Combinatorial Representation Theory

MPI Oberseminar
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Basic Data:

W finite real reflection group

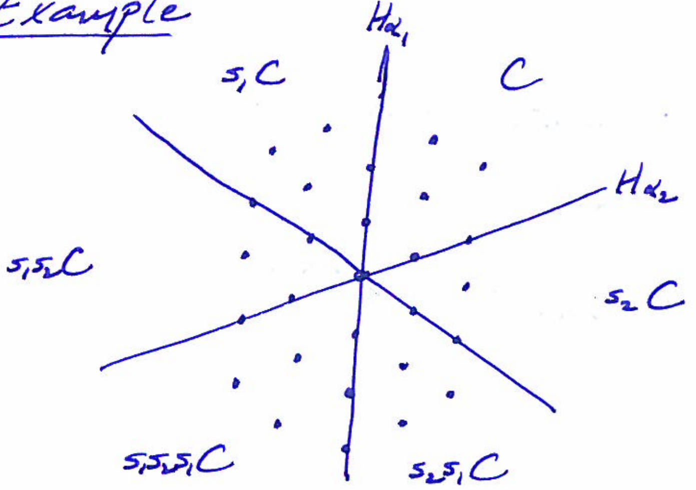
C fix fundamental chamber

P W invariant lattice

Let $H_{\alpha_1}, \dots, H_{\alpha_n}$ be the walls of C

s_i the reflection in H_{α_i}.

Example



$W = \{\text{chambers}\}$

$P^+ = P \cap \overline{C}$

$P^{++} = P \cap C$

$$P^+ \xrightarrow{\sim} P^{++}$$
$$\lambda \longmapsto \lambda + \rho$$

Equivalent data

G complex reductive algebraic group $\qquad GL_n(\mathbb{C})$

$\cup\!|$ $\qquad \cup\!|$

B Borel subgroup $\qquad \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$

$\cup\!|$ $\qquad \cup\!|$

T maximal torus $\qquad \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$

Representation Theory

$$\mathbb{C}[P] = \text{span}\{X^\lambda \mid \lambda \in P\} \quad \text{with} \quad X^\lambda X^\mu = X^{\lambda+\mu}$$

W acts on $\mathbb{C}[P]$ by $wX^\lambda = X^{w\lambda}$.

$$\mathbb{C}[P]^W = \{f \in \mathbb{C}[P] \mid wf = f, \text{ for all } w \in W\}$$

$$\mathbb{C}[P]^{\det} = \{f \in \mathbb{C}[P] \mid wf = \det(w)f, \text{ for all } w \in W\}$$

If $\mathbb{1} = \sum_{w \in W} w$ and $\varepsilon = \sum_{w \in W} \det(w^{-1}) w$ then

$$\begin{array}{ccc} \mathbb{1} \cdot \mathbb{C}[P] = \mathbb{C}[P]^W & \longrightarrow & \mathbb{C}[P]^{\det} = \varepsilon\, \mathbb{C}[P] \\ f & \longmapsto & a_\rho f \\ s_\lambda & \longleftarrow & a_{\lambda+\rho} = \varepsilon X^{\lambda+\rho} \end{array}$$

$$\mathbb{1} \cdot X^\lambda = m_\lambda$$

__Problem__ Describe $K_{\lambda\mu}$ given by

$$s_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu} m_\mu$$

__Hermann Weyl__ (a) P indexes simple T-modules

(b) P^+ indexes simple G-modules, $L(\lambda)$

If $\text{Res}^G_T(L(\lambda)) = \bigoplus_\mu L(\lambda)_\mu$ then

$$s_\lambda = \sum_{\mu \in P} \dim(L(\lambda)_\mu) X^\mu.$$

The affine Hecke algebra $\tilde{H}$

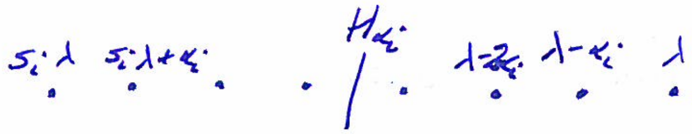
Generators: T_w, $w \in W$, and X^λ, $\lambda \in P$

Relations: $X^\lambda X^\mu = X^{\lambda+\mu}$

$$T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } s_i w > w \quad \left(\begin{matrix} s_i w C \text{ is farther from } C \\ \text{than } wC \end{matrix}\right) \\ T_{s_i w} + (q-q^{-1}) T_w, & \text{if } s_i w < w \quad \left(\begin{matrix} s_i w C \text{ is closer to } C \\ \text{than } wC \end{matrix}\right) \end{cases}$$

If λ is on the positive side of C then

$$T_{s_i} X^\lambda = X^{s_i\lambda} T_{s_i} + (q-q^{-1})(X^{s_i\lambda+\alpha_i} + \cdots + X^{\lambda-\alpha_i} + X^\lambda)$$



$\tilde{H}$ has two bases

$$\{T_w X^\lambda \mid w \in W, \lambda \in P\} \text{ and } \{X^\mu T_v \mid \mu \in P, v \in W\}$$

<u>Problem</u> Describe $c_{w\lambda}^{\mu v}$ given by

$$T_w X^\lambda = \sum_{\mu, v} c_{w\lambda}^{\mu v} X^\mu T_v.$$

<u>Spherical functions</u> on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$

Let $H = \text{span}\{T_w \mid w \in W\}$ and let $\mathbb{1}_0, \varepsilon_0 \in H$ given by

$$\mathbb{1}_0^2 = \mathbb{1}_0 \qquad \text{and} \qquad T_{s_i}\mathbb{1}_0 = q\mathbb{1}_0$$
$$\varepsilon_0^2 = \varepsilon_0 \qquad\qquad T_{s_i}\varepsilon_0 = (-q^{-1})\varepsilon_0$$

Note that

$$\begin{aligned} \tilde{H}\mathbb{1}_0 &\xrightarrow{\sim} \mathbb{C}[P] \\ X^\lambda \mathbb{1}_0 &\longmapsto X^\lambda \end{aligned}$$

Then

$$\begin{array}{ccccc} \mathbb{C}[P]^W = Z(\tilde{H}) & \longrightarrow & \mathbb{1}_0\tilde{H}\mathbb{1}_0 & \longrightarrow & \varepsilon_0 \tilde{H}\mathbb{1}_0 \\ f & \longmapsto & f\mathbb{1}_0 & & \\ & & h & \longmapsto & A_\rho h \\ s_\lambda & \longleftarrow & C'_{n_\lambda} & \longleftarrow & A_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho}\mathbb{1}_0 \\ P_\lambda & \longleftarrow & M_\lambda = \mathbb{1}_0 X^\lambda \mathbb{1}_0 & & \end{array}$$

where C'_{n_λ} is the Kazhdan-Lusztig element for n_λ the maximal length element in $Wt_\lambda W$.

<u>Problem</u> Describe $K_{\lambda\mu}(q)$ given by

$$s_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu}(q) P_\mu$$

$K_T(G/B)$: T-equivariant K-theory of G/B

Let $q^2 = 0$ in $\tilde{H}$ and extend coefficients to

$$R = \mathrm{span}\{e^\lambda \mid \lambda \in P\} \quad \text{with} \quad e^\lambda e^\mu = e^{\lambda+\mu}$$

$$R[P] = R \otimes_{\mathbb{Z}} \mathbb{Z}[P] = R\text{-span}\{X^\lambda \mid \lambda \in P\}$$

$$\tilde{H}_R = R \otimes_{\mathbb{Z}} \tilde{H} = R\text{-span}\{X^\lambda T_w \mid \lambda \in P,\ w \in W\}.$$

The R-algebra homomorphism

$$\begin{matrix} ev: R[P] \longrightarrow R \\ X^\lambda \longmapsto e^\lambda \end{matrix} \quad \text{gives an } R[P]\text{-module} \quad R[\mathcal{O}_{X_1}] \text{ with } X^\lambda[\mathcal{O}_{X_1}] = e^\lambda[\mathcal{O}_{X_1}].$$

Then

$$K_T(G/B) = \tilde{H}_R \otimes_{R[P]} ev = \mathrm{Ind}_{R[P]}^{\tilde{H}_R}(ev) = \tilde{H}_R[\mathcal{O}_{X_1}]$$

has basis

$$\{[\mathcal{O}_{X_w}] \mid w \in W\} \quad \text{with} \quad T_{w^{-1}}[\mathcal{O}_{X_1}] = [\mathcal{O}_{X_w}].$$

The map

$$\begin{matrix} \Phi: R[P] & \longrightarrow & \tilde{H}_R \mathbb{1}_0 & \hookrightarrow & \tilde{H}_R & \longrightarrow & K_T(G/B) \\ X^\lambda & \longmapsto & X^\lambda \mathbb{1}_0 & \longmapsto & X^\lambda \mathbb{1}_0 & \longmapsto & X^\lambda \mathbb{1}_0[\mathcal{O}_{X_1}] = [\mathcal{L}_\lambda] = [G \times_B \mathbb{C}_\lambda] \end{matrix}$$

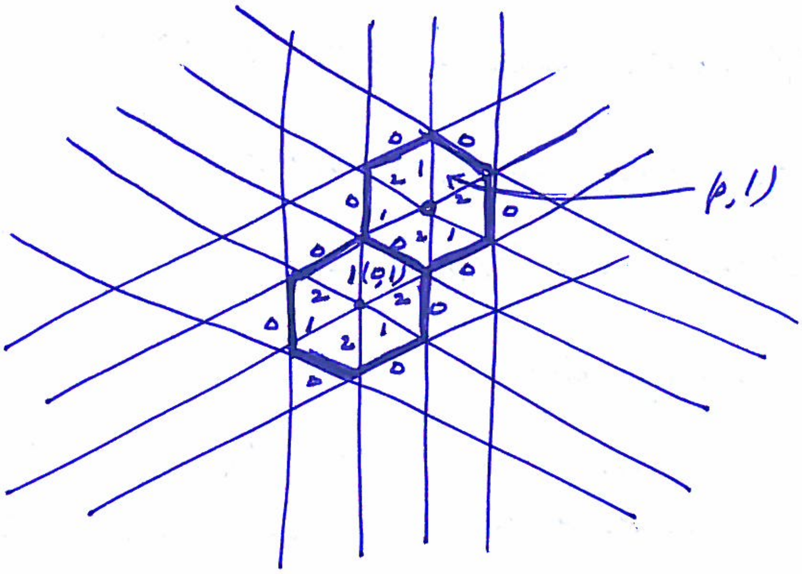
is surjective.

<u>Problem</u> Describe $c_{\lambda w}^v$ given by

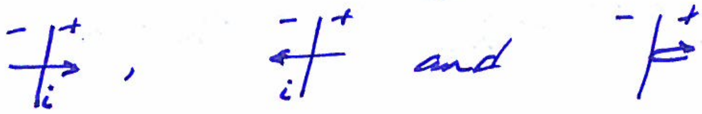
$$[\mathcal{L}_\lambda][\mathcal{O}_{X_w}] = \sum_v c_{\lambda w}^v [\mathcal{O}_{X_v}].$$

Alcove walks $T_w X^\lambda = \sum_{\mu,\nu} c_{w\lambda}^{\mu\nu} X^\mu T_\nu$

Alcove addresses: (μ, ν)



Alcove walks have steps labeled by i $(0 \le i \le n)$



where the positive direction is towards $\infty\rho$.

Fix, $w \in W$ and $\lambda \in P$.

Let $w_1, \ldots, w_r$ be a shortest path to $(0, w)$

Let $l_1, \ldots, l_s$ be a shortest path to $(\lambda, 1)$

Then
$$c_{w\lambda}^{\mu\nu} = \sum_p (q - q^{-1})^{\#\text{ of } \rightleftarrows \text{ in } p}$$

where the sum is over all paths p with steps $w_1, \ldots, w_r, l_1, l_2, \ldots, l_s$ and end (μ, ν).