

Calibrated representations. A finite dimensional \tilde{H} -module M is *calibrated* if $M_t^{\text{gen}} = M_t$, for all $t \in T$.

Proposition 1.10. ([?] Proposition 4.2)

- (a) *An irreducible \tilde{H} -module M is calibrated if and only if $\dim(M_t^{\text{gen}}) = 1$ for all weights t of M .*
- (b) *If M is an irreducible \tilde{H} -module with regular central character t (i.e. $Z(t) = \emptyset$) then M is calibrated.*

Let α_i and α_j be simple roots in R and let R_{ij} be the rank two root subsystem of R which is generated by α_i and α_j . Let W_{ij} be the Weyl group of R_{ij} , the subgroup of W generated by the simple reflections s_i and s_j . A weight $t \in T$ is *calibratable* for R_{ij} if one of the following two conditions holds:

- (a) $t(X^\alpha) \neq 1$ for all $\alpha \in R_{ij}$,
- (b) R_{ij} is of type C_2 or G_2 (assume that α_i is the long root and α_j is the short root), $ut(X^{\alpha_i}) = q^2$ and $ut(X^{\alpha_j}) = 1$ for some $u \in W_{ij}$, and $t(X^{\alpha_i}) \neq 1$ and $t(X^{\alpha_j}) \neq 1$.

A *placed skew shape* is a placed shape (t, J) such that for all $w \in \mathcal{F}^{(t, J)}$ and all pairs α_i, α_j of simple roots in R the weight wt is calibratable for R_{ij} .

Theorem 1.11. ([?] Theorem 3.1 and Proposition 4.1)

- (a) Let (t, J) be a placed skew shape and let $\mathcal{F}^{(t, J)}$ be the set of standard tableaux of shape (t, J) . Define

$$\tilde{H}^{(t, J)} = \mathbb{C}\text{-span}\{v_w \mid w \in \mathcal{F}^{(t, J)}\},$$

so that the symbols v_w are a labeled basis of the vector space $\tilde{H}^{(t, J)}$. Then the following formulas make $\tilde{H}^{(t, J)}$ into an irreducible \tilde{H} -module: For each $w \in \mathcal{F}^{(t, J)}$,

$$X^\lambda v_w = (wt)(X^\lambda)v_w, \quad \text{for } X^\lambda \in X, \text{ and}$$

$$T_i v_w = (T_i)_{ww} v_w + (q^{-1} + (T_i)_{ww}) v_{s_i w}, \quad \text{for } 1 \leq i \leq n,$$

where $(T_i)_{ww} = \frac{q - q^{-1}}{1 - (wt)(X^{-\alpha_i})}$, and we set $v_{s_i w} = 0$ if $s_i w \notin \mathcal{F}^{(t, J)}$.

- (b) If M is an irreducible calibrated representation such that $\text{supp}(M) = \{wt \mid w \in \mathcal{F}^{(t, J)}\}$ for some placed skew shape (t, J) then M is isomorphic to the module $\tilde{H}^{(t, J)}$ constructed in (a).

Remark 1.12. It follows from the results of Rodier [?] that if M is an irreducible \tilde{H} -module with regular central character (i.e. $Z(t) = \emptyset$) then M satisfies the hypothesis of the statement of Theorem 1.11 (b).

Langlands classification. The following discussion follows the work of Evens [?] and [?], §8. For this subsection it is convenient to assume that $q \in \mathbb{R}_{>0}$ and $q \neq 1$. For the general case see [?], §8. Let $t \in T$ and let $t = t_r t_c$ be the polar decomposition of t . Define

$$\nu(t) \in \sum_{i=1}^n \mathbb{R}\alpha_i^\vee \quad \text{by requiring} \quad t_r(X^\lambda) = q^{2\langle \lambda, \nu(t) \rangle}, \quad \text{for all } \lambda \in P.$$

A finite dimensional \tilde{H} -module M is *tempered* if for all weights t of M (as defined in (1.5)) we have

$$\langle \omega_i, \nu(t) \rangle \leq 0, \quad \text{for all } 1 \leq i \leq n.$$

The module M is *square integrable* if $\langle \omega_i, \nu(t) \rangle < 0$ for all $1 \leq i \leq n$ and all weights t of M .

Let I be a subset of the simple roots and let \tilde{H}_I be the subalgebra of \tilde{H} generated by T_i , $i \in I$, and all $X^\lambda \in X$. We shall say that a finite dimensional \tilde{H}_I -module is *tempered* if I is the maximal set such that for all weights t of M ,

$$\langle \omega_i, \nu(t) \rangle \leq 0, \quad \text{for all } i \in I.$$

Theorem 1.13. (see [?]) *Let $I \subseteq \{1, 2, \dots, n\}$ and let \mathcal{T} be an irreducible tempered representation of \tilde{H}_I .*

- (a) $M_{\mathcal{T}, I} = \text{Ind}_{\tilde{H}_I}^{\tilde{H}}(\mathcal{T})$ has a unique irreducible quotient $L_{\mathcal{T}, I}$.
- (b) Every irreducible \tilde{H} -module is isomorphic to $L_{\mathcal{T}, I}$ for some pair (\mathcal{T}, I) .
- (c) If $L_{\mathcal{T}, I} \cong L_{\mathcal{T}', I'}$ then $I = I'$ and $\mathcal{T} \cong \mathcal{T}'$ as \tilde{H}_I -modules.

The *Langlands parameters* of an irreducible \tilde{H} -module M are given by the pair (\mathcal{T}, I) specified by Theorem 1.13 (b).

Classification by indexing triples. Kazhdan and Lusztig [?] (see also the important work of Ginzburg [?]) gave a refinement of the Langlands classification. Let G be the simple complex algebraic group with root system R and weight lattice P . (Although there is a different affine Hecke algebra associated to each simple complex group, in this paper we have considered only the affine Hecke algebras corresponding to the simply connected groups G . For a precise description of how the affine Hecke algebra corresponds to the simple complex group G and the extension of the Langlands classification to the other cases see the recent work of Ram-Ramagge [?] and Reeder [?].)

An *indexing triple* (s, u, ρ) consists of

$$\begin{aligned} & \text{a semisimple element } s \in G, \\ & \text{a unipotent element } u \in G, \end{aligned} \quad \text{such that } sus^{-1} = u^{q^2},$$

and an irreducible representation ρ of the component group $A(s, u) = Z_G(s, u) / Z_G(s, u)^\circ$, where $Z_G(s, u) = Z_G(s) \cap Z_G(u)$. Let $K(\mathcal{B}_{s, u})$ be the K -theory of the variety

$$\mathcal{B}_{s, u} = \{\text{Borel subgroups of } G \text{ containing both } s \text{ and } u\}.$$

By a theorem of Lusztig [?] $K(\mathcal{B}_{s, u})$ is an \tilde{H} -module. The group $A(s, u)$ also acts on $K(\mathcal{B}_{s, u})$ and this action commutes with the action of \tilde{H} . The *standard modules* $M_{s, u, \rho}$ are the \tilde{H} -modules given by the decomposition

$$K(\mathcal{B}_{s, u}) = \bigoplus_{\rho} M_{s, u, \rho} \otimes \rho,$$

where the sum is over all irreducible representations of $A(s, u)$.

Theorem 1.14. [?]

- (a) If $M_{s, u, \rho} \neq 0$ then it has a unique simple quotient $L_{s, u, \rho}$.
- (b) Every simple \tilde{H} -module isomorphic to some $L_{s, u, \rho}$.
- (c) If $L_{s, u, \rho} \cong L_{s', u', \rho'}$ then there is a $g \in G$ such that $s' = gsg^{-1}$, $u' = gug^{-1}$, and $\rho' = \rho$ (identifying $A(s', u')$ and $A(s, u)$).

In this way each irreducible \tilde{H} -module corresponds to a unique (up to conjugation) indexing triple. One can replace u by $n = \ln u$ in the Lie

algebra $\mathfrak{g} = \text{Lie}(G)$ (see [?], Ch. 8) so that an indexing triple is

$$\begin{array}{l} \text{a semisimple element } s \in G, \\ \text{a nilpotent element } n \in \mathfrak{g}, \end{array} \quad \text{such that } \text{Ad}(s)n = q^2n,$$

and an irreducible representation ρ of the component group $A(s, n) = Z_G(s, n)/Z_G(s, n)^\circ$, where $Z_G(s, n) = Z_G(s) \cap Z_G(n)$ and $Z_G(n)$ is taken with respect to the adjoint action of G on \mathfrak{g} . Since the part of $A(s, u)$ coming from the center of G acts trivially on $\mathcal{B}_{s,u}$ the only representations ρ of $A(s, u)$ that appear in $K(\mathcal{B}_{s,u})$ are representations of $A'(n) = Z_{G_s}(n)/Z_{G_s}(n)^0$ where G_s is the simple adjoint group corresponding to the reductive group $Z_G(s)$. We will use this form of the indexing triples in the examples in later sections.

Principal series modules. Let $t \in T$ and let $\mathbb{C}v_t$ be the one dimensional $\mathbb{C}[X]$ -module corresponding to the character $t: X \rightarrow \mathbb{C}^*$. Specifically, $\mathbb{C}v_t$ is the one dimensional vector space with basis $\{v_t\}$ and $\mathbb{C}[X]$ -action given by

$$X^\lambda v_t = t(X^\lambda)v_t, \quad \text{for all } X^\lambda \in X.$$

The *principal series module* corresponding to t is the \tilde{H} -module

$$M(t) = \text{Ind}_{\mathbb{C}X}^{\tilde{H}}(\mathbb{C}v_t).$$

Theorem 1.15. [?]

- (a) Every irreducible \tilde{H} -module M with central character t is a composition factor of the principal series module $M(t)$.
- (b) If $w \in W$ and $t \in T$ then $M(t)$ and $M(wt)$ have the same composition factors.

Theorem 1.16. (Kato's irreducibility criterion [?]) Let $t \in T$ and let $P(t) = \{\alpha > 0 \mid t(X^\alpha) = q^{\pm 2}\}$. The principal series module $M(t)$ is irreducible if and only if $P(t) = \emptyset$.

Remark. Kato actually proves a more general result and thus needs a further condition for irreducibility. We have simplified matters by specifying the weight lattice P in our construction of the affine Hecke algebra. One can use any W -invariant lattice in \mathbb{R}^n and Kato works in this more general situation. When the one uses the weight lattice P , a result of Steinberg [?], 4.2, 5.3 says that the stabilizer W_t of a point $t \in T$ under the action of W is always a reflection group. Because of this Kato's criterion takes a simpler form.

Weights of induced modules. If $I \subseteq \{1, \dots, n\}$ define \tilde{H}_I to be the subalgebra of \tilde{H} generated by $T_i, i \in I$, and all $X^\lambda \in X$.

Lemma 1.17. *Let $t \in T$ such that $t(X^{\alpha_i}) = q^2$ for all $i \in I$ and let $\mathbb{C}v_t$ be the one dimensional \tilde{H}_I -module with basis $\{v_t\}$ and \tilde{H}_I -action given by*

$$T_i v_t = q v_t, \quad \text{for } i \in I,$$

and

$$X^\lambda v_t = t(X^\lambda) v_t, \quad \text{for all } X^\lambda \in X.$$

Let W/W_I be the set of minimal length coset representatives of cosets of W_I in W . Then the weights of the \tilde{H} -module $M = \text{Ind}_{\tilde{H}_I}^{\tilde{H}}(\mathbb{C}v_t)$ are wt , $w \in W/W_I$, and

$$\dim(M_{wt}^{\text{gen}}) = (\# \text{ of } u \in W/W_I \text{ such that } ut = wt).$$

Proof. The module M has basis $\{T_w \otimes v_t \mid w \in W/W_I\}$. By writing $T_w = T_{i_1} \cdots T_{i_p}$ for a reduced word $w = s_{i_1} \dots s_{i_p}$ and inductively using the defining relation (1.2) we get

$$\begin{aligned} X^\lambda(T_w \otimes v_t) &= t(X^{w^{-1}\lambda})(T_w \otimes v_t) + \sum_{v < w} a_v(t)(T_v \otimes v_t) \\ &= (wt)(X^\lambda)(T_w \otimes v_t) + \sum_{v < w} a_v(t)(T_v \otimes v_t), \end{aligned}$$

where the sum is over $v \in W$ which are less than w in Bruhat order and $a_v(t) \in \mathbb{C}$. This shows that the eigenvalues of X^λ on M are $(wt)(X^\lambda)$. The result follows by counting the multiplicity of each eigenvalue. ■

The τ operators. The maps $\tau_i: M_t^{\text{gen}} \rightarrow M_{s_i t}^{\text{gen}}$ defined below are local operators on M in the sense that they act on each weight space M_t^{gen} of M separately. The operator τ_i is only defined on weight spaces M_t^{gen} such that $t(X^{\alpha_i}) \neq 1$.

Proposition 1.18. ([?] Proposition 2.7) *Let $t \in T$ such that $t(X^{\alpha_i}) \neq 1$ and let M be a finite dimensional \tilde{H} -module. Define*

$$\begin{aligned} \tau_i: M_t^{\text{gen}} &\longrightarrow M_{s_i t}^{\text{gen}} \\ m &\longmapsto \left(T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} \right) m. \end{aligned}$$

- (a) *The map $\tau_i: M_t^{\text{gen}} \rightarrow M_{s_i t}^{\text{gen}}$ is well defined.*
- (b) *As operators on M_t^{gen} , $X^\lambda \tau_i = \tau_i X^{s_i \lambda}$, for all $X^\lambda \in X$.*
- (c) *As operators on M_t^{gen} , $\tau_i \tau_i = \frac{(q - q^{-1} X^{\alpha_i})(q - q^{-1} X^{-\alpha_i})}{(1 - X^{\alpha_i})(1 - X^{-\alpha_i})}$.*

(d) Let $1 \leq i \neq j \leq n$ and let m_{ij} be as in (1.1). Then

$$\underbrace{\tau_i \tau_j \tau_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{\tau_j \tau_i \tau_i \cdots}_{m_{ij} \text{ factors}},$$

whenever both sides are well defined operators on M_t^{gen} .

Lemma 1.19. Let $t \in T$ such that $t(X^{\alpha_i}) = 1$ and suppose that M is an \tilde{H} -module such that $M_t^{\text{gen}} \neq 0$. Let W_t be the stabilizer of t under the action of W on T . Assume that $\bar{w} \in W/W_t$ is such that t and $\bar{w}t$ are in the same connected component of $\Gamma(t)$. Let w be a minimal length coset representative of \bar{w} . Then

- (a) $\dim(M_{\bar{w}t}^{\text{gen}}) \geq 2$, and
- (b) If $M_{s_j \bar{w}t}^{\text{gen}} = 0$ then $(\bar{w}t)(X^{\alpha_j}) = q^{\pm 2}$ and $\langle w^{-1}\alpha_j, \alpha_i^\vee \rangle = 0$.

Proof. Let $M(t)$ be the two dimensional principal series module for the affine Hecke algebra $\tilde{H}A_1$ of type A_1 (see §2 central character t_o). Then $M(t) = M(t)_t^{\text{gen}}$ and has basis $\{v_t, T_1 v_t\}$. Let n_t be a nonzero weight vector in M_t . There is a unique $\tilde{H}A_1$ -module homomorphism

$$\begin{array}{ccc} M(t) & \longrightarrow & M \\ v_t & \longmapsto & n_t \end{array}$$

where we view M as an $\tilde{H}A_1$ -module by restriction to the parabolic subalgebra $\tilde{H}_{\{i\}} \subseteq \tilde{H}$. This homomorphism must be an injection since $M(t)$ is irreducible. Thus the vectors $n_t, T_i n_t$ span a two dimensional subspace of M_t^{gen} and $X^\lambda \in X$ acts on this subspace by the matrix

$$\phi_t(X^\lambda) = t(X^\lambda) \begin{pmatrix} 1 & (q - q^{-1})\langle \lambda, \alpha_i^\vee \rangle \\ 0 & 1 \end{pmatrix}.$$

Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression of w . Since t and wt are in the same connected component of $\Gamma(t)$ we can use Proposition 1.18 (c) to show that the map

$$\tau_w = \tau_{i_1} \cdots \tau_{i_p} : M_t^{\text{gen}} \longrightarrow M_{wt}^{\text{gen}}$$

is well defined and bijective. Thus the vectors $\tau_w n_t, \tau_w T_i n_t$ span a two dimensional subspace of M_{wt}^{gen} and, by Proposition 1.18 (b) $X^\lambda \in X$ acts on this subspace by the matrix

$$\phi_{wt}(X^\lambda) = t(X^{w^{-1}\lambda}) \begin{pmatrix} 1 & (q - q^{-1})\langle w^{-1}\lambda, \alpha_i^\vee \rangle \\ 0 & 1 \end{pmatrix}.$$

This proves (a). Then

$$\phi_{wt}(1 - X^{-\alpha_j}) = (1 - t(X^{-w^{-1}\alpha_j})) \begin{pmatrix} 1 & \frac{(q - q^{-1})t(X^{-w^{-1}\alpha_j})}{1 - t(X^{-w^{-1}\alpha_j})} \langle -w^{-1}\alpha_j, \alpha_i^\vee \rangle \\ 0 & 1 \end{pmatrix}.$$

Since $M_{s_j wt}^{\text{gen}} = 0$, $\tau_j: M_{wt}^{\text{gen}} \rightarrow M_{s_j wt}^{\text{gen}}$ is the zero map and so

$$\begin{aligned} \phi_{wt}(T_j) &= \phi_{wt} \left(\frac{q - q^{-1}}{1 - X^{-\alpha_j}} \right) \\ &= \frac{q - q^{-1}}{1 - t(X^{-w^{-1}\alpha_j})} \begin{pmatrix} 1 & \frac{(q - q^{-1})t(X^{-w^{-1}\alpha_j})}{1 - t(X^{-w^{-1}\alpha_j})} \langle w^{-1}\alpha_j, \alpha_i^\vee \rangle \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The relation $T_j^2 = (q - q^{-1})T_j + 1$ is the same as $(T_j - q)(T_j + q^{-1}) = 0$. This relation forces $\phi_{wt}(T_j)$ to have Jordan blocks of size 1 and eigenvalues $\pm q^{\pm 1}$. It follows that $t(X^{w^{-1}\alpha_j}) = q^{\pm 2}$ and $\langle w^{-1}\alpha_j, \alpha_i^\vee \rangle = 0$. ■

2. Classification for A_1

The root system R for A_1 has one simple root α_1 and fundamental weight $\omega_1 = \frac{1}{2}\alpha_1$.

Irreducible representations. Table 2.1 lists the irreducible \tilde{H} -modules by their central characters. The sets $P(t)$ and $Z(t)$ are as given in (1.7) and correspond to the choice of representative for the central character displayed in Figure 2.1. The Langlands parameters usually consist of a pair (\mathcal{T}, I) where I is a subset of $\{1\}$ and \mathcal{T} is a tempered representation for the parabolic subalgebra \tilde{H}_I . In our cases the tempered representation \mathcal{T} of \tilde{H}_I is completely determined by a character $t \in T$. Specifically, \mathcal{T} is the only tempered representation of \tilde{H}_I which has t as a weight. In the labeling in Table 2.1 we have replaced the representation \mathcal{T} by the weight t . The nilpotent element e_{α_1} is a representative of the root space \mathfrak{g}_{α_1} for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. For each calibrated module with central character t we have listed the subset $J \subseteq P(t)$ such that (t, J) is the corresponding placed skew shape (see Theorem 1.11). The abbreviation ‘nc’ indicates modules that are not calibrated.

Central character	$P(t)$	$Z(t)$	Dimension	Langlands parameters	Indexing triple	Calibration set J
t_a	$\{\alpha_1\}$	\emptyset	1	(t_a, \emptyset)	$(t_a, 0, 1)$	\emptyset
			1	tempered	$(t_a, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
t_b	\emptyset	\emptyset	2	(t_b, \emptyset)	$(t_b, 0, 1)$	\emptyset
t_o	\emptyset	$\{\alpha_1\}$	2	tempered	$(t_o, 0, 1)$	nc

Table 2.1. Irreducible representations

Figure 2.1 displays the real parts of the central characters in Table 2.1. If $t \in T$ then the polar decomposition $t = t_r t_c$ determines an element $\mu \in \mathbb{R}^n$ such that $t_r(X^\lambda) = e^{\langle \lambda, \mu \rangle}$ (see (1.3)). For each central character t_p the point labeled by p in Figure 2.1 is the graph of the corresponding $\mu_p \in \mathbb{R}^n$. Assume (for pictorial convenience) that q is a positive real number and let

$$H_{\alpha_1} = \{x \in \mathbb{R} \mid \langle \alpha_1, x \rangle = 0\},$$

and

$$H_{\alpha_1 \pm \delta} = \{x \in \mathbb{R} \mid \langle \alpha_1, x \rangle = \ln(q^{\pm 2})\}.$$

The $|$ marks indicate the (affine) hyperplanes $H_{\alpha_1 \pm \delta}$.

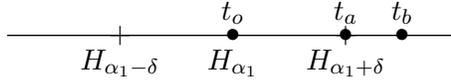


Figure 2.1. Real parts of central characters in Table 2.1

Tempered and square integrable representations. The tempered (resp. square integrable) \tilde{H} -modules are the ones which have all their weight spaces in the closure (resp. interior) of the dotted region of Figure 2.2.

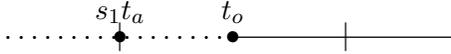


Figure 2.2. Real parts of weights of tempered representations

The irreducible tempered representations with real central character can be indexed by the irreducible representations of the symmetric group S_2 (see [?]). These representations are indexed by the partitions $(2), (1^2)$ of 2. Let e_{α_1} be an element of the root space \mathfrak{g}_{α_1} for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. The two nilpotent orbits in \mathfrak{g} and the corresponding tempered representations of \tilde{H} are as in Table 2.2.

Nilpotent orbit	Indexing triple	Square integrable	W representation
regular	$(t_a, e_{\alpha_1}, 1)$	yes	(1^2)
0	$(t_o, 0, 1)$	no	(2)

Table 2.2. Tempered representations and the Springer correspondence

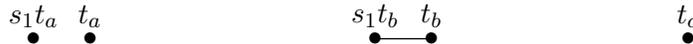


Figure 2.3. Calibration graphs for central characters in Table 2.1

The analysis. *Central character t_a :* There are two one-dimensional representations $\mathbb{C}v_a$ and $\mathbb{C}v_{s_1a}$ with central character t_a . These representations are given explicitly by

$$\begin{aligned} X^\lambda v_a &= t_a(X^\lambda)v_a, \\ T_1 v_a &= qv_a, \end{aligned}$$

and

$$\begin{aligned} X^\lambda v_{s_1a} &= (s_1 t_a)(X^\lambda)v_{s_1a}, \\ T_1 v_{s_1a} &= -q^{-1}v_{s_1a}, \end{aligned}$$

respectively. One uses Theorem 1.15 and the fact that the principal series module $M(t_a)$ is two dimensional to conclude that $\mathbb{C}v_a$ and $\mathbb{C}v_{s_1a}$ are the only irreducible representations of \tilde{H} with central character t_a .

Central character t_b : By Theorem 1.15 and Kato's irreducibility criterion, Theorem 1.16, the only irreducible representation with central character t_b is the principal series module $M(t_b)$. Alternatively, this module can be constructed by applying Theorem 1.11 to the placed skew shape (t_b, \emptyset) .

Central character t_o : The weights given by $t_o(X^{\omega_1}) = \pm 1$ are the *two* central characters $t_o \in T$ which satisfy $P(t) = \emptyset$, $Z(t) = \{\alpha_1\}$. In either case Kato's irreducibility criterion (Theorem 1.16) tells us that the principal series module $M(t_o)$ is irreducible. This module has basis $\{v_t, T_1 v_t\}$ and action given by

$$\phi(X^\lambda) = t(X^\lambda) \begin{pmatrix} 1 & (q - q^{-1})\langle \lambda, \alpha_1^\vee \rangle \\ 0 & 1 \end{pmatrix}$$

and

$$\phi(T_1) = \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}.$$

3. Classification for $A_1 \times A_1$

The affine Hecke algebra of $A_1 \times A_1$ is naturally isomorphic to $\tilde{H}A_1 \otimes \tilde{H}A_1$. The finite dimensional irreducible representations of $\tilde{H}A_1 \otimes \tilde{H}A_1$ are all of the form $M \otimes N$ where M and N are finite dimensional irreducible representations of $\tilde{H}A_1$.

4. Classification for A_2

The root system R for A_2 has simple roots α_1 and α_2 , fundamental weights ω_1 and ω_2 , and

$$\begin{aligned} \langle \alpha_1, \alpha_2^\vee \rangle &= -1 & \omega_1 &= \frac{1}{3}(2\alpha_1 + \alpha_2) & \text{and} & \alpha_1 &= 2\omega_1 - \omega_2 \\ \langle \alpha_2, \alpha_1^\vee \rangle &= -1, & \omega_2 &= \frac{1}{3}(\alpha_1 + 2\alpha_2), & & \alpha_2 &= -\omega_1 + 2\omega_2. \end{aligned}$$

Irreducible representations. Table 4.1 lists the irreducible \tilde{H} -modules by their central characters. The sets $P(t)$ and $Z(t)$ are as given in (1.7) and correspond to the choice of representative for the central character displayed in Figure 4.1. The Langlands parameters usually consist of a pair (\mathcal{T}, I) where I is a subset of $\{1, 2\}$ and \mathcal{T} is a tempered representation for the parabolic subalgebra \tilde{H}_I . In our cases the tempered representation \mathcal{T} of \tilde{H}_I is completely determined by a character $t \in T$. Specifically, \mathcal{T} is the only tempered representation of \tilde{H}_I which has t as a weight. In the labeling in Table 4.1 we have replaced the representation \mathcal{T} by the weight t . The nilpotent elements e_{α_1} and e_{α_2} are representatives of the root spaces \mathfrak{g}_{α_1} and \mathfrak{g}_{α_2} , respectively, where \mathfrak{g} is the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$. For each calibrated module with central character t we have listed the subset $J \subseteq P(t)$ such that (t, J) is the corresponding placed skew shape (see Theorem 1.11). The abbreviation ‘nc’ indicates modules that are not calibrated.

Figure 4.1 displays the real parts of the central characters in Table 4.1. If $t \in T$ then the polar decomposition $t = t_r t_c$ determines an element $\nu \in \mathbb{R}^n$ such that $t_r(X^\lambda) = e^{\langle \lambda, \nu \rangle}$ (see (1.3)).

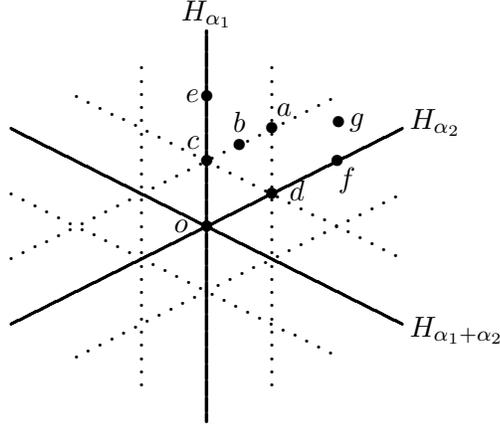


Figure 4.1. Real parts of central characters in Table 4.1

For each central character t_p the point labeled by p in Figure 4.1 is the graph of the corresponding $\nu_p \in \mathbb{R}^n$. Assume (for pictorial convenience) that q is a positive real number and let

$$H_\beta = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = 0\}, \quad \text{and} \quad H_{\beta \pm \delta} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = \ln(q^{\pm 2})\},$$

for each positive root β . The dotted lines display the (affine) hyperplanes $H_{\beta \pm \delta}$.

Central character	$P(t)$ $Z(t)$	Dim.	Langlands parameters	Indexing triple	Calibration set J
t_a	$\{\alpha_1, \alpha_2\}$ \emptyset	1	(t_a, \emptyset)	$(t_a, 0, 1)$	\emptyset
		2	$(s_1 t_a, \{2\})$	$(t_a, e_{\alpha_2}, 1)$	$\{\alpha_2\}$
		2	$(s_2 t_a, \{1\})$	$(t_a, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		1	tempered	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$	$\{\alpha_1, \alpha_2\}$
t_b	$\{\alpha_2\}$ \emptyset	3	(t_b, \emptyset)	$(t_b, 0, 1)$	\emptyset
		3	$(s_2 t_b, \{2\})^\dagger$	$(t_b, e_{\alpha_2}, 1)$	$\{\alpha_2\}$
t_c	$\{\alpha_2, \alpha_1 + \alpha_2\}$ $\{\alpha_1\}$	3	$(t_c, \{1\})$	$(t_c, 0, 1)$	nc
		3	$(s_2 t_c, \{2\})$	$(t_c, e_{\alpha_2}, 1)$	nc
t_d	$\{\alpha_1, \alpha_1 + \alpha_2\}$ $\{\alpha_2\}$	3	$(t_d, \{2\})$	$(t_d, 0, 1)$	nc
		3	$(s_1 t_d, \{1\})$	$(t_d, e_{\alpha_1}, 1)$	nc
t_e	\emptyset $\{\alpha_1\}$	6	$(t_e, \{1\})$	$(t_e, 0, 1)$	nc
t_f	\emptyset $\{\alpha_2\}$	6	$(t_f, \{2\})$	$(t_f, 0, 1)$	nc
t_g	\emptyset \emptyset	6	(t_g, \emptyset)	$(t_g, 0, 1)$	\emptyset
t_o	\emptyset $\{\alpha_1, \alpha_2\}$	6	tempered	$(t_o, 0, 1)$	nc

Table 4.1. Irreducible representations ¹

Tempered and square integrable representations. The tempered (resp. square integrable) \tilde{H} -modules are the ones which have the real parts of all their weights in the closure (resp. interior) of the shaded region of Figure 4.2. Let $t \in T$ be given by $t(X^{-\alpha_1}) = \pm q$, $t(X^{-\alpha_2}) = \pm q$. This is a special case of the central character t_b in Table 4.1. For this particular

¹There is one case when this representation is tempered, see Table 4.2.

special case there is one tempered representation with central character t .

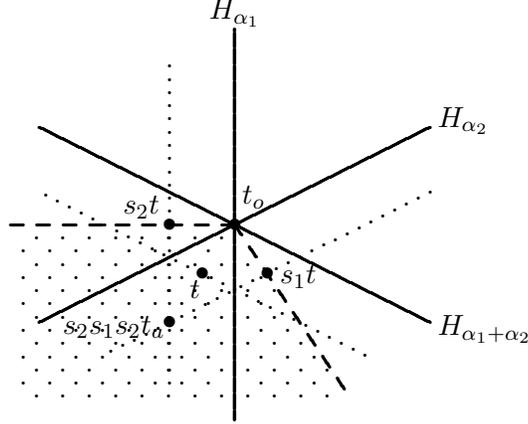
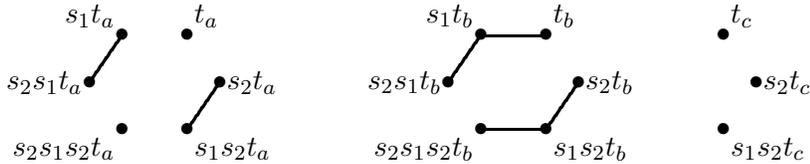


Figure 4.2. Real parts of weights of tempered representations

The irreducible tempered representations with real central character are in one-to-one correspondence with the irreducible representations of the symmetric group S_3 (see [?]). These representations are indexed by the partitions $(3), (21), (1^3)$ of 3. Equivalently, they can be indexed by the pairs (n, ρ) which appear in the Springer correspondence. The n and ρ will also be elements of the indexing triple for the corresponding tempered representation of \tilde{H} . Here n is a nilpotent element of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$ and ρ is an irreducible representation of the component group $Z_G(n)/Z_G(n)^\circ$. In type A the component group is always trivial. For each root $\beta \in R$ let e_β be an element of the root space \mathfrak{g}_β . The three nilpotent orbits in \mathfrak{g} and the corresponding tempered representations of \tilde{H} are as in Table 4.2.

Nilpotent orbit	$Z_G(n)/Z_G(n)^\circ$	Indexing triple	Sq. int.	W rep.
regular	1	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$	yes	(3)
subregular	1	$(s_2 s_1 t, e_{\alpha_2}, 1)$	no	(21)
0	1	$(t_0, 0, 1)$	no	(1^3)

Table 4.2. Tempered representations and the Springer correspondence



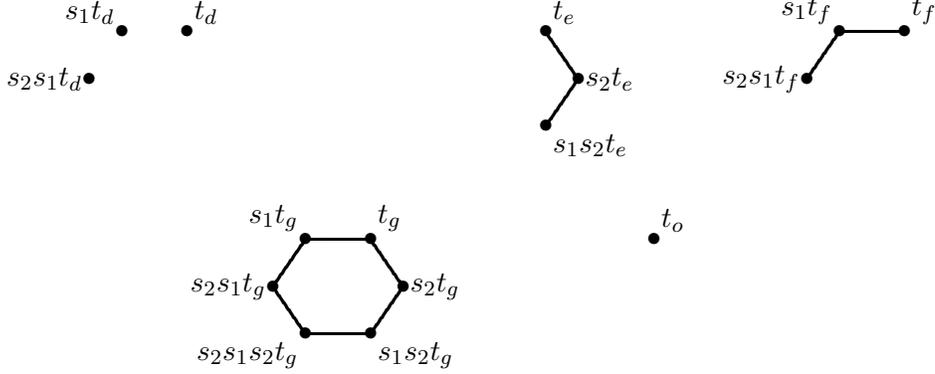


Figure 4.3. Calibration graphs for central characters in Table 4.1

The analysis. *Central characters t_a, t_b, t_g :* Since $Z(t) = \emptyset$ these weights are regular. Thus the representations corresponding to these central characters are in one to one correspondence with the connected components of the calibration graph $\Gamma(t)$ and can be constructed explicitly with the use of Theorem 1.11. Up to isomorphism the principal series module $M(t_g)$ is the only irreducible \tilde{H} -module with central character t_g . The Langlands parameters for each module can be determined from its weight structure and the indexing triple is then determined from the Langlands data by using the induction theorem of Kazhdan and Lusztig (see the discussion in [?], p. 34).

There is one special case of the central character t_b when the irreducible module constructed by applying Theorem 1.11 to the placed skew shape (t_b, \emptyset) is tempered. This happens when $t_b = s_2s_1t$ for the weight $t \in T$ given by $t(X^{-\alpha_1}) = \pm q, t(X^{-\alpha_2}) = \pm q$. The indexing triple and the calibration set for this case are still given by $(t_b, e_{\alpha_2}, 1)$ and $J = \{\alpha_2\}$, respectively.

Central characters t_c and t_d : One can use the defining relations of \tilde{H} to check that the only 1-dimensional representations of \tilde{H} are the ones with central character t_a . Construct two 3-dimensional representations of \tilde{H} by

$$\text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}}(\mathbb{C}v_c) \quad \text{and} \quad \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}}(\mathbb{C}v_{s_2c}),$$

where $\mathbb{C}v_c$ and $\mathbb{C}v_{s_2c}$ are the two one-dimensional representations of $\tilde{H}_{\{2\}}$ given by

$$T_2v_c = qv_c, \quad X^{\alpha_1}v_c = v_c, \quad X^{\alpha_2}v_c = q^2v_c,$$

$$T_2v_{s_2c} = -q^{-1}v_{s_2c}, \quad X^{\alpha_1}v_{s_2c} = q^2v_{s_2c}, \quad X^{\alpha_2}v_{s_2c} = q^{-2}v_{s_2c}.$$

These representations must be irreducible since, if not, they would either have a 1-dimensional submodule or a one dimensional quotient. But there are no 1-dimensional modules with central character t_c .

The central characters t_c and t_d are taken into each other under the automorphism of the Dynkin diagram of A_2 which switches the two nodes and thus these two central characters will produce modules which have the same structure (up to twisting by the automorphism which switches α_1 and α_2). Thus the representations with central character t_d can be obtained from the ones with central character t_c by switching all 1's and 2's and changing all c 's to d 's.

Central characters t_e and t_f : Since $P(t_e) = \emptyset$, Kato's irreducibility criterion (Theorem 1.16) implies that the principal series module $M(t_e)$ is irreducible. By Theorem 1.15 this is the only irreducible with central character t_e . As for the case of t_c and t_d , the central characters t_e and t_f are taken into each other under the automorphism of the Dynkin diagram of A_2 . Thus the irreducible representations with central character t_f can be obtained from the one with central character t_e by switching all 1's and 2's and changing all e 's to f 's.

Central character t_o : Since $P(t_o) = \emptyset$, Kato's irreducibility criterion (Theorem 1.16) implies that the principal series module $M(t_o)$ is irreducible. By Theorem 1.15 this is the only irreducible with central character t_o .

5. Classification for C_2

The root system R for C_2 has simple roots α_1 and α_2 , fundamental weights ω_1 and ω_2 , and

$$\begin{aligned} \langle \alpha_1, \alpha_2^\vee \rangle &= -2 & \omega_1 &= \alpha_1 + \alpha_2 & \text{and} & \alpha_1 &= 2\omega_1 - 2\omega_2 \\ \langle \alpha_2, \alpha_1^\vee \rangle &= -1, & \omega_2 &= \frac{1}{2}\alpha_1 + \alpha_2, & & \alpha_2 &= -\omega_1 + 2\omega_2. \end{aligned}$$

Irreducible representations. Table 5.1 lists the irreducible \tilde{H} -modules by their central characters. We have listed only those central characters t for which the principal series module $M(t)$ is not irreducible (see Theorem 1.16). The sets $P(t)$ and $Z(t)$ are as given in (1.7) and correspond to the choice of representative for the central character displayed in Figure 5.1. The Langlands parameters usually consist of a pair (\mathcal{T}, I) where I is a subset of $\{1, 2\}$ and \mathcal{T} is a tempered representation for the parabolic subalgebra \tilde{H}_I . In our cases the tempered representation \mathcal{T} of \tilde{H}_I is completely determined by a character $t \in T$. Specifically, \mathcal{T} is the only tempered representation of \tilde{H}_I which has t as a weight. In the labeling in Table 5.1 we have replaced the representation \mathcal{T} by the weight t . The notation for the nilpotent elements in the indexing triples is as in Table 5.2. For each calibrated module with central character t we have listed the subset $J \subseteq P(t)$ such that (t, J) is the corresponding placed skew shape (see Theorem 1.11). The abbreviation 'nc' indicates modules that are not calibrated.

Figure 5.1 displays the real parts of the central characters in Table 5.1. If $t \in T$ then the polar decomposition $t = t_r t_c$ determines an element $\mu \in \mathbb{R}^n$ such that $t_r(X^\lambda) = e^{\langle \lambda, \mu \rangle}$ (see (1.3)). For each central character t_p the point

labeled by p in Figure 5.1 is the graph of the corresponding $\mu_p \in \mathbb{R}^n$. Assume (for pictorial convenience) that q is a positive real number and let

$$H_\beta = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = 0\}, \quad \text{and} \quad H_{\beta \pm \delta} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = \ln(q^{\pm 2})\},$$

for each positive root β . The dotted lines display the (affine) hyperplanes $H_{\beta \pm \delta}$.

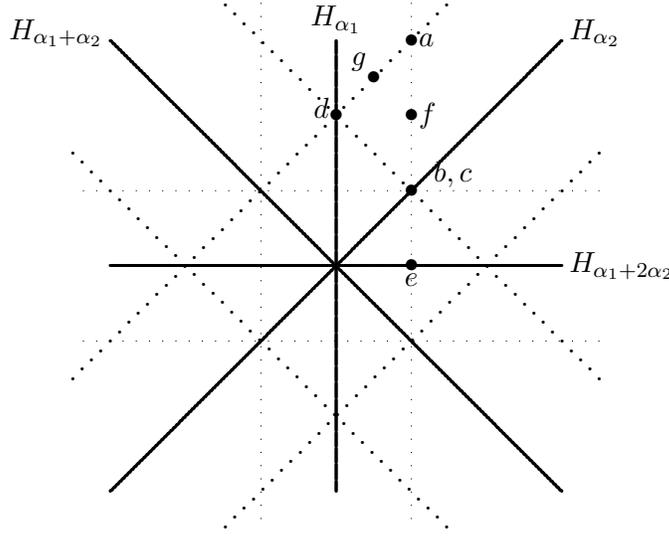


Figure 5.1. Real parts of central characters in Table 5.1

Tempered and square integrable representations. The tempered (resp. square integrable) \tilde{H} -modules are the ones which have the real parts of all their weights in the closure (resp. interior) of the shaded region of Figure 5.2.

The irreducible tempered representations with real central character can be indexed by the irreducible representations of the Weyl group W of type C_2 (see [?], p. 34). Equivalently, these representations can be indexed by the pairs (n, ρ) which appear in the Springer correspondence. The n and ρ will also be elements of the indexing triple for the corresponding tempered representation of \tilde{H} . Here n is a nilpotent element of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, G is the complex simple group over \mathbb{C} of type C_2 and ρ is an irreducible representation of the component group $Z_G(n)/Z_G(n)^\circ$ (see [Ca]). For each root $\beta \in R$ let e_β be an element of the root space \mathfrak{g}_β . The four nilpotent orbits in \mathfrak{g} and the corresponding tempered representations of \tilde{H} are as in Table 5.2. We have used the notation of Carter [?], p. 424 to label the irreducible representations of the Weyl group.

Central char.	$P(t)$ $Z(t)$	Dim.	Langlands parameters	Indexing triple	Cal. set J
t_a	$\{\alpha_1, \alpha_2\}$ \emptyset	1	$(s_1 s_2 s_1 s_2 t_a, \emptyset)$	$(t_a, 0, 1)$	\emptyset
		3	$(s_1 t_a, \{1\})$	$(t_a, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		3	$(s_2 t_a, \{2\})$	$(t_a, e_{\alpha_2}, 1)$	$\{\alpha_2\}$
		1	tempered	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$	$\{\alpha_1, \alpha_2\}$
t_b	$\{\alpha_1, \alpha_1 + \alpha_2,$ $\alpha_1 + 2\alpha_2\}$ $\{\alpha_2\}$	3	$(t_b, \{2\})$	$(t_b, 0, 1)$	nc
		1	$(s_1 t_b, \{1\})$	$(t_b, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		1	tempered	$(t_b, e_{\alpha_1 + \alpha_2}, -1)$	$\{\alpha_1, \alpha_1$ $+ \alpha_2\}$
		3	tempered	$(t_b, e_{\alpha_1 + \alpha_2}, 1)$	nc
t_c	$\{\alpha_1, \alpha_1 + 2\alpha_2\}$ \emptyset	2	$(t_c, \{2\})$	$(t_c, 0, 1)$	\emptyset
		2	$(s_1 t_c, \{1\})$	$(t_c, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		2	$(s_1 s_2 t_c, \{1\})$	$(t_c, e_{\alpha_1 + 2\alpha_2}, 1)$	$\{\alpha_1 + 2\alpha_2\}$
		2	tempered	$(t_c, e_{\alpha_1} + e_{\alpha_1 + 2\alpha_2}, 1)$	$\{\alpha_1, \alpha_1$ $+ 2\alpha_2\}$
t_d	$\{\alpha_2, \alpha_1 + \alpha_2\}$ $\{\alpha_1\}$	4	$(t_d, \{1\})$	$(t_d, 0, 1)$	nc
		4	$(s_2 t_d, \{2\})$	$(t_d, e_{\alpha_2}, 1)$	nc
t_e	$\{\alpha_1\}$ $\{\alpha_1 + 2\alpha_2\}$	4	$(s_2 t_e, \{1\})$	$(t_e, 0, 1)$	nc
		4	tempered	$(t_e, e_{\alpha_1}, 1)$	nc
t_f	$\{\alpha_1\}$ \emptyset	4	(t_f, \emptyset)	$(t_f, 0, 1)$	\emptyset
		4	$(s_1 t_f, \{1\})$	$(t_f, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
t_g	$\{\alpha_2\}$ \emptyset	4	(t_g, \emptyset)	$(t_g, 0, 1)$	\emptyset
		4	$(s_2 t_g, \{2\})$	$(t_g, e_{\alpha_2}, 1)$	$\{\alpha_2\}$

Table 5.1. Irreducible (non principal series) representations

Nilpotent orbit	$Z_G(n)/Z_G(n)^\circ$	Indexing triple	Sq. int.	W rep.
regular	1	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$	yes	$(\emptyset, 11)$
subregular	$\mathbb{Z}/2\mathbb{Z}$	$(t_b, e_{\alpha_1 + \alpha_2}, 1)$	yes	$(1, 1)$
		$(t_b, e_{\alpha_1 + \alpha_2}, -1)$	yes	$(\emptyset, 2)$
minimal	1	$(t_e, e_{\alpha_1}, 1)$	no	$(11, \emptyset)$
0	1	$(t_o, 0, 1)$	no	$(2, \emptyset)$

Table 5.2. Tempered representations and the Springer correspondence

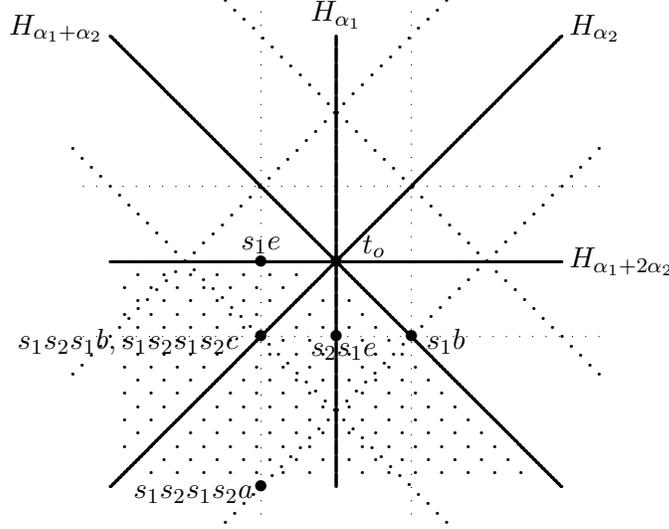
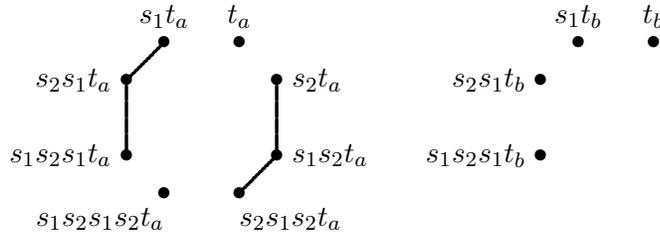


Figure 5.2. Real parts of weights of tempered representations

The only other tempered representation is the square integrable representation $(t_c, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, 1)$. This representation does not have real central character. It is the representation constructed in [Lu3, 4.14, 4.23]. (In Lusztig's notation it is the star of the representation corresponding to the graph $\mathcal{G}' \oplus \mathcal{G}''$.)

The analysis. A general calculation with the defining relations of \tilde{H} shows that there are only four one dimensional \tilde{H} -modules $\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}v_3,$ and $\mathbb{C}v_4$ which have weights $t_a, s_2s_1t_b, s_1t_b$ and $s_1s_2s_1s_2t_a$, respectively. These modules are given explicitly by

$$\begin{array}{llll} T_1v_1 = qv_1, & T_2v_1 = qv_1, & X^{\alpha_1}v_1 = q^2v_1, & X^{\alpha_2}v_1 = q^2v_1, \\ T_1v_2 = qv_2, & T_2v_2 = -q^{-1}v_2, & X^{\alpha_1}v_2 = q^2v_2, & X^{\alpha_2}v_2 = q^{-2}v_2, \\ T_1v_3 = -q^{-1}v_3, & T_2v_3 = qv_3, & X^{\alpha_1}v_3 = q^{-2}v_3, & X^{\alpha_2}v_3 = q^2v_3, \\ T_1v_4 = -q^{-1}v_4, & T_2v_4 = -q^{-1}v_4, & X^{\alpha_1}v_4 = q^{-2}v_4, & X^{\alpha_2}v_4 = q^{-2}v_4. \end{array}$$



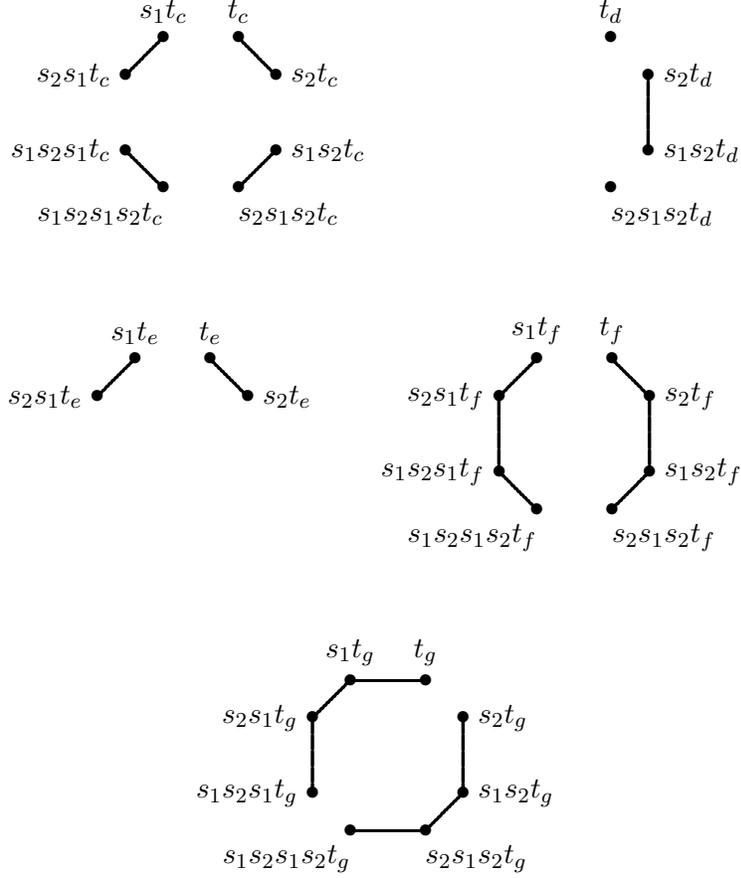


Figure 5.3. Calibration graphs for central characters in Table 5.1

Central character t_a : Since $Z(t_a) = \emptyset$, t_a is regular and thus all irreducible representations with central character t_a are calibrated. They are in one to one correspondence with the connected components of the calibration graph and can be constructed with the use of Theorem 1.11. The Langlands parameters for each module can be determined from its weight structure and the indexing triple is then determined from the Langlands data by using the induction theorem of Kazhdan-Lusztig (see the discussion in [?], p.34).

Central character t_b : We already know from our general computation above, that there are two one-dimensional \tilde{H} -modules with central character t_b . One has weight $s_1 t_b$ and the other has weight $s_2 s_1 t_b$. Let $\mathbb{C}v_b$ and $\mathbb{C}v_{s_1 b}$ be the one dimensional representations of $\tilde{H}_{\{1\}}$ given by

$$\begin{aligned} T_1 v_b &= q v_b, & X^{\alpha_1} v_b &= q^2 v_b, & X^{\alpha_2} v_b &= v_b, \\ T_1 v_{s_1 b} &= -q^{-1} v_{s_1 b}, & X^{\alpha_1} v_{s_1 b} &= q^{-2} v_{s_1 b}, & X^{\alpha_2} v_{s_1 b} &= q^2 v_{s_1 b}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}}(\mathbb{C}v_b) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}}(\mathbb{C}v_{s_1b}).$$

By Lemma 1.17 these modules have weights $\text{supp}(M_1) = \{t_b, s_1t_b, s_2s_1t_b\}$ and $\text{supp}(M_2) = \{s_1t_b, s_2s_1t_b, s_1s_2s_1t_b\}$, respectively. Both M_1 and M_2 are 4 dimensional. By Proposition 1.18 (c) one of the two operators $\tau_2: (M_1)_{s_2s_1t_b} \rightarrow (M_1)_{s_1t_b}$ or $\tau_2: (M_1)_{s_1t_b} \rightarrow (M_1)_{s_2s_1t_b}$ must have nonzero kernel. This implies that M_1 has either a 3 dimensional submodule or a 3 dimensional quotient, call it N_1 , with weights $\{t_b, s_1t_b\}$. By Lemma 1.19, any module P with weights $\{t_b, s_1t_b\}$ must have $\dim(P_{t_b}^{\text{gen}}) \geq 2$ and $\dim(P_{s_1t_b}^{\text{gen}}) \geq 1$. It follows that N_1 is irreducible. A similar argument can be used to show that M_2 has either a 3 dimensional submodule or a 3-dimensional quotient which must be irreducible.

The representation N_1 constructed in the previous paragraph and the 1 dimensional representation with weight s_1t_b are both tempered. One obtains the corresponding indexing triples by comparing the Langlands parameters for these modules with the labelings of the corresponding representations of W in the Springer correspondence. See [?], p. 424, [?], p. 34 and Table 5.2.

Central character t_c : Since $Z(t_c) = \emptyset$, t_c is regular and thus all irreducible representations with central character t_c are calibrated. They are in one to one correspondence with the connected components of the calibration graph and can be constructed with the use of Theorem 1.11. The Langlands parameters for each module can be determined from its weight structure. The only representation for which the indexing triple cannot be determined from the Langlands parameters and the [?] induction theorem (see the discussion in [BM, p. 34]) is the tempered representation. This representation is constructed in [?], 4.14 and 4.23. In Lusztig's notation, it is the star (see [?], 4.23) of the representation corresponding to the graph $\mathcal{G}' \oplus \mathcal{G}''$. The indexing triple for this representation is given in the discussion for B_2 in [?], 2.10.

Central character t_d : Let $\mathbb{C}v_d$ and $\mathbb{C}v_{s_2d}$ be the one dimensional representations of $\tilde{H}_{\{2\}}$ given by

$$\begin{aligned} T_2v_d &= qv_d, & X^{\alpha_1}v_d &= v_d, & X^{\alpha_2}v_d &= q^2v_d, \\ T_2v_{s_2d} &= -q^{-1}v_{s_2d}, & X^{\alpha_1}v_{s_2d} &= q^4v_{s_2d}, & X^{\alpha_2}v_{s_2d} &= q^{-2}v_{s_2d}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}}(\mathbb{C}v_d) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}}(\mathbb{C}v_{s_2d}).$$

By Lemma 1.17 these modules have weights $\text{supp}(M_1) = \{t_d, s_2t_d, s_1s_2t_d\}$ and $\text{supp}(M_2) = \{s_2t_d, s_1s_2t_d, s_2s_1s_2t_d\}$, respectively. Both M_1 and M_2 are 4 dimensional.

Let M be any \tilde{H} -module such that $M_{t_d} \neq 0$. By Lemma 1.19 (a), $\dim(M_{t_d}^{\text{gen}}) \geq 2$. Since $\langle \alpha_2, \alpha_1^\vee \rangle \neq 0$ it follows from Lemma 1.19 (b) that $\dim(M_{s_2t_d}^{\text{gen}}) \geq 1$. Then, by Proposition 1.6, $\dim(M_{s_1s_2t_d}^{\text{gen}}) \geq 1$. Adding these

numbers up we see that $\dim(M) \geq 4$. It follows that M_1 is irreducible. An analogous argument can be applied to conclude that M_2 is irreducible.

Central character t_e : Let $\mathbb{C}v_e$ and $\mathbb{C}v_{s_1e}$ be the one dimensional representations of $\tilde{H}_{\{1\}}$ given by

$$\begin{aligned} T_1 v_e &= qv_e, & X^{\alpha_1} v_e &= q^2 v_e, & X^{\alpha_2} v_e &= q^{-1} v_e, \\ T_1 v_{s_1e} &= -q^{-1} v_{s_1e}, & X^{\alpha_1} v_{s_1e} &= q^{-2} v_{s_1e}, & X^{\alpha_2} v_{s_1e} &= qv_{s_1e}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}}(\mathbb{C}v_e) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}}(\mathbb{C}v_{s_1e}).$$

By Lemma 1.17 these modules have weights $\text{supp}(M_1) = \{t_e, s_2 t_e\}$ and $\text{supp}(M_2) = \{s_1 t_e, s_2 s_1 t_e\}$ respectively. Both M_1 and M_2 are 4 dimensional.

Let M be any \tilde{H} -module such that $M_{t_e} \neq 0$. By Lemma 1.19 (a) and Proposition 1.6, $\dim(M_{t_e}^{\text{gen}}) = \dim(M_{s_2 t_e}^{\text{gen}}) \geq 2$. Thus $\dim(M) \geq 4$. It follows that M_1 is irreducible. An analogous argument can be applied to conclude that M_2 is irreducible.

Central character t_f and t_g : These cases are handled in the same way as for the central character t_a .

6. Classification for G_2

The root system R for G_2 has simple roots α_1 and α_2 , fundamental weights ω_1 and ω_2 , and

$$\begin{aligned} \langle \alpha_1, \alpha_2^\vee \rangle &= -3 & \omega_1 &= 2\alpha_1 + 3\alpha_2 \\ \langle \alpha_2, \alpha_1^\vee \rangle &= -1, & \omega_2 &= \alpha_1 + 2\alpha_2, \end{aligned}$$

and

$$\begin{aligned} \alpha_1 &= -\omega_1 + 2\omega_2 \\ \alpha_2 &= 2\omega_1 - 3\omega_2. \end{aligned}$$

Irreducible representations. Table 6.1 lists the irreducible \tilde{H} -modules by their central characters. We have listed only those central characters t for which the principal series module $M(t)$ is not irreducible (see Theorem 1.16). The sets $P(t)$ and $Z(t)$ are as given in (1.7) and correspond to the choice of representative for the central character displayed in Figure 6.1. The Langlands parameters usually consist of a pair (\mathcal{T}, I) where I is a subset of $\{1, 2\}$ and \mathcal{T} is a tempered representation for the parabolic subalgebra \tilde{H}_I . In our cases the tempered representation \mathcal{T} of \tilde{H}_I is completely determined by a character $t \in T$. Specifically, \mathcal{T} is the only tempered representation of \tilde{H}_I which has t as a weight. In the labeling in Table 6.1 we have replaced the representation \mathcal{T} by the weight t . The notation for the nilpotent elements in the indexing triples is as in Table 6.3.

Table 6.2 lists the irreducible calibrated \tilde{H} -modules. For each module with central character t we have listed the subset $J \subseteq P(t)$ such that (t, J) is the corresponding placed skew shape (see Theorem 1.11). We have listed

only those central characters t for which the principal series module $M(t)$ is not irreducible (see Theorem 1.16).

Central character	$P(t)$ $Z(t)$	Dim.	Langlands parameters	Indexing triple
t_a	$\{\alpha_1, \alpha_2\}$ \emptyset	1	(t_a, \emptyset)	$(t_a, 0, 1)$
		5	$(s_1 t_a, \{1\})$	$(t_a, e_{\alpha_1}, 1)$
		5	$(s_2 t_a, \{2\})$	$(t_a, e_{\alpha_2}, 1)$
		1	tempered	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$
t_b	$\{\alpha_1\}$ \emptyset	6	(t_b, \emptyset)	$(t_b, 0, 1)$
		6	$(s_1 t_b, \{1\})$	$(t_b, e_{\alpha_1}, 1)$
t_c	$\{\alpha_1, \alpha_1 + 3\alpha_2\}$ \emptyset	2	$(t_c, \{2\})$	$(t_c, 0, 1)$
		4	$(s_1 t_c, \{1\})$	$(t_c, e_{\alpha_1}, 1)$
		4	$(s_1 s_2 t_c, \{1\})$	$(t_c, e_{\alpha_1 + 3\alpha_2}, 1)$
		2	tempered	$(t_c, e_{\alpha_1} + e_{\alpha_1 + 3\alpha_2}, 1)$
t_d	$\{\alpha_1, \alpha_1 + 2\alpha_2\}$ \emptyset	3	$(t_d, \{2\})$	$(t_d, 0, 1)$
		3	$(s_1 t_d, \{1\})$	$(t_d, e_{\alpha_1}, 1)$
		3	$(s_2 s_1 s_2 t_d, \{2\})$	$(t_d, e_{\alpha_1 + 2\alpha_2}, 1)$
		3	tempered	$(t_d, e_{\alpha_1} + e_{\alpha_1 + 2\alpha_2}, 1)$
t_e	$\{\alpha_1, \alpha_1 + 2\alpha_2,$ $\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2\}$ $\{\alpha_2\}$	3	$(t_e, \{2\})$	$(t_e, 0, 1)$
		1	$(s_1 t_e, \{1\})$	$(t_e, e_{\alpha_1}, 1)$
		2	$(s_2 s_1 t_e, \{2\})$	$(t_e, e_{\alpha_1 + \alpha_2}, 1)$
		1	tempered	$(t_e, e_{\alpha_1} + e_{\alpha_1 + 2\alpha_2}, (21))$
		3	tempered	$(t_e, e_{\alpha_1} + e_{\alpha_1 + 2\alpha_2}, (3))$
t_f	$\{\alpha_1, 2\alpha_1 + 3\alpha_2\}$ $\{\alpha_1 + 3\alpha_2\}$	6	$(t_f, \{1\})$	$(t_f, 0, 1)$
		6	$(s_1 t_f, \{1\})$	$(t_f, e_{\alpha_1}, 1)$
t_g	$\{\alpha_1\}$ $\{\alpha_1 + 2\alpha_2\}$	6	$(t_g, \{2\})$	$(t_g, 0, 1)$
		6	tempered	$(t_g, e_{\alpha_1}, 1)$
t_h	$\{\alpha_2\}$ \emptyset	6	(t_h, \emptyset)	$(t_h, 0, 1)$
		6	$(s_2 t_h, \{2\})$	$(t_h, e_{\alpha_2}, 1)$
t_i	$\{\alpha_2, \alpha_1 + \alpha_2\}$ $\{\alpha_1\}$	6	$(t_i, \{1\})$	$(t_i, 0, 1)$
		6	$(s_2 t_i, \{2\})$	$(t_i, e_{\alpha_2}, 1)$
t_j	$\{\alpha_2\}$ $\{2\alpha_1 + 3\alpha_2\}$	6	$(t_j, \{1\})$	$(t_j, 0, 1)$
		6	tempered	$(t_j, e_{\alpha_2}, 1)$

Table 6.1. Irreducible (non principal series) representations

Figure 6.1 displays the real parts of the central characters in Table 6.1. If $t \in T$ then the polar decomposition $t = t_r t_c$ determines an element $\nu \in \mathbb{R}^n$ such that $t_r(X^\lambda) = e^{\langle \nu, \lambda \rangle}$ (see (1.3)). For each central character t_p the point labeled by p in Figure 6.1 is the graph of the corresponding $\nu_p \in \mathbb{R}^n$. Assume (for pictorial convenience) that q is a positive real number and let

$$H_\beta = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = 0\}, \quad \text{and} \quad H_{\beta \pm \delta} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = \ln(q^{\pm 2})\},$$

for each positive root β . The dotted lines display the (affine) hyperplanes $H_{\beta \pm \delta}$.

Central character	$P(t)$ $Z(t)$	Dim.	Indexing triple	Calibration set J
t_a	$\{\alpha_1, \alpha_2\}$ \emptyset	1	$(t_a, 0, 1)$	\emptyset
		5	$(t_a, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		5	$(t_a, e_{\alpha_2}, 1)$	$\{\alpha_2\}$
		1	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$	$\{\alpha_1, \alpha_2\}$
t_b	$\{\alpha_1\}$ \emptyset	6	$(t_b, 0, 1)$	\emptyset
		6	$(t_b, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
t_c	$\{\alpha_1, \alpha_1 + 3\alpha_2\}$ \emptyset	2	$(t_c, 0, 1)$	\emptyset
		4	$(t_c, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		4	$(t_c, e_{\alpha_1 + 3\alpha_2}, 1)$	$\{\alpha_1 + 3\alpha_2\}$
		2	$(t_c, e_{\alpha_1} + e_{\alpha_1 + 3\alpha_2}, 1)$	$\{\alpha_1, \alpha_1 + 3\alpha_2\}$
t_d	$\{\alpha_1, \alpha_1 + 2\alpha_2\}$ \emptyset	3	$(t_d, 0, 1)$	\emptyset
		3	$(t_d, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		3	$(t_d, e_{\alpha_1 + 2\alpha_2}, 1)$	$\{\alpha_1 + 2\alpha_2\}$
		3	$(t_d, e_{\alpha_1} + e_{\alpha_1 + 2\alpha_2}, 1)$	$\{\alpha_1, \alpha_1 + 2\alpha_2\}$
t_e	$\{\alpha_1, \alpha_1 + 2\alpha_2,$ $\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2\}$ $\{\alpha_2\}$	1	$(t_e, e_{\alpha_1}, 1)$	$\{\alpha_1\}$
		2	$(t_e, e_{\alpha_1 + \alpha_2}, 1)$	$\{\alpha_1, \alpha_1 + \alpha_2\}$
		1	$(t_e, e_{\alpha_1} + e_{\alpha_1 + 2\alpha_2}, (21))$	$P(t_e) \setminus \{\alpha_1 + 3\alpha_2\}$
t_h	$\{\alpha_2\}$ \emptyset	6	$(t_h, 0, 1)$	\emptyset
		6	$(t_h, e_{\alpha_2}, 1)$	$\{\alpha_2\}$

Table 6.2. Calibrated irreducible (non principal series) representations

Tempered and square integrable representations. The irreducible tempered representations with real central character can be indexed by the irreducible representations of the Weyl group W of type G_2 (see [?], p. 34). Equivalently, these representations can be indexed by the pairs (n, ρ) which appear in the Springer correspondence. The n and ρ will also be elements of the indexing triple for the corresponding tempered representation of \tilde{H} .

Here n is a nilpotent element of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, G is the complex simple group over \mathbb{C} of type G_2 and ρ is an irreducible representation of the component group $Z_G(n)/Z_G(n)^\circ$ (see [Ca]). For each root $\beta \in R$ let e_β be an element of the root space \mathfrak{g}_β . The five nilpotent orbits in \mathfrak{g} and the corresponding tempered representations of \tilde{H} are as in Table 6.3. The notation S_3 denotes the symmetric group on three elements, which has irreducible representations indexed by the partitions $(3), (21), (1^3)$ of 3. We have used the notation of Carter [?], p. 427 to label the irreducible representations of the Weyl group W of type G_2 . The only other tempered representations are the representations labeled by the triples $(t_c, e_{\alpha_1} + e_{\alpha_1+3\alpha_2}, 1)$ and $(t_d, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, 1)$. These representations are square integrable but do not have real central character.

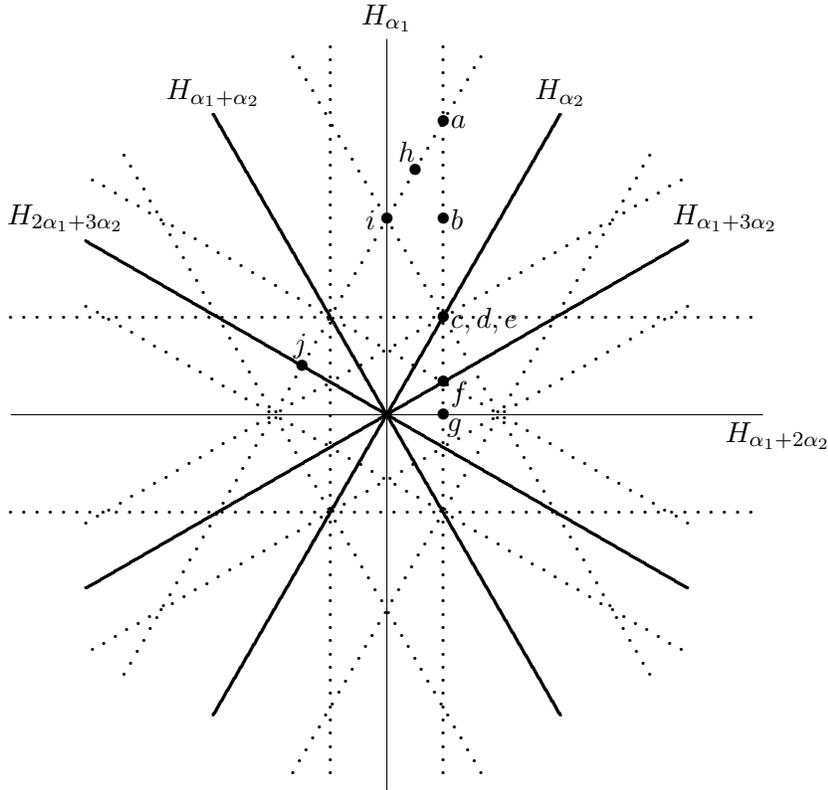
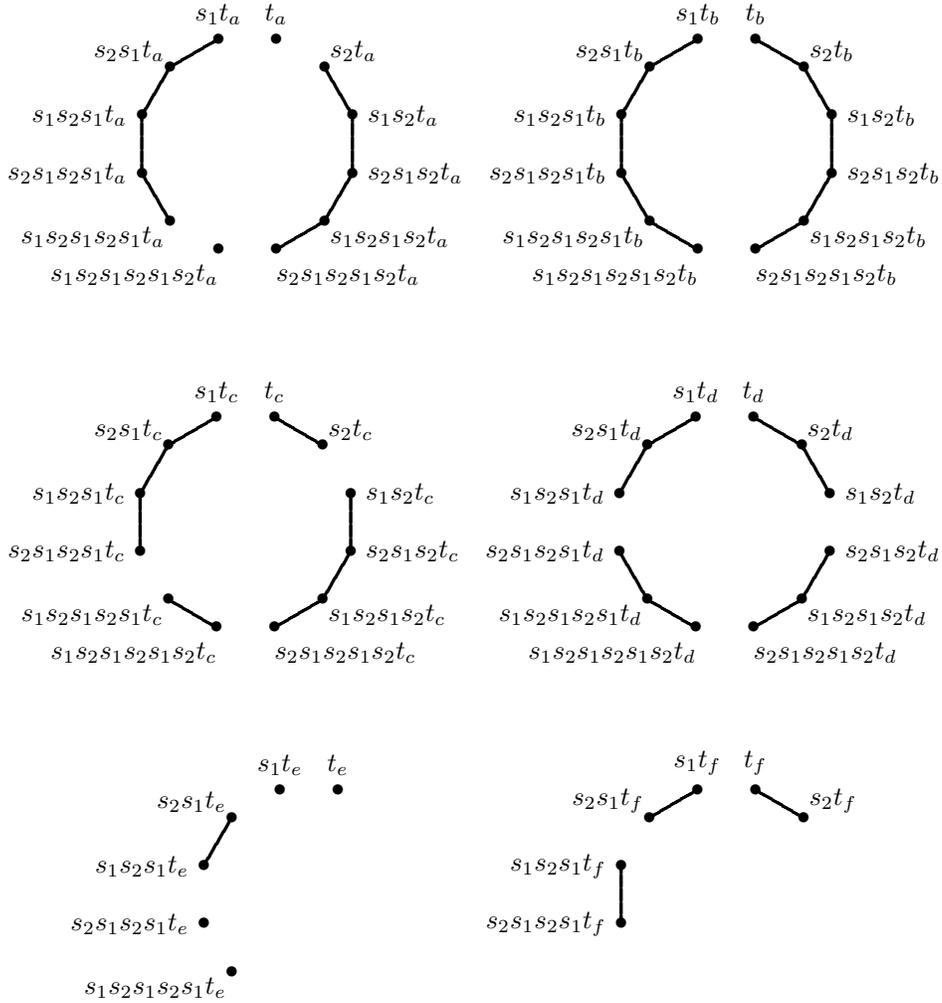


Figure 6.1. Real parts of central characters in Table 6.1

The modules labeled by $(t_c, e_{\alpha_1} + e_{\alpha_1+3\alpha_2}, 1)$, $(t_d, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, 1)$, $(t_e, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, (3))$, $(t_e, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, (21))$ are the ones constructed by Lusztig in [?] 4.20, 4.19, 4.7 and 4.22 respectively. In Lusztig's notation these are the stars (see [?], 4.23) of the modules labeled by the graphs \mathcal{G}'' , \mathcal{G}' , \mathcal{G} and \mathcal{G}''' , respectively.

Nilpotent orbit	$Z_G(n)/Z_G(n)^o$	Indexing triple	Sq. int.	W rep.
regular	1	$(t_a, e_{\alpha_1} + e_{\alpha_2}, 1)$	yes	$\phi_{1,0}$
subregular	S_3	$(t_e, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, (3))$	yes	$\phi_{2,1}$
		$(t_e, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, (21))$	yes	$\phi_{1,3}'$
subminimal	1	$(t_j, e_{\alpha_2}, 1)$	no	$\phi_{2,2}$
minimal	1	$(t_g, e_{\alpha_1}, 1)$	no	$\phi_{1,3}''$
0	1	$(t_o, 0, 1)$	no	$\phi_{1,6}$

Table 6.3. Tempered representations and the Springer correspondence



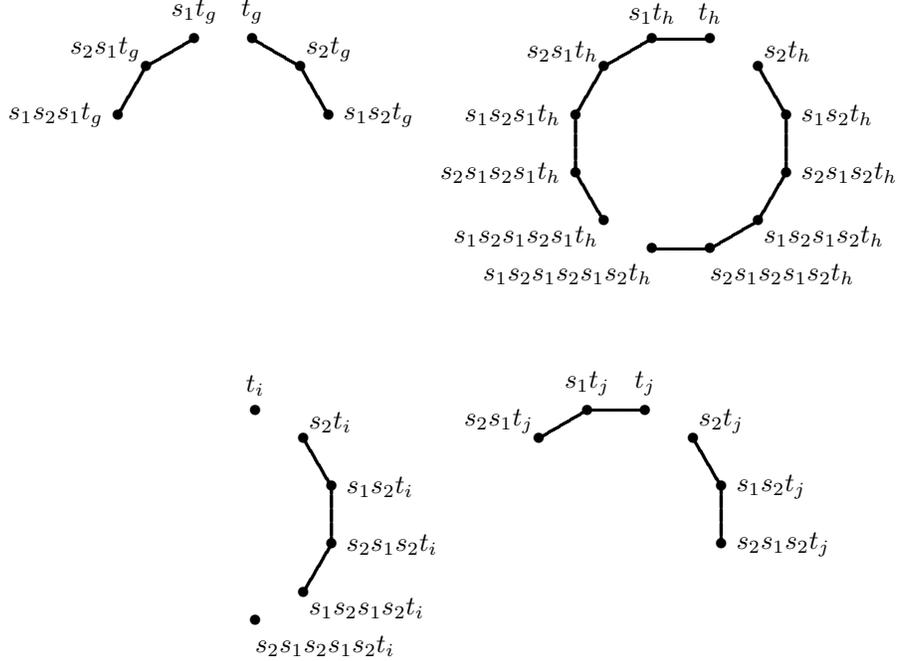


Figure 6.2. Calibration graphs for central characters in Table 6.1

The analysis. *Central characters* t_a, t_b, t_c, t_d, t_h : The central characters t_e, t_f, t_g, t_i and t_j are the only ones which have both $Z(t)$ and $P(t)$ nonempty. The other cases are handled by Theorem 1.16 and Theorem 1.11 as in the cases of central characters t_a, t_b, t_g and t_o for type A_2 .

The Langlands parameters for each module can be determined from its weight structure. The indexing triple is determined from the Langlands data by using the induction theorem of Kazhdan and Lusztig (see the discussion in [?], p. 34). Let us give an example to illustrate the procedure. The Langlands parameters $(s_1s_2t_c, \{1\})$ for the 4 dimensional representation with central character t_c correspond to the indexing triple $(s_2t_c, e_{\alpha_1}, 1)$ which is conjugate to the triple $(t_c, s_2e_{\alpha_1}, 1) = (t_c, e_{\alpha_1+3\alpha_2}, 1)$.

The indexing triples for the tempered representations cannot be determined with the use of the Kazhdan-Lusztig induction theorem. The indexing triples for the tempered representations with real central character are determined from the Springer correspondence, see Table 6.3 and [?], p. 34. The two tempered representations with central characters t_c and t_d do not have real central character. By the last two sentences of [?], 2.10 we know that the indexing triples for these representations contain the subregular nilpotent

and that the component groups are isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively. In both cases the component group acts trivially on $K(\mathcal{B}_{s,u})$ and so $\rho = 1$. The fact that the elements $e_{\alpha_1} + e_{\alpha_1+3\alpha_2}$ and $e_{\alpha_1} + e_{\alpha_1+2\alpha_2}$ are representatives of the subregular nilpotent orbit can be derived from the analysis in [?], Theorem 4.40 or [?]. This determines the triples $(t_c, e_{\alpha_1} + e_{\alpha_1+3\alpha_2}, 1)$ and $(t_d, e_{\alpha_1} + e_{\alpha_1+2\alpha_2}, 1)$.

Central character t_e : Theorem 1.11 applied to the placed skew shapes $(t_e, \{\alpha_1, \alpha_1 + \alpha_2\})$, $(t_e, \{\alpha_1\})$ and $(t_e, \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\})$ produces, respectively, a two dimensional irreducible module M with $\text{supp}(M) = \{s_2s_1t_e, s_1s_2s_1t_e\}$, a one dimensional irreducible module N with $\text{supp}(N) = \{s_1t_e\}$ and a one dimensional irreducible module N^* with $\text{supp}(N^*) = \{s_2s_1s_2s_1t_e\}$. Lusztig [?] Theorem 4.7 constructs a 3-dimensional irreducible \tilde{H} -module P with $\dim(P_{t_e}^{\text{gen}}) = 2$ and $\dim(P_{s_1t_e}^{\text{gen}}) = 1$. In Lusztig's notation this is the module labeled by the graph \mathcal{G} for \tilde{G}_2 .

As described in [?], 4.23 we can twist the module P by an involutive automorphism of \tilde{H} to obtain another 3-dimensional irreducible module P^* which has $\dim((P^*)_{s_2s_1s_2s_1t_e}^{\text{gen}}) = 2$ and $\dim((P^*)_{s_1s_2s_1s_2s_1t_e}^{\text{gen}}) = 1$.

All of the modules M, N, P, N^*, P^* must appear as composition factors of the principal series module $M(t_e)$. By comparing dimensions of weight spaces, any other module Q which appears in a composition series of $M(t_e)$ must have $\text{supp}(Q) \subseteq \{s_2s_1t_e, s_1s_2s_1t_e\}$. Theorem 1.11(b) then implies that Q must be isomorphic to M . Thus Theorem 1.15 implies that M, N, P, N^* , and P^* are (up to isomorphism) all the irreducible modules with central character t_e .

The Langlands parameters for each module are determined from its weight structure. The Kazhdan-Lusztig induction theorem allows us to use the Langlands parameters to determine the indexing triples for the modules which are not tempered. Since t_e is a real central character the indexing triples for the tempered representations can be determined from the Springer correspondence, see Table 6.3. Alternatively, one can get these triples from [?], 2.10 where it is explained that the nilpotent in the indexing triple is subregular, the variety $\mathcal{B}_{s,u}$ (where $s = t_e$ and u is subregular) consists of three disjoint points and a projective line, and the component group is isomorphic to the symmetric group S_3 . The symmetric group S_3 acts trivially on the line and permutes the three points, which implies that the line corresponds to $\rho = (3)$ (trivial representation of S_3) and the three points are split between the isotypic components $\rho = (3)$ and $\rho = (21)$. In this case the standard module $M_{s,u,(1^3)} = 0$. The projective line in $\mathcal{B}_{s,u}$ corresponds to the two dimensional weight space $(P^*)_{t_e}^{\text{gen}}$ in the module P^* .

Central character t_f : Let $\mathbb{C}v_f$ and $\mathbb{C}v_{s_1f}$ be the one dimensional representations of $\tilde{H}_{\{1\}}$ given by

$$\begin{aligned} T_1 v_f &= qv_f, & X^{\alpha_1} v_f &= q^2 v_f, & X^{3\alpha_2} v_f &= q^{-2} v_f, \\ T_1 v_{s_1f} &= -q^{-1} v_{s_1f}, & X^{\alpha_1} v_{s_1f} &= q^{-2} v_{s_1f}, & X^{3\alpha_2} v_{s_1f} &= q^4 v_{s_1f}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}} (\mathbb{C}v_f) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}} (\mathbb{C}v_{s_1f}).$$

By Lemma 1.17 these modules have weights $\text{supp}(M_1) = \{s_2 t_f, t_f, s_1 t_f, s_2 s_1 t_f\}$ and $\text{supp}(M_2) = \{s_1 t_f, s_2 s_1 t_f, s_1 s_2 s_1 t_f, s_2 s_1 s_2 s_1 t_f\}$ respectively. Both M_1 and M_2 are 6 dimensional.

Let M be any \tilde{H} -module such that $M_{s_2 t_f} \neq 0$. By Lemma 1.19 and Proposition 1.6, $\dim(M_{t_f}^{\text{gen}}) = \dim(M_{s_2 t_f}^{\text{gen}}) \geq 2$. Since $\langle s_2 \alpha_1, \alpha_1^\vee \rangle = \langle \alpha_1 + 3\alpha_2, \alpha_1^\vee \rangle \neq 0$ it follows from Lemma 1.19 (b) that $\dim(M_{s_1 t_f}^{\text{gen}}) \geq 1$. Then, by Proposition 1.6, $\dim(M_{s_2 s_1 t_f}^{\text{gen}}) \geq 1$. Adding these numbers up we see that $\dim(M) \geq 6$. It follows that M_1 is irreducible. An analogous argument can be applied to conclude that M_2 is irreducible.

Central character t_g : Let $\mathbb{C}v_g$ and $\mathbb{C}v_{s_1g}$ be the one dimensional representations of $\tilde{H}_{\{1\}}$ given by

$$\begin{aligned} T_1 v_g &= qv_g, & X^{\alpha_1} v_g &= q^2 v_g, & X^{2\alpha_2} v_g &= q^{-2} v_g, \\ T_1 v_{s_1g} &= -q^{-1} v_{s_1g}, & X^{\alpha_1} v_{s_1g} &= q^{-2} v_{s_1g}, & X^{2\alpha_2} v_{s_1g} &= q^2 v_{s_1g}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}} (\mathbb{C}v_g) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{1\}}}^{\tilde{H}} (\mathbb{C}v_{s_1g}).$$

By Lemma 1.17 these modules have weights $\text{supp}(M_1) = \{s_1 s_2 t_g, s_2 t_g, t_g\}$ and $\text{supp}(M_2) = \{s_1 t_g, s_2 s_1 t_g, s_1 s_2 s_1 t_g\}$ respectively. Both M_1 and M_2 are 6 dimensional.

Let M be any \tilde{H} -module such that $M_{s_1 s_2 t_g} \neq 0$. By Lemma 1.19 and Proposition 1.6, $\dim(M_{t_g}^{\text{gen}}) = \dim(M_{s_2 t_g}^{\text{gen}}) = \dim(M_{s_1 s_2 t_g}^{\text{gen}}) \geq 2$. Thus $\dim(M) \geq 6$. It follows that M_1 is irreducible. An analogous argument can be applied to conclude that M_2 is irreducible.

Central character t_i : Let $\mathbb{C}v_i$ and $\mathbb{C}v_{s_2i}$ be the one dimensional representations of $\tilde{H}_{\{2\}}$ given by

$$\begin{aligned} T_2 v_i &= qv_i, & X^{\alpha_1} v_i &= v_i, & X^{\alpha_2} v_i &= q^2 v_i, \\ T_2 v_{s_2i} &= -q^{-1} v_{s_2i}, & X^{\alpha_1} v_{s_2i} &= q^6 v_{s_2i}, & X^{\alpha_2} v_{s_2i} &= q^{-2} v_{s_2i}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}} (\mathbb{C}v_i) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}} (\mathbb{C}v_{s_2i}).$$

An argument similar to that for the central character t_f shows that M_1 and M_2 are irreducible.

Central character t_j : Let $\mathbb{C}v_j$ and $\mathbb{C}v_{s_2j}$ be the one dimensional representations of $\tilde{H}_{\{2\}}$ given by

$$\begin{aligned} T_2v_j &= qv_j, & X^{2\alpha_1}v_j &= q^{-6}v_j, & X^{\alpha_2}v_j &= q^2v_j, \\ T_2v_{s_2j} &= -q^{-1}v_{s_2j}, & X^{2\alpha_1}v_{s_2j} &= q^6v_{s_2j}, & X^{\alpha_2}v_{s_2j} &= q^{-2}v_{s_2j}. \end{aligned}$$

Let

$$M_1 = \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}}(\mathbb{C}v_j) \quad \text{and} \quad M_2 = \text{Ind}_{\tilde{H}_{\{2\}}}^{\tilde{H}}(\mathbb{C}v_{s_2j}).$$

An argument similar to that for the central character t_g shows that M_1 and M_2 are irreducible.

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