Algebraseminar, Vermacrystals, University of Lyon

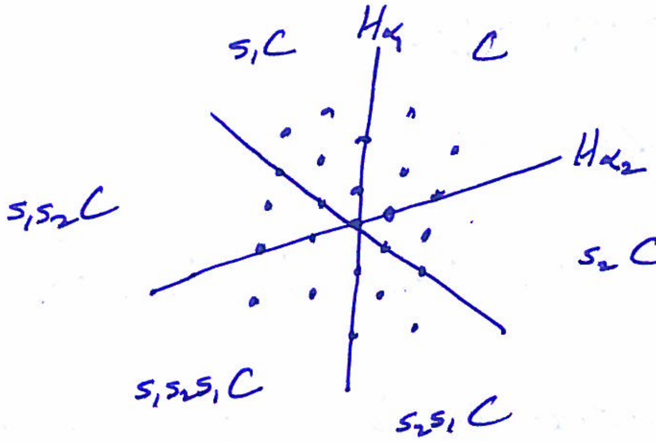
Fundamental data

May 26, 2005

$P = \mathbb{Z}$-lattice

W = a finite group acting on P

C = a fixed fundamental chamber in $\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} P$.



$W \longleftrightarrow \{\text{chambers}\}$.

The quantum group associated to (W,C,P) is the $\mathbb{Q}(q)$ algebra

$$\mathcal{U} = \mathcal{U}_{<0}\mathcal{U}_0\mathcal{U}_{>0}$$

$\mathcal{U}_{<0}$ is generated by $F_1, F_2, \ldots, F_n$

$\mathcal{U}_{>0}$ is generated by $E_1, E_2, \ldots, E_n$

$\mathcal{U}_0 = \mathbb{C}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}]$

with relations ...

The Verma module is the $\mathcal{U}$-module

$$M(\lambda) = \mathcal{U}_{<0} v_\lambda^+ \quad \text{with } E_i v_\lambda^+ = 0,\ K_i v_\lambda^+ = q^{\langle \lambda, \alpha_i^\vee \rangle} v_\lambda^+.$$

$L(\lambda)$ is the unique simple quotient of $M(\lambda)$.

Let $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$ and $[m]! = [m][m-1]\cdots[2]$

Fix a reduced word $w_0 = s_{i_1}\cdots s_{i_N}$.

The negative root vectors are

$$F_{\beta_1} = F_{i_1}, \ F_{\beta_2} = T_{i_1}F_{i_2}, \ \ldots, \ F_{\beta_N} = T_{i_1}\cdots T_{i_{N-1}}F_{i_N}$$

where $T_i : U \to U$ are Lusztig's braid group automorphisms.

Recall:

$$\beta_1 = \alpha_{i_1}, \ \beta_2 = s_{i_1}\alpha_{i_2}, \ \ldots, \ \beta_N = s_{i_1}s_{i_2}\cdots s_{i_{N-1}}\alpha_{i_N}.$$

are the positive roots.

The PBW bases of $U_{<0}$ is

$$\{F^{(\vec{m})} = F_{\beta_1}^{(m_1)}\cdots F_{\beta_N}^{(m_N)} \mid \vec{m} = (m_1,\ldots,m_N) \in \mathbb{Z}_{\geq 0}^N\}$$

where $F_\beta^{(m)} = \dfrac{F_\beta^m}{[m]!}$

The bar involution is the $\mathbb{Q}$ automorphism of U

given by

$$\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{K_i} = K_i^{-1}, \quad \overline{q} = q^{-1}.$$

The *canonical basis* is

$$\mathcal{B} = \{ b^{(\vec{m})} \mid \vec{m} \in \mathbb{Z}^N_{\geq 0} \} \quad \text{given by}$$

(1) $\overline{b^{(\vec{m})}} = b^{(\vec{m})}$

(2) $b^{(\vec{m})} = F^{(\vec{m})} + \sum_{\vec{n} \neq \vec{m}} p_{\vec{n}\vec{m}} F^{(\vec{n})}$ with $p_{\vec{n}\vec{m}} \in q\mathbb{Z}[q]$.

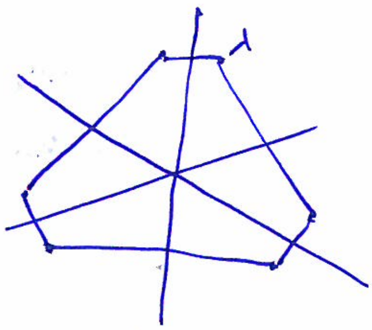
$\{ F^{(\vec{m})} \mid \vec{m} \in \mathbb{Z}^N_{\geq 0} \}$ depends on the choice $w_0 = s_{i_1} \cdots s_{i_N}$

Theorem (Lusztig)

(a) $\mathcal{B}$ does not depend on the choice $w_0 = s_{i_1} \cdots s_{i_N}$

(b) $\mathcal{B}(\lambda) = \{ b^{(\vec{m})} v_\lambda^+ \mid b^{(\vec{m})} v_\lambda^+ \neq 0 \text{ in } L(\lambda) \}$ is a basis of $L(\lambda)$.

The *Weyl polytope* is the convex hull of $\{ w\lambda \mid w \in W \}$,

$[W\lambda] =$ 

Let $b \in B$. The <u>MV polytope</u> of b is

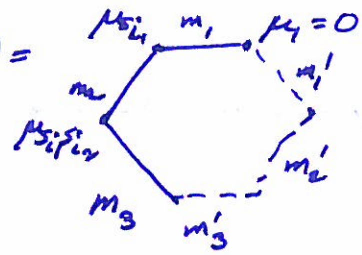
$[b]$ = convex hull of $\{b_w \mid w \in W\}$ where

$$b_{s_{i_1}\cdots s_{i_j}} = -m_1\rho_1 - m_2\rho_2 - \cdots - m_j\rho_j.$$

Then

$$B \overset{\sim}{\longleftrightarrow} \{\text{MV polytopes}\}$$

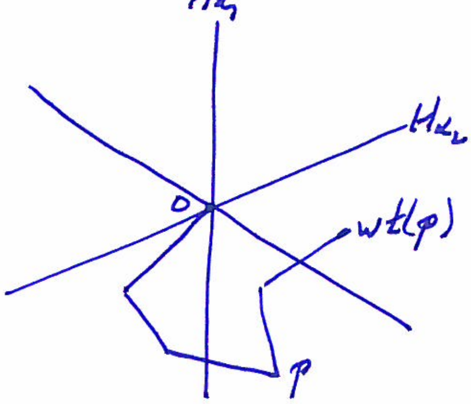
$$b \longmapsto [b] =$$

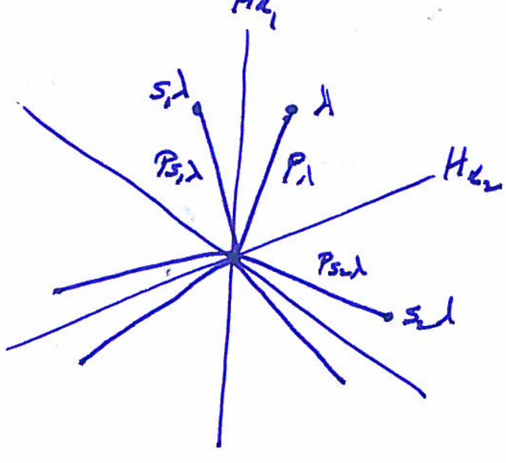


and $\quad B(\lambda) \overset{\sim}{\longleftrightarrow} \{\text{MV polytopes } [b] \mid [b]+\lambda \subseteq [W\lambda]\}.$

Kamnitzer has described the crystal structure (root operators) on $\{$MV polytopes$\}$ to produce a <u>Verma crystal</u>.

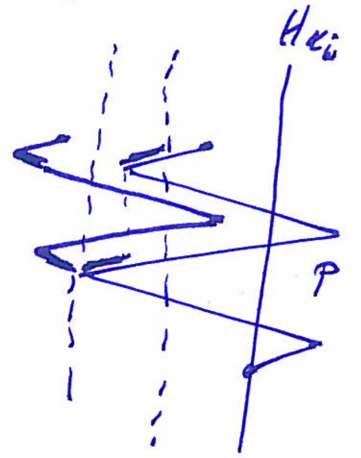
<u>Path crystals</u>

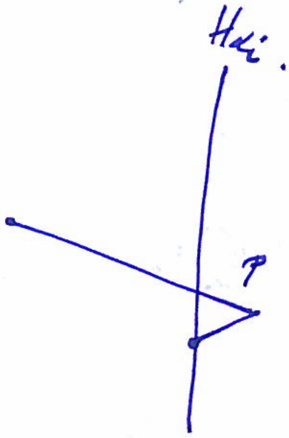




Root operators

$\tilde f_i p$





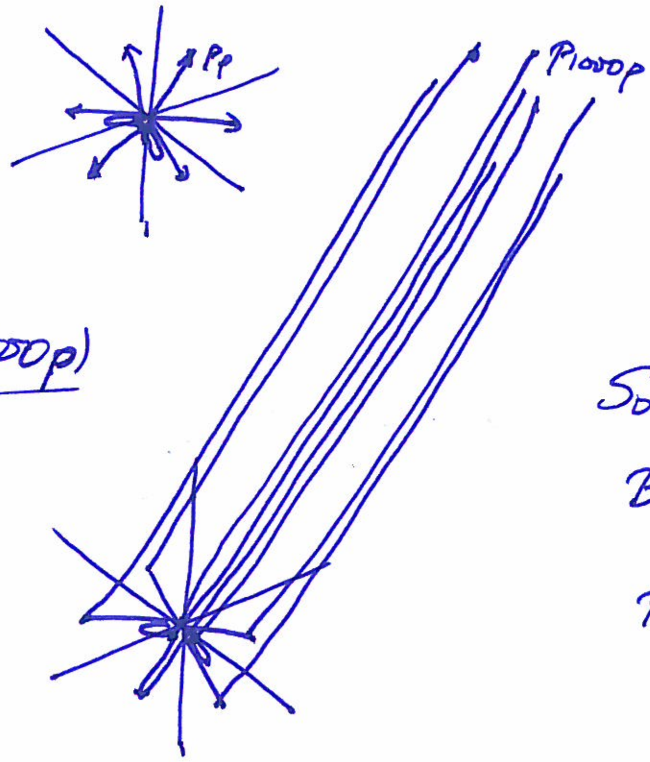
$\tilde f_i p = 0$.

and $\quad \tilde e_i \tilde f_i p = p \quad$ if $\quad \tilde f_i p \neq 0$.

A <u>crystal</u> is a set of paths which is closed under the root operators.

$B(\lambda)$ is the crystal generated by p_λ.

<u>$B(\rho)$</u>



<u>$B(1000\rho)$</u>

So

$$B(\rho) \hookrightarrow B(1000\rho)$$

$$p \longmapsto p \otimes p_{999\rho}.$$

Let
$$C(\lambda) = B(\infty\lambda)$$

View $p_{\infty\lambda} \in C(\lambda)$ as

$$p_{\infty\lambda} : \mathbb{R}_{\geq 0} \longrightarrow \mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^n \quad \text{with} \quad \varphi(1) = \lambda.$$

If $\varphi \in C(\lambda)$ and

$$t = (a, w_1, w_2) = a \quad \text{is a turn of } \varphi$$

then let

u_t be the unique element (up to constant) in $U_{<0}$ s.t.

$$M(w_1(a\lambda - \rho)) \hookrightarrow M(w_2(a\lambda - \rho))$$
$$v^+_{w_1(a\lambda-\rho)} \mapsto u_t \, v^+_{w_2(a\lambda-\rho)}$$

Let
$$u_\varphi = \prod_{\text{turns } t} u_t$$

<u>Theorem</u> (Littelmann)

$\{ u_\varphi \mid \varphi \in C(\lambda) \}$ is a basis of $U_{<0}$

<u>Théorème</u> Let $\lambda \in P^+$ and $w \in W$. Define

$$C(w\circ\lambda) = \{p \in C(\lambda) \mid \text{the } \lambda \text{ turn of } p \text{ is } \geq w\}$$

$$B(w\circ\lambda) = \{p \in C(w\circ\lambda) \mid \text{the tail of } p \text{ is } \lambda \text{ straight}\}$$

$$C(w\circ\lambda)^+_v = \{p \in C(w\circ\lambda) \mid p \text{ is } v\text{-highest weight}\}.$$

Then

(1) $C(w\circ\lambda)$ is a model for $M(w\circ\lambda)$

$B(w\circ\lambda)$ is a model for $L(w\circ\lambda)$

(2) (BGG resolution) There is an exact sequence of crystals

$$0 \to C(w_0\circ\lambda) \to \cdots \to \bigoplus_{\ell(w)=i} C(w\circ\lambda) \to \cdots \to C(\lambda) \to B(\lambda) \to 0$$

(3) Let $\mu \in P^+$. The q-weight multiplicity is

$$K_{\lambda\mu}(t) = \sum_{p \in B(\lambda)_\mu} t^{ch(p)}$$

where $B(\lambda)_\mu = \{p \in B(\lambda) \mid wt(p) = \mu\}$ and $ch(p)$ is the depth of p.

(4) Let $v \in W$. The Kazhdan-Lusztig polynomial is

$$P_{vw}(t) = \sum_{p \in C(w\circ\lambda)^+_v} t^{d(p)}$$

where $d(p)$ is the <u>depth</u> of p.