$\mathcal{M}_{g,n}$ counting and recursions

Algebra, Geometry + Topology seminar
Univ. of Melbourne 4 August 2008.

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Why do I care?

		Witten				
Okounkov	=	Kontsevich	=	$\mathcal{M}_{g,n}$	=	counting points in polytopes
"		"		"		"
symmetric functions		"physics"		Number Theory		algebraic combinatorics

symmetric functions $\longleftrightarrow$ T-equivariant cohomology $\longleftrightarrow$ algebraic combinatorics

T-equivariant cohomology
"
Grothendieck-Lefschetz
Brion-Vergne
Localization at T fixed pts

Grothendieck Dessin d'Enfants Curves over $\overline{\mathbb{Q}}$ $\longleftrightarrow$ Drinfeld QuasiHopf algebras and deformations. $\sim$ Langlands program.

Theorem (Okounkov-Pandharipande)

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{virt}} \prod c_1(\mathcal{L}_i)^{k_i} = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^{2-2\,\text{genus}(X)} \prod_i \frac{p^*_{k_i+1}(\lambda)}{(k_i+1)!}$$

where

$$p^*_k(\lambda) = \sum_i \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^k - \left(-i+\tfrac{1}{2}\right)^k\right) \ldots$$

$p^*_k(\lambda)$ is the central character of a completed k cycle

$\frac{1}{k} C_{(k)}$ + smaller permutations.

Fat graphs

A fat graph $\Gamma = (V, E, O)$ is an undirected graph (V, E) with

a cyclic ordering z_v of the edges that contain v, for each $v \in V$, and

such that

each vertex $v \in V$ is in > 2 edges.

A boundary component of Γ is a sequence of distinct edges

$$(\{v_0, v_1\}, \ldots, \{v_{\ell-1}, v_0\}) \text{ such that}$$

for $i = 1, \ldots, \ell-1$,

$\{v_i, v_{i+1}\}$ is the successor of $\{v_{i-1}, v_i\}$ at v_i;

$\{v_0, v_1\}$ is the successor of $\{v_{\ell-1}, v_0\}$ at v_0

Let $n = \#$ of boundary components of Γ.

$d = \#$ of edges of Γ

$d' = \#$ of vertices of Γ.

The genus and Euler characteristic of Γ are

$$g = \frac{d' - d + n}{2} - 1 \quad \text{and} \quad \chi = d' - d,$$

respectively.

Norbury polynomials

A metric fat graph is a pair (Γ, x) where Γ is a fat graph and $x \in \mathbb{R}^d_{\geq 0}$.

The length of an edge e is x_e and the length of a boundary component $(e_0, \ldots, e_{\ell-1})$ is $x_{e_0} + \cdots + x_{e_{\ell-1}}$.

Let Γ be a fat graph with boundary components labeled $1, 2, \ldots, n$. ~~Let~~ For $b_1, \ldots, b_n \in \mathbb{R}_{\geq 0}$ let

$$P_\Gamma(b_1, \ldots, b_n) = \left\{ \begin{array}{l} \text{metric fat graphs } (\Gamma, x) \text{ with} \\ \text{boundary component lengths } b_1, \ldots, b_n \end{array} \right\}$$

$$= \{ x \in \mathbb{R}^d_{\geq 0} \mid A_\Gamma x = b \} \quad \text{where}$$

$(A_\Gamma)_{eB} =$ # of times e appears in B.

Define

$$N_\Gamma(b_1, \ldots, b_n) = \mathrm{Card}(P_\Gamma(b_1, \ldots, b_n) \cap \mathbb{Z}^r) \quad \text{and}$$

$$N_{g,n}(b_1, \ldots, b_n) = \sum_{\Gamma \in \mathrm{Fat}_{g,n}} \frac{N_\Gamma(b_1, \ldots, b_n)}{|\mathrm{Aut}(\Gamma)|}, \quad \text{where}$$

$$\mathrm{Fat}_{g,n} = \left\{ \begin{array}{l} \text{fat graphs with genus } g \text{ and } n \text{ boundary} \\ \text{components labeled } 1, 2, \ldots, n \end{array} \right\}$$

$$\mathrm{Aut}(\Gamma) = \{ \text{automorphisms of } \Gamma \}.$$

Permutations

Let $\Gamma = (V, E, Z)$ be a fat graph.

Let
$$\bar{E} = \{(v_i; v_j), (v_j; v_i) \mid \{v_i; v_j\} \in E\}$$
so that $(V, \bar{E})$ is a directed graph with $2d$ edges.

Define $\sigma_0, \sigma_{\frac{1}{2}} \in S_{2d}$ by

$$\sigma_{\frac{1}{2}}((v_i; v_j)) = (v_j; v_i), \quad \text{and}$$

$$(\sigma_0 \sigma_{\frac{1}{2}})((v_i; v_j)) = (v_j; v_k), \text{ where } \{v_j; v_k\} \text{ is the successor of } \{v_i; v_j\} \text{ at } v_j.$$

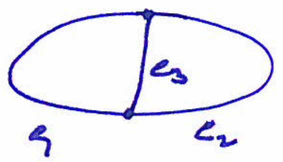
Then, in cycle notation

$$\sigma_\infty^{-1} = \sigma_0 \sigma_{\frac{1}{2}} = (B_1, \ldots, B_n), \quad \text{where}$$

B_i are the boundary components of Γ,

and
$$\sigma_0 \sigma_{\frac{1}{2}} \sigma_\infty = I, \text{ in } S_d.$$

Example 1

$\Gamma =$ 

$Z_1 = (e_1 e_2 e_3)$

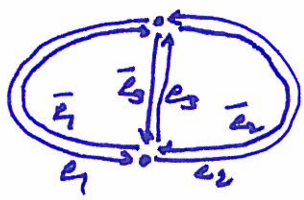
$Z_2 = (e_1 e_3 e_2)$

$B_1 = (e_1 e_2)$

$B_2 = (e_2 e_3)$

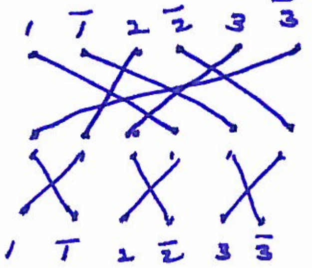
$B_3 = (e_1 e_3)$

$$A_\Gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



$\sigma_0 =$

$\sigma_{\frac{1}{2}} =$

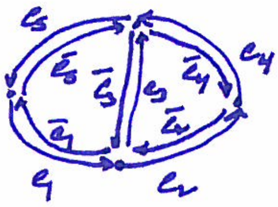


Then

$\sigma_\infty^{-1} = \sigma_0 \sigma_{\frac{1}{2}}$ has cycles $(e_1 e_2)(\bar{e}_1 \bar{e}_3)(\bar{e}_2 \bar{e}_3)$ and

$\sigma_0 \sigma_{\frac{1}{2}} \sigma_\infty = 1$.

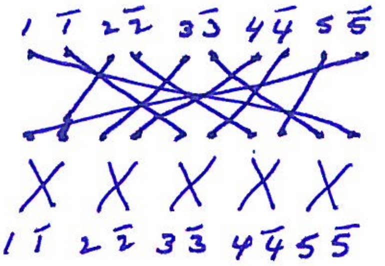
To consider the metric fat graph $(\Gamma; (2,2,1))$ insert vertices in e_1 and e_2



Then

$\sigma_0 =$

$\sigma_{\frac{1}{2}} =$



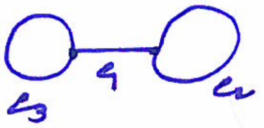
has cycles $(e_1, e_2, e_4, e_5), (\bar{e}_1, \bar{e}_5, \bar{e}_3), (\bar{e}_2, e_3, \bar{e}_4)$

and $(b_1, b_2, b_3) = (4, 3, 3)$.

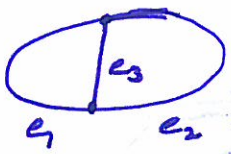
Base case examples

Example $(g,n)=(0,3)$ where $N_{0,3}=1$

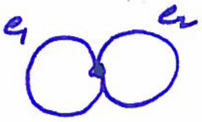
There are 7 labeled fat graphs
3 unlabeled fat graphs

$\Gamma =$ 

$$A_\Gamma = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

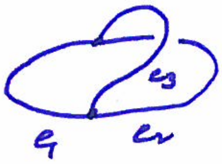
$\Gamma =$ 

$$A_\Gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$\Gamma =$ 

$$A_\Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

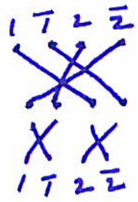
Example $(g,n)=(1,1)$ where $N_{1,1}(b_1)=\frac{1}{48}(b_1^2-4)$

$\Gamma =$ 

$A_\Gamma = (2\ 2\ 2)$ $\qquad |\mathrm{Aut}\,\Gamma| = 6.$

$\Gamma =$ 

$A_\Gamma = (2\ 2)$

$\sigma_0 =$ $\sigma_{\frac{1}{2}} =$ 

$\mathrm{Aut}\,\Gamma = \mathbb{Z}/4\mathbb{Z}$

Relation to $\mathcal{M}_{g,n}$

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{c} \text{genus } g \text{ curves with} \\ n \text{ marked points} \end{array} \right\}$$

curve = Riemann surface, oriented, conformal class...
connected, compact....

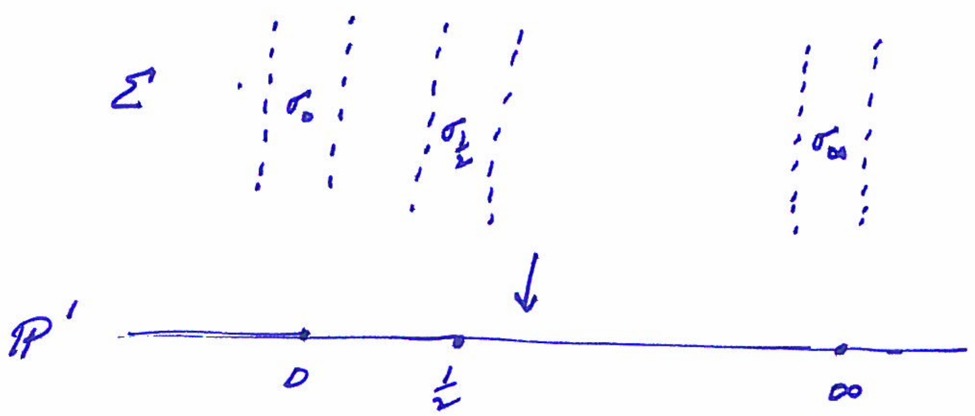
$$\mathcal{M}_{g,n}^{\text{comb}}(b_1, \ldots, b_n) = \bigcup_{\Gamma \in \mathrm{Fat}_{g,n}} \mathcal{P}_\Gamma(b_1, \ldots, b_n) \cong \mathcal{M}_{g,n}.$$

Relation to branched covers of $\mathbb{P}^1$

A <u>branched cover</u> of $\mathbb{P}^1$ is a map

$$\begin{array}{c} \Sigma \\ \downarrow \\ \mathbb{P}^1 \end{array} \qquad \text{where } \Sigma \text{ is} \ldots \quad \cancel{\text{such that}} \ldots$$

Three permutations $\sigma_0, \sigma_{\frac{1}{2}}, \sigma_\infty \in S_{2d}$ such that



$\sigma_0 \sigma_{\frac{1}{2}} \sigma_\infty = 1$ specify a degree d branched cover of $\mathbb{P}^1$.