The affine Hecke algebra H

Generators: $X^{\varepsilon_1}, T_1, \ldots, T_{d-1}$

Relations: $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $\quad T_i T_j = T_j T_i$ if $j \neq i \pm 1$

$T_1 X^{\varepsilon_1} T_1 X^{\varepsilon_1} = X^{\varepsilon_1} T_1 X^{\varepsilon_1} T_1$ $\quad T_i X^{\varepsilon_1} = X^{\varepsilon_1} T_i$ if $i \neq 1$

Murphy elements X^{ε_i}

Define

$$X^{\varepsilon_i} = T_{i-1} \cdots T_2 T_1 X^{\varepsilon_1} T_1 T_2 \cdots T_{i-1}$$

for $i = 1, \ldots, d$. Then

$$X^{\varepsilon_i} X^{\varepsilon_j} = X^{\varepsilon_j} X^{\varepsilon_i}, \qquad \text{for } 1 \leq i, j \leq d,$$

and

$$\mathbb{C}[X] = \mathbb{C}[X^{\pm\varepsilon_1}, \ldots, X^{\pm\varepsilon_d}]$$

is a commutative subalgebra of H

Calibrated modules

An irreducible finite dimensional H-module M is ***integrally calibrated*** if

$$M = \bigoplus_{\gamma \in \mathbb{Z}^d} M_\gamma$$

where

$$M_\gamma = \{ m \in M \mid X^{\varepsilon_i} m = t^{\gamma_i/2} m \text{ for } i = 1, \ldots, d \}$$

for $\gamma = (\gamma_1, \ldots, \gamma_d)$

Theorem (Cherednik)

The irreducible integrally calibrated H-modules are H^λ where

λ is a skew shape

$$H^\lambda = \mathrm{span}\left\{ v_T \;\middle|\; \begin{array}{l} T \text{ is a standard} \\ \text{tableau of shape } \lambda \end{array} \right\}$$

$$X^{\varepsilon_i} v_T = t^{\frac{1}{2}\gamma_i} v_T,$$

$$T_i v_T = \left(\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - t^{\frac{1}{2}(c(T(i)) - c(T(i+1)))}} \right) v_T$$

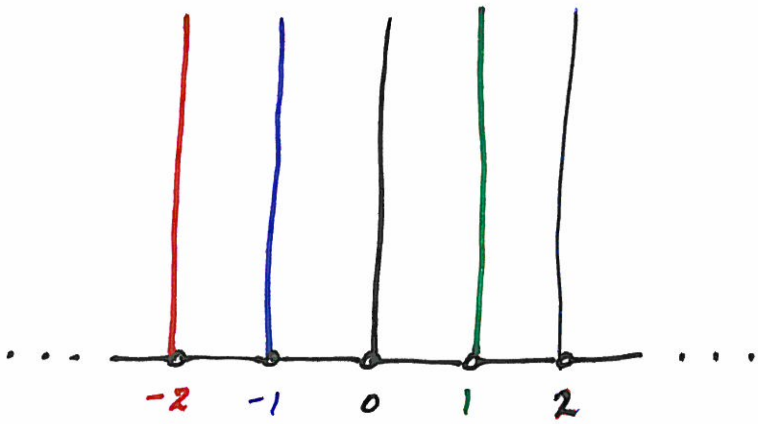
$$+ \left(t^{-\frac{1}{2}} + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - t^{\frac{1}{2}(c(T(i)) - c(T(i+1)))}} \right) v_{s_i T}$$

where

$s_i T$ is T with i and $i+1$ switched

$v_{s_i T} = 0$ if $s_i T$ is not shape λ

$\Gamma =$

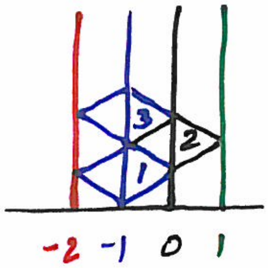


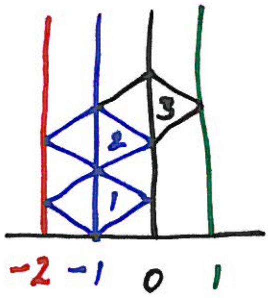
A <u>standard tableau</u> T is obtained by

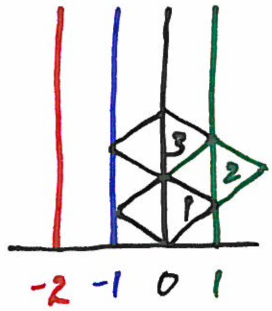
place d beads on the runners

label the i^{th} bead with i

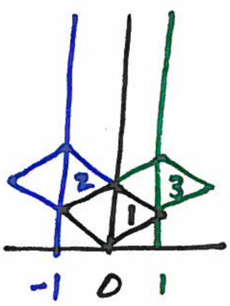
Examples:

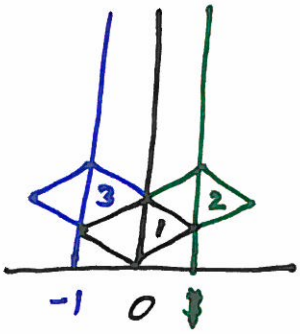






have different shapes and



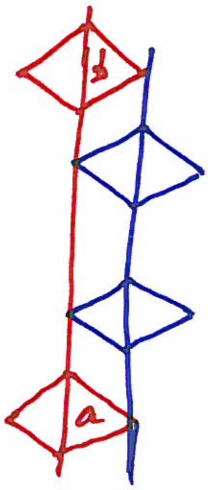


are different standard tableaux of the <u>same</u> shape.

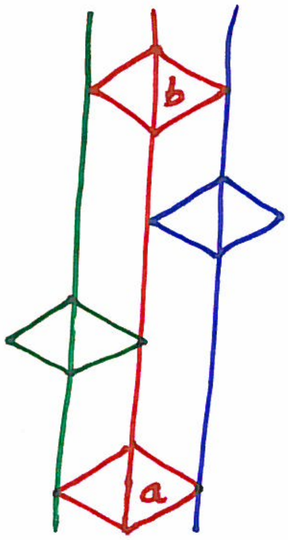
A <u>skew shape</u> is λ such that any two beads on the same runner are separated by two beads

if

then



or



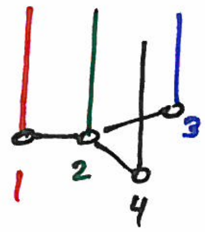
$c(T(i))$ is the colour of bead i's runner

m_c = # of beads on runner c

The Khovanov-Lauda algebra R_α

Γ is a graph with

vertices/runners coloured $1, \dots, n$



Fix $\alpha = (m_1, \dots, m_n)$, $m_i \in \mathbb{Z}_{\geq 0}$

Let $d = m_1 + \dots + m_n$

The symmetric group S_d acts on

$$I^\alpha = \left\{ (\gamma_1, \dots, \gamma_d) \;\middle|\; \begin{array}{l} \gamma_i \in \{1, 2, \dots, n\} \\ \text{colour } j \text{ appears } m_j \text{ times} \end{array} \right\}$$

The Khovanov-Lauda algebra R_α has

Generators:

$$e_\gamma, \quad y_1 e_\gamma, \dots, y_d e_\gamma, \quad \psi_1 e_\gamma, \dots \psi_{d-1} e_\gamma$$

for each $\gamma \in I^\alpha$

$\mathbb{Z}$-grading:

$\deg(e_\gamma) = 0, \qquad \deg(y_r e_\gamma) = 2$

$$\deg(\psi_r e_\gamma) = \begin{cases} -2, & \text{if } \gamma_r = \gamma_{r+1}, \\ 1, & \text{if } \underset{\gamma_r}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\gamma_{r+1}}{\circ} \\ 0, & \text{if } \underset{\gamma_r}{\circ} \quad \underset{\gamma_{r+1}}{\circ} \end{cases}$$

Relations:

$$e_\gamma^2 = e_\gamma, \qquad 1 = \sum_{\gamma \in I^\alpha} e_\gamma$$

$$y_s y_r = y_r y_s, \quad y_r e_\gamma = e_\gamma y_r, \quad \psi_r e_\gamma = e_{s_r \gamma} \psi_r$$

$$y_s \psi_r e_\gamma = \psi_r y_s e_\gamma, \qquad \text{if } s \neq r, r+1$$

$$y_r \psi_r e_\gamma = \begin{cases} (\psi_r y_{r+1} - 1) e_\gamma, & \text{if } \gamma_r = \gamma_{r+1}, \\ \psi_r y_{r+1} e_\gamma, & \text{otherwise} \end{cases}$$

$$y_{r+1} \psi_r e_\gamma = \begin{cases} (\psi_r y_r + 1) e_\gamma, & \text{if } \gamma_r = \gamma_{r+1}, \\ \psi_r y_r e_\gamma, & \text{otherwise} \end{cases}$$

$$\psi_r^2 e_\gamma = \begin{cases} 0, & \text{if } \gamma_r = \gamma_{r+1} \\ e_\gamma, & \text{if } \overset{\circ}{\gamma_r} \quad \overset{\circ}{\gamma_{r+1}} \\ (y_r + y_{r+1}) e_\gamma, & \text{if } \overset{\circ\!-\!-\!\circ}{\gamma_r \ \gamma_{r+1}} \end{cases}$$

$$\psi_r \psi_{r+1} \psi_r e_\gamma = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e_\gamma, & \text{if } \gamma_r = \gamma_{r+2}, \ \overset{\circ\!-\!-\!\circ}{\gamma_r \ \gamma_{r+1}} \\ \psi_{r+1} \psi_r \psi_{r+1}, & \text{otherwise} \end{cases}$$

A **$\mathbb{Z}$-graded vector space** is a vector space V with a direct sum decomposition

$$V = \bigoplus_{k \in \mathbb{Z}} V[k]$$

An irreducible finitedimensional $\mathbb{Z}$-graded R_α-module M is **homogeneous**, or **pure**, if

$$M = M[0]$$

$$\left(\text{For a } \mathbb{Z}\text{-graded } R_\alpha\text{-module } M \quad R_\alpha[k] \cdot M[l] \subseteq M[k+l] \right)$$

Theorem (Kleshchev–Ram)

The irreducible homogeneous R_α-modules are R_α^λ where

λ is a skew shape

$$R_\alpha^\lambda = \mathrm{span}\left\{ v_T \;\middle|\; \begin{array}{l} T \text{ is a standard} \\ \text{tableau of shape } \lambda \end{array} \right\}$$

and the action is given by

$$e_\gamma v_T = \begin{cases} v_T, & \text{if } \gamma = (c(T(1)), \ldots, c(T(d))), \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_r e_\gamma v_T = \begin{cases} v_{s_r T}, & \text{if } s_r T \text{ is shape } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

$$y_r e_\gamma v_T = 0.$$