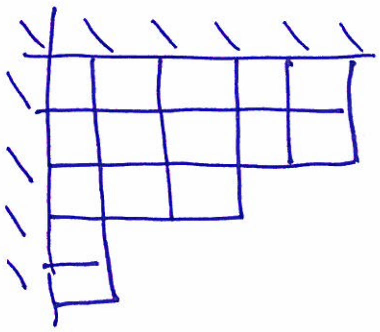
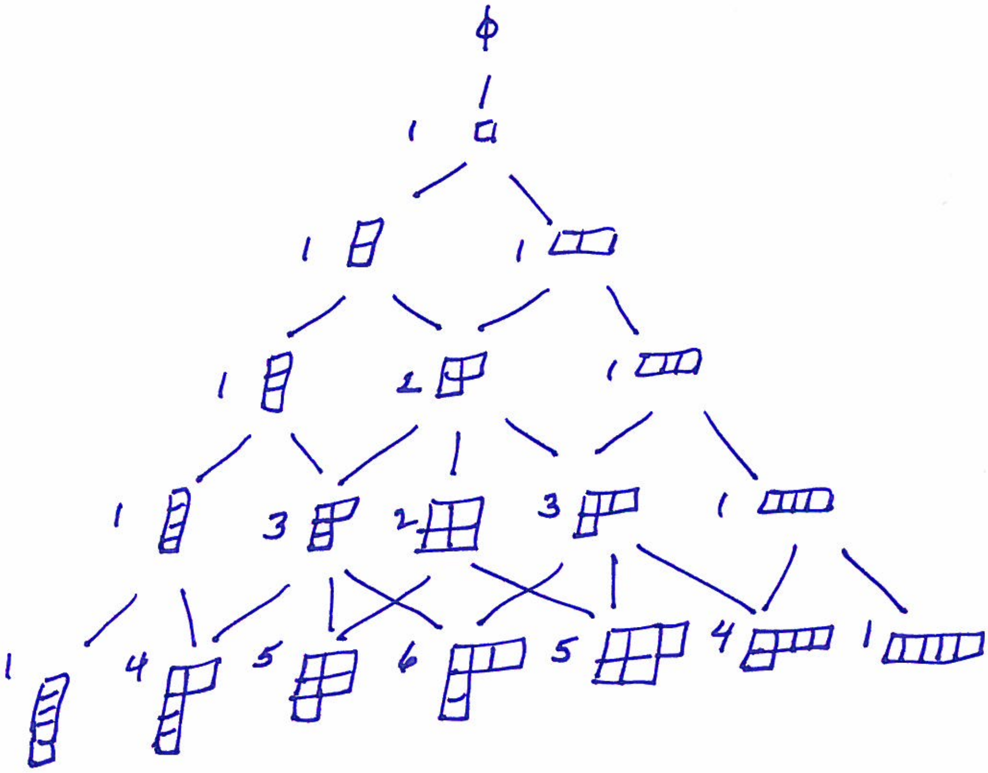
Generalizing Pascal's Triangle, Melbourne Univ. Math. Soc. Lunchtime Seminar 09.19.2008

A partition is a collection of boxes in a corner.





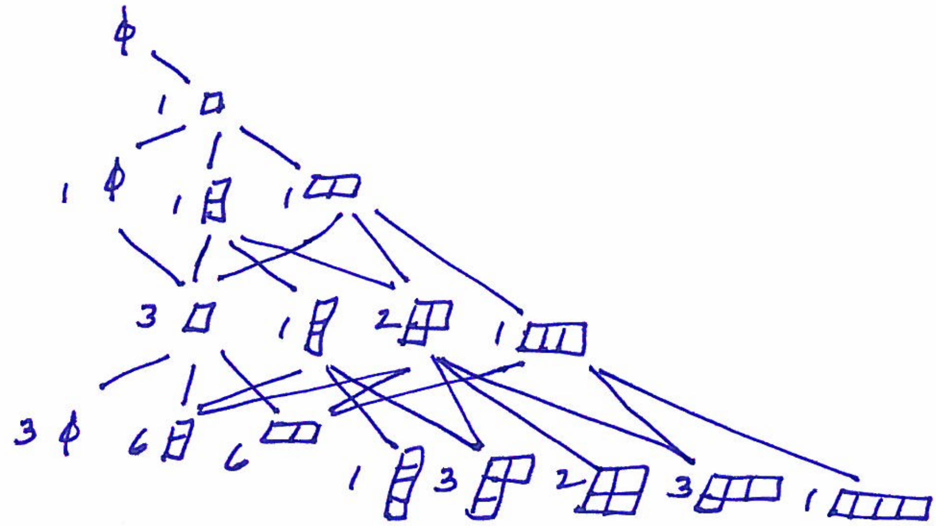
$$1^2 = 1$$

$$1^2 + 1^2 = 2$$

$$1^2 + 2^2 + 1^2 = 6$$

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24$$

$$1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 120$$



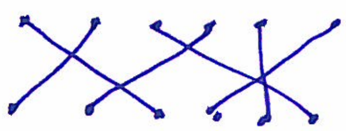
$1^2 = 1$

$1^2 + 1^2 + 1^2 = 3$

$3^2 + 1^2 + 2^2 + 1^2 = 15$

$3^2 + 6^2 + 6^2 + 1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 105$

<u>Graphs</u>: k vertices on top
k vertices on bottom
edges top to bottom
each vertex in one edge

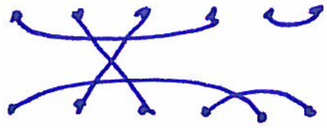


These are also called <u>bijections</u> or <u>permutations</u> or <u>automorphisms</u>.

$S_k = \{ \text{bijections } w : \{1, \ldots, k\} \to \{\cdot, \cdot, \ldots, \cdot\} \}$

$\mathrm{Card}(S_k) = k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1.$

<u>Graphs:</u> k vertices on top
k vertices on bottom
each vertex in one edge

 $\in B_6$

$$\mathrm{Card}(B_6) = 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1.$$

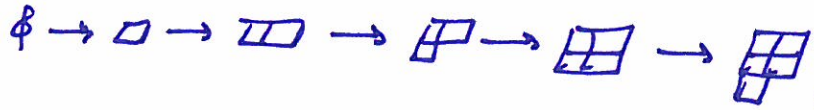
<u>Why is $1^2+3^2+2^2+3^2+1^2=24$?</u>

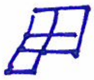
Let
$$\dim(S_k^\lambda) = (\#\text{ of paths } \phi \to \lambda).$$

Why is
$$\sum_{\lambda \vdash k} \dim(S_k^\lambda)^2 = k!$$
where the sum is over partitions with k boxes.

A <u>standard</u> <u>tableau</u> is a filling T of the boxes of λ with $1, 2, \ldots, k$ such that

(a) the rows of T increase (left to right)
(b) the columns of T increase (top to bottom).

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array}$$ is  ,

a path from ϕ to . So

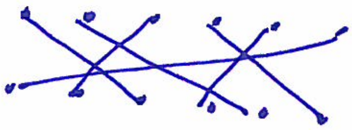
$$\dim(S_k^\lambda) = \left(\begin{array}{c}\text{\# of standard tableaux of} \\ \text{shape } \lambda\end{array}\right)$$

Why is

$$\sum_{\lambda \vdash k} \dim(S_k^\lambda)^2 = k! \ ?$$

Why is

$$k! = \sum_{\lambda \vdash k} \dim(S_k^\lambda)^2.$$

 $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 6 & 4 & 1 \end{pmatrix}$

We need a <u>bijection</u>

$$S_k \longrightarrow \coprod_{\lambda \vdash k} (\mathcal{T}^\lambda \times \mathcal{T}^\lambda)$$

permutations   pairs of standard tableaux

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 6 & 4 & 1 \end{pmatrix}$$

$$\emptyset \qquad\qquad \emptyset$$

$$\begin{array}{|l} \hline 3 \end{array}$$

$$\begin{array}{|l} \hline 3\ 5 \end{array} \qquad \begin{array}{|l} \hline 1\ 2 \end{array}$$

$$\begin{array}{|l} \hline 2\ 5 \\ 3 \end{array} \qquad \begin{array}{|l} \hline 1\ 2 \\ 3 \end{array}$$

$$\begin{array}{|l} \hline 2\ 5\ 6 \\ 3 \end{array} \qquad \begin{array}{|l} \hline 1\ 2\ 4 \\ 3 \end{array}$$

$$\begin{array}{|l} \hline 2\ 4\ 6 \\ 3\ 5 \end{array} \qquad \begin{array}{|l} \hline 1\ 2\ 4 \\ 3\ 5 \end{array}$$

$$\begin{array}{|l} \hline 1\ 4\ 6 \\ 2\ 5 \\ 3 \end{array} \qquad \begin{array}{|l} \hline 1\ 2\ 4 \\ 3\ 5 \\ 6 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 6 & 4 & 1 \end{pmatrix} \text{ “=” } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 2 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix} \text{ "=" } \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & \cdot \\ 3 & 3 & 5 & 2 & 2 & 2 & 3 & 6 & 6 & \end{pmatrix}$$

$$\begin{pmatrix} \cdot & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 \\ & 2 & 2 & 4 & 4 & 4 & 5 & 5 & 1 & 5 \end{pmatrix}$$

and this "equals"

$$\begin{array}{cc}
\phi & \phi \\
\vdots & \vdots \\
\begin{array}{|l} \hline 3 \end{array} & \begin{array}{|l} \hline 1 \end{array} \\
\vdots & \vdots \\
\begin{array}{|ll} \hline 3 & 3 \end{array} & \begin{array}{|ll} \hline 1 & 1 \end{array} \\
\vdots & \vdots \\
\begin{array}{|lll} \hline 3 & 3 & 5 \end{array} & \begin{array}{|lll} \hline 1 & 1 & 2 \end{array} \\
\vdots & \vdots \\
\begin{array}{|lll} \hline 2 & 3 & 5 \\ 3 & & \end{array} & \begin{array}{|lll} \hline 1 & 1 & 2 \\ 3 & & \end{array} \\
\vdots & \vdots \\
\begin{array}{|lll} \hline 2 & 2 & 5 \\ 3 & 3 & \end{array} & \begin{array}{|lll} \hline 1 & 1 & 2 \\ 3 & 3 & \end{array}
\end{array}$$

etc.

So we have a bijection

$$\left\{ \begin{matrix} \text{matrices} \\ \text{with entries} \\ \text{in } \mathbb{Z}_{\geq 0} \end{matrix} \right\} \longrightarrow \bigsqcup_{\lambda} C^{\lambda} \times C^{\lambda}$$

where

$$C^\lambda = \left\{ \begin{array}{c} \text{column strict tableaux of} \\ \text{shape } \lambda \end{array} \right\}$$

and a <u>column</u> <u>strict</u> <u>tableau</u> <u>of</u> <u>shape</u> <u>λ</u> is a filling of the boxes of λ such that

(a) rows are weakly increasing (left to right)

(b) columns are strictly increasing (top to bottom).

Now

$$\sum_{\substack{\text{matrices} \\ A}} \left(\prod_{i,j} (x_i y_j)^{a_{ij}} \right) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

since

$$\frac{1}{1-x_i y_j} = 1 + x_i y_j + (x_i y_j)^2 + (x_i y_j)^3 + \cdots$$

i.e.

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\substack{\text{matrices } A \\ \text{entries in } \mathbb{Z}_{\geq 0}}} \left(\prod_{\substack{\text{entries} \\ a_{ij} \text{ in } A}} (x_i y_j)^{a_{ij}} \right)$$

The <u>Schur</u> <u>function</u> is

$$s_\lambda = \sum_{\substack{\text{column strict} \\ \text{tableaux } T \text{ of} \\ \text{shape } \lambda}} x_1^{\#1\text{'s in } T} \cdot x_2^{\#2\text{'s in } T} \cdots$$

Theorem

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda} s_\lambda(x_1, x_2, \dots)\, s_\lambda(y_1, y_2, \dots)$$

This theorem contains

$$k! = \sum_{\lambda} \dim(S^\lambda)^2$$

by looking only at matrices with 0,1 entries.