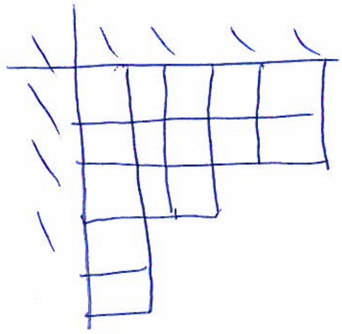
Beads on Runners, Colloquium, Monash University

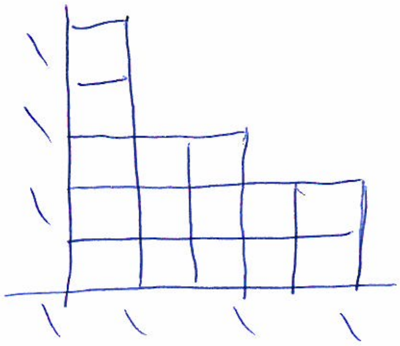
6 November 2008

Partitions

A partition is a collection of boxes in a corner.



English

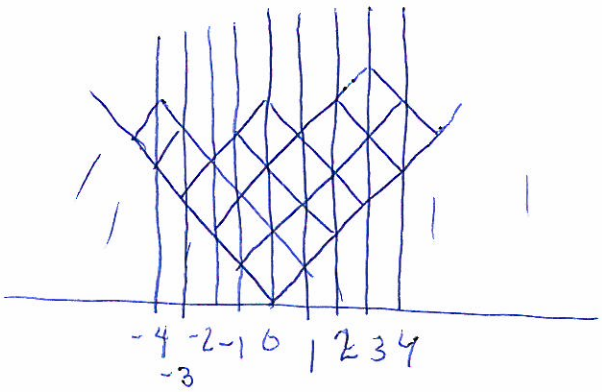


French

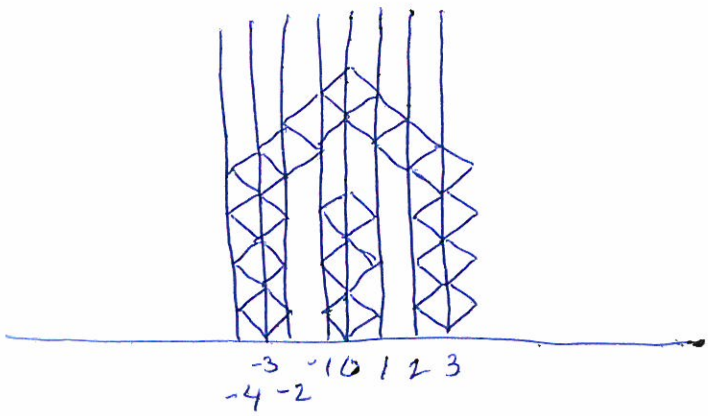
American: $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \cdots \geq \lambda_\ell$

λ_i = # of boxes in row i.

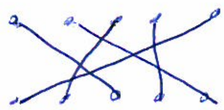
Russian



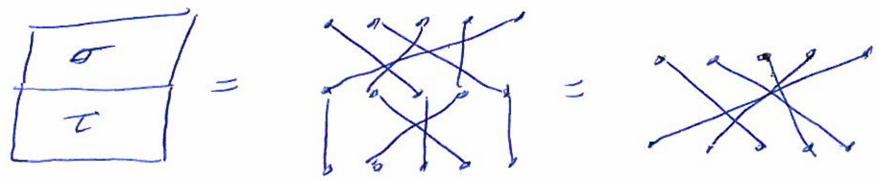
Fun



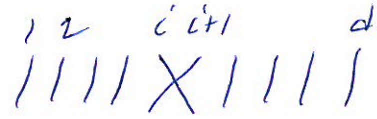
The symmetric group S_d

 $\in S_5$ and $\mathrm{Card}(S_5) = 5!$

with product

$\sigma\tau =$ 

or let

$s_i =$  for $i = 1, \ldots, d-1$

Theorem The symmetric group S_d is presented by generators $s_1, \ldots, s_{d-1}$ with relations

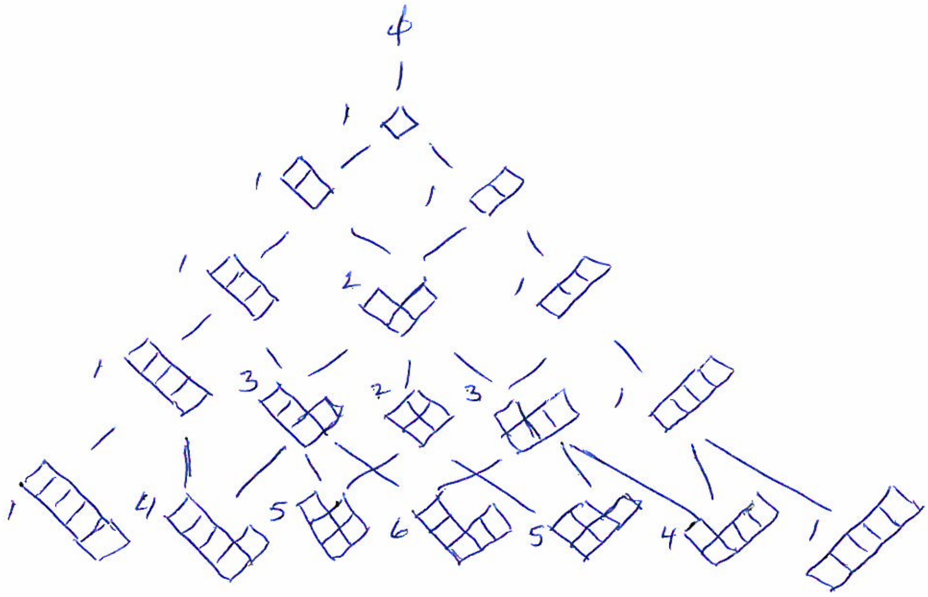
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{and} \quad s_i s_j = s_j s_i \quad \text{for } j \neq i, i\pm 1$$

$$s_i^2 = 1.$$

A representation of S_d is an S_d-module, i.e. a vector space V with an action of S_d by linear transformations.

An irreducible representation has no submodules.

The Bratelli diagram $\hat{S}$



Let

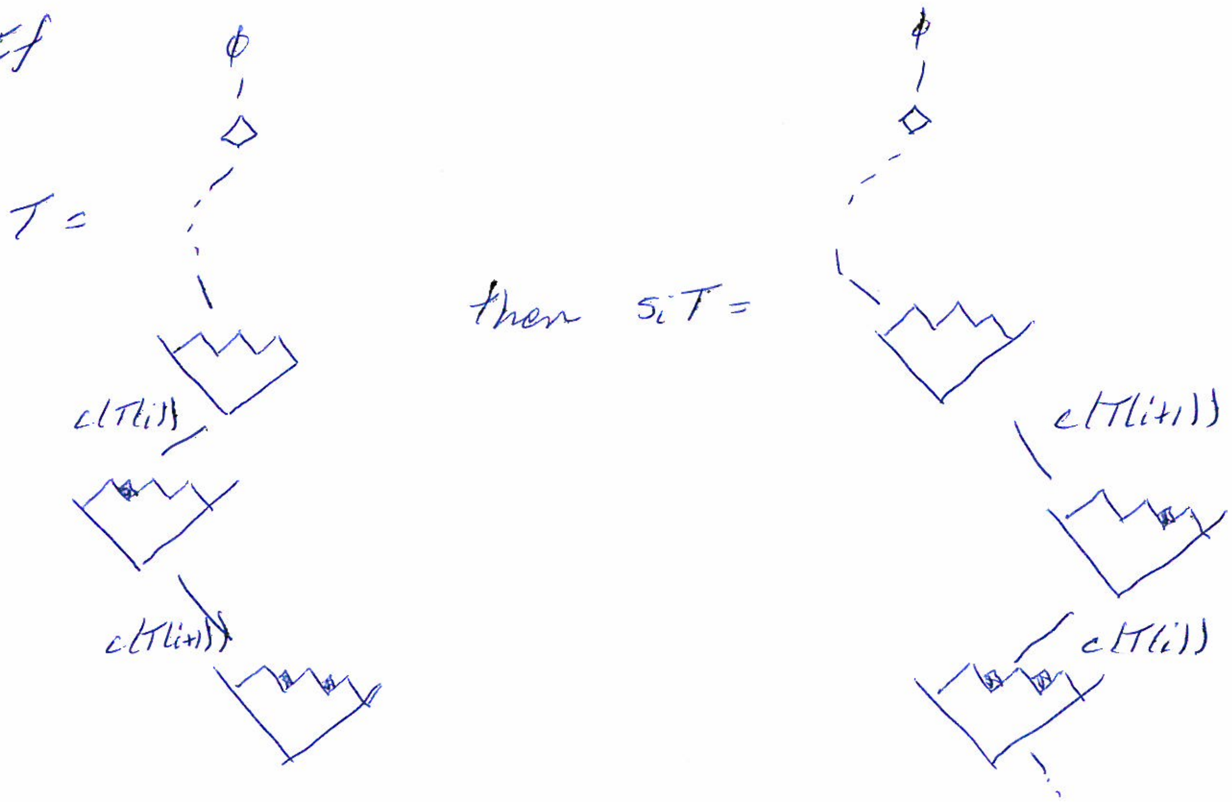
$$\hat{S}_d = \{ \lambda \mid \lambda \text{ is on level } d \}$$

$$\hat{S}^\lambda = \{\text{paths to } \lambda \text{ on } \hat{S}\}$$

$$= \{ T = (\phi \xrightarrow{c(T(1))} \diamond \xrightarrow{c(T(2))} \cdots \xrightarrow{c(T(d))} \lambda) \}$$

where $c(T(i))$ = color of bead added at level i.

If



<u>Theorem</u> (Young) For $\lambda \in \hat{S}_d$ let

$$V^\lambda = \text{span}\{ v_T \mid T \in \hat{S}^\lambda \}$$ so that

the symbols v_T form a basis of V^λ.

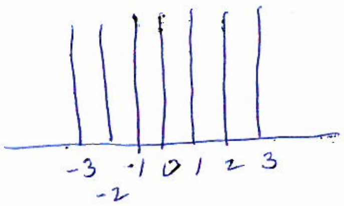
Define an action of S_d on V^λ by

$$s_i v_T = \left(\frac{1}{c(T(i)) - c(T(i+1))} \right) v_T + \left(1 + \frac{1}{c(T(i)) - c(T(i+1))} \right) v_{s_i T}$$

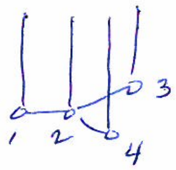
where $v_{s_i T} = 0$ if $s_i T$ does not exist.

These are the irreducible representations of S_d.

<u>Generalization</u> (jt. with Kleshchev).

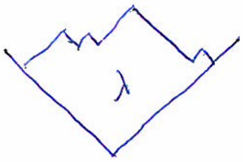


gets replaced by



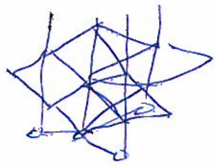
a Dynkin diagram

S_d the symmetric group gets replaced by R_α a Khovanov-Lauda algebra



partitions

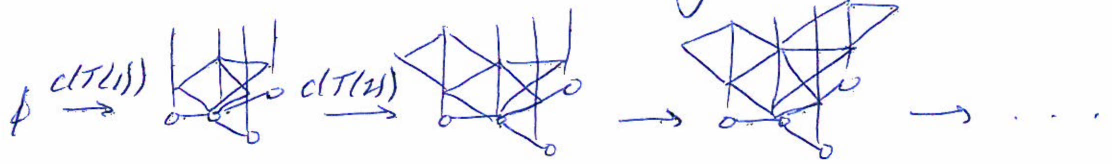
get replaced by



$= \lambda$.

skew shapes

A path in the Bratelli diagram $\hat{R}$

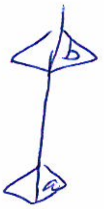


$$c(T(1)) = 2, \quad c(T(2)) = 1, \quad c(T(3)) = 3, \quad \dots$$

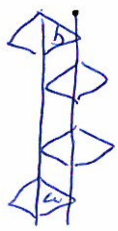
If $T = \phi \xrightarrow{c(T(1))} \Diamond \to \cdots \to \gamma \xrightarrow{c(T(i))} \nu \xrightarrow{c(T(i+1))} \mu \to \cdots \to \lambda$

then $s_i T = \phi \xrightarrow{c(T(1))} \Diamond \to \cdots \to \gamma \xrightarrow{c(T(i+1))} \tau \xrightarrow{c(T(i))} \mu \to \cdots \to \lambda.$

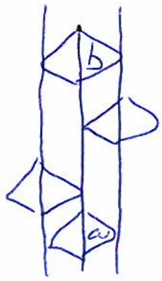
A skew shape is λ such that any two beads on the same runner are separated by two beads, i.e. if



then



or



Theorem (Kleshchev-Ram) For a skew shape λ let

$$V^\lambda = \mathrm{span}\{ v_T \mid T = (\phi \to \Diamond \to \cdots \to \lambda)\}$$

and define an action of R_α on V^λ by

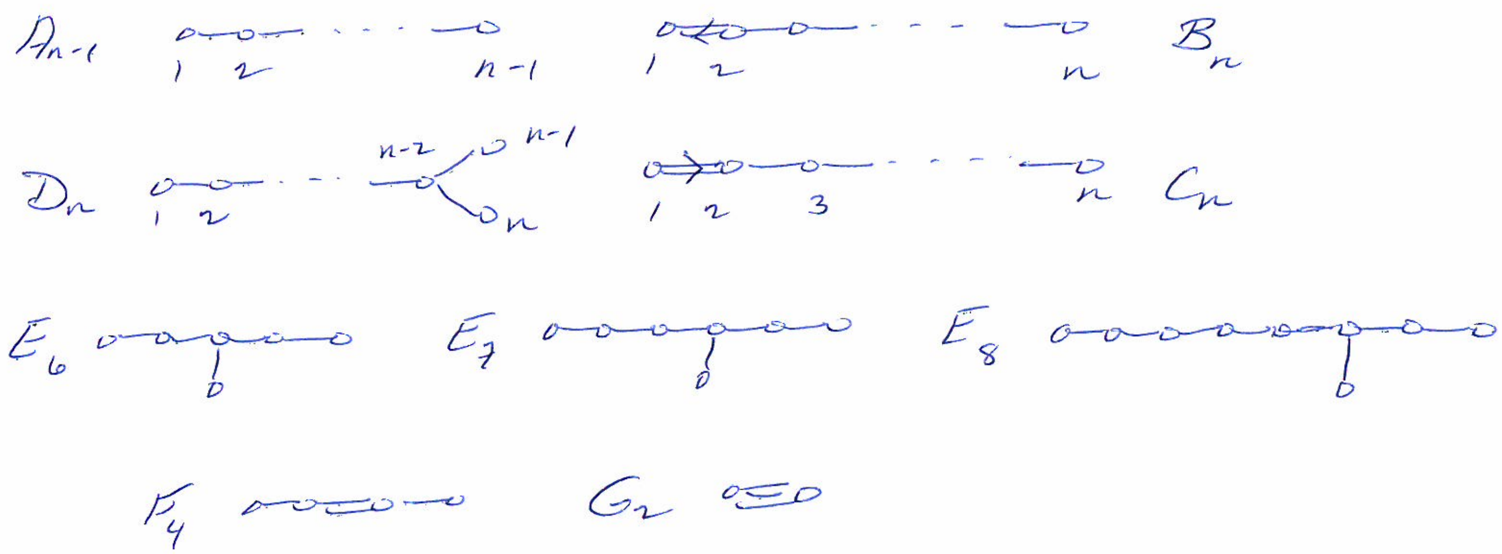
$$e_\gamma v_T = \begin{cases} v_T, & \text{if } \gamma = (c(T(1)), \dots, c(T(d))), \\ 0, & \text{otherwise,} \end{cases}$$

$$\psi_i v_T = \begin{cases} v_{s_i T}, & \text{if } s_i T \text{ exists,} \\ 0, & \text{otherwise} \end{cases} \quad \text{and } y_r v_T = 0$$

for $i = 1, \dots, d-1$, and $r = 1, \dots, d$. These are the irreducible homogeneous R_α-modules.

<u>Theorem</u> (Khovanov–Lauda) The category of representations of R_α is a categorification of quantum groups.

The <u>Dynkin</u> <u>diagrams</u> are



<u>Theorem</u>

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Dynkin}\\ \text{diagrams}\end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \left\{\begin{matrix}\text{compact connected}\\ \text{simply connected}\\ \text{Lie groups}\end{matrix}\right\} \arrow[d, leftrightarrow] \\
\left\{\begin{matrix}\text{enveloping algebras}\\ U\mathfrak{g} \text{ of complex}\\ \text{ss Lie algebras}\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{complex semisimple}\\ \text{Lie algebras } \mathfrak{g}\end{matrix}\right\}
\end{tikzcd}$$

<u>Example</u> A_1

$$\begin{tikzcd}
\circ \arrow[r, mapsto] & SU_2 \arrow[d] \\
U\mathfrak{sl}_2 & \mathfrak{sl}_2 \arrow[l, mapsto]
\end{tikzcd}$$

where $U\mathfrak{sl}_2$ is given by generators x, y, h with relations $xy - yx = h$, $hx - xh = 2x$ and $hy - yh = -2y$

Deformations

classical	deformed
n	$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)}$
$\binom{n}{k}$	$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ with $[n]! = [n][n-1]\cdots[2][1]$
$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^k$	$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^? x^k y^{n-k}$
if $yx = xy$	if $yx = q^2 xy$.
S_d the symmetric group	H_d the Iwahori-Hecke algebra
$U\mathfrak{g}$	$U_q\mathfrak{g}$ - the quantum group.

The Iwahori-Hecke algebra H_d is given by generators $T_1, \dots, T_{d-1}$ and relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i \text{ if } j \neq i, i\pm 1,$$

$$T_i^2 = (q - q^{-1}) T_i + 1.$$

The quantum group $U_q\mathfrak{sl}_2$ is given by generators E, F, K with relations $KK^{-1} = 1$,

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F.$$

Categorifications

after		before
Vector spaces	$\xrightarrow{\dim}$	$\mathbb{Z}_{\geq 0}$
V	$\longmapsto$	n
graded vector space	$\xrightarrow{\text{qdim}}$	$\mathbb{Z}_{\geq 0}[q^{\pm 1}]$
$V = \bigoplus_{i \in \mathbb{Z}} V_i$	$\longmapsto$	$\sum_{i \in \mathbb{Z}} \dim(V_i) q^i$.
Representations of symmetric groups		$\hat{S}$
Representations of Khovanov-Lauda algebras R_α	$\xrightarrow{\text{character}}$	$U_q\mathfrak{g}$

the algebraic object on the right is the "shadow" of the category on the left.