A combinatorial formula for Macdonald polynomials

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville VIC 3010 Australia
aram@unimelb.edu.au
and
Department of Mathematics
University of Wisconsin
Madison, WI 53706

Martha Yip
Department of Mathematics
University of Wisconsin
Madison, WI 53706
yip@math.wisc.edu

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Abstract

Abstract. In this paper we use the combinatorics of alcove walks to give uniform combinatorial formulas for Macdonald polynomials for all Lie types. These formulas resemble the formulas of Haglund-Haiman-Loehr for Macdonald polynoimals of type GL_n. At $q = 0$ these formulas specialize to the formula of Schwer for the Macdonald spherical function in terms of positively folded alcove walks and at $q = t = 0$ these formulas specialize to the formula for the Weyl character in terms of the Littelmann path model (in the positively folded gallery form of Gaussent-Littelmann).

1 Introduction

The Macdonald polynomials were introduced in the mid 1980s [Mac1] [Mac2] as a remarkable family of orthogonal polynomials generalizing the spherical functions for a p-adic group, the Weyl characters, the Jack polynomials and the zonal polynomials. In the early 1990s Cherednik [Ch] introduced the double affine Hecke algebra (the DAHA) and used it as a tool to prove conjectures of Macdonald. The DAHA is a fundamental tool for studying Macdonald polynomials. Using the DAHA, the nonsymmetric Macdonald polynomials E_μ can be constructed by applying products of "intertwining operators" $\tau_i^\vee$ to the generator $\mathbf{1}$ of the polynomial representation of the DAHA (see [Hai, Prop. 6.13]), and the symmetric Macdonald polynomials P_μ can then be constructed from the E_μ by "symmetrizing" (see [Mac3, Remarks after (6.8)]).

Of recent note in the theory of Macdonald polynomials has been the success of Haglund-Haiman-Loehr in giving, in the type GL_n case, explicit combinatorial formulas for the expansion of Macdonald polynomials in terms of monomials. These formulas were conjectured by J. Haglund and proved by Haglund-Haiman-Loehr in [HHL1] and [HHL2]. The papers [GR] and [Hai] are excellent survey articles discussing these developments.

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Following a key idea of C. Schwer [Sc], the paper [Ra] developed a combinatorics for working in the affine Hecke algebra, the *alcove walk model*. It turns out that this combinatorics is the ideal tool for expansion of products of intertwining operators in the DAHA. These expansions, when applied to the generator of the polynomial representation of the DAHA, give formulas for the Macdonald polynomials for all root systems.

Our formulas resemble the formulas obtained by Haglund-Haiman-Loehr [HHL1] [HHL2] in type GL_n. The form of the terms is the same but our formula has many more terms. Lenart [Le] has explained how to combine terms in our formula to obtain a formula similar to the Haglund-Haiman-Loehr (HHL) one. The individual terms of our formula have, in general, many more factors appearing in the denominators than the terms in the HHL formula. In the GL_n case there is an integral form of the Macdonald polynomials obtained by multiplying by a factor. This factor cancels all denominators in the HHL formula but not in ours and so the "integrality" of the integral form is not directly visible from our formula in the way that it is visible in the HHL formula. To our knowledge, a suitable definition of an integral form of the Macdonald polynomials is still unknown in the general root system setting.

At $q = 0$ the symmetric Macdonald polynomials are the *Hall-Littlewood polynomials* or the *Macdonald spherical functions*. These are the spherical functions for G/K, where G is a p-adic group and K is a maximal compact subgroup. The work of Schwer [Sc, Thm. 1.1] provided fomulas for the expansion of the Macdonald spherical functions in terms of positively folded alcove walks. See [Ra, Thm. 4.2(a)] for a description of the Schwer formula in terms of the alcove walk model. The formula for Macdonald polynomials which we give in Theorem 3.4 reduces to the Schwer formula at $q = 0$.

At $q = t = 0$ the symmetric Macdonald polynomials are the *Weyl characters* or *Schur functions*. In this case our formula for the Macdonald polynomial specializes to the formula for the Weyl character in terms of the Littelmann path model (in the maximal dimensional positively folded gallery form of Gaussent-Littelmann [GL, Cor. 1 p. 62], or the λ-chain form of Lenart-Postnikov [LP1] [LP2]).

It is interesting to note that, in the formulas for the symmetric Macdonald polynomials, the negative folds and the positive folds play an equal role. It is known [GL] that the alcove walks with only positive folds contain detailed information about the geometry of Mirković-Vilonen intersections in the loop Grassmannian. It is tantalizing to wonder whether the alcove walks with both positive and negative folds play a similar role in the geometry of flag varieties for reductive groups over two dimensional local fields and whether the expansions of Macdonald polynomials in this paper are shadows of geometric decompositions.

The papers [GL], [LP1], [LP2], and [Ra] explain how the combinatorics of alcove walks is almost equivalent to the combinatorics of crystal bases and Kashiwara operators (at least for the positively folded alcove walks of maximal dimension). Our expansions of Macdonald polynomials in terms of alcove walks give insight into possible relationships between Macdonald polynomials and crystal and canonical bases.

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2 Double Weyl groups, braid groups and Hecke algebras

In this section we review the basic definitions and notations for affine Weyl groups and double affine Hecke algebras following the expositions in [Ra], [Ch], [Mac4] and [Hai]. Following the definitions we prove Theorem 2.2, a formula for the expansion of products of intertwining operators in the DAHA. This formula is a "lift into the DAHA" of the expansions of Macdonald polynomials given in Section 3.

2.1 Double affine Weyl groups

Let $\mathfrak{h}_{\mathbb{Z}}$ be a $\mathbb{Z}$-lattice with an action of a finite subgroup W_0 of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections. Then W_0 acts on $\mathfrak{h}_{\mathbb{Z}}^*$ by

$$\langle w\mu, \lambda^\vee\rangle = \langle \mu, w^{-1}\lambda^\vee\rangle, \qquad \text{where} \quad \langle \lambda^\vee, \mu\rangle = \mu(\lambda^\vee) \ \text{ for } \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}},\ \mu \in \mathfrak{h}_{\mathbb{Z}}^*. \tag{2.1}$$

Let $R^+ \subseteq \mathfrak{h}_{\mathbb{Z}}^*$ and $(R^\vee)^+ \subseteq \mathfrak{h}_{\mathbb{Z}}$ denote fixed choices of the positive roots and the positive coroots so that the reflections s_α in W_0 act on $\mathfrak{h}_{\mathbb{Z}}$ and on $\mathfrak{h}_{\mathbb{Z}}^*$ by

$$s_\alpha\lambda = \lambda - \langle \lambda, \alpha^\vee\rangle\alpha \qquad \text{and} \qquad s_\alpha\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha\rangle\alpha^\vee, \qquad \text{respectively.} \tag{2.2}$$

The groups

$$X = \{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\} \quad \text{and} \quad Y = \{Y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} \tag{2.3}$$

with

$$X^\mu X^\nu = X^{\mu+\nu} \quad \text{and} \quad Y^{\lambda^\vee}Y^{\sigma^\vee} = Y^{\lambda^\vee+\sigma^\vee} \tag{2.4}$$

are the groups $\mathfrak{h}_{\mathbb{Z}}^*$ and $\mathfrak{h}_{\mathbb{Z}}$ respectively, except written multiplicatively, and the semidirect product

$$W_0 \ltimes (X \times Y) = \{X^\mu w Y^{\lambda^\vee} \mid w \in W_0, \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} \tag{2.5}$$

has additional relations

$$wX^\mu = X^{w\mu}w \quad \text{and} \quad wY^{\lambda^\vee} = Y^{w\lambda^\vee}w, \tag{2.6}$$

for $w \in W_0$, $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ and $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$.

Assume that the action of W_0 on $\mathfrak{h}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$ is irreducible. The *double affine Weyl group* $\widetilde{W}$ is the universal central extension of $W_0 \ltimes (X \times Y)$. If e is the smallest integer such that $\langle \lambda^\vee, \mu\rangle \in \frac{1}{e}\mathbb{Z}$ for all $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$ and $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ then $\widetilde{W}$ is presented by

$$\widetilde{W} = \{q^k X^\mu w Y^{\lambda^\vee} \mid k \in \tfrac{1}{e}\mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0\}$$

with (2.4), (2.6) and

$$X^\mu Y^{\lambda^\vee} = q^{\langle \lambda^\vee, \mu\rangle} Y^{\lambda^\vee} X^\mu, \qquad \text{for } \mu \in \mathfrak{h}_{\mathbb{Z}}^*,\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}. \tag{2.7}$$

The subgroup $\{q^k X^\mu Y^{\lambda^\vee} \mid k \in \frac{1}{e}\mathbb{Z},\ \mu \in \mathfrak{h}_{\mathbb{Z}}^*,\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$ is a Heisenberg group and

$$W = \{X^\mu w \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0\} \qquad \text{and} \quad W^\vee = \{wY^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0\} \tag{2.8}$$

are *affine Weyl groups* inside $\widetilde{W}$. Letting

$$q = X^\delta = Y^{-d} \tag{2.9}$$

and extending the notation of (2.6) gives actions of $W^\vee$ on $\mathfrak{h}_\mathbb{Z}^* \oplus \mathbb{Z}\delta$ and W on $\mathfrak{h}_\mathbb{Z} \oplus \mathbb{Z}d$ with

$$Y^{\lambda^\vee}\mu = \mu - \langle \mu, \lambda^\vee\rangle\delta \qquad \text{and} \qquad X^\mu\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \mu\rangle d. \tag{2.10}$$

Let $\varphi \in R$ be the highest root and $\varphi^\vee \in R^\vee$ the highest coroot and let

$$s_0 = Y^{\varphi^\vee} s_\varphi \qquad \text{and} \qquad s_0^\vee = X^\varphi s_{\varphi^\vee}. \tag{2.11}$$

Let

$$\alpha_0 = -\varphi + \delta, \quad \alpha_0^\vee = -\varphi^\vee + d, \qquad \langle d, \mu\rangle = 0, \quad \langle \lambda^\vee, \delta\rangle = 0, \qquad \langle d, \delta\rangle = 0, \tag{2.12}$$

so that

$$s_0\mu = \mu - \langle \mu, \alpha_0^\vee\rangle\alpha_0 \qquad \text{and} \qquad s_0^\vee\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha_0\rangle\alpha_0^\vee. \tag{2.13}$$

The *alcoves* of $\mathfrak{h}_\mathbb{R}^* = \mathbb{R} \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}^*$ are the connected components of

$$\mathfrak{h}_\mathbb{R}^* \backslash \left(\bigcup_{\alpha^\vee \in (R^\vee)^+,\, j \in \mathbb{Z}} \mathfrak{h}^{\alpha^\vee + jd} \right) \qquad \text{where} \qquad \mathfrak{h}^{\alpha^\vee + jd} = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \alpha^\vee\rangle = -j\}. \tag{2.14}$$

The action of $W = \{X^\mu w \mid \mu \in \mathfrak{h}_\mathbb{Z}^*, w \in W_0\}$ on $\mathfrak{h}_\mathbb{R}^*$ given by

$$X^\mu \cdot \nu = \nu + \mu \qquad \text{and} \qquad w \cdot \nu = w\nu, \qquad \text{for } w \in W_0,\ \mu \in \mathfrak{h}_\mathbb{Z}^* \text{ and } \nu \in \mathfrak{h}_\mathbb{R}^*, \tag{2.15}$$

sends alcoves to alcoves; $s_0^\vee, \ldots, s_n^\vee$ are the reflections in the walls $\mathfrak{h}^{\alpha_0^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ of the fundamental alcove

$$1 = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \alpha_i^\vee\rangle \geq 0, \text{ for } i = 0, 1, \ldots, n\}; \qquad \text{and} \tag{2.16}$$

$$\ell(v) = (\text{number of hyperplanes between 1 and } v) \tag{2.17}$$

is the *length* of $v \in W$. Let $\Omega^\vee$ be the set of length zero elements of W. The affine Weyl group W has an alternate presentation by generators $s_0^\vee, s_1^\vee, \ldots, s_n^\vee$ and $\Omega^\vee$ with relations

$$(s_i^\vee)^2 = 1, \qquad \underbrace{s_i^\vee s_j^\vee \cdots}_{m_{ij}^\vee} = \underbrace{s_j^\vee s_i^\vee \cdots}_{m_{ij}^\vee}, \qquad \text{and} \qquad g^\vee s_i^\vee (g^\vee)^{-1} = s_{\sigma^\vee(i)}^\vee, \quad \text{for } g^\vee \in \Omega^\vee, \tag{2.18}$$

where $\pi/m_{ij}^\vee$ is the angle between $\mathfrak{h}^{\alpha_i^\vee}$ and $\mathfrak{h}^{\alpha_j^\vee}$ and $\sigma^\vee$ denotes the permutation of the $\mathfrak{h}^{\alpha_i^\vee}$ induced by the action of $g^\vee$. If $\Omega^\vee \times \mathfrak{h}_\mathbb{R}^*$ is $|\Omega^\vee|$ copies of $\mathfrak{h}_\mathbb{R}^*$ (sheets), with $\Omega^\vee$ acting by switching sheets then there is a bijection

$$W \longleftrightarrow \{\text{alcoves in } \Omega^\vee \times \mathfrak{h}_\mathbb{R}^*\} \tag{2.19}$$

and we will often identify $v \in W$ with the corresponding alcove in $\Omega^\vee \times \mathfrak{h}_\mathbb{R}^*$. The pictures illustrating this bijection in type SL_3 are displayed in the appendix.

The *periodic orientation* is the orientation of the hyperplanes $\mathfrak{h}^{\alpha^\vee + kd}$ such that

$$\begin{aligned} &\text{(a) 1 is on the positive side of } \mathfrak{h}^{\alpha^\vee} \text{ for } \alpha^\vee \in (R^\vee)^+, \\ &\text{(b) } \mathfrak{h}^{\alpha^\vee + kd} \text{ and } \mathfrak{h}^{\alpha^\vee} \text{ have parallel orientations.} \end{aligned} \tag{2.20}$$

The pictures in the appendix illustrate the periodic orientation for type SL_3.

A similar "pictorial" viewpoint applies to the group $W^\vee$ acting on $\Omega\times\mathfrak{h}_\mathbb{R}$ where $\mathfrak{h}_\mathbb{R}=\mathbb{R}\otimes_\mathbb{Z}\mathfrak{h}_\mathbb{Z}$ and Ω is the set of length zero elements of $W^\vee$. Then $W^\vee$ has an alternate presentation by generators $s_0, s_1, \ldots, s_n$ and Ω with relations

$$s_i^2=1, \qquad \underbrace{s_is_j\cdots}_{m_{ij}} = \underbrace{s_js_i\cdots}_{m_{ij}}, \qquad \text{and} \qquad gs_ig^{-1}=s_{\sigma(i)}, \quad \text{for } g\in\Omega, \tag{2.21}$$

where π/m_{ij} is the angle between $\mathfrak{h}^{\alpha_i}$ and $\mathfrak{h}^{\alpha_j}$ and σ denotes the permutation of the $\mathfrak{h}^{\alpha_i}$ induced by the action of g.

2.2 Double affine braid groups

The *double affine braid group* $\tilde{\mathcal{B}}$ is the group generated by $T_0,\ldots,T_n$, Ω and X with relations

$$\underbrace{T_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_i\cdots}_{m_{ij}}, \quad gT_ig^{-1}=T_{\sigma(i)}, \quad gX^\mu = X^{g\mu}g, \tag{2.22}$$

for $g\in\Omega$, and

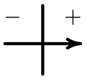
$$\begin{array}{ll} T_iX^\mu = X^{s_i\mu}T_i, & \text{if } \langle\mu,\alpha_i^\vee\rangle = 0, \\ T_iX^\mu T_i = X^{s_i\mu}, & \text{if } \langle\mu,\alpha_i^\vee\rangle = 1, \end{array} \qquad \text{for } i=0,1,\ldots,n, \tag{2.23}$$

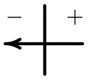
where the action of $W^\vee$ on $\mathfrak{h}_\mathbb{Z}^*\oplus\mathbb{Z}\delta$ is as in (2.10). The element

$$q = X^\delta \quad \text{is in the center of } \tilde{\mathcal{B}}. \tag{2.24}$$

For $w\in W^\vee$, view a reduced word $w=gs_{i_1}\cdots s_{i_\ell}$ as a minimal length path p from the fundamental alcove to w in $\mathfrak{h}_\mathbb{R}$ and define

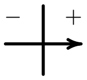
$$Y^w = g(T_{i_1})^{\epsilon_1}\cdots(T_{i_\ell})^{\epsilon_\ell}, \qquad \text{with} \qquad \epsilon_k = \tag{2.25}$$

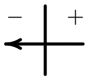
$+1$, if the kth step of p is  ,

-1, if the kth step of p is  ,

with respect to the *periodic orientation* (see (2.20) and the pictures in the appendix). For $v\in W$, view a reduced word $v=g^\vee s_{i_1}^\vee\cdots s_{i_\ell}^\vee$ as a minimal length path $p^\vee$ from the fundamental alcove to v in $\mathfrak{h}_\mathbb{R}^*$ and define

$$X^v = g^\vee(T_{i_1}^\vee)^{\epsilon_1^\vee}\cdots(T_{i_\ell}^\vee)^{\epsilon_\ell^\vee}, \quad \text{with} \quad \epsilon_k^\vee = \tag{2.26}$$

-1, if the kth step of $p^\vee$ is  ,

$+1$, if the kth step of $p^\vee$ is  ,

Let $T_i^\vee = T_i$, for $i=1,2,\ldots,n$,

$$g^\vee = X^{\omega_g}T^\vee_{w_gw_0}, \qquad (T_0^\vee)^{-1} = X^\varphi T^\vee_{s_\varphi}, \qquad g = Y^{\omega_g^\vee}T^{-1}_{w_0w_g}, \qquad T_0 = Y^{\varphi^\vee}T^{-1}_{s_\varphi}. \tag{2.27}$$

where φ and $\varphi^\vee$ are as in (2.11) and, using the action in (2.15), $\omega_g = g^\vee\cdot 0$ and w_g is the longest element of the stabilizer of ω_g in W_0.

The following theorem, discovered by Cherednik [Ch, Thm. 2.2], is proved in [Mac4, 3.5-3.7], in [Io], and in [Hai, 4.13-4.18].

Theorem 2.1. *(Duality) Let* $Y^d = q^{-1}$. *The double affine braid group* $\tilde{\mathcal{B}}$ *is generated by* $T_0^\vee, T_1^\vee, \ldots, T_n^\vee$, $\Omega^\vee$ *and* Y *with relations*

$$\underbrace{T_i^\vee T_j^\vee \cdots}_{m_{ij}^\vee} = \underbrace{T_j^\vee T_i^\vee \cdots}_{m_{ij}^\vee}, \qquad g^\vee T_i^\vee (g^\vee)^{-1} = T_{\sigma^\vee(i)}^\vee, \qquad g^\vee Y^{\lambda^\vee} = Y^{g^\vee \lambda^\vee} g^\vee, \tag{2.28}$$

for $g^\vee \in \Omega^\vee$, *and*

$$\begin{matrix} T_i^\vee Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} T_i^\vee, & \textit{if } \langle \lambda^\vee, \alpha_i \rangle = 0, \\ (T_i^\vee)^{-1} Y^{\lambda^\vee} (T_i^\vee)^{-1} = Y^{s_i^\vee \lambda^\vee}, & \textit{if } \langle \lambda^\vee, \alpha_i \rangle = 1, \end{matrix} \qquad \textit{for } i = 0, 1, \ldots, n, \tag{2.29}$$

where the action of W *on* $\mathfrak{h}_\mathbb{Z} \oplus \mathbb{Z}d$ *is as in* (2.10).

2.3 Double affine Hecke algebras

In the following, for simplicity of exposition we shall assume that we are not in the special case of [Mac4, (2.1.6)] where the root system is type C_n and $\mathfrak{h}_\mathbb{Z}$ is the (co)root lattice. All our results and proofs are valid in this special case but the definition of the double affine Hecke algebra and the formulas in (2.32) and (2.34) may need some slight modification. See Remark 2.3 for details.

Let $R^\vee = (R^\vee)^+ \cup (-(R^\vee)^+)$ be the set of coroots and fix parameters $c_{\beta^\vee}$, indexed by $\beta^\vee \in R^\vee + \mathbb{Z}d$, such that for all $w \in W$ and $\beta^\vee \in R^\vee + \mathbb{Z}d$,

$$c_{\beta^\vee} = c_{w\beta^\vee}. \qquad \text{Set} \quad t_{\beta^\vee} = q^{c_{\beta^\vee}} \quad \text{and} \quad t_i = t_{\alpha_i^\vee}. \tag{2.30}$$

The *double affine Hecke algebra* $\widetilde{H}$ is the group algebra $\mathbb{C}\widetilde{\mathcal{B}}$ of the double braid group with the additional relations

$$T_i^2 = (t_i^{1/2} - t_i^{-1/2}) T_i + 1, \qquad \text{for } i = 0, 1, \ldots, n. \tag{2.31}$$

The double affine Hecke algebra $\widetilde{H}$ has bases

$$\{T_w X^\mu \mid w \in W,\ \mu \in \mathfrak{h}_\mathbb{Z}^* \oplus \mathbb{Z}\delta\}, \qquad \{Y^{\lambda^\vee} T_w^\vee \mid w \in W^\vee,\ \lambda^\vee \in \mathfrak{h}_\mathbb{Z} \oplus \mathbb{Z}d\},$$

and

$$\{q^k X^\mu T_w Y^{\lambda^\vee} \mid w \in W_0,\ \lambda^\vee \in \mathfrak{h}_\mathbb{Z}, \mu \in \mathfrak{h}_\mathbb{Z}^*, k \in \tfrac{1}{e}\mathbb{Z}\}$$

(see [Hai, Prop. 5.4 and Cor. 5.8]).

In the presence of (2.31) the relations (2.29) are equivalent to

$$T_i^\vee Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} T_i^\vee + (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) \frac{Y^{\lambda^\vee} - Y^{s_i^\vee \lambda^\vee}}{1 - Y^{-\alpha_i^\vee}}, \qquad \text{for } i = 0, 1, \ldots, n. \tag{2.32}$$

In turn (2.32) is equivalent to

$$\tau_i^\vee Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} \tau_i^\vee, \qquad \text{for } i = 0, 1, \ldots, n, \tag{2.33}$$

where

$$\tau_i^\vee = T_i^\vee + \frac{t_i^{-\frac{1}{2}}(1 - t_i)}{1 - Y^{-\alpha_i^\vee}} = (T_i^\vee)^{-1} + \frac{t_i^{-\frac{1}{2}}(1 - t_i) Y^{-\alpha_i^\vee}}{1 - Y^{-\alpha_i^\vee}}. \tag{2.34}$$

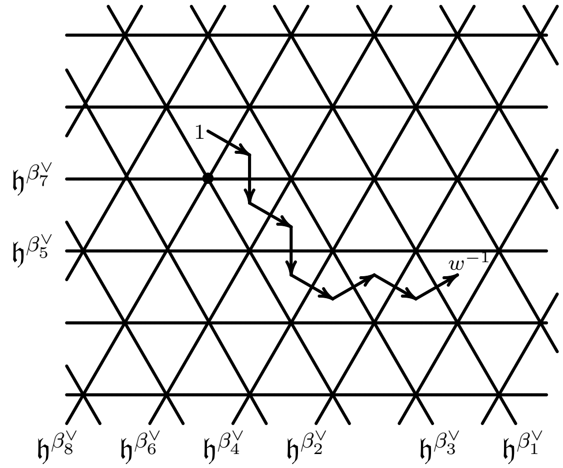
Using that the $\tau_i^\vee$ satisfy the braid relations and that

$$g^\vee Y^{\lambda^\vee} = Y^{g^\vee \lambda^\vee} g^\vee, \qquad \text{write} \qquad \tau_w^\vee Y^{\lambda^\vee} = Y^{w\lambda^\vee} \tau_w^\vee, \quad \text{for } w \in W.$$

Let $w \in W$ and let $w = s_{i_1}^\vee \cdots s_{i_\ell}^\vee$ be a reduced word for w. For $k = 1, \ldots, \ell$ let

$$\beta_k^\vee = s_{i_\ell}^\vee s_{i_{\ell-1}}^\vee \cdots s_{i_{k+1}}^\vee \alpha_{i_k}^\vee \qquad \text{and} \qquad t_{\beta_k^\vee} = t_{i_k}, \tag{2.35}$$

so that the sequence $\beta_\ell^\vee, \beta_{\ell-1}^\vee, \ldots, \beta_1^\vee$ is the sequence of labels of the hyperplanes crossed by the walk $w^{-1} = s_{i_\ell}^\vee s_{i_{\ell-1}}^\vee \cdots s_{i_1}^\vee$. For example, in Type A_2, with $w = s_2^\vee s_0^\vee s_1^\vee s_2^\vee s_1^\vee s_0^\vee s_2^\vee s_1^\vee$ the picture is



Let $v \in W^\vee$. An *alcove walk* of type $i_1, \ldots, i_\ell$ beginning at v is a sequence of steps, where a step of type j is

positive j-crossing   negative j-crossing   positive j-fold   negative j-fold

Let $\mathcal{B}(v, \vec{w})$ be the set of alcove walks of type $\vec{w} = (i_1, \ldots, i_\ell)$ beginning at v. For a walk $p \in \mathcal{B}(v, \vec{w})$ let

$$\begin{aligned} f^+(p) &= \{k \mid \text{the } k\text{th step of } p \text{ is a positive fold}\}, \\ f^-(p) &= \{k \mid \text{the } k\text{th step of } p \text{ is a negative fold}\}, \end{aligned} \tag{2.36}$$

and

$$\mathrm{end}(p) = \text{endpoint of } p \quad \text{(an element of } W\text{)}. \tag{2.37}$$

Theorem 2.2. *Let $v, w \in W$, let $w = s_{i_1}^\vee \cdots s_{i_\ell}^\vee$ be a reduced word for w and let $\beta_\ell^\vee, \ldots, \beta_1^\vee$ be as defined in* (2.35). *Then, in $\widetilde{H}$,*

$$X^v \tau_w^\vee = \sum_{p \in \mathcal{B}(v, \vec{w})} X^{\mathrm{end}(p)} \left(\prod_{k \in f^+(p)} \frac{t_{\beta_k^\vee}^{-1/2}(1 - t_{\beta_k^\vee})}{1 - Y^{-\beta_k^\vee}} \right) \left(\prod_{k \in f^-(p)} \frac{t_{\beta_k^\vee}^{-1/2}(1 - t_{\beta_k^\vee}) Y^{-\beta_k^\vee}}{1 - Y^{-\beta_k^\vee}} \right),$$

where the sum is over all alcove walks of type $\vec{w} = (i_1, \ldots, i_\ell)$ beginning at v.

Proof. The proof is by induction on the length of w, the base case being the formulas in (2.34). To do the induction step let $p \in \mathcal{B}(v, \vec{w})$,

$$F^+(p) = \left(\prod_{k \in f^+(p)} \frac{t_{\beta_k^\vee}^{-1/2}(1 - t_{\beta_k^\vee})}{1 - Y^{-\beta_k^\vee}} \right), \qquad F^-(p) = \left(\prod_{k \in f^-(p)} \frac{t_{\beta_k^\vee}^{-1/2}(1 - t_{\beta_k^\vee}) Y^{-\beta_k^\vee}}{1 - Y^{-\beta_k^\vee}} \right)$$

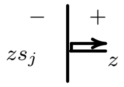
and let

$$p_1, p_2 \in \mathcal{B}(v, \vec{w}s_j) \text{ be the two extensions of } p \text{ by a step of type } j$$

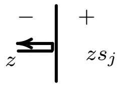
(by a crossing and a fold, respectively). Let $z = \mathrm{end}(p)$. By induction, a term in $X^v \tau_w^\vee \tau_j^\vee$ is

$$\begin{aligned}
&X^z F^+(p) F^-(p) \tau_j^\vee = X^z \tau_j^\vee \big(s_j F^+(p)\big)\big(s_j F^-(p)\big) \\
&= \begin{cases} X^z \left(T_j^\vee + \dfrac{t_j^{-1/2}(1-t_j)}{1 - Y^{-\alpha_j^\vee}} \right) \big(s_j F^+(p)\big)\big(s_j F^-(p)\big), & \text{if } X^{zs_j} = X^z T_j^\vee, \\ X^z \left((T_j^\vee)^{-1} + \dfrac{t_j^{-1/2}(1-t_j) Y^{-\alpha_j^\vee}}{1 - Y^{-\alpha_j^\vee}} \right) \big(s_j F^+(p)\big)\big(s_j F^-(p)\big), & \text{if } X^{zs_j} = X^z (T_j^\vee)^{-1}, \end{cases} \\
&= X^{\mathrm{end}(p_1)} F^+(p_1) F^-(p_1) + X^{\mathrm{end}(p_2)} F^+(p_2) F^-(p_2).
\end{aligned}$$

The last step of p_2 is



if $X^{zs_j} = X^z T_j^\vee$, and



if $X^{zs_j} = X^z (T_j^\vee)^{-1}$.

□

Remark 2.3. In some special cases when the affine root system is nonreduced (see [Mac4, (2.1.6)]) the formulas in (2.32) and (2.34) need modification and the definition of the double affine Hecke algebra may need an additional relation. The most involved of these cases is type $(C_n^\vee, C_n)$ (see [Mac4, (1.4.3)]) where the double affine Hecke algebra needs additional parameters $u_0^{1/2}$ and $u_n^{1/2}$ (in the notation of [Mac4], $u_0^{1/2} = \tau_0'$ and $u_n^{1/2} = \tau_n'$) and additional relations

$$(T_0' - u_0^{\frac{1}{2}})(T_0' + u_0^{-\frac{1}{2}}) = 0 \quad \text{and} \quad (T_0^\vee - u_n^{\frac{1}{2}})(T_0^\vee + u_n^{-\frac{1}{2}}), \quad \text{where } T_0' = q^{-\frac{1}{2}} X^{\varepsilon_1} T_0^{-1}, \tag{2.38}$$

and the formulas for $\tau_n^\vee$ and $\tau_0^\vee$ need to be changed to

$$\tau_n^\vee = T_n + \frac{t_n^{-\frac{1}{2}}(1-t_n) + t_0^{-\frac{1}{2}}(1-t_0) Y^{-\varepsilon_n^\vee}}{1 - Y^{-2\varepsilon_n^\vee}} = T_n^{-1} + \frac{(t_n^{-\frac{1}{2}}(1-t_n) + t_0^{-\frac{1}{2}}(1-t_0) Y^{\varepsilon_n^\vee}) Y^{-2\varepsilon_n^\vee}}{1 - Y^{-2\varepsilon_n^\vee}}, \tag{2.39}$$

and

$$\tau_0^\vee = T_0^\vee + \frac{u_n^{-\frac{1}{2}}(1-u_n) + u_0^{-\frac{1}{2}}(1-u_0) q^{\frac{1}{2}} Y \varepsilon_1^\vee}{1 - q^{-1} Y^{-2\varepsilon_1^\vee}} \tag{2.40}$$

$$= (T_0^\vee)^{-1} + \frac{(u_n^{-\frac{1}{2}}(1-u_n) + u_0^{-\frac{1}{2}}(1-u_0) q^{-\frac{1}{2}} Y^{-\varepsilon_1^\vee}) q^{-1} Y^{-2\varepsilon_1^\vee}}{1 - q^{-1} Y^{-2\varepsilon_1^\vee}}. \tag{2.41}$$

The statement and proof of the analogue of Theorem 2.2 for this case is the same, except with the factors associated to the 0-folds and n-folds replaced by the rational functions in Y which appear in the expressions of $\tau_0^\vee$ and $\tau_n^\vee$ in Equations (2.40), (2.41), and (2.39).

3 Macdonald polynomials

In this section we use Theorem 2.2 to give expansions of the nonsymmetric Macdonald polynomials E_μ (Theorem 3.1) and the symmetric Macdonald polynomials P_μ (Theorem 3.4).

Let $\widetilde{H}$ be the double affine Hecke algebra (defined in (2.31)) and let H be the subalgebra of $\widetilde{H}$ generated by $T_0, \ldots, T_n$ and Ω. The *polynomial representation* of $\widetilde{H}$ is

$$\mathbb{C}[X] = \mathrm{Ind}_H^{\widetilde{H}}(\mathbf{1}) = \mathbb{C}\text{-span}\{q^k X^\mu \mathbf{1} \mid k \in \tfrac{1}{e}\mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*\} \tag{3.1}$$

with

$$T_i \mathbf{1} = t_i^{1/2}\mathbf{1} \qquad \text{and} \qquad g\mathbf{1} = \mathbf{1}, \quad \text{for } g \in \Omega. \tag{3.2}$$

The monomials $X^\mu \mathbf{1}$, $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, form a $\mathbb{C}[q^{\pm 1/e}]$-basis of $\mathbb{C}[X]$. Another favourite $\mathbb{C}[q^{\pm 1/e}]$-basis of $\mathbb{C}[X]$ is the basis of *nonsymmetric Macdonald polynomials*

$$\{E_\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}, \qquad \text{where} \quad E_\mu = \tau^\vee_{X^\mu m}\mathbf{1} \tag{3.3}$$

with $X^\mu m$ the minimal length element in the coset $X^\mu W_0$. Note that $\tau^\vee_w \mathbf{1} = 0$ for $w \in W_0$ since $\tau^\vee_i \mathbf{1} = 0$ for $i = 1, 2, \ldots, n$.

If $\mathfrak{h}_{\mathbb{Z}}^+$ is the set of dominant integral coweights (analogous to $(\mathfrak{h}_{\mathbb{Z}}^*)^+$ defined in (3.8)), $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+$ and $Y^{\lambda^\vee} = s_{i_1}\cdots s_{i_\ell}$ is a reduced word, then

$$Y^{\lambda^\vee}\mathbf{1} = T_{i_1}\cdots T_{i_\ell}\mathbf{1} = t_{i_1}^{\frac{1}{2}}\cdots t_{i_\ell}^{\frac{1}{2}}\mathbf{1} = q^{\frac{1}{2}(c_{i_1}+\cdots+c_{i_\ell})}\mathbf{1} = q^{\frac{1}{2}\sum_{\alpha\in R^+} c_\alpha \langle \lambda^\vee, \alpha\rangle}\mathbf{1} = q^{\langle \lambda^\vee, \rho_c\rangle}\mathbf{1},$$

since $\langle \lambda^\vee, \alpha\rangle$ is the number of hyperplanes parallel to $\mathfrak{h}^\alpha$ which are between $Y^{\lambda^\vee}$ and 1. If $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$ then $\lambda^\vee = \mu^\vee - \nu^\vee$ for some $\mu^\vee$, $\nu^\vee \in \mathfrak{h}_{\mathbb{Z}}^+$ and so, for all $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$,

$$Y^{\lambda^\vee}\mathbf{1} = q^{\langle \lambda^\vee, \rho_c\rangle}\mathbf{1}, \qquad \text{where} \quad \rho_c = \tfrac{1}{2}\sum_{\alpha\in R^+} c_\alpha \alpha. \tag{3.4}$$

More generally, if $X^\mu m$ is the minimal length element of the coset $X^\mu W_0$ then

$$\begin{aligned} Y^{\lambda^\vee}E_\mu &= Y^{\lambda^\vee}\tau^\vee_{X^\mu m}\mathbf{1} = \tau^\vee_{X^\mu m} Y^{m^{-1}X^{-\mu}\lambda^\vee}\mathbf{1} = \tau^\vee_{X^\mu m} Y^{m^{-1}(\lambda^\vee + \langle \lambda^\vee, \mu\rangle d)}\mathbf{1} \\ &= \tau^\vee_{X^\mu m} Y^{m^{-1}\lambda^\vee} q^{-\langle \lambda^\vee, \mu\rangle}\mathbf{1} = q^{\langle m^{-1}\lambda^\vee, \rho_c\rangle - \langle \lambda^\vee, \mu\rangle}\tau^\vee_{X^\mu m}\mathbf{1} = q^{\langle \lambda^\vee, m\rho_c - \mu\rangle}E_\mu \\ &= q^{\langle \lambda^\vee, X^{-\mu}m\cdot\rho_c\rangle}E_\mu, \end{aligned}$$

where, in the last line, the action of W on $\mathfrak{h}_{\mathbb{Z}}^*$ is as in (2.15). Thus the E_μ are eigenvectors for the action of the $Y^{\lambda^\vee}$ on the polynomial representation $\mathbb{C}[X]$.

Retain the notation of (2.36-2.37) so that if $w = s^\vee_{i_1}\cdots s^\vee_{i_\ell}$ is a reduced word then $\mathcal{B}(v, \vec{w})$ denotes the set of alcove walks of type $\vec{w} = (i_1, \ldots, i_\ell)$ beginning at v. For $p \in \mathcal{B}(v, \vec{w})$ define the *weight* $\mathrm{wt}(p)$ and the *final direction* $\varphi(p)$ of p by

$$X^{\mathrm{end}(p)} = X^{\mathrm{wt}(p)}T^\vee_{\varphi(p)}, \qquad \text{with } \mathrm{wt}(p) \in \mathfrak{h}_{\mathbb{Z}}^* \text{ and } \varphi(p) \in W_0. \tag{3.5}$$

In other words, $\mathrm{wt}(p)$ is the "hexagon where p ends". For $w \in W$ define

$$t_w^{1/2} = t_{i_1}^{1/2}\ldots t_{i_\ell}^{1/2}, \qquad \text{if } w = s^\vee_{i_1}\cdots s^\vee_{i_\ell} \text{ is a reduced word.} \tag{3.6}$$

If $\beta^\vee_k = s^\vee_{i_\ell}\cdots s^\vee_{i_{k+1}}\alpha^\vee_{i_k}$ are as defined in (2.35) then, by (3.4),

$$Y^{-\beta^\vee_k}\mathbf{1} = Y^{-(-\gamma^\vee + jd)}\mathbf{1} = q^j q^{\langle \gamma^\vee, \rho_c\rangle}\mathbf{1}, \qquad \text{if } \beta^\vee_k = -\gamma^\vee + jd$$

with $\gamma^\vee \in R^\vee$, $j \in \mathbb{Z}$. By (2.30) and the definition of ρ_c in (3.4), the constant $q^{\langle \gamma^\vee, \rho_c\rangle}$ is a monomial in the symbols $t_i^{1/2}$. To simplify the notation for these constants write $q^j q^{\langle \gamma^\vee, \rho_c\rangle} = q^{\langle -\beta^\vee_k, \rho_c\rangle}$ so that

$$Y^{-\beta^\vee_k}\mathbf{1} = q^{\langle -\beta^\vee_k, \rho_c\rangle}\mathbf{1}. \tag{3.7}$$

Theorem 3.1. *Let* $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ *and let* $w = X^\mu m$ *be the minimal length element in the coset* $X^\mu W_0$. *Fix a reduced word* $\vec{w} = s_{i_1}^\vee \cdots s_{i_\ell}^\vee$ *for* w *and let* $\beta_\ell^\vee, \ldots, \beta_1^\vee$ *be as defined in* (2.35). *With notations as in (3.5-3.7) the nonsymmetric Macdonald polynomial*

$$E_\mu = \sum_{p \in \mathcal{B}(\vec{\mu})} X^{\mathrm{wt}(p)} t_{\varphi(p)}^{\frac{1}{2}} \left(\prod_{k \in f^+(p)} \frac{t_{\beta_k^\vee}^{-\frac{1}{2}}(1 - t_{\beta_k^\vee})}{1 - q^{\langle -\beta_k^\vee, \rho_c \rangle}} \right) \left(\prod_{k \in f^-(p)} \frac{t_{\beta_k^\vee}^{-\frac{1}{2}}(1 - t_{\beta_k^\vee}) q^{\langle -\beta_k^\vee, \rho_c \rangle}}{1 - q^{\langle -\beta_k^\vee, \rho_c \rangle}} \right),$$

where the sum is over the set $\mathcal{B}(\vec{\mu}) = \mathcal{B}(1, \vec{w})$ *of alcove walks of type* $i_1, \ldots, i_\ell$ *beginning at* 1.

Proof. Since $E_\mu = \tau_{X^\mu m}^\vee \mathbf{1}$,

$$X^{\mathrm{end}(p)} \mathbf{1} = X^{\mathrm{wt}(p)} T_{\varphi(p)}^\vee \mathbf{1} = X^{\mathrm{wt}(p)} t_{\varphi(p)}^{\frac{1}{2}} \mathbf{1} \qquad \text{and} \qquad Y^{\lambda^\vee} \mathbf{1} = q^{\langle \lambda^\vee, \rho_c \rangle} \mathbf{1},$$

applying the formula for $\tau_{X^\mu m}^\vee$ in Theorem 2.2 to $\mathbf{1}$ gives the formula in the statement. □

Remark 3.2. From the expansion of E_μ in Theorem 3.1, the nonsymmetric Macdonald polynomial E_μ has top term $t_m^{1/2} X^\mu$, where $X^\mu m$ is the minimal length representative of the coset $X^\mu W_0$. This term is the term corresponding to the unique alcove walk in $\mathcal{B}(\vec{\mu})$ with no folds.

Remark 3.3. If $w = X^\mu m = s_{i_1}^\vee \cdots s_{i_\ell}^\vee$ is a reduced word for the minimal length element of the coset $X^\mu W_0$ then $w^{-1} = s_{i_\ell}^\vee \cdots s_{i_1}^\vee$ is a walk from 1 to w^{-1} which stays completely in the dominant chamber. This has the effect that the roots $\beta_\ell^\vee, \ldots, \beta_1^\vee$ are all of the form $-\gamma^\vee + jd$ with $\gamma^\vee \in (R^\vee)^+$ (positive coroots) and $j \in \mathbb{Z}_{>0}$. The *height* of a coroot $\gamma^\vee$ is

$$\mathrm{ht}(\gamma^\vee) = \langle \gamma^\vee, \rho \rangle, \qquad \text{where } \rho = \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

In the case that all the parameters are equal ($t_i = t = q^c$ for $i = 0, \ldots, n$) the values which appear in Theorem 3.1,

$$q^{\langle -\beta_k^\vee, \rho_c \rangle} = q^{\langle \gamma^\vee - jd, \rho_c \rangle} = q^j t^{\mathrm{ht}(\gamma^\vee)}, \qquad \text{have positive exponents (in } \mathbb{Z}_{>0}).$$

The set of *dominant integral weights* is

$$(\mathfrak{h}_{\mathbb{Z}}^*)^+ = \{\mu \in \mathfrak{h}_{\mathbb{Z}}^* \mid \langle \mu, \alpha_i^\vee \rangle \geq 0 \text{ for } i = 1, \ldots, n\}. \tag{3.8}$$

Recall the notation for $t_w^{1/2}$ from (3.6). For $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$, the *symmetric Macdonald polynomial* (see [Mac3, Remarks after (6.8)]) is

$$P_\mu = \mathbf{1}_0 E_\mu \mathbf{1} \qquad \text{where} \quad \mathbf{1}_0 = \sum_{w \in W_0} t_{w_0 w}^{-\frac{1}{2}} T_w, \tag{3.9}$$

so that $T_i \mathbf{1}_0 = t_i^{1/2} \mathbf{1_0}$ for $i = 1, 2 \ldots n$, and $\mathbf{1}_0$ has top term T_{w_0} with coefficient 1. The symmetric Macdonald polynomials are W_0-symmetric polynomials in X^μ which are eigenvectors for the action of W_0-symmetric polynomials in the $Y^{\lambda^\vee}$.

Theorem 3.4. *Let* $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$ *and let* $X^\mu m = s_{i_1}^\vee \cdots s_{i_\ell}^\vee$ *be a reduced word for the minimal length element* $X^\mu m$ *in the coset* $X^\mu W_0$. *Let* $\beta_\ell^\vee, \ldots, \beta_1^\vee$ *be as defined in* (2.35) *and let*

$$\mathcal{P}(\vec{\mu}) = \bigcup_{v \in W_0} \mathcal{B}(v, \vec{w})$$

be the set of alcove walks of type $\vec{w} = (i_1, \ldots, i_\ell)$ beginning at an element $v \in W_0$. Then the symmetric Macdonald polynomial

$$P_\mu = \sum_{p \in \mathcal{P}(\vec{\mu})} X^{\mathrm{wt}(p)} t_{\varphi(p)}^{\frac{1}{2}} t_{w_0\iota(p)}^{-\frac{1}{2}} \left(\prod_{k \in f^+(p)} \frac{t_{\beta_k^\vee}^{-\frac{1}{2}}(1 - t_{\beta_k^\vee})}{1 - q^{\langle -\beta_k^\vee, \rho_c \rangle}} \right) \left(\prod_{k \in f^-(p)} \frac{t_{\beta_k^\vee}^{-\frac{1}{2}}(1 - t_{\beta_k^\vee}) q^{\langle -\beta_k^\vee, \rho_c \rangle}}{1 - q^{\langle -\beta_k^\vee, \rho_c \rangle}} \right),$$

where $\iota(p)$ is the initial alcove of the path p.

Proof. The expression

$$\mathbf{1}_0 = \sum_{v \in W_0} t_{w_0 v}^{-\frac{1}{2}} X^v, \qquad \text{gives} \qquad P_\mu \mathbf{1} = \mathbf{1}_0 E_\mu \mathbf{1} = \sum_{v \in W_0} t_{w_0 v}^{-\frac{1}{2}} X^v \tau_{X^\mu m}^\vee \mathbf{1},$$

which is computed by the same method as in Theorem 2.2 and Theorem 3.1. □

Remark 3.5. The *Hall-Littlewood polynomials* or *Macdonald spherical functions* are $P_\mu(0, t)$ and the *Schur functions* or *Weyl characters* are $s_\mu = P_\mu(0, 0)$. In the first case the formula in Theorem 3.4 reduces to the formula for the Macdonald spherical functions in terms of positively folded alcove walks as given in [Sc, Thm. 1.1] (see also [Ra, Thm. 4.2(a)]). In the case $q = t = 0$, the formula in Theorem 3.4 reduces to the formula for the Weyl characters in terms of maximal dimensional positively folded alcove walks (*the Littelmann path model*) as given in [GL, Cor. 1 p. 62], or the λ-chain formulation of [LP1],[LP2].

When μ is not a regular weight, the formula for P_μ in Theorem 3.4 has an alternate formulation as a sum over paths p whose initial alcove $\iota(p)$ is in the minimal coset representatives W^μ of W_0/W_μ, where $W_\mu = \{w \in W_0 \mid w\mu = \mu\}$ is the stabilzer of μ. To see this, suppose $s_i\mu = \mu$ for some $i \in \{1, \ldots, n\}$. Then $\langle \mu, \alpha_i^\vee \rangle = 0$ implies $Y^{-\alpha_i^\vee} E_\mu \mathbf{1} = t_i^{-1} E_\mu \mathbf{1}$. Further, let $X^\mu m$ be the minimal length element in the coset $X^\mu W_0$. Then $s_i X^\mu m = X^{s_i\mu} s_i m = X^\mu m s_j$ for some $j \in \{1, \ldots, n\}$, so $\tau_i^\vee E_\mu \mathbf{1} = \tau_i^\vee \tau_{X^\mu m}^\vee \mathbf{1} = \tau_{X^\mu m}^\vee \tau_j^\vee \mathbf{1} = 0$. Therefore, $T_i E_\mu \mathbf{1} = \left(\tau_i^\vee - \frac{t_i^{-\frac{1}{2}} - t_i^{\frac{1}{2}}}{1 - Y^{-\alpha_i^\vee}} \right) E_\mu \mathbf{1} = t_i^{\frac{1}{2}} E_\mu \mathbf{1}$, and

$$\begin{aligned} P_\mu \mathbf{1} = \mathbf{1}_0 E_\mu \mathbf{1} &= \sum_{w \in W_0} t_{w_0 w}^{-\frac{1}{2}} T_w E_\mu \mathbf{1} = t_{w_0}^{\frac{1}{2}} \left(\sum_{v \in W^\mu} t_{w_0 v}^{-\frac{1}{2}} T_v \right) \left(\sum_{u \in W_\mu} t_{w_0 u}^{-\frac{1}{2}} T_u \right) E_\mu \mathbf{1} \\ &= W_\mu(t) \sum_{v \in W^\mu} t_{w_0 v}^{-\frac{1}{2}} X^v \tau_{X^\mu m}^\vee \mathbf{1}, \end{aligned}$$

where $W_\mu(t) = \sum_{w \in W_\mu} t_w$ is the Poincaré polynomial of W_μ.

Remark 3.6. In order to derive the Haglund-Haiman-Loehr formula for P_μ in the type GL_n case, one would need to start with a similar formula that is based on the alcove walk from the fundamental alcove to the alcove X^μ instead of using the formula in Theorem 3.4. Such a formula is equivalent to expanding the sum

$$P_\mu \mathbf{1} = t_m^{\frac{1}{2}} \sum_{v \in W_0} t_{w_0 v}^{-\frac{1}{2}} X^v \tau_{X^\mu m}^\vee T_m^{-1} \mathbf{1}$$

in terms of the monomial basis.

Alternatively, let $w_\mu = s_{i_1}\cdots s_{i_l}$ be the longest element in W_μ, so that $w_0 w_\mu$ is the shortest element in the coset $w_0 W_\mu$. Renormalizing P_μ,

$$\begin{aligned}\left(\prod_{j=1}^{l} t_{i_j}^{\frac12} + \frac{t_{i_j}^{-\frac12} - t_{i_j}^{\frac12}}{1-q^{-\langle \alpha_{i_j}^\vee, X^{-\mu} m\cdot\rho_c\rangle}}\right) P_\mu \mathbf{1} &= \mathbf{1}_0 \left(\prod_{j=1}^{l} t_{i_j}^{\frac12} + \frac{t_{i_j}^{-\frac12} - t_{i_j}^{\frac12}}{1-q^{-\langle \alpha_{i_j}^\vee, X^{-\mu} m\cdot\rho_c\rangle}}\right) E_\mu \mathbf{1} \\ &= \mathbf{1}_0 \prod_{j=1}^{l}\left(T_{i_j}^{\frac12} + \frac{t_{i_j}^{-\frac12} - t_{i_j}^{\frac12}}{1-Y^{-\alpha_{i_j}^\vee}}\right) E_\mu \mathbf{1} = \mathbf{1}_0 \tau^\vee_{w_0 w_\mu} E_\mu \mathbf{1} = \mathbf{1}_0 E_{w_0 w_\mu \mu}\mathbf{1},\end{aligned}$$

and expanding the right hand side in the same way as in the proof of Theorem 3.4 will produce a formula based on the alcove walk from the fundamental alcove to the alcove $X^{w_0 w_\mu \mu}$. This walk has the same length as the walk from the fundamental alcove to the alcove X^μ.

4 Examples

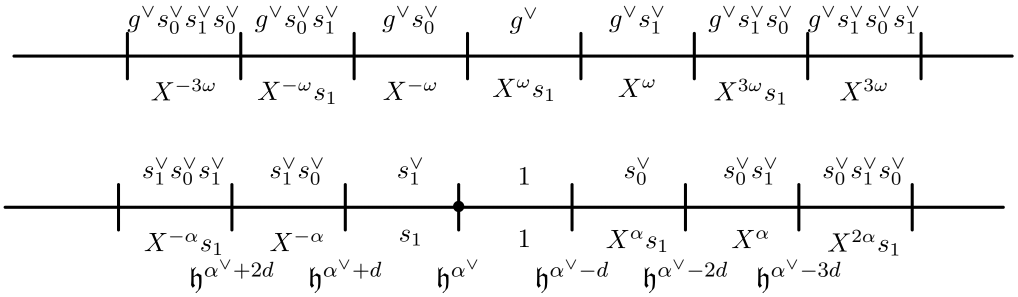
4.1 Type A_1

The Weyl group $W_0 = \langle s_1 \mid s_1^2 = 1\rangle$ has order two and acts on the lattices

$$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\omega^\vee \quad \text{and} \quad \mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\omega \qquad \text{by} \qquad s_1\omega^\vee = -\omega^\vee \quad \text{and} \quad s_1\omega = -\omega, \tag{4.1}$$

and

$$\varphi^\vee = \alpha^\vee = 2\omega^\vee, \qquad \varphi = \alpha = 2\omega, \quad \text{and} \quad \langle \omega^\vee, \alpha\rangle = 1. \tag{4.2}$$



The double affine braid group $\widetilde{\mathcal{B}}$ is generated by T_0, T_1, g, X^ω, and $q^{1/2}$, with relations

$$\begin{gathered} T_0 = gT_1g^{-1}, \qquad g^2 = 1, \qquad q = X^\delta, \\ gX^\omega = q^{1/2}X^{-\omega}g, \qquad T_1X^\omega T_1 = X^{-\omega}, \quad \text{and} \quad T_0X^{-\omega}T_0 = q^{-1}X^\omega. \end{gathered} \tag{4.3}$$

In the double affine braid group

$$g = Y^{\omega^\vee}T_1^{-1}, \qquad T_0 = Y^{\varphi^\vee}T_1^{-1}, \qquad g^\vee = X^\omega T_1, \qquad (T_0^\vee)^{-1} = X^\varphi T_1^\vee. \tag{4.4}$$

At this point, the following Proposition, which is the Type A_1 case of Theorem 2.1, is easily proved by direct computation.

Proposition 4.1. *(Duality). Let $Y^d = q^{-1}$. The double affine braid group $\widetilde{\mathcal{B}}$ is generated by $T_0^\vee, T_1^\vee, g^\vee, Y^{\omega^\vee}$ and $q^{1/2}$ with relations*

$$\begin{gathered} Y^d = q^{-1}, \qquad (g^\vee)^2 = 1, \qquad T_0^\vee = g^\vee T_1^\vee (g^\vee)^{-1}, \\ g^\vee Y^{\omega^\vee} = q^{-1/2}Y^{-\omega^\vee}g^\vee, \quad T_1^{-1}Y^{\omega^\vee}T_1^{-1} = Y^{-\omega^\vee}, \quad \textit{and} \quad (T_0^\vee)^{-1}Y^{-\omega^\vee}(T_0^\vee)^{-1} = qY^{\omega^\vee}. \end{gathered}$$

The double affine Hecke algebra $\widetilde{H}$ is $\mathbb{C}\widetilde{\mathcal{B}}$ with the additional relations

$$T_i^2 = (t^{1/2} - t^{-1/2})T_i + 1, \quad \text{for } i = 0, 1, \qquad \text{and} \qquad t_0 = t_1 = t = q^c. \tag{4.5}$$

Using (4.5), the relations in Proposition (4.1) give

$$g^\vee Y^{\omega^\vee} = q^{-1/2} Y^{-\omega^\vee} g^\vee, \qquad T_1 Y^{\omega^\vee} = Y^{-\omega^\vee} T_1 + (t^{1/2} - t^{-1/2}) \frac{Y^{\omega^\vee} - Y^{-\omega^\vee}}{1 - Y^{-\alpha^\vee}}, \quad \text{and}$$

$$T_0^\vee Y^{\omega^\vee} = q^{-1} Y^{-\omega^\vee} T_0^\vee + (t^{1/2} - t^{-1/2}) \left(\frac{Y^{\omega^\vee} - q^{-1} Y^{-\omega^\vee}}{1 - qY^{\alpha^\vee}} \right).$$

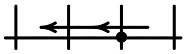
With $Y^{-\alpha_0^\vee} = qY^{\alpha^\vee}$ and $Y^{-\alpha_1^\vee} = Y^{-\alpha^\vee}$, then

$$\tau_g^\vee = g^\vee, \qquad \text{and} \qquad \tau_i^\vee = T_i^\vee - (t^{1/2} - t^{-1/2}) \left(\frac{1}{1 - Y^{-\alpha_i^\vee}} \right), \qquad \text{for } i = 0, 1.$$

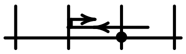
To illustrate Theorem 2.2, note that $X^{-2\omega} = s_1^\vee s_0^\vee$ is a reduced word and

$$\begin{aligned}
\tau_1^\vee \tau_0^\vee &= \left(T_1^\vee + \frac{t^{-1/2}(1-t)}{1 - Y^{-\alpha_1^\vee}} \right) \tau_0^\vee \\
&= T_1^\vee T_0^\vee + T_1^\vee \frac{t^{-1/2}(1-t)}{1 - Y^{-\alpha_0^\vee}} + (T_0^\vee)^{-1} \frac{t^{-1/2}(1-t)}{1 - Y^{-s_0\alpha_1^\vee}} + \left(\frac{t^{-1/2}(1-t)}{1 - Y^{-s_0\alpha_1^\vee}} \right) \left(\frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1 - Y^{-\alpha_0^\vee}} \right) \\
&= X^{-2\omega} + T_1^\vee \frac{t^{-1/2}(1-t)}{1 - Y^{-\alpha_0^\vee}} + X^{2\omega} T_1^\vee \frac{t^{-1/2}(1-t)}{1 - Y^{-s_0\alpha_1^\vee}} + \left(\frac{t^{-1/2}(1-t)}{1 - Y^{-s_0\alpha_1^\vee}} \right) \left(\frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1 - Y^{-\alpha_0^\vee}} \right).
\end{aligned}$$

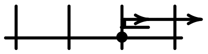
The corresponding paths in $\mathcal{B}(1, \overrightarrow{-2\omega}) = \mathcal{B}(\overrightarrow{-2\omega})$ are



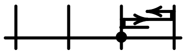
$$X^{-2\omega}$$



$$T_1^\vee \frac{t^{-1/2}(1-t)}{1 - Y^{-\alpha_0^\vee}}$$



$$X^{2\omega} T_1^\vee \frac{t^{-1/2}(1-t)}{1 - Y^{-s_0\alpha_1^\vee}}$$



$$\frac{t^{-1/2}(1-t)}{1 - Y^{-s_0\alpha_1^\vee}} \frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1 - Y^{-\alpha_0^\vee}}$$

The polynomial representation is defined by

$$T_i \mathbf{1} = t^{1/2} \mathbf{1}, \qquad \text{and} \qquad g\mathbf{1} = \mathbf{1}.$$

In this case

$$\rho_c = \frac{1}{2} c\alpha \qquad \text{and} \qquad W^0 = \{X^{-\ell\omega} \mid \ell \in \mathbb{Z}_{\geq 0}\} \cup \{X^{\ell\omega} s_1^\vee \mid \ell \in \mathbb{Z}_{>0}\}, \tag{4.6}$$

is the set of minimal length coset representatives of $W^\vee / W_0$.

Applying the expansion of $\tau_1^\vee \tau_0^\vee$ to $\mathbf{1}$ and using

$$Y^{-\alpha_0^\vee} \mathbf{1} = qY^{\alpha^\vee} \mathbf{1} = qq^c \mathbf{1} = qt\mathbf{1}, \qquad \text{and} \qquad Y^{-s_0\alpha_1^\vee} \mathbf{1} = Y^{\alpha^\vee + 2d} \mathbf{1} = q^2 Y^{\alpha^\vee} \mathbf{1} = q^2 t,$$

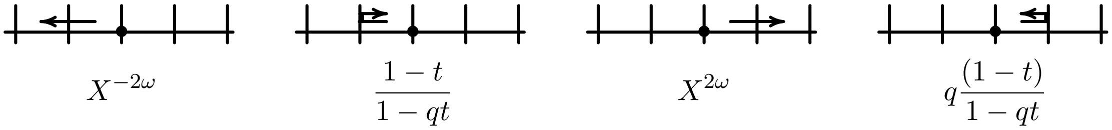
gives

$$\begin{aligned}
E_{-2\omega} &= \tau_1^\vee \tau_0^\vee \mathbf{1} \\
&= X^{-2\omega} + t^{1/2} \frac{t^{-1/2}(1-t)}{1 - qt} + X^{2\omega} t^{1/2} \frac{t^{-1/2}(1-t)}{1 - q^2 t} + \left(\frac{t^{-1/2}(1-t)}{1 - q^2 t} \right) \left(\frac{t^{-1/2}(1-t)qt}{1 - qt} \right) \\
&= X^{-2\omega} + \frac{1-t}{1-qt} + X^{2\omega} \frac{1-t}{1-q^2 t} + \left(\frac{1-t}{1-q^2 t} \right) \left(\frac{(1-t)q}{1-qt} \right).
\end{aligned}$$

Since $\mathbf{1}_0 = T_1^\vee + t^{-1/2}$ the symmetric Macdonald polynomial $P_{2\omega} = \mathbf{1}_0 E_{2\omega} = \mathbf{1}_0 \tau_0^\vee \mathbf{1}$ is

$$\begin{aligned}
P_{2\omega} &= \mathbf{1_0} E_{2\omega} = (T_1^\vee + t^{-1/2})\tau_0^\vee \mathbf{1} \\
&= \left(T_1^\vee T_0^\vee + T_1^\vee \frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_0^\vee}} + t^{-1/2}(T_0^\vee)^{-1} + t^{-1/2}\frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}} \right)\mathbf{1} \\
&= \left(X^{-2\omega} + t^{1/2}\frac{t^{-1/2}(1-t)}{1-qt} + t^{-1/2}X^{2\omega}T_1^\vee + t^{-1/2}\frac{t^{-1/2}(1-t)qt}{1-qt} \right)\mathbf{1} \\
&= \left(X^{-2\omega} + \frac{1-t}{1-qt} + X^{2\omega} + \frac{(1-t)q}{1-qt} \right)\mathbf{1} = \left(X^{2\omega} + X^{-2\omega} + (1+q)\frac{1-t}{1-qt} \right)\mathbf{1}.
\end{aligned}$$

The corresponding paths in $\mathcal{P}(\overrightarrow{2\omega})$ are



4.2 Type A_2

The Weyl group $W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle$ acts on the lattices

$$\mathfrak{h}_\mathbb{Z} = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee \qquad \text{and} \qquad \mathfrak{h}_\mathbb{Z}^* = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \tag{4.7}$$

where s_1 and s_2 are the reflections in the hyperplanes determined by

$$\alpha_1^\vee = 2\omega_1^\vee - \omega_2^\vee, \quad \alpha_2^\vee = -\omega_1^\vee + 2\omega_2^\vee, \quad \alpha_1 = 2\omega_1 - \omega_2, \quad \text{and} \quad \alpha_2 = -\omega_1 + 2\omega_2, \tag{4.8}$$

with $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}$, and $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. In this case,

$$\varphi^\vee = \alpha_1^\vee + \alpha_2^\vee, \qquad \text{and} \qquad \varphi = \alpha_1 + \alpha_2. \tag{4.9}$$

The double affine braid group $\widetilde{\mathcal{B}}$ is generated by $T_0, T_1, T_2, g, X^{\omega_1}, X^{\omega_2}$, and $q^{1/3}$, with relations

$$\begin{gathered}
T_iT_jT_i = T_jT_iT_j, \qquad \text{for } i \neq j, \\
X^\mu X^\lambda = X^{\mu+\lambda} = X^\lambda X^\mu, \qquad \text{for } \mu, \lambda \in \mathfrak{h}_\mathbb{Z}^*, \\
T_1X^{\omega_2} = X^{\omega_2}T_1, \quad T_2X^{\omega_1} = X^{\omega_1}T_2, \quad T_1X^{\omega_1}T_1 = X^{-\omega_1+\omega_2}, \quad T_2X^{\omega_2}T_2 = X^{\omega_1-\omega_2}, \qquad (4.10) \\
g^3 = 1, \qquad gX^{\omega_1} = q^{1/3}X^{-\omega_1+\omega_2}g, \quad gX^{\omega_2} = q^{2/3}X^{-\omega_1}g, \\
gT_0g^{-1} = T_1, \qquad gT_1g^{-1} = T_2, \qquad gT_2g^{-1} = T_0.
\end{gathered}$$

The formula (2.27) gives

$$g = Y^{\omega_1^\vee}T_1^{-1}T_2^{-1}, \qquad g^2 = Y^{\omega_2^\vee}T_2^{-1}T_1^{-1}, \qquad T_0 = Y^{\varphi^\vee}T_1^{-1}T_2^{-1}T_1^{-1}, \tag{4.11}$$

$$g^\vee = X^{\omega_1}T_1^\vee T_2^\vee, \qquad (g^\vee)^2 = X^{\omega_2}T_2^\vee T_1^\vee, \qquad (T_0^\vee)^{-1} = X^\varphi T_1^\vee T_2^\vee T_1^\vee. \tag{4.12}$$

At this point, the following Proposition, which is the Type A_2 case of Theorem 2.1, is easily proved by direct computation.

Proposition 4.2. *(Duality). Let* $Y^d = q^{-1}$. *The double affine braid group* $\widetilde{\mathcal{B}}$ *is generated by* $T_0^\vee$, $T_1^\vee$, $T_2^\vee$, $g^\vee$, $Y^{\omega_1^\vee}$, $Y^{\omega_2^\vee}$ *and* $q^{1/3}$, *with relations*

$$(T_1^\vee)^{-1}Y^{\omega_1^\vee}(T_1^\vee)^{-1} = Y^{-\omega_1^\vee+\omega_2^\vee}, \quad (T_2^\vee)^{-1}Y^{\omega_2^\vee}(T_2^\vee)^{-1} = Y^{\omega_1^\vee-\omega_2^\vee},$$

$$(T_1^\vee)^{-1}Y^{\omega_2^\vee} = Y^{\omega_2^\vee}(T_1^\vee)^{-1}, \quad (T_2^\vee)^{-1}Y^{\omega_1^\vee} = Y^{\omega_1^\vee}(T_2^\vee)^{-1},$$

$$(g^\vee)^3 = 1, \quad g^\vee Y^{\omega_1^\vee} = q^{-1/3}Y^{-\omega_1^\vee+\omega_2^\vee}g^\vee, \quad g^\vee Y^{\omega_2^\vee} = q^{-2/3}Y^{-\omega_1^\vee}g^\vee,$$

$$g^\vee T_0^\vee (g^\vee)^{-1} = T_1^\vee, \qquad g^\vee T_1^\vee (g^\vee)^{-1} = T_2^\vee \qquad \text{and} \qquad g^\vee T_2^\vee (g^\vee)^{-1} = T_0^\vee.$$

To give a concrete example of Theorem 3.4 let us compute the symmetric Macdonald polynomial P_ρ where $\rho = \alpha_1 + \alpha_2$. Since

$$\mathbf{1_0} = X^{s_1s_2s_1} + t^{-1/2}X^{s_1s_2} + t^{-1/2}X^{s_2s_1} + t^{-2/2}X^{s_1} + t^{-2/2}X^{s_2} + t^{-3/2},$$

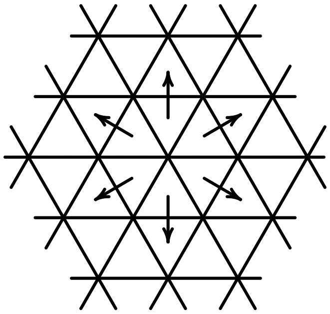
and $X^\rho m = s_0^\vee$ is the minimal length element of the coset $X^\rho W_0$,

$$\begin{aligned}
P_\rho &= \mathbf{1}_0 E_\rho = \mathbf{1}_0 \tau_0^\vee \mathbf{1} \\
&= \left(X^{s_1s_2s_1} + t^{-1/2}X^{s_1s_2} + t^{-1/2}X^{s_2s_1}\right)\left(T_0^\vee + \frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_0^\vee}}\right)\mathbf{1} \\
&\qquad + \left(t^{-2/2}X^{s_1} + t^{-2/2}X^{s_2} + t^{-3/2}\right)\left((T_0^\vee)^{-1} + \frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}}\right)\mathbf{1} \\
&= \left(X^{s_1s_2s_1s_0} + t^{-1/2}X^{s_1s_2s_0} + t^{-1/2}X^{s_2s_1s_0} + t^{-2/2}X^{s_1s_0} + t^{-2/2}X^{s_2s_0} + t^{-3/2}X^{s_0}\right)\mathbf{1} \\
&\qquad + \left(X^{s_1s_2s_1} + t^{-1/2}X^{s_1s_2} + t^{-1/2}X^{s_2s_1}\right)\frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_0^\vee}}\mathbf{1} \\
&\qquad + \left(t^{-2/2}X^{s_1} + t^{-2/2}X^{s_2} + t^{-3/2}\right)\frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}}\mathbf{1}.
\end{aligned}$$

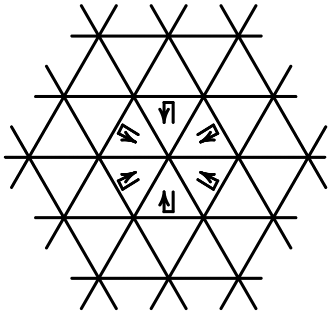
Since $Y^{-\alpha_0^\vee}\mathbf{1} = Y^{\varphi^\vee - d}\mathbf{1} = qY^{\alpha_1^\vee+\alpha_2^\vee}\mathbf{1} = qt^2\mathbf{1}$,

$$\begin{aligned}
P_\rho &= \left(\begin{array}{l} X^{w_0\rho} + t^{-1/2}X^{s_1s_2\rho}T_2^\vee + t^{-1/2}X^{s_2s_1\rho}T_1^\vee \\ \quad +t^{-2/2}X^{s_1\rho}T_2^\vee T_1^\vee + t^{-2/2}X^{s_2\rho}T_1^\vee T_2^\vee + t^{-3/2}X^\rho T_1^\vee T_2^\vee T_1^\vee \end{array}\right)\mathbf{1} \\
&\qquad + \left(T_1^\vee T_2^\vee T_1^\vee + t^{-1/2}T_1^\vee T_2^\vee + t^{-1/2}T_2^\vee T_1^\vee\right)\frac{t^{-1/2}(1-t)}{1-qt^2}\mathbf{1} \\
&\qquad + \left(t^{-2/2}T_1^\vee + t^{-2/2}T_2^\vee + t^{-3/2}\right)\frac{t^{-1/2}(1-t)qt^2}{1-qt^2}\mathbf{1} \\
&= (X^{w_0\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1\rho} + X^{s_2\rho} + X^\rho)\,\mathbf{1} \\
&\qquad + \left(t^{3/2} + t^{1/2} + t^{1/2}\right)\frac{t^{-1/2}(1-t)}{1-qt^2}\mathbf{1} + \left(t^{-1/2} + t^{-1/2} + t^{-3/2}\right)\frac{t^{-1/2}(1-t)qt^2}{1-qt^2}\mathbf{1} \\
&= \left(X^{w_0\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1\rho} + X^{s_2\rho} + X^\rho + (t + 2 + 2qt + q)\frac{1-t}{1-qt^2}\right)\mathbf{1}.
\end{aligned}$$

The set $\mathcal{P}(\vec{\rho})$ contains 12 alcove walks,



and



The Hall-Littlewood polynomial and the Weyl character are

$$P_\rho(0,t) = m_\rho + (2+t)(1-t) \qquad \text{and} \qquad s_\rho = P_\rho(0,0) = m_\rho + 2,$$

where $m_\rho = X^{w_0\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1\rho} + X^{s_2\rho} + X^\rho$.

The expression $X^{s_1s_2\rho}s_2 = s_1^\vee s_2^\vee s_0^\vee$ is a reduced word for the minimal length element in the coset $X^{s_1s_2\rho}W_0$ and Theorem 2.2 is illustrated by

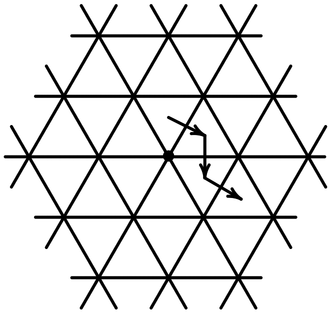
$$\begin{aligned}
\tau_1^\vee\tau_2^\vee\tau_0^\vee &= \left(T_1^\vee + \frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_1^\vee}}\right)\tau_1^\vee\tau_0^\vee = \left(T_1^\vee\tau_2^\vee + \tau_2^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-s_2\alpha_1^\vee}}\right)\tau_0^\vee \\
&= \left(T_1^\vee T_2^\vee + T_1^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_2^\vee}} + T_2^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-s_2\alpha_1^\vee}} + \frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_2^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_2\alpha_1^\vee}}\right)\tau_0^\vee \\
&= T_1^\vee T_2^\vee\tau_0^\vee + T_1^\vee\tau_0^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-s_0\alpha_2^\vee}} + T_2^\vee\tau_0^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-s_0s_2\alpha_1^\vee}} + \tau_0^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-s_0\alpha_2^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0s_2\alpha_1^\vee}} \\
&= T_1^\vee T_2^\vee T_0^\vee + T_1^\vee T_2^\vee\frac{t^{-1/2}(1-t)}{1-Y^{-\alpha_0^\vee}} + T_1^\vee(T_0^\vee)^{-1}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0\alpha_2^\vee}} \\
&\qquad + T_1^\vee\frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0\alpha_2^\vee}} + T_2^\vee(T_0^\vee)^{-1}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0s_2\alpha_1^\vee}} \\
&\qquad + T_2^\vee\frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0s_2\alpha_1^\vee}} + (T_0^\vee)^{-1}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0\alpha_2^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0s_2\alpha_1^\vee}} \\
&\qquad + \frac{t^{-1/2}(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0\alpha_2^\vee}}\frac{t^{-1/2}(1-t)}{1-Y^{-s_0s_2\alpha_1^\vee}},
\end{aligned}$$

where the eight terms in this expansion correspond to the eight alcove walks in $\mathcal{B}(1, s_1^\vee s_2^\vee s_0^\vee) = \mathcal{B}(\overrightarrow{s_1s_2\rho})$ pictured below. Applying the expansion of $\tau_1^\vee\tau_2^\vee\tau_0^\vee$ to $\mathbf{1}$ and using

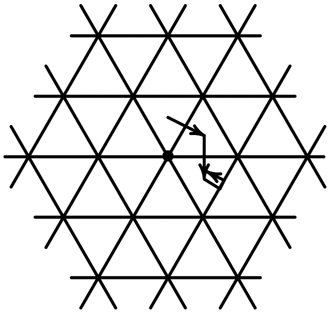
$$\begin{gathered}
Y^{-\alpha_0^\vee}\mathbf{1} = Y^{\varphi^\vee-d}\mathbf{1} = qt^2\mathbf{1}, \qquad Y^{-s_0\alpha_2^\vee}\mathbf{1} = Y^{\alpha_1^\vee-d}\mathbf{1} = qt\mathbf{1}, \\
\text{and} \qquad Y^{-s_0s_2\alpha_1^\vee}\mathbf{1} = Y^{\varphi^\vee-2d}\mathbf{1} = q^2t^2\mathbf{1},
\end{gathered} \tag{4.13}$$

computes

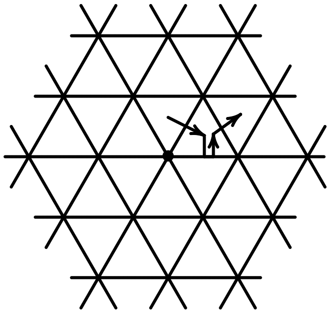
$$\begin{aligned}
E_{s_1s_2\rho} = \Bigg(& X^{s_1s_2\rho}t^{1/2} + t\frac{t^{-1/2}(1-t)}{1-qt^2} + X^{s_1\rho}t\frac{t^{-1/2}(1-t)}{1-qt} + t^{1/2}\frac{t^{-1/2}(1-t)qt^2}{1-qt^2}\frac{t^{-1/2}(1-t)}{1-qt} \\
& + X^{s_2\rho}t\frac{t^{-1/2}(1-t)}{1-q^2t^2} + t^{1/2}\frac{t^{-1/2}(1-t)qt^2}{1-qt^2}\frac{t^{-1/2}(1-t)}{1-q^2t^2} \\
& + X^{\rho}t^{3/2}\frac{t^{-1/2}(1-t)}{1-qt}\frac{t^{-1/2}(1-t)}{1-q^2t^2} + \frac{t^{-1/2}(1-t)qt^2}{1-qt^2}\frac{t^{-1/2}(1-t)}{1-qt}\frac{t^{-1/2}(1-t)}{1-q^2t^2}\Bigg)\mathbf{1}. \\
= t^{1/2}\Bigg(& X^{s_1s_2\rho} + \frac{(1-t)}{1-qt^2} + X^{s_1\rho}\frac{(1-t)}{1-qt} + t\frac{(1-t)q}{1-qt^2}\frac{(1-t)}{1-qt} + X^{s_2\rho}\frac{(1-t)}{1-q^2t^2} \\
& + t\frac{(1-t)q}{1-qt^2}\frac{(1-t)}{1-q^2t^2} + X^{\rho}\frac{(1-t)}{1-qt}\frac{(1-t)}{1-q^2t^2} + \frac{(1-t)q}{1-qt^2}\frac{(1-t)}{1-qt}\frac{(1-t)}{1-q^2t^2}\Bigg)\mathbf{1}.
\end{aligned}$$



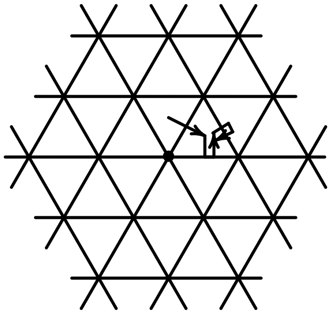
$X^{s_1s_2\rho}t^{1/2}$



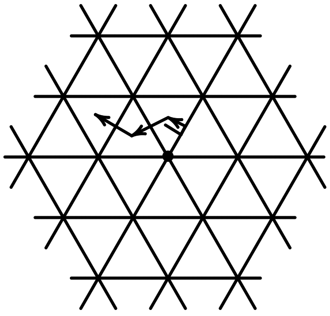
$t^{1/2}\frac{(1-t)}{1-qt^2}$



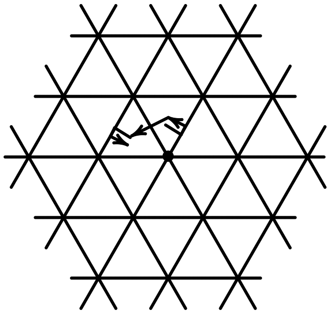
$X^{s_1\rho}t^{1/2}\frac{(1-t)}{1-qt}$



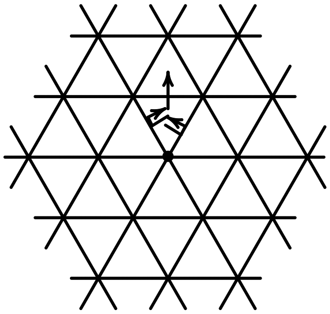
$t^{3/2}\frac{(1-t)}{1-qt}\frac{(1-t)q}{1-qt^2}$



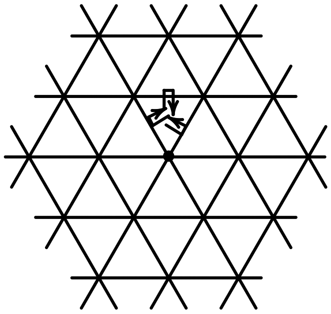
$X^{s_2\rho}t^{1/2}\frac{(1-t)}{1-q^2t^2}$



$t^{3/2}\frac{(1-t)q}{1-qt^2}\frac{(1-t)}{1-q^2t^2}$



$$X^\rho t^{1/2}\frac{(1-t)}{1-qt}\frac{(1-t)}{1-q^2t^2}$$



$$t^{3/2}\frac{(1-t)}{1-qt}\frac{(1-t)q}{1-qt^2}\frac{(1-t)}{1-q^2t^2}$$

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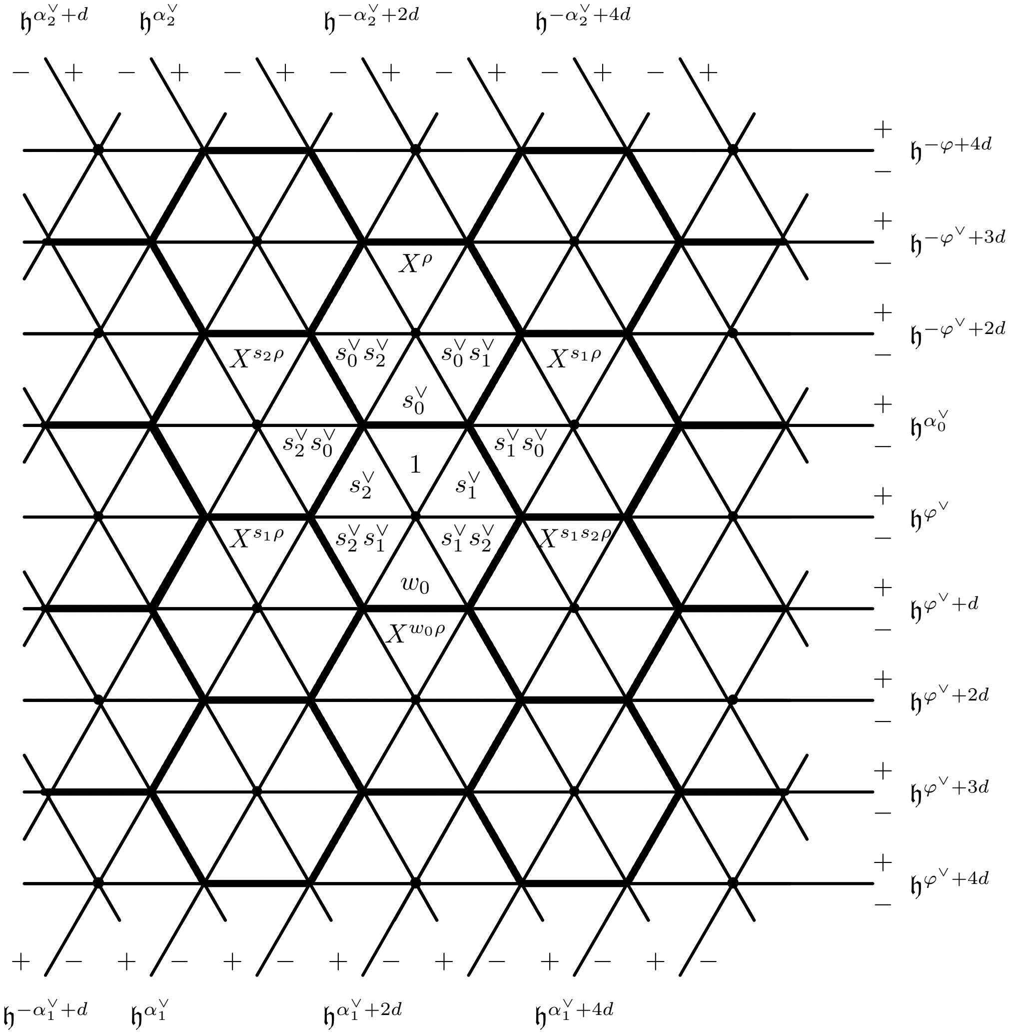
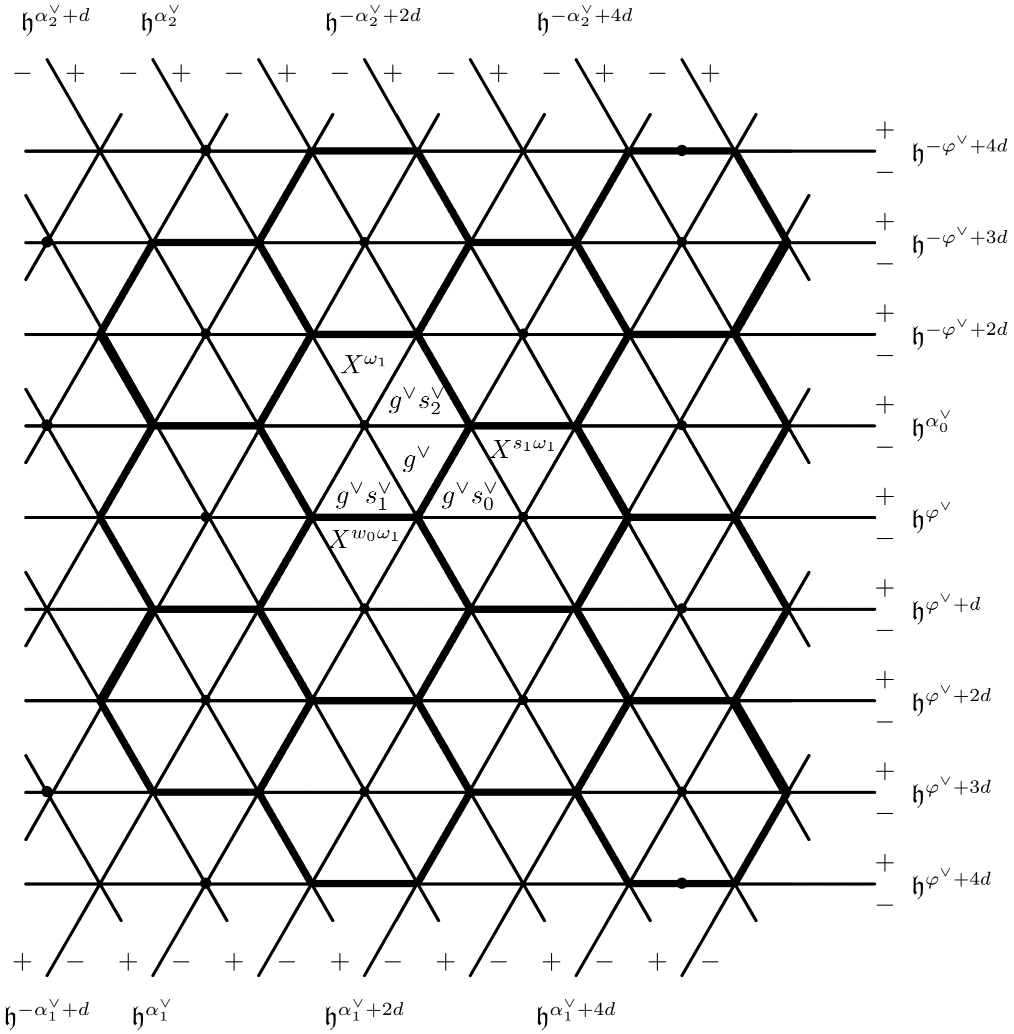
5 Appendix: The bijection between W and alcoves in type SL_3

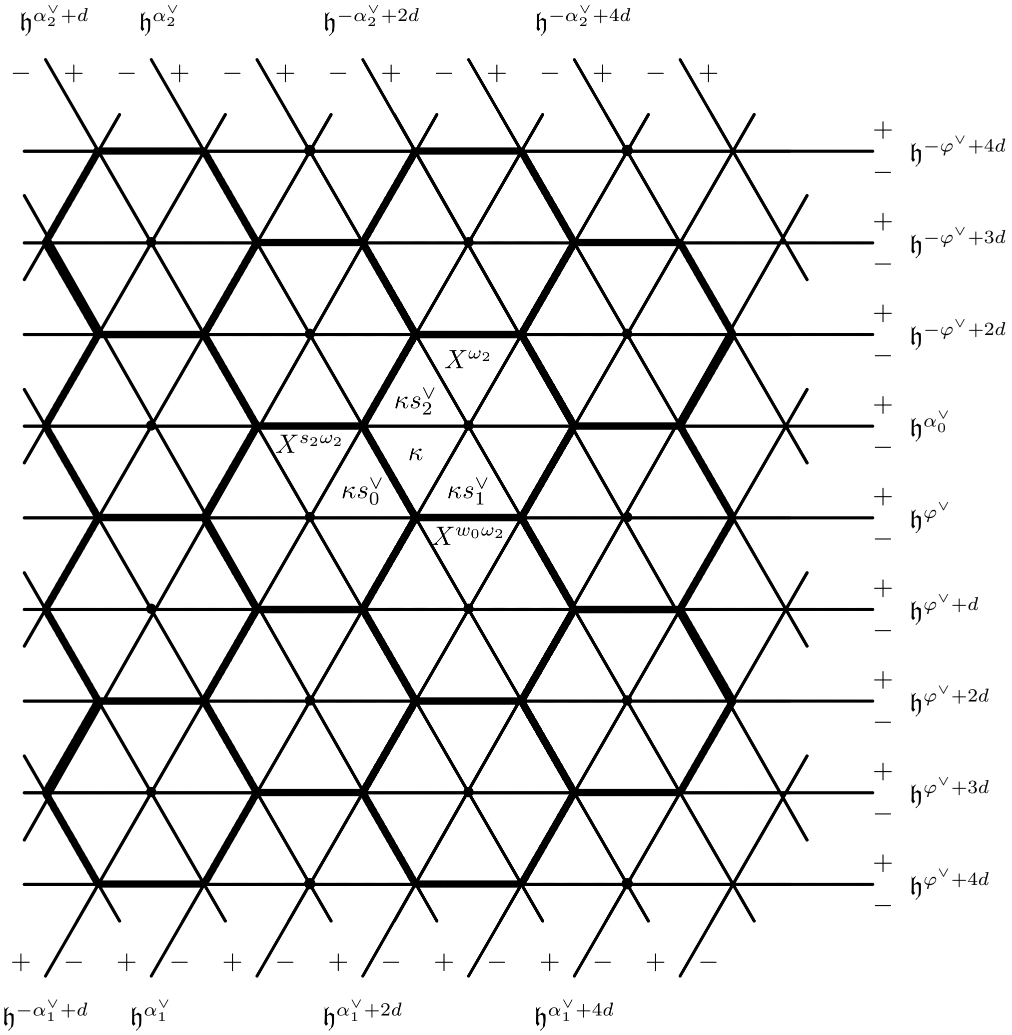
The following pictures illustrate the bijection of (2.19) for type SL_3. In this case, $\Omega^\vee = \{1, g^\vee, (g^\vee)^2\} \cong \mathbb{Z}/3\mathbb{Z}$, and $\Omega^\vee \times \mathfrak{h}^*_{\mathbb{R}}$ has 3 sheets. The alcoves are the triangles and the (centres of) hexagons are the elements of $\mathfrak{h}^*_{\mathbb{Z}}$.



Sheet 1


$\mathfrak{h}^{\alpha_2^\vee+d}$
$\mathfrak{h}^{\alpha_2^\vee}$
$\mathfrak{h}^{-\alpha_2^\vee+2d}$
$\mathfrak{h}^{-\alpha_2^\vee+4d}$
$-\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +$
$\mathfrak{h}^{-\varphi^\vee+4d}$
$\mathfrak{h}^{-\varphi^\vee+3d}$
$\mathfrak{h}^{-\varphi^\vee+2d}$
$\mathfrak{h}^{\alpha_0^\vee}$
$\mathfrak{h}^{\varphi^\vee}$
$\mathfrak{h}^{\varphi^\vee+d}$
$\mathfrak{h}^{\varphi^\vee+2d}$
$\mathfrak{h}^{\varphi^\vee+3d}$
$\mathfrak{h}^{\varphi^\vee+4d}$
X^{ω_1}
$g^\vee s_2^\vee$
$g^\vee$
$X^{s_1\omega_1}$
$g^\vee s_1^\vee$
$g^\vee s_0^\vee$
$X^{w_0\omega_1}$
$+\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -$
$\mathfrak{h}^{-\alpha_1^\vee+d}$
$\mathfrak{h}^{\alpha_1^\vee}$
$\mathfrak{h}^{\alpha_1^\vee+2d}$
$\mathfrak{h}^{\alpha_1^\vee+4d}$

Sheet $g^\vee$



Sheet $\kappa = (g^\vee)^2$