Universal Verma Modules and Translation, Geometry and Combinatorics in Representation Theory, Seoul, Korea ①

$U = U_q\mathfrak{gl}_N$ and Universal Verma modules

22 Sept. 2009

$$U = U^{<0} U^0 U^{>0} \quad \text{or} \quad U = U^{<0} \mathbb{K} U^{>0}$$

where

$U^{<0}$ is generated by $F_1, \ldots, F_{N-1}$,

$U^{>0}$ is generated by $E_1, \ldots, E_{N-1}$,

$U^0 = \mathbb{Q}(q)[L_1^{\pm 1}, \ldots, L_N^{\pm 1}]$ or $\mathbb{K} = \mathbb{Q}(q)(L_1, \ldots, L_N)$

with relations

$$E_i F_j - F_j E_i = \left(\frac{L_i L_{i+1}^{-1} - L_{i+1} L_i^{-1}}{q - q^{-1}}\right) \delta_{ij}, \quad \text{etc.}$$

The <u>universal Verma module</u> $M = U^{<0} \mathbb{K} m^+$ is the U-module generated by m^+ with

$$E_i m^+ = 0, \quad \text{for } i = 1, \ldots, N-1.$$

<u>Remark</u> For $\mu = (\mu_1, \ldots, \mu_N)$

$$\begin{array}{ccc} \sigma_\mu : \mathbb{K} \longrightarrow \mathbb{K} & \text{and} & ev_\mu : \mathbb{K} \longrightarrow \mathbb{Q}(q) \\ L_i \longmapsto q^{\mu_i} L_i & & L_i \longmapsto q^{\mu_i} \end{array}$$

are automorphisms and 'characters', respectively.

$$M(\mu) = ev_\mu(M) = U^{<0} m^+$$

is the Verma module of highest weight μ.

Translation by V

Let V be the $U = U_q\mathfrak{gl}_N$-module with basis

$b_1, \ldots, b_N$ and

$$L_i b_j = q^{\delta_{ij}} b_j, \quad E_i b_j = \delta_{j\,i+1} b_i, \quad F_i b_j = \delta_{ji} b_{i+1}.$$

Find $U^{>0}$-invariants in $M \otimes V$:

$$n_1 = m^+ \otimes b_1,$$

$$n_2 = m^+ \otimes b_2 + u_{21} n_1, \qquad \text{with } E_i n_k = 0,$$

$$n_3 = m^+ \otimes b_3 + u_{32} n_2 + u_{31} n_3,$$

$$\vdots$$

for $i = 1, \ldots, N-1$, and $k = 1, \ldots, N$.

<u>Theorem</u> Let $[x] = \dfrac{x - x^{-1}}{q - q^{-1}}$, $\quad L_{ij} = L_i L_j^{-1}$,

$\varepsilon_j = (0, \ldots, 0, \overset{j^{\text{th}}}{1}, 0, \ldots 0)$ and $\rho = (N-1, N-2, \ldots, 2, 1, 0)$,

$F_{i\,i+1} = F_i$ and $F_{ij} = q F_{i+1\,j} F_{i,i+1} - F_{i,i+1} F_{i+1,j}$.

Then

$$u_{kj} = \sum_{A \subseteq \{j+1, \ldots, k-1\}} F_{ja_1} F_{a_1 a_2} \cdots F_{a_\ell k}\, \sigma_\rho \left(\frac{[L_{ja_1}] \cdots [L_{ja_\ell}] \cdot L_j \cdots L_k}{\sigma_{\varepsilon_{j+1}}([L_{j\,j+1}]) \cdots \sigma_{\varepsilon_k}([L_{jk}])\, L_{ja_1} \cdots L_{ja_\ell}} \right) q^{-2\ell - m - 1}$$

Proof: Use $E_i n_k = 0$ to write relation for u_{kj}.
Steal the formula from Brundan "Modular branching rules and the Mullineaux map" (generalization of Kleschev's thesis?) //.

Contravariant forms

A symmetric bilinear form $\langle , \rangle$ on a U-module P is contravariant if

$$\langle u p_1, p_2 \rangle = \langle p_1, \tau(u) p_2 \rangle \quad \text{for } u \in U,\ p_1, p_2 \in P,$$

where $\tau : U \to U$ is the algebra antiautomorphism coalgebra automorphism given by

$$\tau(E_i) = -F_i L_i L_{i+1}^{-1}, \quad \tau(F_i) = -L_i^{-1} L_{i+1} E_i, \quad \tau(L_i) = L_i.$$

Let $\langle \, \rangle_M : M \otimes M \to \mathbb{K}$ and $\langle , \rangle_V : V \otimes V \to \mathbb{Q}(q)$ be contravariant forms with $\langle m^+, m^+ \rangle_M = 1$ and $\langle v^+, v^+ \rangle_V = 1$ and define $\langle \, \rangle : (M \otimes V) \times (M \otimes V) \to \mathbb{K}$ by

$$\langle m_1 \otimes v_1, m_2 \otimes v_2 \rangle = \langle m_1, m_2 \rangle_M \langle v_1, v_2 \rangle_V.$$

Shadows of highest weight vectors

Theorem

(a) $$\langle n_k, n_k\rangle = \sigma_p\left(\frac{[L_{1k}][L_{2k}]\cdots[L_{k-1\,k}]}{\sigma_{\varepsilon_1}([L_{1k}])\,\sigma_{\varepsilon_2}([L_{2k}])\cdots\sigma_{\varepsilon_{k-1}}([L_{k-1\,k}])}\right)$$

(b) Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition,

$$\begin{aligned} ev_\mu : \mathbb{K} &\to \mathbb{Q}(q) \\ L_i &\mapsto q^{\mu_i} \end{aligned} \quad \text{and} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Then

$$ev_\mu(\langle n_k, n_k\rangle) = \prod_{1\le j<k} \frac{[(\mu_k - k) - (\mu_j - j) - 1]}{[(\mu_k - k) - (\mu_j - j)]}$$

(c) Let Φ_ℓ be the ℓth cyclotomic polynomial and $|\cdot|_{\Phi_\ell} : \mathbb{Q}(q) \to \mathbb{Z}$ the corresponding valuation.

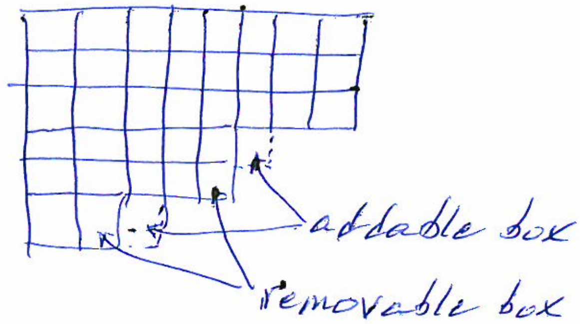
$$\left|\frac{\text{num}}{\text{denom}}\right| = \left(\#\text{ of } \Phi_\ell \text{ in num}\right) - \left(\#\ \Phi_\ell \text{ in denom}\right).$$

Let $c = \mu_k - k + 1$ = content of addable box in kth row.

Then

$$|ev_\mu(\langle n_k, n_k\rangle)| = \left(\begin{matrix}\#\text{ of removable boxes}\\ \text{of content } c\\ \text{above } k\text{th row}\end{matrix}\right) - \left(\begin{matrix}\#\text{ of addable boxes}\\ \text{of content } c \text{ above}\\ k\text{th row}\end{matrix}\right)$$

where contents are taken mod ℓ.



<u>Remark 1</u> Misra-Miwa Fock space is

$$\mathcal{F} = \mathbb{Z}[v,v^{-1}]\text{-span}\{ |\lambda\rangle \mid \lambda \text{ is a partition} \}$$

with operators

$$f_i |\mu\rangle = \sum_{\substack{\lambda = \mu + \varepsilon_k \\ \lambda/\mu = \square \text{ of content } c}} v^{|ev_\mu(\langle n_k, n_k\rangle)|_{\Phi_\ell}} |\lambda\rangle .$$

<u>Remark 1b</u> Ryan-Hansen J. Alg. 2005 sees the 'hook formula' on (b) from forms $\langle\ \rangle$ on Specht modules and makes connection to MM Fock space.

By ... [Arakawa-Suzuki] [Suzuki] [Orellana-R]

$(M \otimes V \otimes \cdots \otimes V)^+_\lambda$ = hw vectors of weight λ in $M \otimes V^{\otimes k}$

is an $\hat{H}_k$-module (affine Hecke algebra) and

$\langle\ \rangle_{M \otimes V^{\otimes k}}$ ~~becomes a form~~ transfers.

Remark 2 (a) is a special case of a formula in Jantzen's thesis:

Let $U = U_q\mathfrak{g}$, and $V = L(\nu)$ the fin. dim. irreducible of hw. ν. Let $b_1, \dots, b_d$ be a basis of $L(\nu)$ with

$$i < j \quad \text{if} \quad wt(b_i) > wt(b_j).$$

Find $U^{>0}$-invariants in $M \otimes V$:

$$n_1 = m^+ \otimes b_1$$

$$n_2 = m^+ \otimes b_2 + u_{21} n_1, \qquad \text{with} \quad E_i n_k = 0.$$

$$\vdots$$

Let $L(\nu)_\lambda = \mathrm{span}\{b_\ell, \dots, b_m\}$. Then

$$\frac{\det(\langle n_i, n_j\rangle)_{\ell \le i,j \le m}}{\det(\langle b_i, b_j\rangle)_{\ell \le i,j \le m}} = \prod_{\lambda \le \tau \le \nu} \sigma_\rho\left(\frac{[K_{\lambda-\tau}]}{\tau([K_{\tau-\lambda}])}\right)^{\dim(L(\nu)_\tau)}.$$

<u>Remark 2b</u> Gabber-Joseph "Toward the Kazhdan-Lustig conjective" explain that the operators

$$C_{s_i}|w\circ\lambda\rangle = v^{|\langle n_w, n_w\rangle|_{[K_\lambda]}}|w\circ\lambda\rangle + v^{|\langle n_{ws_i}, n_{ws_i}\rangle|_{[K_\lambda]}}|ws_i\circ\lambda\rangle$$

for $V = L(w\circ\lambda - ws_i\circ\lambda)$, $\Phi^\mu_\beta = \sigma_\mu\left(\frac{q^r K_\beta - q^{-r} K_\beta}{q-q^{-1}}\right)$

give an action of the Hecke algebra on the Grothendieck group of $\mathcal{O}$.