A path model formula for Macdonald Polynomials

Algebra seminar
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①

The double affine Weyl group $\tilde{W}$

$$\left.\begin{matrix}\mathfrak{h}_{\mathbb{Z}}\\ \mathfrak{h}_{\mathbb{Z}}^*\end{matrix}\right\}\text{ dual lattices,}\qquad \langle\,,\,\rangle:\mathfrak{h}_{\mathbb{Z}}\times\mathfrak{h}_{\mathbb{Z}}^*\longrightarrow\mathbb{Z}$$

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by

$$R^+=\{s\in W_0\mid s\text{ is a reflection}\}$$

A reflection is $s\in GL(\mathfrak{h}_{\mathbb{Z}})$ with exactly one eigenvalue not equal to 1

W_0 acts on $\mathfrak{h}_{\mathbb{Z}}^*$ by

$$\langle w\mu,\lambda^\vee\rangle=\langle\mu,w^{-1}\lambda\rangle\quad\text{for } \mu\in\mathfrak{h}_{\mathbb{Z}}^*,\ \lambda^\vee\in\mathfrak{h}_{\mathbb{Z}},\ w\in W_0.$$

The double affine Weyl group is

$$\tilde{W}=\{q^k X^\mu w Y^{\lambda^\vee}\mid k\in\mathbb{Z},\ \mu\in\mathfrak{h}_{\mathbb{Z}}^*,\ \lambda^\vee\in\mathfrak{h}_{\mathbb{Z}},\ w\in W_0\}$$

with

$$q\in Z(\tilde{W}),\quad X^\mu X^\nu=X^{\mu+\nu},\quad Y^{\lambda^\vee}Y^{\sigma^\vee}=Y^{\lambda^\vee+\sigma^\vee}$$

$$X^\mu Y^{\lambda^\vee}=q^{\langle\mu,\lambda^\vee\rangle}Y^{\lambda^\vee}X^\mu$$

$$wX^\mu=X^{w\mu}w\quad\text{and}\quad wY^{\lambda^\vee}=Y^{w\lambda^\vee}w.$$

The affine Weyl group

$$W^\vee = \{ X^\mu w \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0 \}.$$

$W^\vee$ acts on $\{ q^k Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$ by conjugation

Write $\quad Y^{v\lambda^\vee} = v Y^{\lambda^\vee} v^{-1}$, for $v \in W^\vee$

$W^\vee$ acts on $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{h}_{\mathbb{Z}}^* \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$(X^\mu w) \cdot \nu = \mu + w\nu, \quad \text{for } \nu \in \mathfrak{h}_{\mathbb{R}}^*.$$

Let C be a fundamental region for this action such that $0 \in \bar{C}$ (closure of C). Let

$\mathfrak{h}^{\alpha_0^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ be the walls of C

$s_0^\vee, \ldots, s_n^\vee$ the reflections in these walls

$W^\vee$ is presented by generators $s_0^\vee, \ldots, s_n^\vee$

with

$$(s_i^\vee)^2 = 1, \quad \underbrace{s_i^\vee s_j^\vee s_i^\vee \cdots}_{m_{ij}^\vee \text{ factors}} = \underbrace{s_j^\vee s_i^\vee s_j^\vee \cdots}_{m_{ij}^\vee \text{ factors}}$$

where $\pi/m_{ij}^\vee = \mathfrak{h}^{\alpha_i^\vee} \measuredangle \mathfrak{h}^{\alpha_j^\vee}$.

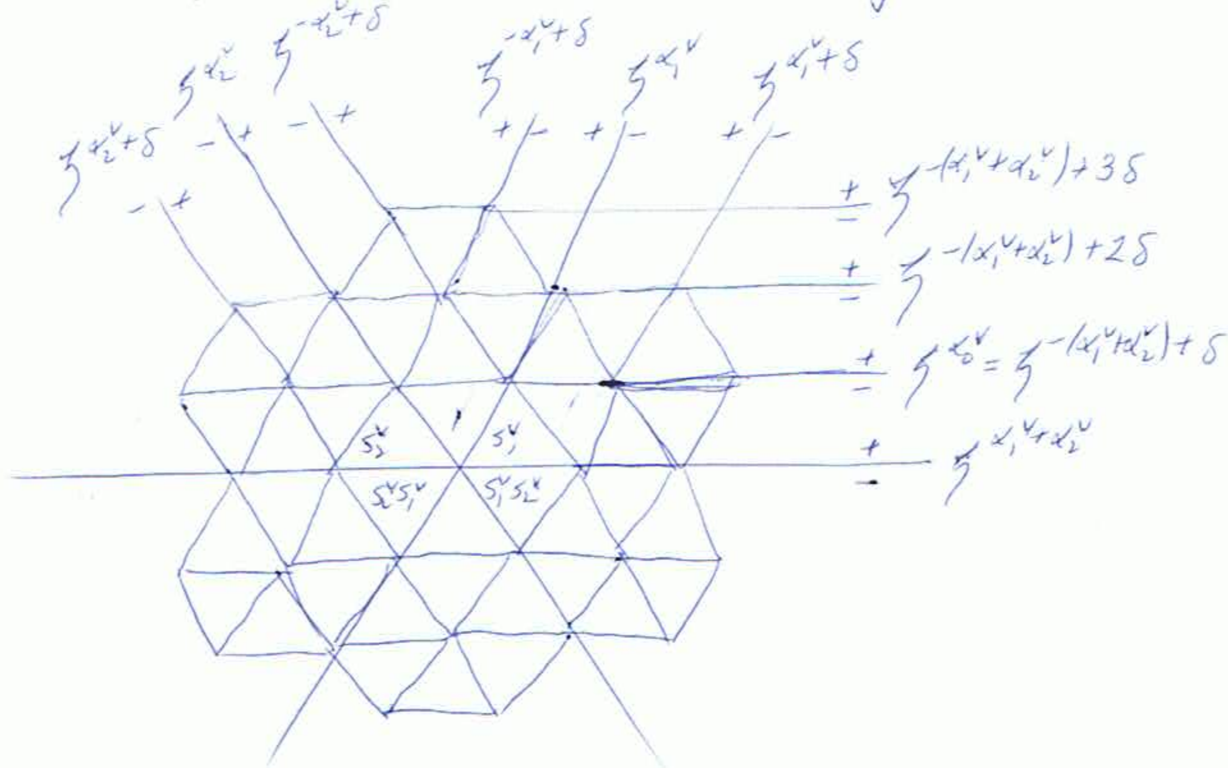
Example $(PGL_3^\vee, PGL_3)$

$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee$ $\qquad \langle \alpha_1, \alpha_1^\vee \rangle = 2, \quad \langle \alpha_1, \alpha_2^\vee \rangle = -1$

$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ $\qquad \langle \alpha_2, \alpha_1^\vee \rangle = -1, \quad \langle \alpha_2, \alpha_2^\vee \rangle = 2$

$$W \overset{1-1}{\longleftrightarrow} \{\text{alcoves}\}$$

$W_0 \longleftrightarrow \left\{ \begin{matrix} \text{alcoves in the} \\ 0\text{-hexagon} \end{matrix} \right\}$ $\qquad \mathfrak{h}_{\mathbb{Z}}^* \longleftrightarrow \left\{ \begin{matrix} \text{centers of} \\ \text{hexagons} \end{matrix} \right\}$



The periodic orientation has

(1) 1 on the positive side of $\mathfrak{h}^{\alpha^\vee}$ for $\alpha^\vee \in R^+$

(2) $\mathfrak{h}^{\alpha^\vee + j\delta}$ and $\mathfrak{h}^{\alpha^\vee}$ have parallel orientations.

Macdonald polynomials

Let $\mu \in \mathfrak{h}^{+}_{\mathbb{Z}}$ and $p_\mu = s^\vee_{i_1} \cdots s^\vee_{i_\ell}$

a minimal length walk to the μ hexagon.

Let

$$\gamma^\vee_\ell + j_\ell \delta,\ \gamma^\vee_{\ell-1} + j_{\ell-1}\delta, \ldots, \gamma^\vee_1 + j_1 \delta$$

be the hyperplanes crossed by $\mathrm{rev}(p_\mu)$.

The <u>nonsymmetric</u> <u>Macdonald</u> <u>polynomial</u> is

$$E_\mu = \sum_{\substack{\text{foldings } p \\ \text{of } p_\mu}} X^{wt(p)} t^{\frac{1}{2}\ell(\varphi(p))} \prod_{k \in F^+(p)} f_k^+ \prod_{k \in F^-(p)} f_k^-$$

where $\mathrm{end}(p) = X^{wt(p)}_{\varphi(p)}$

$$F^+(p) = \{ k \mid k^{\text{th}} \text{ step of } p \text{ is } \overset{+}{\Leftarrow}\!|^{-} \}$$

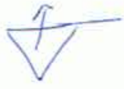
$$F^-(p) = \{ k \mid k^{\text{th}} \text{ step of } p \text{ is } {}^{-}|\!\overset{+}{\Rightarrow} \}$$

$$f_k^+ = \frac{t^{-\frac{1}{2}}(1-t)}{1 - t^{ht(\gamma_k^\vee)} q^{j_k}}, \qquad f_k^- = \frac{t^{-\frac{1}{2}}(1-t)\, t^{ht(\gamma_k^\vee)} q^{j_k}}{1 - t^{ht(\gamma_k^\vee)} q^{j_k}}$$

The <u>symmetric</u> <u>Macdonald</u> <u>polynomial</u>

$$P_\mu = \sum_{w \in W_0} t^{\frac{1}{2}(\mathrm{Card}(R^+) - \ell(w))} \sum_{\substack{\text{foldings } p \\ \text{of } w p_\mu}} X^{wt(p)} t^{\frac{1}{2}\ell(\varphi(p))} \prod_{k \in F^+(p)} f_k^+ \prod_{k \in F^-(p)} f_k^-$$

Examples

$p_\rho =$  $= s_0^\vee$ $\qquad$ $\mathrm{rev}(p_\rho) =$  $\zeta^{-(\alpha_1^\vee+\alpha_2^\vee)+\delta}$

$p_{s_1\rho} =$  $= s_1^\vee s_0^\vee$ $\qquad$ $\mathrm{rev}(p_{s_1\rho}) =$ 

$\zeta^{-(\alpha_1^\vee+\alpha_2^\vee)+\delta} = \zeta^{-\gamma_2^\vee+j_2\delta}$

$\zeta^{-\alpha_2^\vee+\delta} = \zeta^{-\gamma_1^\vee+j_1\delta}.$

$E_\rho =$  $+$  $= X^\rho t^{3/2} + X^0 t^{0/2} f_1^-$

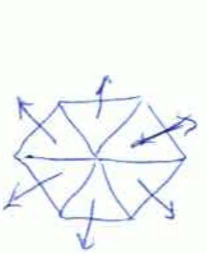
$$= X^\rho t^{3/2} + X^0 \frac{t^{-\frac{1}{2}}(1-t)\, t^2 q}{1-t^2 q}$$

$E_{s_1\rho} =$  $+$  $+$  $+$ 

$$= X^{s_1\rho} t^{2/2} + X^0 t^{\frac{1}{2}} \frac{t^{-\frac{1}{2}}(1-t)\, t^2 q}{1-t^2 q} + X^\rho t^{3/2} \frac{t^{-\frac{1}{2}}(1-t)}{1-tq} +$$

$$+ X^0 \frac{t^{-\frac{1}{2}}(1-t)}{1-tq} \frac{t^{-\frac{1}{2}}(1-t)\, t^2 q}{1-t^2 q}$$

P_ρ is a sum over

 and 

The Weyl character formula

W_0 acts on

$$\mathbb{C}[X] = \mathrm{span}\{X^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\} \quad \text{by} \quad wX^\mu = X^{w\mu}$$

Then

$$\mathbb{C}[X]^{W_0} = \{f \in \mathbb{C}[X] \mid wf = f \text{ for all } w \in W_0\}$$

$$\mathbb{C}[X]^{\det} = \{f \in \mathbb{C}[X] \mid wf = \det(w) f \text{ for all } w \in W_0\}.$$

Let

$$\varepsilon_0 = \sum_{w \in W_0} w \quad \text{and} \quad e_0 = \sum_{w \in W_0} \det(w^{-1}) w.$$

Then $\mathbb{C}[X]^{\det}$ is a free $\mathbb{C}[X]^{W_0}$-module of rank 1:

$$\begin{array}{ccc} \varepsilon_0 \mathbb{C}[X] = \mathbb{C}[X]^{W_0} & \xrightarrow{\sim} & \mathbb{C}[X]^{\det} = e_0 \mathbb{C}[X] \\ f & \longmapsto & a_\rho f. \end{array}$$

Weyl character $\quad s_\mu \longleftarrow a_{\mu+\rho} = e_0 X^{\mu+\rho} \quad$ naive basis

naive basis $\quad m_\mu = \varepsilon_0 X^\mu$

Then $\quad s_\mu = P_\mu(0,0).$

The double affine Hecke algebra $\tilde{H}$

$\tilde{W}$ is generated by $s_0^\vee, \ldots, s_n^\vee$ and $y^{\lambda^\vee}$ and q

with $(s_i^\vee)^2 = 1$, $s_i^\vee s_j^\vee s_i^\vee \cdots = s_j^\vee s_i^\vee s_j^\vee \cdots$, $y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee}$

$q \in Z(\tilde{W})$ and $s_i^\vee y^{\lambda^\vee} = y^{s_i^\vee \lambda^\vee} s_i$.

$\tilde{H}$ is generated by $T_0^\vee, \ldots, T_n^\vee$, and $y^{\lambda^\vee}$ and q

with $(T_i^\vee)^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1$, $\underbrace{T_i^\vee T_j^\vee T_i^\vee \cdots}_{m_{ij}^\vee \text{ factors}} = \underbrace{T_j^\vee T_i^\vee T_j^\vee \cdots}_{m_{ij}^\vee \text{ factors}}$

$q \in Z(\tilde{W})$, $y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee}$, $\tau_i^\vee y^{\lambda^\vee} = y^{s_i \lambda^\vee} \tau_i^\vee$

where
$$\tau_i^\vee = T_i^\vee - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - y^{-\alpha_i^\vee}}.$$

$\tilde{H}$ has a basis $\{ q^k X^\mu T_w y^{\lambda^\vee} \}$

$\cup$

$H = \text{span}\{ T_w y^{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$

$\cup$

$H_0 = \text{span}\{ T_w \mid w \in W_0 \}$

Weyl character formula for Macdonald polynomials

The trivial representation of H is $\mathrm{span}\{\mathbb{1}\}$ with

$$T_i \mathbb{1} = t^{\frac{1}{2}} \mathbb{1} \quad \text{and} \quad Y^{\lambda^\vee} \mathbb{1} = t^{\langle \lambda^\vee, \rho \rangle} \mathbb{1}$$

for $i = 1, \ldots, n$ and $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$.

The polynomial representation of $\tilde{H}$ is

$$\mathrm{Ind}_H^{\tilde{H}}(\mathbb{1}) = \tilde{H}\mathbb{1} = \mathrm{span}\{q^k X^\mu \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$$

The Macdonald polynomials are the simultaneous eigenvectors of $Y^{\lambda^\vee}$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$:

$$E_\mu = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbb{1} \quad \text{if} \quad t_\mu = s_{i_1}^\vee \cdots s_{i_\ell}^\vee$$

is a minimal length walk to μ.

Let $\mathbb{1}_0, \varepsilon_0 \in H_0$ such that

$$T_i \mathbb{1}_0 = t^{\frac{1}{2}} \mathbb{1}_0, \quad T_i \varepsilon_0 = -t^{-\frac{1}{2}} \varepsilon_0, \quad \text{for } i = 1, \ldots, n$$

Then

$$\begin{array}{ccc} \mathbb{C}[X]^{W_0}\mathbb{1} = \mathbb{1}_0 \tilde{H}\mathbb{1} & \xrightarrow{\sim} & \varepsilon_0 \tilde{H}\mathbb{1} \\ f & \longmapsto & A_\rho f \\ P_\mu(q, qt) & \longleftarrow\!\shortmid & A_{\mu+\rho} = \varepsilon_0 E_{\mu+\rho} \end{array}$$

naive basis $P_\mu(q,t) = \mathbb{1}_0 E_\mu$