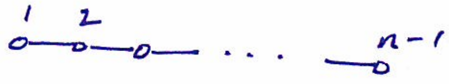
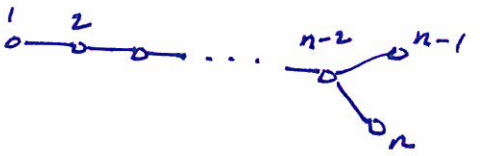
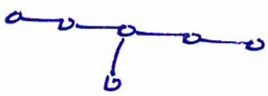
The Glass Bead Game: A short talk for the vacation scholars at Univ. of Melbourne 18.12.2009 ①

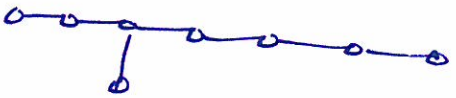
The simply laced Dynkin diagrams are

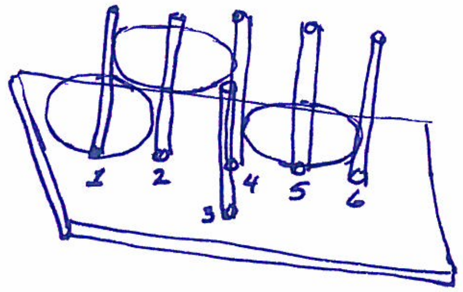
A_{n-1} 

D_n 

E_7 

E_6 

E_8 



The glass bead game.

A sequence $(i_1, i_2, \ldots, i_\ell)$ represents a placement of ℓ beads on runners $i_1, i_2, \ldots, i_\ell$.

Fix an end configuration of beads λ, the shape

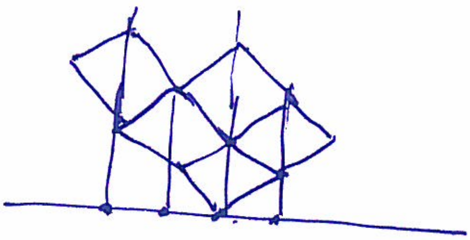
A standard tableau of shape λ is

$T = (i_1, i_2, \ldots, i_\ell)$

such that the resulting bead configuration is λ.

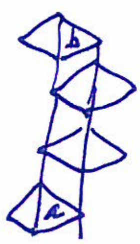
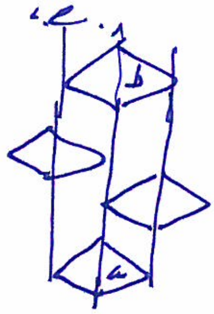
<u>Example</u>

The shape

$\lambda =$  has 5 standard tableaux

5 4	3 5	5 4	4 5	4 5
2 3	2 4	3 2	3 2	2 3
1	1	1	1	1

A <u>skew shape</u> is a shape such that any two beads on the same runner are separated by two beads, ie.

if  then  or 

Let λ be a skew shape.

Make a vector space

R^λ with basis $\left\{ v_T \;\middle|\; \begin{array}{l} T \text{ is a standard} \\ \text{tableau of shape } \lambda \end{array} \right\}$

Define operators on V:

$$e_S v_T = \begin{cases} v_T, & \text{if } S = T \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_j v_T = \begin{cases} v_{s_j T}, & \text{if } s_j T \text{ is a standard tableau of shape } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

where
$$s_j(i_1, i_2, \ldots, i_j, i_{j+1}, \ldots, i_\ell) = (i_1, i_2, \ldots, i_{j+1}, i_j, \ldots, i_\ell).$$

These operators are solutions to the

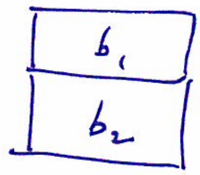
<u>Quiver</u> <u>Yang-Baxter</u> <u>equations</u>:

Orient the edges of D. Then

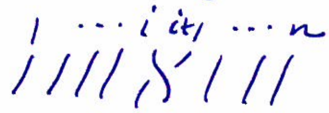
$$(\psi_{j+1}\psi_j\psi_{j+1} - \psi_j\psi_{j+1}\psi_j)e_T = \begin{cases} e_T, & \text{if } i_{j+2} = i_j \text{ and } i_j \to i_{j+1} \\ -e_T, & \text{if } i_{j+2} = i_j \text{ and } i_{j+1} \to i_j \\ 0, & \text{otherwise} \end{cases}$$

if $T = (i_1, i_2, \ldots, i_\ell)$

<u>Remark</u> The braid group has elements

 and product $b_1 b_2 =$ 

The braid group is generated by <u>simple</u> <u>twists</u>

$T_i =$  which satisfy

$$T_{j+1} T_j T_{j+1} = T_j T_{j+1} T_j$$

(the braid relation,
the Coxeter relation,
the Artin relation,
the Yang-Baxter equation)

Let $b_1, \ldots, b_n$ be letters. Let

$b_T = b_{i_1} \cdots b_{i_\ell}$ for a standard tableau $T = (i_1, \ldots i_\ell)$.

Then

$$\sum_T b_T,$$ where the sum is over all standard tableau of shape λ

is a <u>canonical</u> <u>basis</u> <u>element</u> in the

<u>quantum group</u>