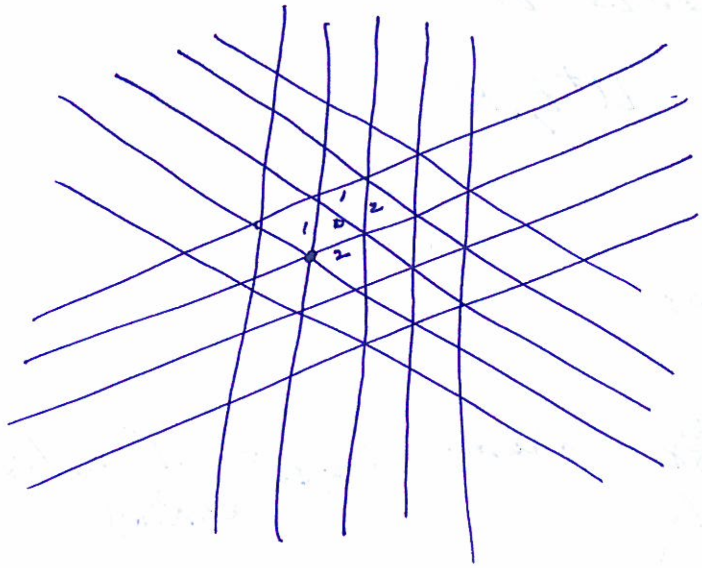
Picturing Hecke algebras and loop groups

University of Roma 1
9/29/05
Algebra + Geometry Seminar.

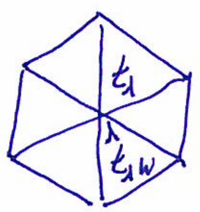
Alcoves $\,{}_1\!\triangleright^{0}_{2}\,$ tile $\mathbb{R}^n = \mathfrak{h}^*_{\mathbb{R}}$



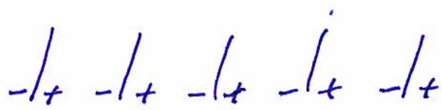
$W_{aff} = \left\{ \begin{matrix} \text{alcoves in} \\ \mathfrak{h}^*_{\mathbb{R}} \end{matrix} \right\}$ $\qquad$ $\tilde{W} = \left\{ \begin{matrix} \text{alcoves in} \\ \mathfrak{h}^*_{\mathbb{R}} \times \Omega \end{matrix} \right\}$

$\Omega = \left\{ \begin{matrix} \text{automorphisms} \\ \text{of } {}_1\!\triangleright^{0}_{2} \end{matrix} \right\}$ $\qquad$ $W = \left\{ \begin{matrix} \text{alcoves in} \\ 0 \text{ hexagon} \end{matrix} \right\}$

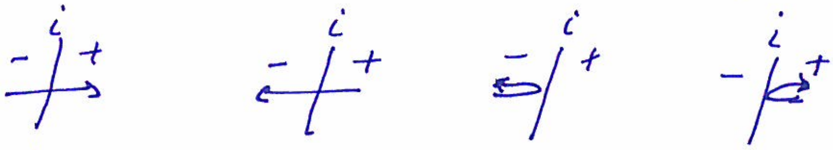
$Q = \left\{ \begin{matrix} \text{centers of} \\ \text{hexagons in } \mathfrak{h}^*_{\mathbb{R}} \end{matrix} \right\}$ $\qquad$ $P = \left\{ \begin{matrix} \text{centers of hexagons} \\ \text{in } \mathfrak{h}^*_{\mathbb{R}} \times \Omega \end{matrix} \right\}$



$\lambda \in P$, $w \in W$.

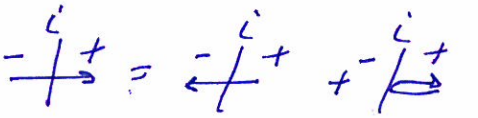
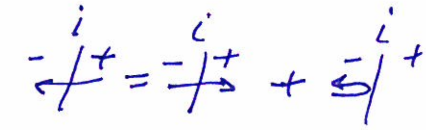
periodic orientation 

The alcove walk algebra $\tilde{\mathcal{A}}$ has generators



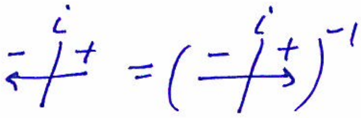
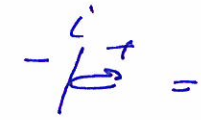
positive i-crossing, negative i-crossing, neg. i-fold, pos. i-fold

with relations

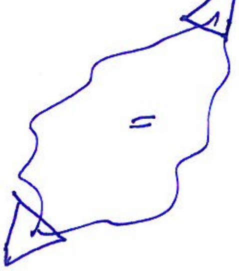
 and 

The alcove walk algebra has basis
{alcove walks}.

The affine Hecke algebra $\tilde{H}$ has additional relations

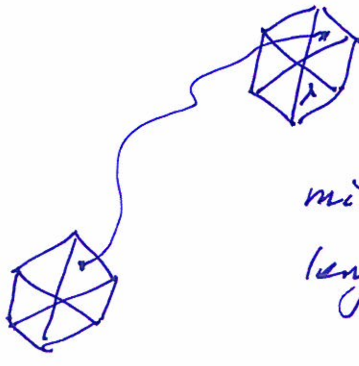
 and  $= (q-q^{-1})$

and



for nonfolded walks.

Let

$T_w^{-1} =$  and $X^\lambda =$ 

minimal length

Let

$$\mathbf{1}_0 = \frac{1}{W_0(q^{-2})} \sum_{w\in W} q^{-\ell(w)} T_w^{-1} \quad \text{so that}$$

$$\mathbf{1}_0^2 = \mathbf{1}_0 \quad \text{and} \quad T_{s_i}\mathbf{1}_0 = q\,\mathbf{1}_0 .$$

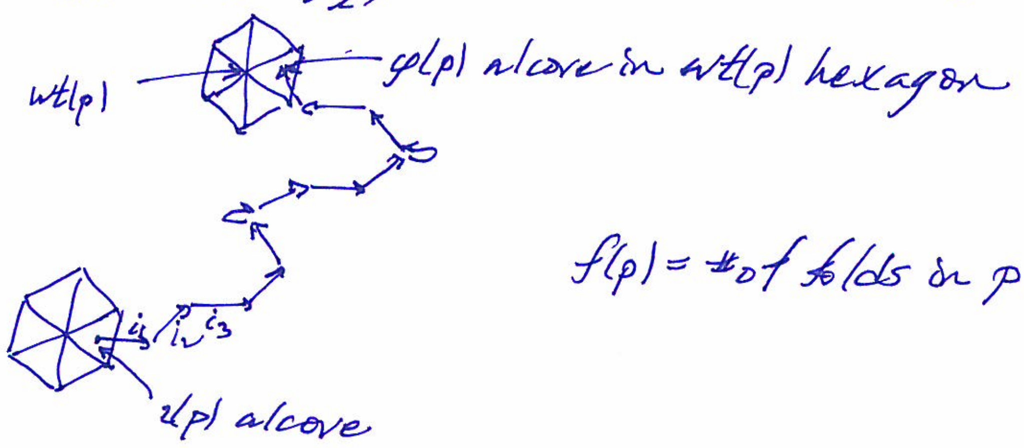
The <u>Macdonald spherical function</u> is $P_\lambda(X;q^{-2})$ given by

$$P_\lambda(X;q^{-2})\mathbf{1}_0 = \frac{W_0(q^{-2})}{W_\lambda(q^{-2})}\mathbf{1}_0 X^\lambda \mathbf{1}_0, \quad \text{for } \lambda\in P^+.$$

The <u>Weyl character</u> is $s_\lambda(X) = P_\lambda(X,q^{-2})\big|_{q^{-2}=0}$.

<u>Theorem</u> Let $\lambda\in P^+$, $t_\lambda = s_{i_1}\cdots s_{i_\ell}$ a minimal length walk to t_λ. Let

$$B(\lambda) = \left\{ \begin{array}{l}\text{positively folded walks from the 0 hexagon}\\ \text{of type } (i_1,\dots,i_\ell)\end{array}\right\}$$



$f(p) = \#$ of folds in p.

$$P_\lambda(X;q^{-2}) = \sum_{p\in B(\lambda)} q^{-(\ell(\iota(p))+\ell(\varphi(p))-f(p))}(1-q^{-2})^{f(p)} X^{wt(p)}$$

<u>Chevalley groups $G(\mathbb{F})$</u>

Let $\mathbb{F}$ be a $\mathfrak{p}$-adic field

$$\begin{array}{l}\mathbb{F}\\ \cup\!\shortmid\\ \mathcal{O} \twoheadrightarrow k\end{array} \qquad \text{where } k = \mathcal{O}/\mathfrak{p}$$

<u>Example:</u> $\mathbb{F} = \mathbb{C}((t)) = \{t^k g \mid k \in \mathbb{Z}, g \in \mathcal{O}^*\}$

$$\mathcal{O} = \mathbb{C}[[t]] = \{ f = \sum_{i\in\mathbb{Z}_{\geq 0}} f_i t^i \mid f_i \in \mathbb{C}\}$$

$$k = \mathbb{C}$$

since

$$\mathfrak{p} = (t) = t\,\mathbb{C}[[t]] \quad \text{and} \quad \mathcal{O}^* = \{f \in \mathcal{O} \mid f_0 \neq 0\}.$$

The group $G(\mathbb{F})$ has a triangular decomposition

$$G = U_{\mathbb{F}}^- H_{\mathbb{F}} U_{\mathbb{F}}^+ .$$

where

$$U_{\mathbb{F}}^+ = \{x_{\beta_1}(f_1) \cdots x_{\beta_N}(f_N) \mid f_i \in \mathbb{F}\}$$

$$H_{\mathbb{F}} = \{h_{\varepsilon_1}(g_1) \cdots h_{\varepsilon_n}(g_n) \mid g_i \in \mathbb{F}^*\}$$

$$U_{\mathbb{F}}^- = \{x_{-\beta_N}(f_N) \cdots x_{-\beta_1}(f_1) \mid f_i \in \mathbb{F}\}.$$

where

$$R^+ = \{\beta_1, \ldots, \beta_N\} \quad \text{and} \quad P = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$$

Then

$$\begin{array}{ccc}
G = G(\mathbb{F}) & & \\
\cup\!| \quad\ \ \cup\!| & & \\
K = G(\mathcal{O}) & \twoheadrightarrow & G(k) \\
\cup\!| & & \cup\!| \\
I & \longrightarrow & B(k)
\end{array}$$

with

$$K = U_{\mathcal{O}}^{-} H_{\mathcal{O}} U_{\mathcal{O}}^{+}, \quad I = U_{\mathbb{F}}^{-} H_{\mathcal{O}} U_{\mathcal{O}}^{+}, \quad B = H_{\mathbb{F}} U_{\mathbb{F}}^{+}$$

The loop Grassmanian is G/K

The loop flag variety is G/I.

(1) Bruhat decomposition: $G = \bigsqcup_{w \in W} BwB$

(2) Iwahori decomposition: $G = \bigsqcup_{w \in \tilde{W}} IwI$

(3) Iwasawa decomposition: $G = \bigsqcup_{w \in \tilde{W}} BwI$

(4) Cartan decomposition: $K = \bigsqcup_{w \in W} IwI$

which imply

$$G = \bigsqcup_{\lambda \in P^+} K t_\lambda K, \quad G = \bigsqcup_{\mu \in P} U^+ t_\mu K, \quad G = BK.$$

The Mirković–Vilonen cycles of type λ are the irreducible components of

$$K t_\lambda K \cap U^+ t_\mu K.$$

Coset representatives:

$$IwI = \{ x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) wI \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

and

$$IwI \cap U^+ vI = \bigsqcup_{p \in B(w)_v} \left\{ x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) wI \;\middle|\; \begin{array}{l} c_i \in \mathbb{F} \text{ if } p_i = \\ c_i \in \mathbb{F}^* \text{ if } p_i = \\ c_i \neq 0 \text{ if } p_i = \end{array} \right\}$$

i.e.