Poles, Strings, Braids, Lattices, Colloquium, La Trobe University

1 May 2009

Reflection group

$$W_0 = O_n(\mathbb{Z}) = \{ w \in M_n(\mathbb{Z}) \mid ww^t = 1 \}$$

is presented by

$$s_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \quad \text{and} \quad s_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

for $i = 1, \ldots, n-1$ with relations

$$\overset{s_1}{\circ} \!-\! \overset{s_2}{\circ} \!-\! \cdots \!-\! \circ \!=\! \overset{s_n}{\circ} \qquad \text{and} \quad s_i^2 = 1, \text{ for } i = 1, \ldots, n$$

where $\overset{a}{\circ}\ \overset{b}{\circ}$ means $ab = ba$, $\overset{a}{\circ}\!=\!\overset{b}{\circ}$ means $abab = baba$.
$\overset{a}{\circ}\!-\!\overset{b}{\circ}$ means $aba = bab$,

Dual lattices

W_0 acts on

$$\mathfrak{h}_{\mathbb{Z}} = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i^\vee \quad \text{and} \quad \mathfrak{h}_{\mathbb{Z}}^* = \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$$

with $\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}$. For

$$\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \quad \text{let} \quad x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$$

$$\lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee \qquad y^{\lambda^\vee} = y_1^{\lambda_1} \cdots y_n^{\lambda_n}.$$

Then

$$X = \langle x_1^{\pm 1}, \ldots, x_n^{\pm 1} \rangle \cong \mathfrak{h}_{\mathbb{Z}}^*, \quad Y = \langle y_1^{\pm 1}, \ldots, y_n^{\pm 1} \rangle \cong \mathfrak{h}_{\mathbb{Z}}$$

and

$$\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \quad \text{and} \quad \mathbb{C}[Y] = \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}].$$

The double affine Weyl group (DAWG)

$$\tilde{W} = \{ q^{k/2} X^{\mu} w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$$

with

$$X^{\mu} X^{\nu} = X^{\mu+\nu}, \quad Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}, \quad q^{\frac{1}{2}} \in Z(\tilde{W})$$

$$w X^{\mu} = X^{w\mu} w, \quad w Y^{\lambda^\vee} = Y^{w\lambda^\vee} w,$$

$$X^{\mu} Y^{\lambda^\vee} = q^{\langle \mu, \lambda^\vee \rangle} Y^{\lambda^\vee} X^{\mu}$$

so that $x_i y_i = q y_i x_i$ and $x_i y_j = y_j x_i$ if $i \neq j$.

Let

$$s_{\varepsilon_1} = s_1 \cdots s_n \cdots s_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$s_0^\vee = X^{\varepsilon_1} s_{\varepsilon_1}, \quad s_0 = s_{\varepsilon_1} Y^{-\varepsilon_1^\vee}, \quad s_0^{\vee\vee} = q^{-\frac{1}{2}} s_0^\vee s_{\varepsilon_1} s_0$$

<u>Theorem</u> $\tilde{W}$ is presented by $q^{\frac{1}{2}}, s_0, s_0^\vee, s_1, \ldots, s_n$ and relations

$$q^{\frac{1}{2}} \in Z(\tilde{W})$$

$$s_0^\vee s_1 s_0 s_1 = s_1 s_0 s_1 s_0^\vee$$

and

$$(s_0^\vee)^2 = (s_0')^2 = s_0^2 = s_1^2 = \cdots = s_n^2 = 1.$$

The double affine braid group $\tilde{\mathcal{B}}$.

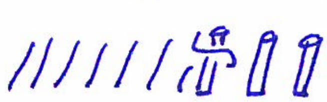
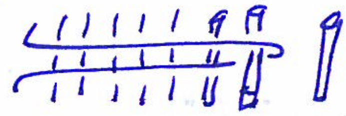
$\tilde{\mathcal{B}}$ is generated by $q^{\frac{1}{2}}, T_0, T_0^\vee, T_1, \dots, T_n$

with relations

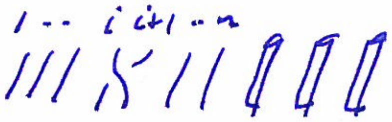
$q^{\frac{1}{2}} \in Z(\tilde{\mathcal{B}})$

$T_0^\vee T_1^{-1} T_0 T_1 = T_1^{-1} T_0 T_1 T_0^\vee$.

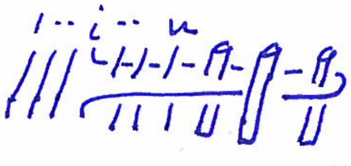
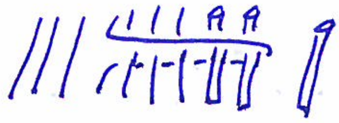
Theorem $\tilde{\mathcal{B}}$ is the 3-pole braid group on n-strands, generated by

$T_n =$  $T_0 =$ 

$T_0^\vee =$  $q^{\frac{1}{2}} \in Z(\tilde{\mathcal{B}})$

$T_i =$  for $i = 1, \dots, n-1$.

Let

$x_i =$  $y_i =$ 

Then

$X = \langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \rangle$ and $Y = \langle y_1^{\pm 1}, \dots, y_n^{\pm 1} \rangle$

are abelian subgroups of $\tilde{\mathcal{B}}$.

The involution $\iota : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ given by

$\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$, $\iota(T_0) = (T_0^\vee)^{-1}$, $\iota(T_i) = T_i^{-1}$ for $i = 1, \dots, n$

has $\iota(X^{\varepsilon_i}) = Y^{\varepsilon_i^\vee}$ and is called duality.

<u>The double affine Hecke algebra</u> (DAHA)

Let $t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_n^{\frac{1}{2}}$ be constants.

$\tilde{H}$ is the algebra generated by

$q^{\frac{1}{2}}, T_0, T_0^\vee, T_1, \ldots, T_n$ and relations (*) and

$$T_n^2 = (t_n^{\frac{1}{2}} - t_n^{-\frac{1}{2}})T_n + 1, \quad T_0^2 = (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})T_0 + 1$$

$$(T_0^\vee)^2 = (u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}})T_0^\vee + 1, \quad (T_0')^2 = (u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}})T_0' + 1$$

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \quad \text{for } i = 1, \ldots, n-1$$

where $T_0' = q^{-\frac{1}{2}}(T_0^\vee)^{-1}T_1^{-1}\cdots T_n^{-1}\cdots T_1^{-1}T_0^{-1}$.

<u>Theorem</u> (Cherednik-Sahi) $\tilde{H}$ has basis

$$\{q^{k/2}X^\mu T_w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathbb{Z}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathbb{Z}_{\mathbb{Z}}\}$$

where $T_w = T_{i_1}\cdots T_{i_\ell}$ if $w = s_{i_1}\cdots s_{i_\ell}$ is a minimal length expression of w in $s_1, \ldots, s_n$

Recall

$$X^\mu = X_1^{\mu_1}\cdots X_n^{\mu_n} \quad \text{and} \quad Y^{\lambda^\vee} = Y_1^{\lambda_1}\cdots Y_n^{\lambda_n}.$$

Orthogonal polynomials P_λ.

Let $\mathbb{1}$ be such that

$$T_n \mathbb{1} = t_n^{\frac{1}{2}} \mathbb{1}, \quad T_i \mathbb{1} = t^{\frac{1}{2}} \mathbb{1}, \quad Y^{\varepsilon_j^\vee} \mathbb{1} = t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-j} \mathbb{1}$$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. Then

$$\tilde{H}\mathbb{1} \text{ has basis } \{ q^{k/2} X^\mu \mathbb{1} \mid k \in \mathbb{Z}, \mu \in \mathbb{Z}^n \}$$

so that $y_1, \ldots, y_n$ act on

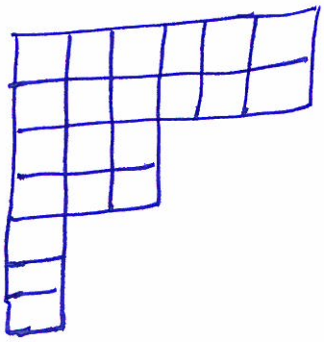
$\mathbb{C}[q^{\frac{1}{2}}][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\mathbb{1}$ as "q-difference operators".

The Koornwinder polynomials P_λ, (or Askey-Wilson polynomials when $n = 1$) are the simultaneous eigenvectors of

$$y_1^2 + \cdots + y_n^2 \quad \text{on} \quad \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_0},$$

the space of polynomials invariant under permutations of $x_1, \ldots, x_n$ and $x_i \mapsto x_i^{-1}$.

The P_λ are indexed by partitions

$\lambda =$  with $\leq n$ rows.

The future

Let

$$C(X) = \text{"some kind of" functions on } X.$$

We think

$$\tilde{H} = C(G) \quad \text{for some space } G$$

$$\tilde{H}\mathbb{1} = C(G/K) \quad \text{for some space } G/K$$

and

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = C(\Gamma \backslash G/K)$$

for some space $\Gamma \backslash G/K$.

I think

$$G \text{ is "a quotient of" } \mathrm{Map}\left(\begin{matrix}\text{elliptic}\\ \text{curve}\end{matrix}, \begin{matrix}\text{unitary}\\ \text{group}\end{matrix}\right).$$