A path model formula for Macdonald Polynomials University of Paris
Double cosets
6 March 2009 ①

$\mathbb{Q}_p$ or $\mathbb{C}((t)) = \{a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + \cdots \mid a_i \in \mathbb{C}, \ell \in \mathbb{Z}\}$

$\cup\!\!\!\mid \qquad \cup\!\!\!\mid$

$\mathbb{Z}_p$ or $\mathbb{C}[[t]] = \{a_0 + a_1 t + \cdots \mid a_i \in \mathbb{C}\}$

$G_0(\mathbb{C})$ = complex reductive algebraic group.

$$\begin{array}{ccccc} G & = & G_0(\mathbb{C}((t))) & & \\ \cup\!\!\!\mid & & \cup\!\!\!\mid & & \\ K & = & G_0(\mathbb{C}[[t]]) & \xrightarrow{t=0} & G_0(\mathbb{C}) \\ \cup\!\!\!\mid & & \cup\!\!\!\mid & & \cup\!\!\!\mid \\ I & = & \Phi^{-1}(B) & \longrightarrow & B_0 = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \end{array}$$

and

$$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+} K t_{\lambda^\vee} K \qquad G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^- t_{\mu^\vee} K$$

$$G = \bigsqcup_{w \in W} I w I \qquad G = \bigsqcup_{v \in W} U^- v I$$

where $U^- = \left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\right\}$, $\mathfrak{h}_{\mathbb{Z}} = \{\text{cocharacters of } G_0\}$

$$W = W_0 \ltimes \mathfrak{h}_{\mathbb{Z}} = \{w y^{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

W_0 the Weyl group of G,

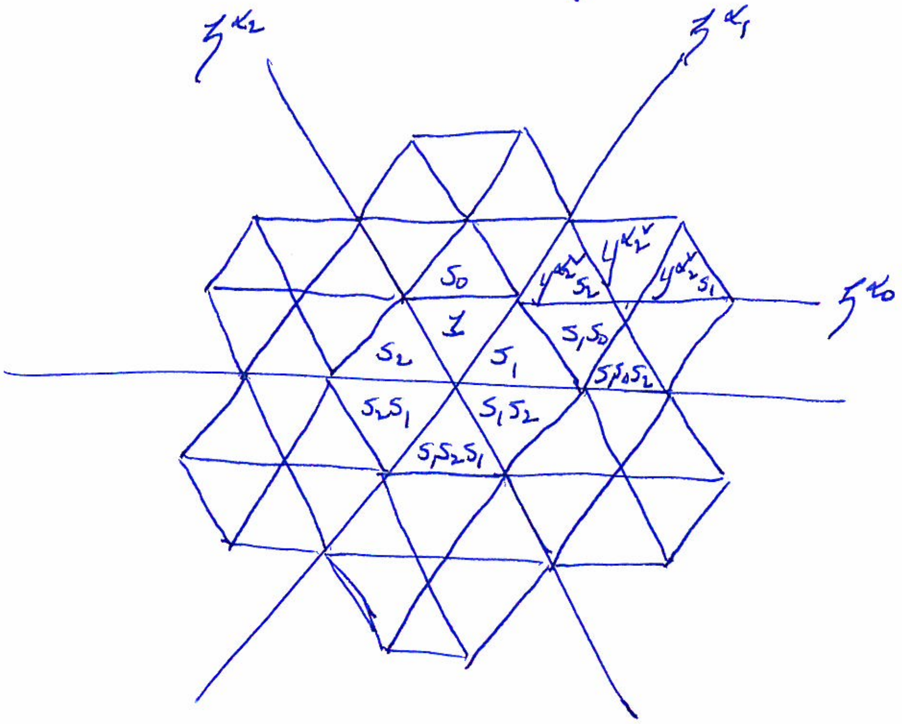
$$y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee}, \qquad w y^{\lambda^\vee} = y^{w\lambda^\vee} w.$$

The affine Weyl group

W acts on $\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$ by

$$(w y^{\lambda^\vee})(v^\vee) = w(\lambda^\vee + v^\vee), \quad \text{for } v^\vee \in \mathfrak{h}_{\mathbb{R}}.$$

Let 1 be a fundamental region (alcove) with $0 \in \bar{1}$.



Let $\mathfrak{h}^{\alpha_0}, \ldots, \mathfrak{h}^{\alpha_n}$ be the walls of 1

$s_0, \ldots, s_n$ the corresponding reflections

W is generated by $s_0, \ldots, s_n$ with

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}$$

for $\pi / m_{ij} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}$.

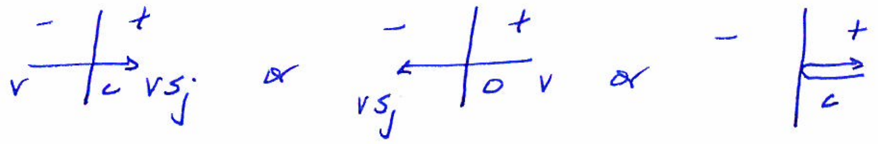
Positively folded alcove walks

The *periodic orientation* has

(a) 1 on the $+$ side of $\mathfrak{h}^{\alpha}$ if $\mathfrak{h}^{\alpha}$ goes through 0

(b) $\mathfrak{h}^{\alpha+k\delta}$ and $\mathfrak{h}^{\alpha}$ have parallel orientations

A *step of type* j is

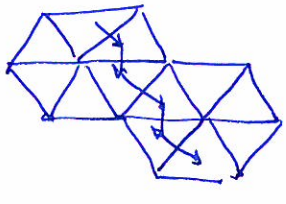


$c \in \mathcal{C}$ $\qquad\qquad\qquad\qquad$ $c \in \mathcal{C}^{\times}$

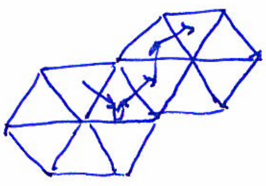
Theorem (Parkinson-Ram-Schwer, d'après Gaussent-Littelmann)

Let $v, w \in W$, $w = s_{i_1} \cdots s_{i_\ell}$ a minimal length path to w.

$$I w I \cap U^- v I \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{positively folded walks} \\ \text{of type } (i_1, \ldots, i_\ell) \text{ beginning} \\ \text{at } 1, \text{ ending at } v \end{array} \right\}$$

Example If $w =$  $= s_1 s_2 s_0 s_1 s_2$

then

 is positively folded

Macdonald Polynomials

Let $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $p_{\lambda^\vee} = s_{i_1} \cdots s_{i_\ell}$ a minimal length walk to the $\lambda^\vee$-hexagon. Let

$$\mathfrak{h}^{-\beta_\ell + j_\ell \delta}, \ \mathfrak{h}^{-\beta_{\ell-1} + j_{\ell-1}\delta}, \ \dots, \ \mathfrak{h}^{-\beta_1 + j_1 \delta}$$

be the hyperplanes crossed by rev(p_μ).

Theorem (Ram-Yip) The nonsymmetric Macdonald polynomial

$$E_{\lambda^\vee} = \sum_{\substack{\text{all foldings } p \\ \text{of } p_{\lambda^\vee}}} y^{wt(p)} t^{\frac{1}{2}\ell(\varphi(p))} \Big(\prod_{k \in F^+(p)} f_k^+\Big)\Big(\prod_{k \in F^-(p)} f_k^-\Big)$$

where $\text{end}(p) = y^{wt(p)} \varphi(p)$

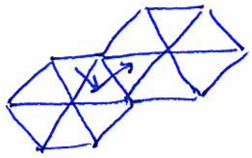
$$F^+(p) = \{ k \mid k^{\text{th}} \text{ step of } p \text{ is } {}^{-}\big|\overset{+}{\hookleftarrow} \}$$

$$F^-(p) = \{ k \mid k^{\text{th}} \text{ step of } p \text{ is } \hookrightarrow\big|^{+} \}$$

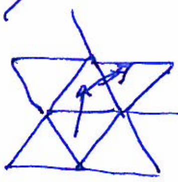
$$f_k^+ = \frac{t^{-\frac{1}{2}}(1-t)}{1 - t^{ht(\beta_k)} q^{j_k}} \quad \text{and} \quad f_k^- = \frac{t^{-\frac{1}{2}}(1-t)\, t^{ht(\beta_k)} q^{j_k}}{1 - t^{ht(\beta_k)} q^{j_k}}$$

with

$$ht(\beta) = c_1 + \cdots + c_n \quad \text{if} \quad \beta = c_1\alpha_1 + \cdots + c_n \alpha_n.$$

<u>Example</u> $\lambda^\vee = \alpha_2^\vee$, $p_{\alpha_2^\vee} =$  $= s_1 s_0$

$\zeta^{-\beta_1 + j_1\delta} = \zeta^{-\alpha_2 + \delta}$

$\mathrm{rev}(p_{\alpha_2^\vee}) =$  $\zeta^{-(\alpha_1+\alpha_2)+\delta} = \zeta^{-\beta_2 + j_2\delta}$

$E_{\alpha_2^\vee} =$  $+$  $+$  $+$ 

$$= y^{\alpha_2^\vee} t^{2/2} + y^0 t^{1/2} f_2^- + y^{\alpha_1^\vee+\alpha_2^\vee} t^{3/2} f_1^+ + y^0 t^0 f_1^+ f_2^-$$

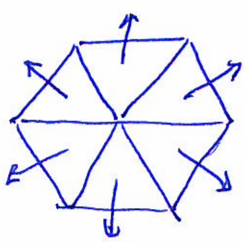
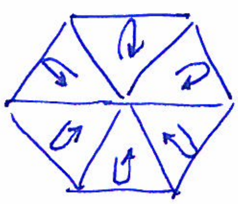
where

$$f_1^+ = \frac{t^{-\frac12}(1-t)}{1-tq} \quad \text{and} \quad f_2^- = \frac{t^{-\frac12}(1-t)t^2 q}{1-t^2 q}$$

The <u>symmetric</u> <u>Macdonald</u> <u>polynomial</u> is

$$P_{\lambda^\vee}(q,t) = \sum_{w \in W_0} \sum_{\substack{\text{foldings } p \\ \text{of } w p_{\lambda^\vee}}} t^{\frac12 \ell(w)} y^{wt(p)} t^{\frac12 \ell(\varphi(p))} \Big(\prod_{k \in F^+(p)} f_k^+\Big)\Big(\prod_{k \in F^-(p)} f_k^-\Big)$$

<u>Example</u> $\lambda^\vee = \rho^\vee = \alpha_1^\vee + \alpha_2^\vee$, $p_{\rho^\vee} =$  $= s_0$

$P_{\rho^\vee}(q,t) =$  $+$ 

Spherical functions

$P_{\lambda^\vee}(q,t)$ = Macdonald polynomial

$P_{\lambda^\vee}(0,p^{-1})$ = Spherical function for $G_0(\mathbb{Q}_p)/G_0(\mathbb{Z}_p)$

$P_{\lambda^\vee}(0,0)$ = Weyl character for $G_0(\mathbb{C})$

Recall

$$\begin{array}{ccccc} G(\mathbb{Q}_p) & \supseteq & G_0(\mathbb{Z}_p) & & \\ \| & & \| & & \\ G & \supseteq & K & \supseteq & I \end{array} \quad \text{Iwahori subgroup.}$$

The affine Hecke algebra is

$$H = \{f : G \to \mathbb{C} \mid f(h_1 g h_2) = f(g) \text{ for } h_1, h_2 \in I\}$$

with basis

$$\{\chi_{IwI} \mid w \in W\}$$

Let

$$\mathbb{1}_0 = \sum_{w \in W_0} \chi_{IwI} \quad \text{and} \quad \varepsilon_0 = \sum_{w \in W_0} (-p^{-1})^{\ell(w)} \chi_{IwI}$$

The spherical Hecke algebra is

$$\mathbb{1}_0 H \mathbb{1}_0 = C_c(K \backslash G / K).$$

⑦

The picture

$$\mathbb{C}[Y]^{W_0}\mathbb{1}_0 = \mathbb{1}_0 H \mathbb{1}_0 \longrightarrow \varepsilon_0 H \mathbb{1}_0$$

$$P_{\lambda^\vee}(0,p^{-1})\mathbb{1}_0 = \mathbb{1}_0 X_{I t_\lambda I}\mathbb{1}_0$$

$$f\mathbb{1}_0 \longmapsto A_{\rho^\vee} f \mathbb{1}_0$$

$$P_{\lambda^\vee}(0,0)\mathbb{1}_0 \longleftarrow\!\shortmid A_{\lambda^\vee+\rho^\vee} = \varepsilon_0 X_{I t_{\lambda^\vee+\rho^\vee} I}\mathbb{1}_0$$

has a (q,t) analogue

$$\mathbb{C}[Y]^{W_0}\mathbb{1} = \mathbb{1}_0 \tilde{H}\mathbb{1} \xrightarrow{\sim} \varepsilon_0 \tilde{H}\mathbb{1}$$

$$P_{\lambda^\vee}(q,t)\mathbb{1} = \mathbb{1}_0 E_{\lambda^\vee}\mathbb{1}$$

$$f\mathbb{1} \longmapsto A_{\rho^\vee}(q,t)\mathbb{1}$$

$$P_{\lambda^\vee}(q,qt)\mathbb{1} \longleftarrow\!\shortmid A_{\lambda^\vee+\rho^\vee}(q,t) = \varepsilon_0 E_{\lambda^\vee+\rho^\vee}\mathbb{1}$$

where

$\tilde{H}$ is the <u>double</u> <u>affine</u> <u>Hecke</u> <u>algebra</u>

$\tilde{H}\mathbb{1}$ is the <u>polynomial</u> <u>representation</u> of $\tilde{H}$