Data for quiver Hecke algebras

(1) F is the free algebra generated by f_i, $i \in I$.

$$Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i \quad \text{with } \deg(f_{i_1} \cdots f_{i_d}) = \alpha_{i_1} + \cdots + \alpha_{i_d}.$$

The symmetric group ~~act~~ $S_d = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ acts on

$$I^d = \{\text{words of length } d\} = \{u \in I^* \mid \ell(u) = d\}$$

(by rearrangements) with orbit decomposition

$$I^d = \bigsqcup_{\substack{\alpha \in Q^+ \\ ht(\alpha) = d}} I^\alpha \quad \text{where } I^\alpha = \{u \in I^* \mid \deg(u) = \alpha\}.$$

(2) Fix a symmetric bilinear form $\cancel{B}\langle\ ,\ \rangle : Q^+ \times Q^+ \to \mathbb{Z}$

given by values $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$

so that

$$A = (\langle \alpha_i^\vee, \alpha_j \rangle) \quad \text{with} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

is a Cartan matrix for a symmetrizable Kac-Moody Lie algebra

Γ is the graph with vertices I and

edges $\overset{i}{\circ}\!-\!\overset{j}{\circ}$ if $\langle \alpha_i, \alpha_j \rangle \neq 0$.

Fix an orientation $\varepsilon_{ij} = \begin{cases} +1, & \text{if } \overset{i}{\circ}\!\rightarrow\!\overset{j}{\circ} \\ -1, & \text{if } \overset{i}{\circ}\!\leftarrow\!\overset{j}{\circ} \end{cases}$ and set

$$Q_{ij}(u,v) = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \langle \alpha_i, \alpha_j \rangle = 0 \\ \varepsilon_{ij}(v^{-\langle \alpha_j^\vee, \alpha_i \rangle} - u^{-\langle \alpha_i^\vee, \alpha_j \rangle}) & \text{if } i \neq j \text{ and } \langle \alpha_i^\vee, \alpha_j \rangle \neq 0 \end{cases}$$

Quiver Hecke algebras R_α, $\alpha \in Q^+$

R_α is the assoc. $\mathbb{Z}$-graded algebra given by generators

$$e_u,\ x_1 e_u, \dots, x_d e_u,\ \tau_1 e_u, \dots, \tau_{d-1} e_u, \quad u \in I^\alpha$$

with degrees

$$\deg(e_u) = 0, \quad \deg(x_i e_u) = \langle u_i ; u_i \rangle, \quad \deg(\tau_i e_u) = -\langle u_i ; u_{i+1} \rangle$$

where u_i is the i^{th} letter in u,

and relations

$$e_u e_v = \delta_{uv}, \quad \sum_{u \in I^\alpha} e_u = 1, \quad x_i x_j = x_j x_i, \quad x_i e_u = e_u x_i$$

$$\tau_i e_u = e_{\sigma_i(u)} \tau_i, \qquad \tau_i \tau_j = \tau_j \tau_i, \text{ if } j \neq i, i \pm 1$$

$$\tau_i^2 e_u = Q_{u_i ; u_{i+1}}(x_i ; x_{i+1}) e_u,$$

$$(\tau_{i+1} \tau_i \tau_{i+1} - \tau_{i+1} \tau_i \tau_{i+1}) e_u$$

$$= \begin{cases} \dfrac{1}{x_{i+2} - x_i} \left(Q_{u_{i+2}, u_{i+1}}(x_{i+2}, x_{i+1}) - Q_{u_{i+1}, u_i}(x_{i+1}, x_i) \right), & \text{if } u_i = u_{i+2} \\ 0, & \text{if } u_i \neq u_{i+2} \end{cases}$$

$$\tau_i x_j e_u = \begin{cases} x_{\sigma_i(j)} \tau_i e_u - \varepsilon_{ij} e_u, & \text{if } u_i = u_{i+1}, \text{ and} \\ x_{\sigma_i(j)} \tau_i e_u, & \text{if } u_i \neq u_{i+1}. \end{cases}$$

<u>Note</u>: $x_i = \sum_{u \in I^\alpha} x_i e_u$, and $\tau_i = \sum_{u \in I^\alpha} \tau_i e_u$.

Structure of R_α

For each $\sigma \in S_d$ fix a reduced word

$\sigma = \sigma_{i_1} \cdots \sigma_{i_\ell}$ and set $\tau_\sigma = \tau_{i_1} \cdots \tau_{i_\ell}$.

Theorem (Khovanov-Lauda, Rouquier).

R_α has basis $\{ x_1^{n_1} \cdots x_d^{n_d} \tau_\sigma e_u \mid u \in I^\alpha, \sigma \in S_d, n_1, \ldots, n_d \in \mathbb{Z}_{\geq 0} \}$

If $\alpha \in Q^+$ and $\beta \in Q^+$ then

$$I^{\alpha+\beta} = \bigcup_{\sigma \in S_{k+\ell}/S_k \times S_\ell} \sigma(I^\alpha \cdot I^\beta)$$ and there is a

homomorphism

$$\begin{aligned}
R_\alpha \otimes R_\beta &\longrightarrow R_{\alpha+\beta} \\
e_u \otimes e_v &\longmapsto e_{uv} \\
x_i e_u \otimes e_v &\longmapsto x_i e_{uv} \\
e_u \otimes x_j e_v &\longmapsto x_{j+k} e_{uv} \\
\tau_i e_u \otimes e_v &\longmapsto \tau_i e_{uv} \\
e_u \otimes \tau_j e_v &\longmapsto \tau_{j+k} e_{uv}
\end{aligned}$$

where $k = ht(\alpha)$

Theorem (Khovanov-Lauda) $R_{\alpha+\beta}$ is a free (right) $R_\alpha \otimes R_\beta$-module with basis

$$\{ \tau_\sigma 1_{\alpha\beta} \mid \sigma \in S_{k+\ell}/S_k \times S_\ell \} \quad \text{with} \quad 1_{\alpha\beta} = \sum_{\substack{u \in I^\alpha \\ v \in I^\beta}} e_{uv}$$

For $M \in R_\alpha$-mod and $N \in R_\beta$-mod

$$M \circ N = \mathrm{Ind}_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (M \otimes N).$$

Graded characters

R_α-mod is the category of finite dimensional $\mathbb{Z}$-graded R_α-modules:

$$M = \bigoplus_{i\in\mathbb{Z}} M[i] \quad \text{with} \quad R_\alpha[j]\,M[i] \subseteq M[i+j],$$

where $R_\alpha[j] = \{\text{elements of degree } j \text{ in } R_\alpha\}$.

$$M = \bigoplus_{i\in\mathbb{Z}} \bigoplus_{u\in I^\alpha} M_u[i] \quad \text{with} \quad M_u[i] = e_u M[i].$$

The graded character of M is

$$\mathrm{gch}(M) = \sum_{i\in\mathbb{Z}} \sum_{u\in I^\alpha} \dim(M_u[i])\, q^i f_u$$

Then

$$\mathrm{gch}(M\circ N) = \mathrm{gch}(M)\circ \mathrm{gch}(N)$$

where the right hand side is $\langle\ \rangle$-shuffle product.

Theorem Let $K(R_\alpha\text{-mod})$ be the Grothendieck group of R_α-mod.

$$\begin{array}{ccc} \bigoplus_{\alpha\in Q^+} K(R_\alpha\text{-mod}) & \longrightarrow & U^- \\ M & \longmapsto & \mathrm{gch}(M) \end{array} \quad \text{is an algebra isomorphism.}$$

$$L(g) \longmapsto b_g^*$$

where $\{b_g^* \mid g\in G\}$ is the dual canonical basis of U^-

$\{L(g) \mid g\in G\}$ are the simple R_α-modules in R_α-mod.

$$\bigoplus_{\alpha \in Q^+} K(R_\alpha\text{-mod}) \longrightarrow \mathcal{U}^-$$

$$M \longmapsto \mathrm{gch}(M)$$

$$L(g) \longleftarrow b_g^*$$

$$\Delta(g) \longleftarrow E_g^*$$

where

$L(g)$, $g \in G$, are the simple R_α-modules,

if $l \in L$ then $\Delta(l) = L(l)$ and

$$\Delta(g) = \Delta(l_1) \circ \cdots \circ \Delta(l_k) \quad \text{if} \quad g = l_1 \cdots l_k$$

with $l_1, \dots, l_k \in G \cap L$ and $l_1 \geq \cdots \geq l_k$.

<u>Theorem</u> $\Delta(g)$ has unique simple quotient $L(g)$.

Projective R_α-modules

$\operatorname{Proj} R_\alpha$ is the category of finitely generated $\mathbb{Z}$-graded projective R_α-modules.

Let $P(i)$ be the unique indecomposable projective R_{α_i}-module:

$$P(i) = \operatorname{span}\{ x_1^n e_{\alpha_1} \mid n \in \mathbb{Z}_{\geq 0} \} \cong \mathbb{C}[x_1]$$

For $P \in \operatorname{Proj} R_\alpha$ and $Q \in \operatorname{Proj} R_\beta$ define

$$PQ = \operatorname{Ind}_{R_\alpha \otimes R_\beta}^{R_{\alpha+\beta}} (P \otimes Q).$$

Let $K(\operatorname{Proj} R_\alpha)$ be the Grothendieck group of $\operatorname{Proj} R_\alpha$.

<u>Theorem</u> (Khovanov-Lauda, Rouquier)

$$\begin{aligned} \mathcal{U}^- &\longrightarrow \bigoplus_{\alpha \in Q^+} K(\operatorname{Proj} R_\alpha) \\ p_i &\longmapsto P(i) \end{aligned}$$

is an algebra isomorphism.

Then

$$b_g \longmapsto P(g)$$

where $\{b_g \mid g \in G\}$ is the canonical basis of $\mathcal{U}^-$

and $\{P(g) \mid g \in G\}$ are the indecomposables in $\operatorname{Proj} R_\alpha$.