The free Lie algebra $\mathcal{L}$

I an alphabet, $I^* = \{\text{words in } I\}$.

$\mathcal{P}$ = free algebra generated by p_i, $i \in I$,

$\mathcal{L}$ is generated by p_i, $i \in I$, under $[\,]: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$,

$$[u, v] = uv - q^{\langle \deg(u), \deg(v) \rangle} vu$$

$\mathcal{P}$ is the enveloping algebra of $\mathcal{L}$.

Choose a <u>total order</u> on I and

use <u>lexicographic order</u> on I^*.

$$L = \{\ell \in I^* \mid \ell \le \ell_2 \text{ if } \ell = \ell_1 \ell_2\}$$

is the set of <u>Lyndon words</u>. For $\ell \in L$ define

$$[\ell] = [[\ell_1], [\ell_2]], \quad \text{if } \ell = \ell_1 \ell_2 \text{ with } \ell_1 \in L \text{ of max. length} \ne \ell.$$

and $[\ell] = \ell$ if $\ell \in I$.

<u>Proposition</u> (a) If $u \in I^*$ then u has a unique factorization $u = \ell_1 \ell_2 \cdots \ell_k$ with $\ell_1, \dots, \ell_k \in L$, $\ell_1 \ge \cdots \ge \ell_k$.

(b) $\mathcal{L}$ has basis $\{[\ell] \mid \ell \in L\}$

$\mathcal{P}$ has basis $\{[\ell_1] \cdots [\ell_k] \mid \ell_1, \dots, \ell_k \in L, \ \ell_1 \ge \cdots \ge \ell_k\}$.

The quantum group U^-

$\mathcal{P}$ is graded by

$$Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\,\alpha_i \quad \text{with} \quad \deg(p_{i_1} \cdots p_{i_\ell}) = \alpha_{i_1} + \cdots + \alpha_{i_\ell}$$

Fix a symmetric bilinear form $\langle\ \rangle : Q^+ \times Q^+ \to \mathbb{Z}$

given by values $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$.

The dual of $\mathcal{P}$ is

$\mathcal{F}$ = free algebra generated by f_i, $i \in I$

with

$$\langle p_{\underline{i}}, f_{\underline{j}} \rangle = \delta_{\underline{i}\underline{j}}, \quad \text{if } p_{\underline{i}} = p_{i_1} \cdots p_{i_k} \text{ and } f_{\underline{j}} = f_{j_1} \cdots f_{j_\ell}.$$

The <u>$\langle\ \rangle$-shuffle product</u> on $\mathcal{F}$ is

$$u \circ v = \sum_{\sigma \in S_{k+\ell}/S_k \times S_\ell} q^{wt(\sigma, u, v)}\, \sigma(uv), \quad \text{where} \quad \begin{array}{l} k = \ell(u) \\ \ell = \ell(v) \end{array}$$

the sum is over shuffles of u with v,

$$wt(\sigma, u, v) = \sum_{\substack{1 \le i < j \le k+\ell \\ \sigma(i) > \sigma(j)}} -\langle u_i, v_{k-j} \rangle$$

where u_i is the i^{th} letter in u, and

v_{k-j} is the $(k-j)^{th}$ letter in v.

U^- is the $\circ$-subalgebra of $\mathcal{F}$ generated by f_i, $i \in I$.

Good Lyndon words

$$\begin{aligned} U^- &\hookrightarrow F \\ f_i &\mapsto f_i \end{aligned} \quad \text{and} \quad \begin{aligned} P &\twoheadrightarrow U^- \\ p_i &\mapsto f_i \end{aligned}$$

and the restriction of $\langle\ \rangle : P \times F \to \mathbb{C}$ to U^-

$\langle , \rangle : U^- \times U^- \to \mathbb{C}$ is nondegenerate

The good words and the good Lyndon words are

$$G = \{ g \in I^* \mid g \text{ is the maximal word in an element of } U^- \}$$

$$GL = G \cap L$$

Proposition (Lalonde-Ram, Leclerc)

(a) Assume that

$$(\langle \alpha_i^\vee, \alpha_j \rangle) \quad \text{with} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

is a symmetrizable Cartan matrix. Let R^+ be the positive roots of the corresponding Kac-Moody Lie algebra $\mathfrak{g}$.

$$\begin{aligned} GL &\longrightarrow R^+ \\ l &\longmapsto \deg(l) \end{aligned} \quad \text{is a bijection with}$$

inverse $l(\beta) \longleftarrow\!\!| \ \beta$ given by

$$l(\beta) = \max \left\{ l(\beta_1) l(\beta_2) \;\middle|\; \begin{array}{c} \beta_1, \beta_2 \in R^+,\ \beta_1 + \beta_2 = \beta \\ l(\beta_1) < l(\beta_2) \end{array} \right\}$$

(b) PBW basis and canonical basis of U^-

$$U^- \text{ has basis } \{E_g \mid g \in G\},$$

where

$$E_g = [l_1][l_2]\cdots[l_k] \quad \text{if} \quad g = l_1 \cdots l_k$$

with $l_1, \ldots, l_k \in L$ and $l_1 \geq \cdots \geq l_k$.

(c) The ordering on R^+ induced by lex order on GL determines a reduced decomposition of w_0,

$$w_0 = s_{i_1} \cdots s_{i_N}, \quad \text{so that} \quad R^+ = \{\beta_1 < \cdots < \beta_N\}$$

with

$$\beta_1 = \alpha_{i_1}, \ \beta_2 = s_{i_1}\alpha_{i_2}, \ \ldots, \ \beta_N = s_{i_N}\alpha_{i_N}.$$

Let $\bar{\ }: U^- \to U^-$ be the automorphism given by

$$\bar{f_i} = f_i \quad \text{and} \quad \bar{q} = q^{-1}.$$

The canonical basis of U^- is $\{b_g \mid g \in G\}$ given by

$$\bar{b_g} = b_g \quad \text{and} \quad b_g = E_g + \sum_{\substack{h \in G \\ h > g}} c_{gh} E_h,$$

with $c_{gh} \in q\mathbb{Z}[q]$.

Dual PBW and dual canonical bases

The <u>dual</u> <u>PBW</u> <u>basis</u> of U^- is $\{E_g^* \mid g \in G\}$ given by $\langle E_g, E_h^* \rangle = \delta_{gh}$.

The <u>dual</u> <u>canonical</u> <u>basis</u> of U^- is $\{b_g^* \mid g \in G\}$ given by $\langle b_g, b_h^* \rangle = \delta_{gh}$.

<u>Theorem</u> (a) (Lusztig) $E_h^* = (\text{const})\, E_h$.

(b) (Leclerc). $\{b_g^* \mid g \in G\}$ is characterised by

$$b_g^* = E_g^* + \sum_{\substack{h \in G \\ h \neq g}} d_{gh} E_h^*, \quad \text{with } d_{gh} \in q\mathbb{Z}[q],$$

and

$$b_g^* = \sum_{h \in \mathbb{I}^*} a_{gh} h \quad \text{with} \quad a_{gh} \in \mathbb{Z}[(q+q^{-1})].$$

U^- has bases

<u>PBW basis</u>	$\{E_g \mid g \in \vec{G}\}$	$\{E_g^* \mid g \in \vec{G}\}$	<u>dual PBW basis</u>
<u>canonical basis</u>	$\{b_g \mid g \in \vec{G}\}$	$\{b_g^* \mid g \in \vec{G}\}$	<u>dual canonical basis</u>