Lecture in the Reading seminar 20.08.2009, Melbourne Univ. ①

Why I care about p-compact groups

(1) Symmetric functions (for $(S_n, \mathbb{Z}^n)$)

3 formulas for the Schur function.

(2) Symmetric functions (for $(W_0, \mathfrak{h}^*_{\mathbb{Z}})$)

3 formulas for the Weyl character.

(3) The Chevalley classification

The Borel-Weil-Bott formula.

(4) Flag varieties: $K(G/B)$ and $H^*(G/B)$

Pieri-Chevalley formulas.

(5) The classification of p-compact groups.

The Clark-Ewing formula.

Alternate title of this talk:

<u>My</u> <u>life</u> <u>in</u> <u>Mathematics</u>

Point of the talk:

The p-compact group corresp. to $(W_0, \mathfrak{h}^*_{\mathbb{Z}_p})$

is

the set of Littelmann paths corresp. to $(W_0, \mathfrak{h}^*_{\mathbb{Z}_p})$.

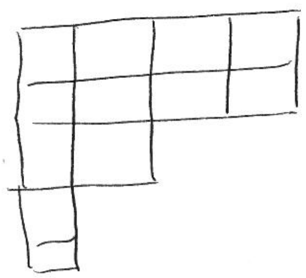
<u>Symmetric functions</u> <u>for $(S_n, \mathbb{Z}^n)$</u> (my life: circa 1988).

S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$ by permuting $x_1, \ldots, x_n$.

The <u>ring of symmetric functions</u> is

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \{ f \in \mathbb{C}[x_1, \ldots, x_n] \mid wf = f, \text{ for } w \in S_n \}.$$

A <u>partition</u> <u>with $\leq n$ rows</u> is a collection of boxes in a corner

$\lambda =$  $= (4, 4, 2, 1, 1)$

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

where $\lambda_i = \#$ of boxes in row i.

<u>Theorem</u> $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ has basis

$$\{ s_\lambda \mid \lambda \text{ is a partition with } \leq n \text{ rows} \}$$

where

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$$

3-formulas for the Schur function

$$s_\lambda = \frac{\sum_{w\in S_n} \det(w)\, w(x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n})}{\sum_{w\in S} \det(w)\, w(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1} x_n^{0})}$$

$$= \sum_{\substack{T \text{ colstrict} \\ \text{shape } \lambda}} x^{wt(T)}$$

$$= Tr\left(\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}, L(\lambda)\right)$$

where

A column strict tableau T of shape λ filled from $\{1, 2, \ldots, n\}$ is a filling T of the boxes of λ such that

(a) rows increase weakly (left to right),

(b) columns increase strictly (top to bottom)

and $x^{wt(T)} = x_1^{\#1\text{'s in }T} x_2^{\#2\text{'s in }T} \cdots x_n^{\#n\text{'s in }T}$.

and $L(\lambda)$ is the finite dimensional irreducible $Gl_n(\mathbb{C})$ module with a vector $v \in L(\lambda)$ such that

$$\begin{pmatrix} x_1 & & * \\ & \ddots & \\ 0 & & x_n \end{pmatrix} v = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} v.$$

Symmetric functions for $(W_0, \mathfrak{h}_\mathbb{Z}^*)$

A Weyl group is a $\mathbb{Z}$-reflection group.

A $\mathbb{Z}$-reflection group is a pair $(W_0, \mathfrak{h}_\mathbb{Z}^*)$ where

$\mathfrak{h}_\mathbb{Z}^*$ is a representation of W_0 (a $\mathbb{Z}W_0$-module)

W_0 is a finite group

such that

$$W_0 \subseteq GL(\mathfrak{h}_\mathbb{Z}^*) = GL_n(\mathbb{Z}) \subseteq GL_n(\bar{\mathbb{Q}})$$

is generated by reflections.

(a reflection is a matrix with all but one eigenvalue equal to 1)

The group algebra of $\mathfrak{h}_\mathbb{Z}^*$ is

$$\mathbb{C}[X] = \text{span}\{X^\lambda \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*\} \text{ with } X^\lambda X^\mu = X^{\lambda+\mu}$$

has a W_0-action given by

$$wX^\lambda = X^{w\lambda}, \quad \text{for } w \in W_0, \ \lambda \in \mathfrak{h}_\mathbb{Z}^*.$$

W_0 acts on $\mathbb{R}^n = \mathbb{R} \otimes \mathfrak{h}_\mathbb{Z}^* = \mathbb{R}\text{-span}\{\omega_1, \ldots, \omega_n\}$

Let C be a fundamental region for W_0 acting on $\mathbb{R}^n$,

$$(\mathfrak{h}_\mathbb{Z}^*)^+ = \mathfrak{h}_\mathbb{Z}^* \cap \bar{C} \quad \text{and} \quad (\mathfrak{h}_\mathbb{Z}^*)^{++} = \mathfrak{h}_\mathbb{Z}^* \cap C$$

Three formulas for the Weyl character

The ring of symmetric functions is

$$\mathbb{C}[X]^{W_0} = \{ f \in \mathbb{C}[X] \mid wf = f \text{ for } w \in W_0 \}.$$

Let $\rho \in \mathfrak{h}_{\mathbb{Z}}^*$ be such that

$$\begin{array}{ccc} (\mathfrak{h}_{\mathbb{Z}}^*)^+ & \longrightarrow & (\mathfrak{h}_{\mathbb{Z}}^*)^{++} \\ \lambda & \longmapsto & \lambda+\rho \end{array} \quad \text{is a bijection.}$$

Then

$\mathbb{C}[X]^{W_0}$ has basis $\{ s_\lambda \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+ \}$

where

$$s_\lambda = \frac{\sum'_{w \in W_0} \det(w)\, w X^{\lambda+\rho}}{\sum'_{w \in W_0} \det(w)\, w X^{\rho}}$$

$$= \sum'_{p \in B(\vec{\lambda})} X^{wt(p)}$$

$$= Tr(\cdot, L(\lambda))$$

where G is a compact Lie
reductive complex algebraic group

corresponding to $(W_0, \mathfrak{h}_{\mathbb{Z}}^*)$.

$L(\lambda)$ is an irreducible G-module and $\mathrm{Tr}(\cdot, L(\lambda)) : T \longrightarrow \mathbb{C}$

$$t \longmapsto \mathrm{Tr}(t, L(\lambda))$$

where T is a maximal torus of G.

<u>Chevalley's Classification</u>

There is an equivalence of categories

$$\left\{\begin{array}{c}\text{complex reductive} \\ \text{algebraic} \\ \text{groups } G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\mathbb{Z}\text{-reflection} \\ \text{groups } (W_0, \mathfrak{h}_{\mathbb{Z}}^*)\end{array}\right\}$$

$$\begin{array}{ccc} G & \longleftarrow\!\!\shortmid & (W_0, \mathfrak{h}_{\mathbb{Z}}^*) \\ \cup\!\mathsf{I} & & \\ B & \rightsquigarrow & \text{choice of } C \\ \cup\!\mathsf{I} & & \\ T & & \end{array}$$

Borel subgroup. choice of B

maximal torus in G — T

The <u>flag variety</u> is G/B.

<u>Weyl's theorem</u> The category of sko representations of G is a categorification of $\mathbb{C}[X]^{W_0}$ for which s_λ corresponds to $L(\lambda)$.

<u>The Borel-Weil-Bott formula</u>

$$H^0(G/B, \mathcal{L}_\lambda) \cong L(\lambda),$$

where $\mathcal{L}_\lambda = G \times_B \mathbb{C}v$, where $\mathbb{C}v$ is the 1-dimensional B-module given by $bv = X^\lambda(b)v$, for $b \in B$.

The flag variety

$$G = \bigsqcup_{w \in W_0} BwB$$

The Schubert varieties are

$$X_w = \overline{BwB} \quad \text{in } G/B, \quad \text{for } w \in W_0.$$

Then

$\{[X_w] \mid w \in W_0\}$ is a basis of $H^*(G/B)$

$\{[\mathcal{O}_{X_w}] \mid w \in W_0\}$ is a basis of $K(G/B)$.

The map

$$\begin{aligned} \mathbb{C}[X] &\longrightarrow K(G/B) \\ X^\lambda &\longmapsto [G \times_B \mathbb{C}v_\lambda] \end{aligned}$$

is surjective, with kernel $\mathbb{C}[X]^{W_0}$

The map
$$\begin{aligned} S(\mathfrak{h}^*) &\longrightarrow H^*(G/B) \\ \lambda &\longmapsto c_1(\mathcal{L}_\lambda) \end{aligned}$$

is surjective, with kernel $S(\mathfrak{h}^*)^{W_0}$.

So

$$K(G/B) \simeq \frac{\mathbb{C}[X]}{\mathbb{C}[X]^{W_0}} \quad \text{and} \quad H^*(G/B) \simeq \frac{S(\mathfrak{h}^*)}{S(\mathfrak{h}^*)^{W_0}}.$$

Chevalley, about 1954, gave a formula: In $H^*(G/B)$

$$c_1(\mathcal{L}_\lambda)\cdot[X_w] = \sum_{v\in W_0} c_{\lambda,w}^{v}\,[X_v],$$

where $c_{\lambda,w}^{v} = \ldots$

H. Pittie and I showed, in $K(G/B)$

$$[\mathcal{L}_\lambda]\cdot[\mathcal{O}_{X_w}] = \sum_{v\in W_0} d_{\lambda,w}^{v}\,[\mathcal{O}_{X_v}],$$

where $d_{\lambda,w}^{v}$ = # of paths $p\in\mathcal{P}(\vec{\lambda})$ with initial direction $\le w$ and final direction v.

So it seems possible to understand everything about

$H^*(G/B)$ and $K(G/B)$

purely from the knowledge of $(W_0, \mathfrak{h}_{\mathbb{Z}}^*)$.