Combinatorial Representation Theory II – Crystals

Crystals

University of Zaragoza, Feb. 24, 2005

G = a complex connected reductive algebraic group

$\cup\!|$

B = a Borel subgroup

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T = a maximal torus

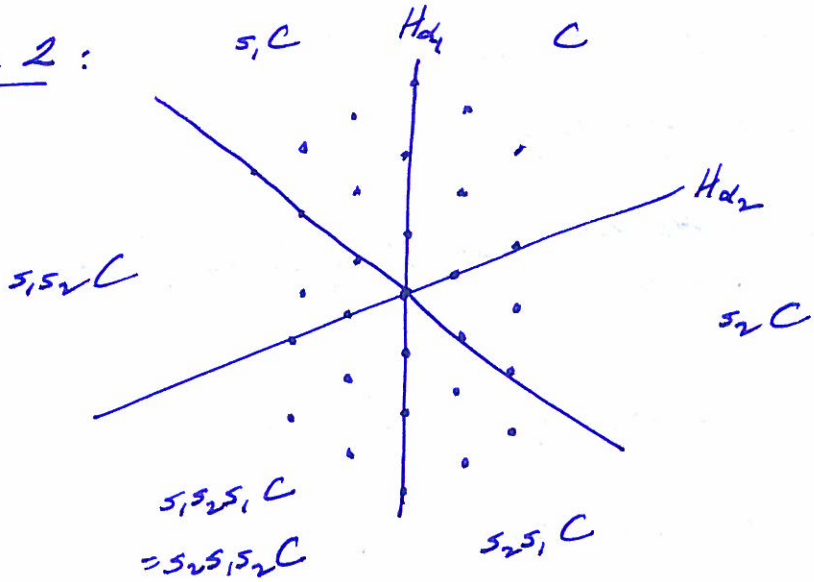
Equivalent data:

W = finite reflection group

C = a fixed fundamental chamber

P = a W-invariant lattice.

Example 2:



Example 1:

$G = GL_n(\mathbb{C})$ $\qquad$ $W = S_n$ acting on $\mathbb{R}^n = \sum_{i=1}^{n} \mathbb{R}\varepsilon_i$

$\cup\!|$

$B = \left\{\begin{pmatrix} * & \cdots & * \\ 0 & \ddots & * \end{pmatrix}\right\}$ $\qquad$ Reflections: s_{ij}, transposes ε_i and ε_j

$\cup\!|$ $\qquad$ $C = \{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)\}$

$T = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$ $\qquad$ $P = \mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$

<u>Theorem</u> (a) There is a bijection

$$\left\{ \begin{matrix} \text{Finite dimensional} \\ \text{simple } T\text{-modules} \end{matrix} \right\} \longleftrightarrow P$$
$$X^\mu : T \to \mathbb{C} \quad \longleftarrow\!\!| \ \mu$$

(b) There is a bijection

$$P^+ \longleftrightarrow \left\{ \begin{matrix} \text{Finite dimensional} \\ \text{simple } G\text{-modules} \end{matrix} \right\}$$
$$\lambda \longmapsto L(\lambda)$$

where $P^+ = P \cap \bar{C}$, $\bar{C}$ is the closure of C.

Let $L(\lambda)$ be a simple G-module

$$\mathrm{Res}^G_T (L(\lambda)) = \bigoplus_{\mu \in P} L(\lambda)_\mu, \text{ where}$$

$L(\lambda)_\mu = \{ m \in L(\lambda) \mid tm = X^\mu(t) m, \text{ for } t \in T \}$.

The <u>character</u> of $L(\lambda)$ is

$$s_\lambda = \sum_\mu \dim(L(\lambda)_\mu) X^\mu$$

an element of $\mathbb{C}[P] = \mathrm{span}\{ X^\mu \mid \mu \in P \}$ with $X^\lambda X^\mu = X^{\lambda+\mu}$.

<u>Goal</u>: The crystal is an index set

$$\hat{L}(\lambda) = \bigsqcup_\mu \hat{L}(\lambda)_\mu \longleftrightarrow \text{basis of } L(\lambda) = \bigoplus_\mu L(\lambda)_\mu$$

such that

$$s_\lambda = \sum_{p \in \hat{L}(\lambda)} X^{wt(p)}, \text{ where } wt(p) = \mu \text{ if } p \in \hat{L}(\lambda)_\mu.$$

The affine Hecke algebra

Let $H_{\alpha_1}, \dots, H_{\alpha_n}$ be the walls of C

s_i the reflection in H_{α_i}.

The *positive side* of H_{α_i} is the side towards C.

The *affine Hecke algebra* $\tilde{H}$ is given by generators X^λ, $\lambda \in P$ and T_w, $w \in W$ and relations

$$X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda,$$

$$T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } s_i w > w, \\ q^{-2} T_{s_i w} + (1-q^{-2}) T_w, & \text{if } s_i w < w. \end{cases}$$

If λ is on the positive side of H_{α_i} then

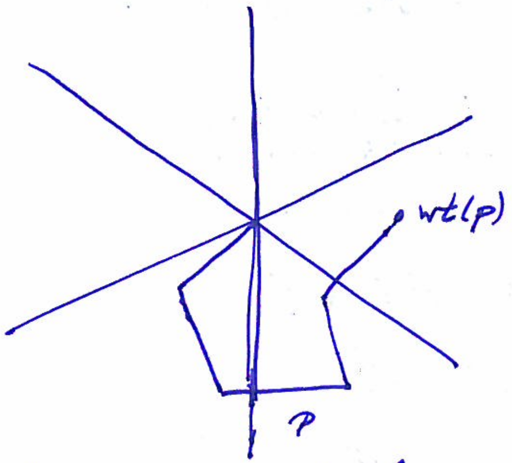
$$X^\lambda T_{s_i} = T_{s_i} X^{s_i \lambda} + (1-q^{-2})(X^{s_i\lambda+\alpha_i} + \dots + X^{\lambda-\alpha_i} + X^\lambda)$$

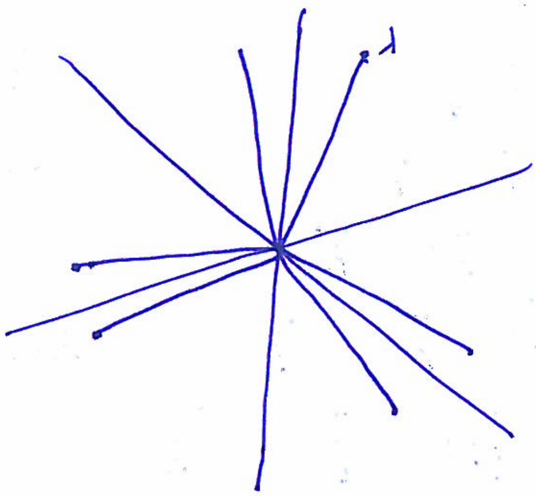
Problem: Find $c_{\lambda w}^{v\mu}$ such that

$$X^\lambda T_w = \sum_{v,\mu} c_{\lambda w}^{v\mu} T_v X^\mu.$$

Idea: The crystal is the solution to this problem at $q^{-2}=0$.

The Path model



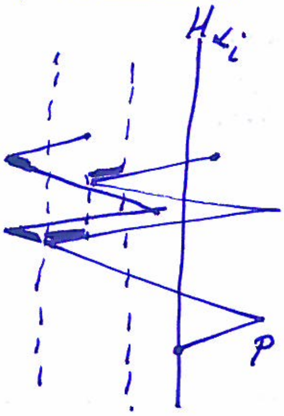


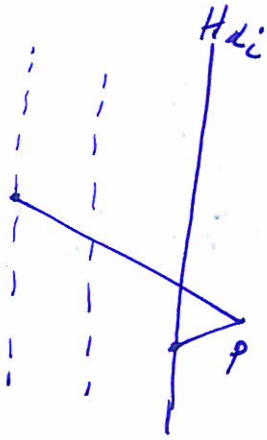
$wt(p)$ = endpoint of p
$\iota(p)$ = initial direction of p
$\varphi(p)$ = final direction of p

straight line path p_λ and directions

Root operators

$\tilde{f}_i p$





$\tilde{f}_i p = 0$

and define $\tilde{e}_i$ by

$$\tilde{e}_i \tilde{f}_i p = p \quad \text{if} \quad \tilde{f}_i p \neq 0.$$

A *crystal* is a set of paths closed under $\tilde{e}_i, \tilde{f}_i$.
Let

$$\hat{\mathcal{L}}(\lambda) = \text{crystal generated by } p_\lambda$$

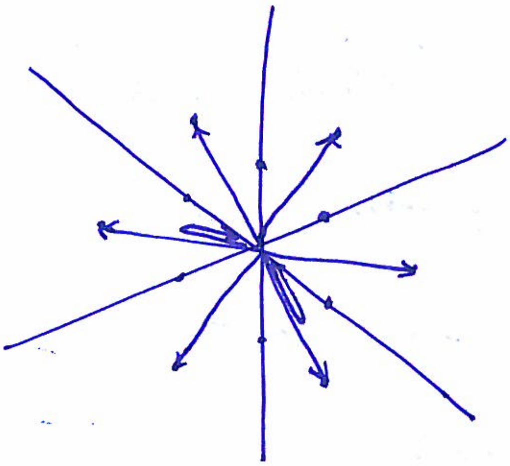
Theorem

$$s_\lambda = \sum_{p \in \hat{\mathcal{L}}(\lambda)} X^{wt(p)}.$$

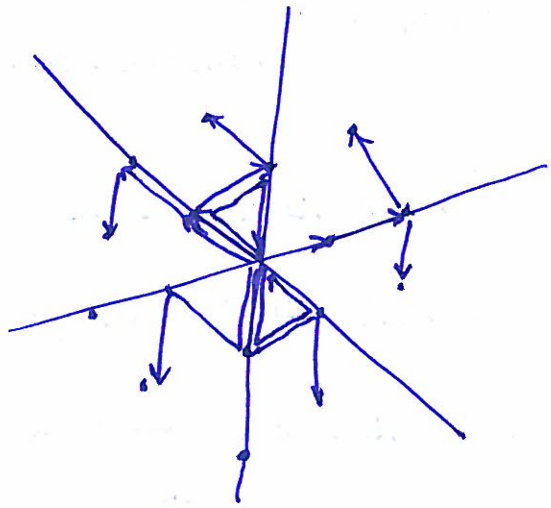
Theorem (Pittie–Ram) Let $q^{-2}=0$ in $\tilde{H}$. Let $\lambda \in P^+$ and $w \in W$. Then

$$X^{\lambda} T_{w^{-1}} = \sum_{\substack{p \in \hat{\mathcal{L}}(\lambda) \\ \iota(p) \le w}} T_{\varphi(p)^{-1}} X^{wt(p)}$$

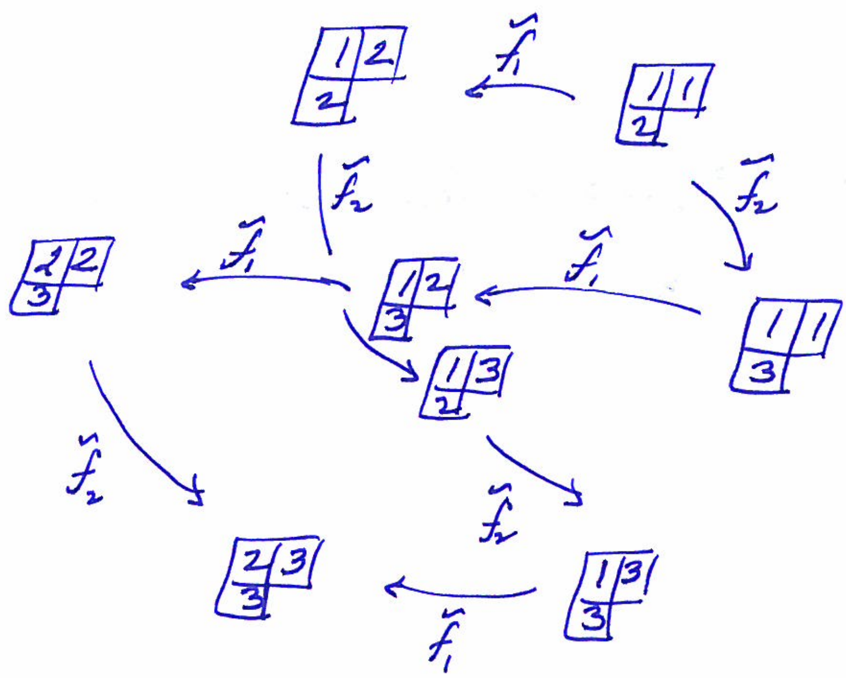
Example $\hat{\mathcal{L}}(\rho)$ where $\rho = \omega_1 + \omega_2 = 2\varepsilon_1 + \varepsilon_2 =$



The crystal generated by p_ρ

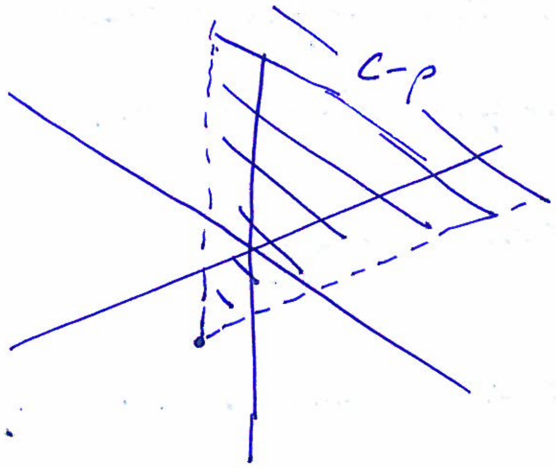


The crystal generated by



Branching / Littlewood-Richardson rules

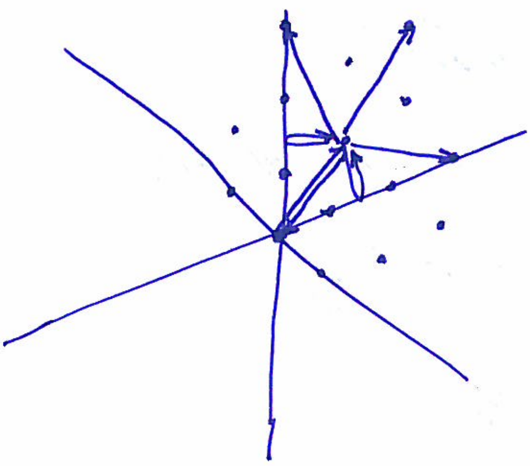
A **highest weight path** is a path $p \subseteq C - \rho$.



Theorem Let B be a crystal. Then

$$\mathrm{char}\, B = \sum_{\substack{p \in B \\ p \subseteq C-\rho}} s_{wt(p)}$$

Example Highest weight paths in $\hat{L}(\rho) \otimes \hat{L}(\rho)$

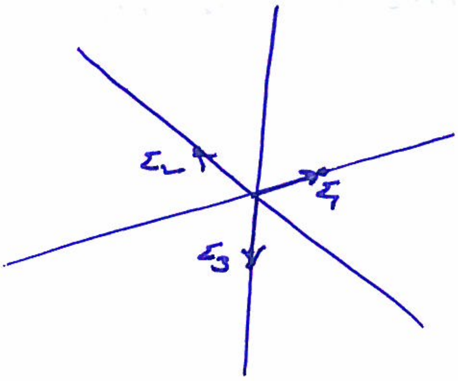


So

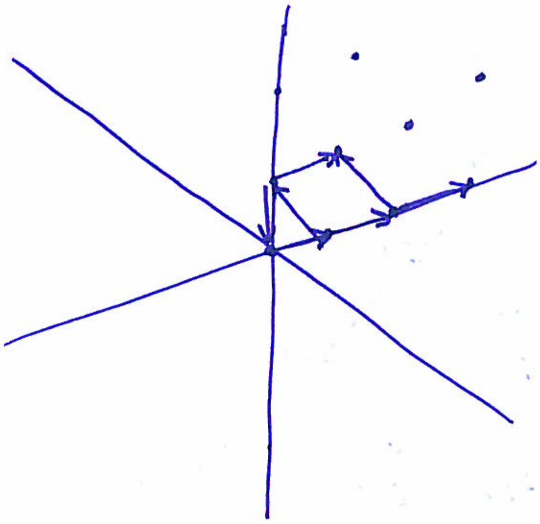
$$s_{(2,1)} s_{(2,1)} = s_0 + 2 s_{(2,1)} + s_{(3)} + s_{(3,3)} + s_{(4,2)}$$

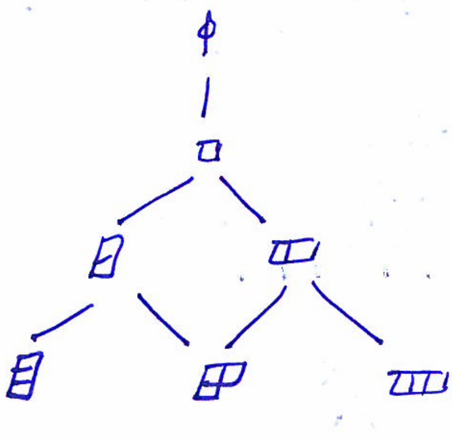
Example Highest weight paths in $\hat{L}(\varepsilon_1)^{\otimes k}$

$\hat{L}(\varepsilon_1) =$



$\hat{L}(\varepsilon_1)^{\otimes k}$





These paths give us a tower.