The Affine Weylgroup

$$Q = \sum_{i \in I} \mathbb{Z}\alpha_i \quad \text{and} \quad \mathfrak{h}^* = \sum_{i \in I} \mathbb{R}\alpha_i$$

with a symmetric bilinear form $\langle\ \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{R}$

given by values $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$

so that

$$A = (\langle \alpha_i^\vee, \alpha_j \rangle) \quad \text{with} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

is the Cartan matrix of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. Let $R^+ = \{\text{positive roots of } \mathfrak{g}\}$.

The Weyl group $W_0 = \langle s_i \mid i \in I \rangle \subseteq GL(\mathfrak{h}^*)$

with

$$\begin{aligned} s_i : \mathfrak{h}^* &\longrightarrow \mathfrak{h}^* \\ \lambda &\longmapsto \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i . \end{aligned}$$

The affine Weyl group $W = W_0 \ltimes Q = \left\{ w X^\mu \;\middle|\; \begin{array}{l} w \in W_0 \\ \mu \in Q \end{array} \right\}$

with

$$\begin{aligned} X^\mu : \mathfrak{h}^* &\longrightarrow \mathfrak{h}^* \\ \lambda &\longmapsto \lambda + \mu . \end{aligned}$$

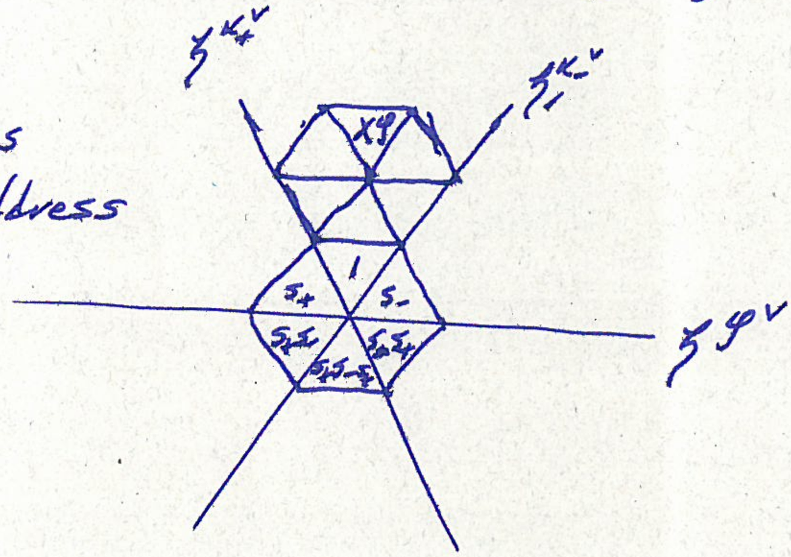
The alcoves are the connected components of

$$\mathfrak{h}^* \setminus \Big(\bigcup_{\substack{\alpha \in R^+ \\ j \in \mathbb{Z}}} \mathfrak{h}^{\alpha^\vee + j d} \Big) \quad \text{where}$$

$$\mathfrak{h}^{\alpha^\vee + jd} = \{ \lambda \in \mathfrak{h}^* \mid \langle \alpha^\vee, \lambda \rangle + j = 0 \}.$$

<u>Example</u> $\mathfrak{h}^* = \mathbb{R}\text{-span}\{\alpha_+, \alpha_-\}$, $\varphi = \alpha_+ + \alpha_-$

Each alcove has two types of address wX^{λ} and $s_{i_1} \cdots s_{i_\ell}$.



$R^+ = \{\alpha_+, \alpha_-, \alpha_+ + \alpha_-\}$ since $G \cap L = \{+, -, +-\}$ if $+ < -$.

W is generated by s_0 and s_i, $i \in I$

where $s_0 = X^\varphi s_\varphi$.

and

$$W \longleftrightarrow \{\text{alcoves}\}$$

$$W_0 \longleftrightarrow \left\{\begin{array}{c}\text{alcoves in the}\\ \text{0-hexagon}\end{array}\right\}$$

$$Q \longleftrightarrow \{\text{hexagons}\}.$$

Chevalley groups $G^\vee(\mathbb{F})$

$G^\vee(\mathbb{F})$ is generated by "elementary matrices"

$$x_{\alpha^\vee}(f) \text{ and } x_{-\alpha^\vee}(f), \qquad f \in \mathbb{F}, \quad \alpha \in R^+$$

with relations

see Steinberg (or Parkinson-Ram-Schwer)

where

$$x_{\alpha^\vee}(f)\, x_{-\alpha^\vee}(-f^{-1})\, x_{\alpha^\vee}(f) = h_\alpha(f)\, n_{\alpha^\vee}$$

and $h_\lambda(f)\, h_\mu(f) = h_{\lambda+\mu}(f)$, for $\lambda, \mu \in Q$.

The <u>loop group</u> is $G(\mathbb{C}((t)))$ where

$$\mathbb{C}((t)) = \{ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \mid a_i \in \mathbb{C},\ \ell \in \mathbb{Z} \}$$

Define

$$x_{\alpha^\vee + j\delta}(c) = x_{\alpha^\vee}(ct^j)$$

$$t_\lambda = h_\lambda(t^{-1}), \qquad \text{for } \lambda \in Q$$

$$n_{\alpha^\vee + j\delta} = x_{\alpha^\vee + j\delta}(1)\, x_{-\alpha^\vee - j\delta}(-1)\, x_{\alpha^\vee + j\delta}(1)$$

Let

$$x_0(c) = x_{-\varphi^\vee + \delta}(c), \qquad x_i(c) = x_{\alpha_i^\vee}(c),$$

$$n_0 = n_{-\varphi^\vee + \delta}(c), \qquad n_i = n_{\alpha_i^\vee}$$

Let

$$\mathbb{C}[[t]] = \{ a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C} \}.$$

<u>Example</u> $I = \{+,-\}$, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$G^\vee(\mathbb{F}) = SL_3(\mathbb{F})$ is generated by

$$x_+(f) = \begin{pmatrix} 1 & f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad x_-(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \quad x_{+-}(f) = \begin{pmatrix} 1 & 0 & f \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

$$x_{-\alpha_+^\vee}(f) = \begin{pmatrix} 1 & & \\ f & 1 & \\ 0 & 0 & 1 \end{pmatrix} \quad x_{-\alpha_-^\vee}(f) = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ f & 0 & 1 \end{pmatrix} \quad x_{-\alpha_+^\vee-\alpha_-^\vee}(f) = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ f & 0 & 1 \end{pmatrix}$$

$Q = \{m\alpha_+ + n\alpha_- \mid m,n \in \mathbb{Z}\}$ and

$$t_\lambda = h_{m\alpha_+ + n\alpha_-}(t^{-1}) = \begin{pmatrix} t^{-m} & 0 & 0 \\ 0 & t^{m-n} & 0 \\ 0 & 0 & t^{n} \end{pmatrix}, \text{ if } \lambda = m\alpha_+ + n\alpha_-$$

$$x_0(c) = x_{-\varphi^\vee+d}(c) = x_{-\varphi^\vee}(ct) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ct & 0 & 1 \end{pmatrix}$$

$$n_0 = \begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & 1 & 0 \\ t & 0 & 0 \end{pmatrix} \quad n_+ = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

MV intersections and MV cycles

$$\begin{array}{ccc} G^\vee = G^\vee(\mathbb{C}((t))) & & \\ \cup\!\!\mid & & \\ K = G^\vee(\mathbb{C}[[t]]) & \xrightarrow{t=0} & G^\vee(\mathbb{C}) \\ \cup\!\!\mid & & \cup\!\!\mid \\ I = \text{Iwahori subgroup} & \longrightarrow & B^\vee = \left\langle \begin{array}{c} x_{\alpha^\vee}(c) \\ h_\lambda(c) \end{array} \middle| \begin{array}{c} \alpha \in R^+ \\ c \in \mathbb{C} \\ \lambda \in Q \end{array} \right\rangle \end{array}$$

G/K is the loop Grassmannian

G/I is the affine flag variety

$$G^\vee = \bigsqcup_{w \in W} IwI \qquad G^\vee = \bigsqcup_{\lambda \in \mathfrak{h}^+_{\mathbb{Z}}} Kt_\lambda K$$

$$G^\vee = \bigsqcup_{v \in W} U^- vI \qquad G^\vee = \bigsqcup_{\mu \in \mathfrak{h}_{\mathbb{Z}}} U^- t_\mu K$$

where $U^- = \langle x_{-\alpha^\vee}(f) \mid \alpha \in R^+,\ f \in \mathbb{C}((t)) \rangle$

$$\mathfrak{h}_{\mathbb{Z}} = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \}$$

$$\mathfrak{h}^+_{\mathbb{Z}} = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \}$$

The MV intersections are

$$IwI \cap U^- vI \quad \text{and} \quad K^\vee t_\lambda K^\vee \cap U^- t_\mu K^\vee$$

and the MV cycles are the irreducible components of

$$\overline{K^\vee t_\lambda K^\vee \cap U^- t_\mu K^\vee} \quad \text{in } G^\vee/K^\vee$$

Points in $IwI \cap U^- vI$

Let $w \in W$ be an alcove and

$w = s_{i_1} \cdots s_{i_\ell}$ a minimal length walk to w

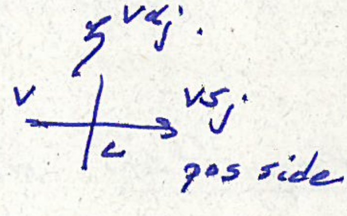
Theorem (Steinberg) The points of IwI are

$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} I$ with $c_1, \ldots, c_\ell \in \mathbb{C}$.

The ~~straightening~~ folding algorithm:

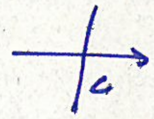
Case 1:

$x_{\gamma_1}(c_1') \cdots x_{\gamma_r}(c_r') n_v x_j(c) n_j^{-1} b$



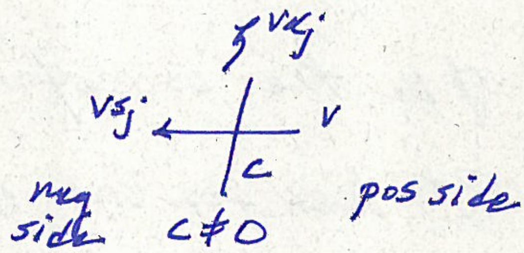
is replaced by

$x_{\gamma_1}(c_1') \cdots x_{\gamma_r}(c_r') x_{v\alpha_j}(c) n_{vs_j} b'$



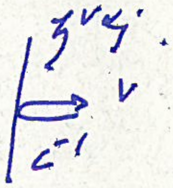
Case 2:

$x_{\gamma_1}(c_1') \cdots x_{\gamma_r}(c_r') n_v x_j(c) n_j^{-1} b$



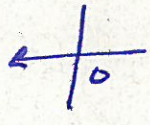
is replaced by

$x_{\gamma_1}(c_1') \cdots x_{\gamma_r}(c_r') x_{-v\alpha_j}(c^{-1}) n_v b'$



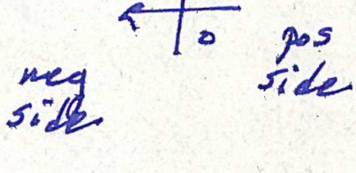
Case 3:

$x_{\gamma_1}(c_1') \cdots x_{\gamma_r}(c_r') n_v x_j(0) n_j^{-1} b$



is replaced by

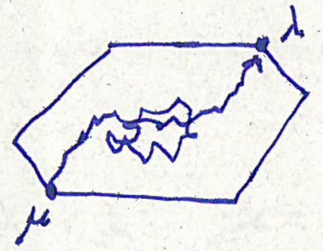
$x_{\gamma_1}(c_1') \cdots x_{\gamma_r}(c_r') x_{v\alpha_j}(0) n_{vs_j} b'$



The resulting path (without labels) is a <u>Littelmann path</u> and

$$IwI \cap U^- vI = \left\{ \begin{array}{l} \text{points of } IwI \text{ whose} \\ \text{folding ends in } v \end{array} \right\}$$

The <u>MV polytope</u> of $IwI \cap U^- vI$ is the support of the folded paths in $IwI \cap U^- vI$. (similarly for $Kt_\lambda K \cap U^- t_\mu K$

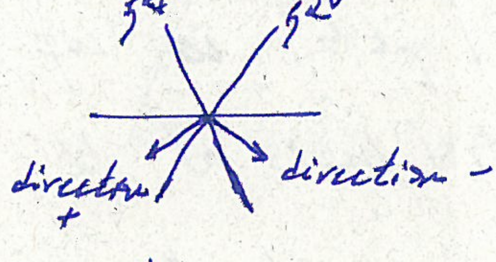


Dual canonical bases

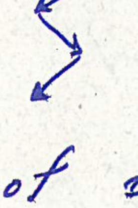
Let

$$b_g^* = \sum_{h \in I^*} a_{gh} h$$

and draw the word $h = i_1 \cdots i_\ell$ as a path in $\mathfrak{h}^*$.



So that $+-+ =$



Then the support of b_g^* is an MV polytope

$$\begin{array}{ccc} \{b_g^* \mid g \in G\} & \longrightarrow & \{\text{MV polytopes}\} \\ b_g^* & \longmapsto & \text{support of } b_g^* \end{array}$$

is a bijection.