Notes for talk 19.11.2009

References

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(2) Duistermaat and Heckman, On the variation in the cohomology in the symplectic form of the reduced phase space, Invent. Math 69 (1982) 259-268

(3) Witten, Supersymmetry and Morse Theory, J. Diff. Geometry, 17 (1982) 661-692.

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Equivariant cohomology

Let T be a group and let

ET be a contractible space on which T acts freely.

The Borel construction is the functor?

$$\{T\text{-spaces}\} \longrightarrow \left\{\begin{matrix}\text{fibre bundles}\\ \text{on } BT\end{matrix}\right\}$$
$$M \longmapsto ET \times_T M$$

where $BT = ET \times_T pt$ and $ET \times_T M = \dfrac{ET \times M}{(xt, m) = (x, tm)}$.

Remark: If T is a compact Lie group acting smoothly on M then $ET \times_T M$ is homotopy equivalent to M/T.

The equivariant cohomology is

$$H_T^*: \{T\text{-spaces}\} \longrightarrow \{H_T^*(pt)\text{-modules}\}$$
$$M \longmapsto H_T^*(M) = H^*(ET \times_T M)$$

Remarks: (a) If $T = (S^1)^n$ then $H_T^*(pt) = \mathbb{C}[x_1, \ldots, x_n]$

(b) If G is a compact Lie group then

$$H_G^*(pt) = \mathbb{C}[f_1, \ldots, f_n] \simeq H_T^*(pt)^W$$

and $H_G^*(M) \simeq H_T^*(M)^W$,

where T is a maximal torus of G and W is the Weyl group of G

<u>K-theory</u>

Let M be a T-space. If M is nice then

$K_T(M)$ = Grothendieck group of T-equivariant vector bundles on M

= Grothendieck group of T-equivariant coherent sheaves on M.

The point: If you allow yourself denominators the <u>Chern</u> <u>character</u> gives an isomorphism

$$ch : K_T(M) \longrightarrow H_T^*(M)^\wedge$$

where $H_T^*(M)^\wedge$ is a completion of $H_T^*(M)$

<u>Examples:</u> (1) $K_T(pt)$ = Grothendieck group of T-modules.

$K(pt) = K_{\{1\}}(pt)$ = Grothendieck group of vector spaces

(2) If $T = (\mathbb{C}^*)^n$ then $K_T(pt) = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

We have isomorphisms: If T is a subgroup of G then

$$H_T^*(M) \simeq H_G^*(G \times_T M) \text{ and } K_T(M) \simeq K_G(G \times_T M).$$

Pushforwards

In K-theory: (a) If $f : M \to N$ is a proper ~~map~~ T-equivariant morphism then there is a morphism

$$f_* : K_T(M) \longrightarrow K_T(N)$$

(b) If $f : M \to N$ is a T-equiv. morphism then there is a pullback morphism

$$f^* : K_T(N) \longrightarrow K_T(M).$$

The Umkehrungs homomorphism: If $f : M \to N$ is a proper T-equivariant map there is a pushforward

$$f_* : H_T^*(M) \longrightarrow H_T^{*+(\dim N - \dim M)}(N)$$

which satisfies:

$$(f \circ g)_* = f_* \circ g_*, \qquad f_*(v\, f^*(u)) = (f_* v)\, u$$

and

If $f : M \to N$ is a fibration then

f_* corresponds to integration over the fibre.

~~Localization~~ Riemann-Roch

Let $f: N \to M$ be a morphism. Then

$$ch(f^*(E)) = f^*(ch(E)) \quad \text{and}$$

$$f_*(\mathrm{Todd}_N \cdot ch(E)) = \mathrm{Todd}_M \cdot ch(f_*(E)).$$

ie.

$$\begin{array}{ccc} K_T(M) & \xrightarrow{f^*} & K(N) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H_T^*(M) & \xrightarrow[f^*]{} & H_T^*(N) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_T(N) & \xrightarrow{f_*} & K_T(M) \\ {\scriptstyle \mathrm{Todd}_N ch}\downarrow & & \downarrow{\scriptstyle \mathrm{Todd}_M \cdot ch} \\ H_T^*(N) & \xrightarrow[f_*]{} & H_T^*(M) \end{array}$$

Thom isomorphism

If $f: N \hookrightarrow M$ is an inclusion of manifolds and ν_N is the normal bundle to N in M then

$$f_*: H_T^{*-(\dim M - \dim N)}(N) \xrightarrow[\sim]{\text{Thom isomorphism}} H^*_{T,c}(\nu_N \cancel{M, M-N}) \xrightarrow{j^*} H^*_T(M)$$

$$c \longleftarrow c\cdot\Phi_N$$

The Thom class of ν_N is $1\cdot\Phi_N$

and the Euler class of ν_N is $f_* f^* 1$

the most important pushforward is

$\pi_*: H^*_T(M) \longrightarrow H^*_T(pt)$ coming from $M \xrightarrow{\pi} pt$.

This is the equivariant Euler characteristic of M.

It corresponds to integrating over the fiber in the fibration

$$\begin{array}{c} ET\times_T M \\ \downarrow \\ BT = ET\times_T pt. \end{array}$$

Localisation

Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a vector bundle with a T-action.

i.e.
T acts on E, and on each fiber by a linear action.

* See [CG] §5.11 and (5.11.9) and Cor. 6.1.17.

Integration formula: If $\pi: M \to pt$ then

$\pi_*: H_T^*(M) \to H_T^*(pt)$ is given by

$$\pi_* \varphi = \sum_P \pi_*^P \left(\frac{i_P^* \varphi}{E(\nu_P)} \right)$$

where the sum is over the connected components P of M^T, $i_P: P \hookrightarrow M$, and

$E(\nu_P) = \prod_{\lambda_j \in \nu_P} \lambda_j$, ~~where λ_j are the irreduci~~ if $\nu_P = \bigoplus_j X^{\lambda_j}$ as a T-module.

Here $\nu_{M^T} = \bigoplus_P \nu_P$ is the normal bundle to $i: M^T \to M$

In the notation of [CG] line after (5.11.4)

$T_{M^A} M = \bigoplus_j N_{\lambda_j}$ is the weight decomposition of the normal bundle, $\lambda_a = \bigoplus_{\lambda_j} \left(\sum_i (-\lambda_j(t))^i \Lambda^i N_{\lambda_j}^\vee \right)$

Counting points in polytopes

A *toric variety* is a normal variety with a Taction with a dense orbit.

There is a bijection

$$\left\{\begin{matrix}\text{integer}\\ \text{polytopes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{pairs } (X,\mathcal{L}) \text{ where}\\ X \text{ is a toric variety}\\ \mathcal{L} \text{ is an ample line bundle on } X\end{matrix}\right\}$$

An *integer polytope* is the convex hull of a finite number of points of $\mathfrak{t}^*_{\mathbb{Z}}$ in $\mathbb{R}^n$.

Theorem (Erhart) There is a polynomial

$$i_P(t) = a_0(P) + a_1(P)t + \cdots + a_n(P)t^n$$

such that

$$i_P(k) = \mathrm{Card}(\mathfrak{t}^*_{\mathbb{Z}} \cap kP) \text{ for } k \in \mathbb{Z}_{\geq 0}.$$

~~And~~ Theorem

$$i_P(k) = \chi(X, \mathcal{L}^{\otimes k})$$

Moment maps

T-action on M preserve a symplectic form ω on M.

$\omega \in \Omega^2(M)$ is closed and

$\frac{\omega^n}{n!}$ is nowhere 0 on M.

Define

$$\mu : \mathfrak{t} \longrightarrow \{\omega\text{-invariant vector fields on } M\}.$$

The moment map is

$$\Phi : M \longrightarrow \mathfrak{t}^*.$$

Then the support

(a) $\Phi_*\left(\frac{\omega^n}{n!}\right)$ is a convex polytope

(b) the measure $\Phi_*\left(\frac{\omega^n}{n!}\right)$ is a piecewise polynomial measure on $\mathbb{R}^l$.

Example (1) G a compact Lie group. Then

$$H_G^*(pt) = \mathbb{C}[f_1, \ldots, f_n] \hookrightarrow H_T^*(M)^W$$

where T is a maximal torus of G and W is the Weyl gp of G.

Umkehrungs homomorphism If $f: N \to M$ is a map of compact oriented manifolds then the "push forward"

$$f_*: H^*(N) \longrightarrow H^{*+(\dim M - \dim N)}(M)$$

satisfies

(a) $(f \circ g)_* = f_* \circ g_*$

(b) $f_*(v f^*(u)) = (f_* v) u$

(c) If $f: N \to M$ is a fibration then f_* corresponds to integration over the fibre.

If $f: N \hookrightarrow M$ is an inclusion of manifolds the Thom isomorphism is

$$H^*(M, M-N) \xrightarrow{\Phi_N} H_c^*(\nu_N)$$

where ν_N is the normal bundle to N in M,

The Thom class is $\Phi_N 1$.

The Euler class is $f^* f_* 1$.

Remark If $\pi: M \to pt$ then $\pi_*: H_T^*(M) \to H_T(pt)$ corresps. to integrating over the fiber in $ET \times_T M$

This is really Euler [illegible]

Localization $H_T^*(M^T) \hookrightarrow H_T^*(M)$

If $T = (\mathbb{C}^*)^n$ then $H_T^*(pt) = \mathbb{C}[x_1, \ldots, x_n]$.

and $H_T^*(M)$ is a $\mathbb{C}[x_1, \ldots, x_n]$ module.

Localization is a functor

$$\{\mathbb{C}[x_1, \ldots, x_n]\text{-modules}\} \longrightarrow \{\text{sheaves on } \mathbb{C}^\ell\}$$
$$M \longmapsto \mathcal{M}$$

where the stalk at f of $\mathcal{M}$ is $\mathcal{M}_f = M \otimes_{\mathbb{C}[x_1, \ldots x_n]} \mathbb{C}[x_1, \ldots x_n]_f$

The support of M is

$$\mathrm{supp}(M) = \bigcap_f V_f \quad \text{where } V_f = \{v \in \mathbb{C}^n \mid f(v_1, \ldots, v_n) = 0\}.$$

where the intersection is over $f \in \mathbb{C}[x_1, \ldots, x_n]$ s.t. $f \cdot M = 0$.

Then $\mathrm{supp}(M) \subseteq \mathfrak{t}_{\mathbb{C}}$ where $\mathfrak{t}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(T)$.