The mysteries of symmetry, Colloquium ANU 20 Nov. 2008

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A formula for Macdonald polynomials $P_\mu(q^{\frac12}, t^{\frac12})$

$t^{\frac12}$ and $q^{\frac12}$ are constants, $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$,

$$P_\mu = \sum_{w \in W_0} \sum_{\substack{\text{foldings } p \\ \text{of } w p_\mu}} t^{\frac12 \ell(\iota(p))} \Big(\prod_{k \in F^+(p)} f_k^+\Big)\Big(\prod_{k \in F^-(p)} f_k^-\Big) X^{\mathrm{wt}(p)} t^{\frac12 \ell(\varphi(p))},$$

$W_0 = \{$alcoves in the 0-octagon$\}$,

p_μ, a minimal length walk to the μ-octagon,

$t^{\frac12 \ell(\iota(p))} = t^{\ell/2}$ if p begins at $s_{i_1} \cdots s_{i_\ell}$

$X^{\mathrm{wt}(p)} t^{\frac12 \ell(\varphi(p))} = X^{\nu} t^{r/2}$ if p ends at $X^{\nu} s_{j_1} \cdots s_{j_\ell}$

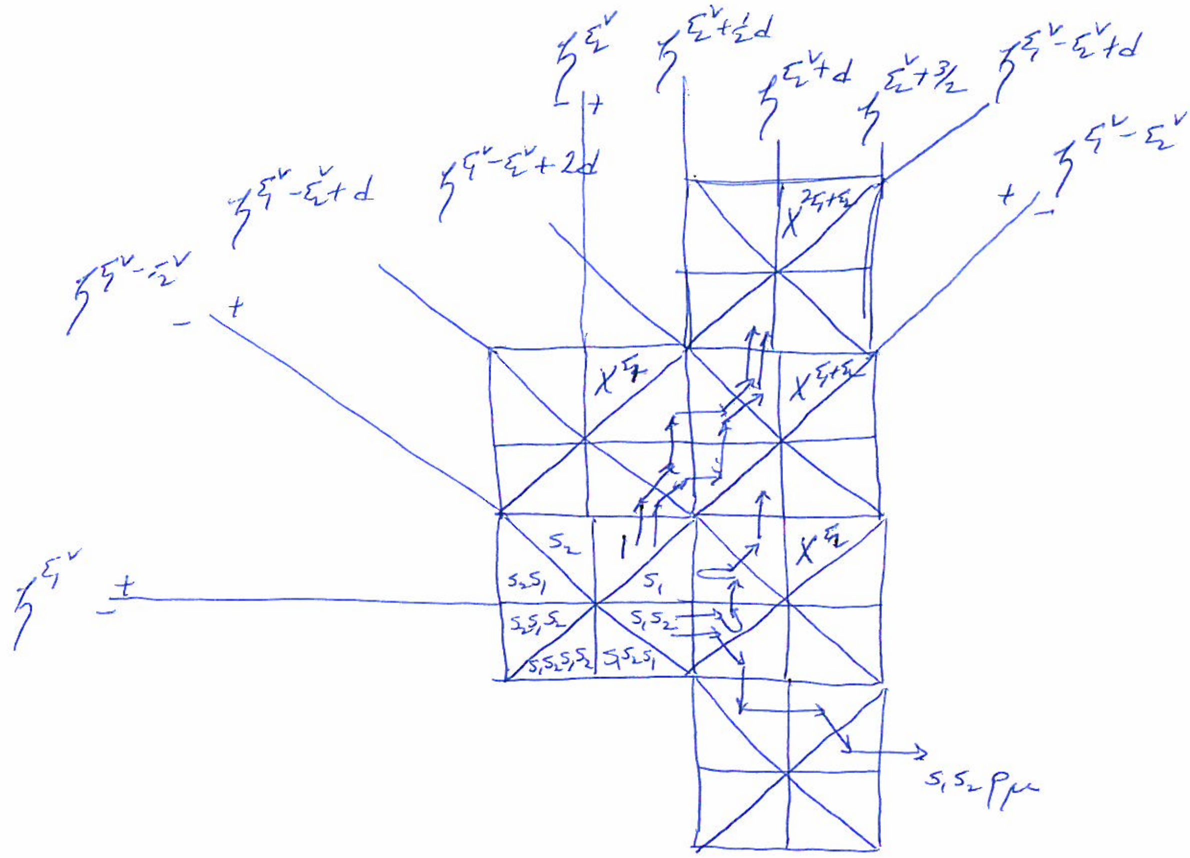
$F^+(p) = \{ k \mid k^{\text{th}}$ step of p is a positive fold $\ - \!\mid\!\!\rightarrow + \}$

$F^-(p) = \{ k \mid k^{\text{th}}$ step of p is a negative fold $\ - \leftarrow\!\!\mid + \}$

$$f_k^+ = \frac{t^{-\frac12}(1-t) + t^{-\frac12}(1-t)\, q^{j_k} y^{\beta_k^\vee}}{1 - q^{2j_k} y^{2\beta_k^\vee}}, \qquad y^{\varepsilon_i^\vee} = t^{n-i+1},$$

where

$\beta_m^\vee + j_m d, \ldots, \beta_2^\vee + j_2 d, \beta_1^\vee + j_1 d$ are the hyperplanes crossed by $\mathrm{rev}(p_\mu)$.



$\mathfrak{h}^*_{\mathbb{Z}} = \sum_{i=1}^{2} \mathbb{Z}\varepsilon_i$, $\quad X^\mu X^\nu = X^{\mu+\nu}$ for $\mu, \nu \in \mathfrak{h}^*_{\mathbb{Z}}$.

$\mu = 2\varepsilon_1+\varepsilon_2$. $\quad X^{2\varepsilon_1+\varepsilon_2} = (X^{\varepsilon_1})^2 X^{\varepsilon_2} = X_1^2 X_2$ if $X_i = X^{\varepsilon_i}$.

$(\mathfrak{h}^*_{\mathbb{Z}})^+ = \{\mu \mid X^\mu$ is in the dominant chamber$\}$

$p_\mu = s_0 s_1 s_2 s_0 s_1 s_0$

p begins at $s_1 s_2$ and ends at $X^{\varepsilon_1+\varepsilon_2}_{s_1 s_2 s_1 s_2}$

$F^+(p) = \{2, 4\} \qquad F^-(p) = \emptyset$

$$f_2^+ = \frac{t^{-\frac12}(1-t) + t^{-\frac12}(1-t)\, q^2 y^{\varepsilon_1^\vee+\varepsilon_2^\vee}}{1-q^4 y^{2(\varepsilon_1^\vee+\varepsilon_2^\vee)}}, \qquad y^{\varepsilon_1^\vee} = t^2, \quad y^{\varepsilon_2^\vee} = t^1.$$

The p term of P_μ is $t^{\frac12\cdot 2} f_2^+ f_4^+ X^{\varepsilon_1+\varepsilon_2} t^{\frac12\cdot 4}$.

The double affine Weyl group $\tilde{W}$

W_0, a finite reflection group, acts on

$$\left.\begin{matrix}\mathfrak{h}_{\mathbb{Z}}^* \\ \mathfrak{h}_{\mathbb{Z}}\end{matrix}\right\} \text{ dual lattices. } \langle\,,\,\rangle : \mathfrak{h}_{\mathbb{Z}}^* \times \mathfrak{h}_{\mathbb{Z}} \longrightarrow \mathbb{Z}$$

so that $\langle w\mu, \lambda^\vee\rangle = \langle \mu, w^{-1}\lambda^\vee\rangle$, for $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $w \in W_0$

$$\tilde{W} = \{ q^k X^\mu w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0 \}$$

with

$$q \in Z(\tilde{W}), \quad X^\mu X^\nu = X^{\mu+\nu}, \quad Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}$$

$$X^\mu Y^{\lambda^\vee} = q^{\langle \mu, \lambda^\vee\rangle} Y^{\lambda^\vee} X^\mu$$

$$w X^\mu = X^{w\mu} w \quad \text{and} \quad w Y^{\lambda^\vee} = Y^{w\lambda^\vee} w.$$

Example $W_0 = S_n$ acts on $\mathfrak{h}_{\mathbb{Z}}^* = \sum_{i=1}^n \mathbb{Z}\varepsilon_i$, $\mathfrak{h}_{\mathbb{Z}} = \sum_{i=1}^n \mathbb{Z}\varepsilon_i^\vee$

$$\mathrm{span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\} = \mathbb{C}[X^{\pm\varepsilon_1}, \ldots, X^{\pm\varepsilon_n}] = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$

$$\mathrm{span}\{Y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} = \mathbb{C}[Y^{\pm\varepsilon_1^\vee}, \ldots, Y^{\pm\varepsilon_n^\vee}] = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$$

$\langle \varepsilon_i, \varepsilon_j^\vee\rangle = \delta_{ij}$ gives $X^{\varepsilon_i} Y^{\varepsilon_j^\vee} = q^{\delta_{ij}} Y^{\varepsilon_j^\vee} X^{\varepsilon_i}$

analogous to $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$

with $[x_i, \frac{\partial}{\partial x_j}] = x_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} x_i = \delta_{ij}$.

The double affine Hecke algebra $\tilde{H}$

$$\tilde{H} = \mathrm{span}\{q^k X^\mu T_w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0\}$$

is a deformation of $\tilde{W}$ ($\tilde{W}$ is $\tilde{H}$ at $t^{\frac{1}{2}}=1$):

$$T_{s_i}^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_{s_i} + 1 \quad \text{in } \tilde{H}.$$

The group

$$W^\vee = \{w y^{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} \text{ acts on}$$

$$X = \{q^k X^\mu \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$$

by

$$z \circ X^\mu = z X^\mu z^{-1}, \quad \text{for } z \in W^\vee, \mu \in \mathfrak{h}_{\mathbb{Z}}^*.$$

This action deforms to an action of

$$H^\vee = \mathrm{span}\{T_w Y^{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} \text{ on}$$

$$\mathbb{C}[X] = \mathrm{span}\{q^k X^\mu \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}\}$$

The Macdonald polynomials are the simultaneous eigenvectors of the $Y^{\lambda^\vee}$ on $\mathbb{C}[X]$.

Note: $\mathbb{C}[X]$ is a fake:

$$\mathbb{C}[X] \cong \tilde{H}\mathbb{1} \quad \text{where}$$

$$Y^{\varepsilon_i^\vee}\mathbb{1} = t^{n-i+1}\mathbb{1} \quad \text{and} \quad T_w\mathbb{1} = q^{\frac{1}{2}\ell}$$

if $w = s_{i_1}\cdots s_{i_\ell}$ is a minimal length path to w.

<u>Loop Groups</u> $G_0^\vee = GL_n$

$$\mathbb{C}((q)) = \{a_{-\ell}q^{-\ell} + a_{-\ell+1}q^{-\ell+1} + \cdots \mid a_i \in \mathbb{C},\ \ell \in \mathbb{Z}\}$$
$$\cup\!|$$
$$\mathbb{C}[[q]] = \{a_0 + a_1 q + a_2 q^2 + \cdots \mid a_i \in \mathbb{C}\}$$

$$\begin{array}{ccccc}
G & = & G_0^\vee(\mathbb{C}((q))) & & \\
\cup\!| & & \cup\!| & & \\
K & = & G_0^\vee(\mathbb{C}[[q]]) & \xrightarrow{q=0} & G_0^\vee(\mathbb{C}) \\
\cup\!| & & \cup\!| & & \cup\!| \\
I & = & \Phi^{-1}(B) & \longrightarrow & B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}
\end{array}$$

G/K is the <u>loop</u> <u>Grassmannian</u>

G/I is the <u>affine</u> <u>flag</u> <u>variety</u>

Let

$$U^- = \left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\right\}$$

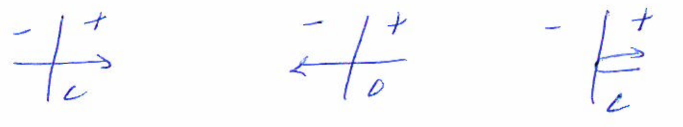
$$G_0 = \bigsqcup_{w \in W} IwI \quad \text{and} \quad G = \bigsqcup_{v \in W} U^- wI$$

where $W = \{X^\mu w \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*,\ w \in W_0\}$.

Theorem (Parkinson-Ram-Schwer)

$$IwI \cap U^- vI \longleftrightarrow \left\{ \begin{array}{l} \text{labeled foldings } p \text{ of } p_w \\ \text{that end in } v \text{ and have} \\ \text{only positive folds} \end{array} \right\}$$

where p_w is a minimal length walk to w, and the legal labels are



$c \in \mathbb{C}$ $\qquad\qquad\qquad$ $c \in \mathbb{C}^*$

Replace $\mathbb{C}$ by $\mathbb{F}_t$, the finite field with t elements.

G acts on

$$C(G/I) = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(gb) = f(g) \text{ for} \\ \text{all } b \in I \end{array} \right\}$$

and

$$\mathrm{End}_G(C(G/I)) \cong H^\vee = \mathrm{span}\{ X^\mu T_w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}},\ w \in W_0 \}$$

$$\mathrm{End}_G(C(G/K)) \cong \mathbb{1}_0 H^\vee \mathbb{1}_0$$

where $\mathbb{1}_0 \in \mathrm{span}\{T_w \mid w \in W_0\}$ such that

$$T_w \mathbb{1}_0 = t^{\frac{1}{2}\ell} \mathbb{1}_0 \quad \text{if} \quad w = s_{i_1} \cdots s_{i_\ell}$$

is a minimal length path to w.